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(continued after index)

Kendall Atkinson Weimin Han

Theoretical Numerical Analysis

A Functional Analysis Framework

With 25 Illustrations

Springer

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Mathematics Subject Classification (2000): 65-01, 73V05, 45L05, 4601

Library of Congress Cataloging-in-Publication Data
Atkinson, Kendall E.
Theoretical numerical analysis: a functional analysis framework / Kendall Atkinson, Weimin Han.
p. cm.—(Texts in applied mathematics; 39)
Includes bibliographical references and index.
1. Functional analysis. I. Han, Weimin. II. Title. III Series.
QA320.A85 2001
515—dc21 00–061920

Printed on acid-free paper.

Production managed by Michael Koy; manufacturing supervised by Joe Quatela.
Typeset by The Bartlett Press, Inc., Marietta, GA.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4684-9301-6 ISBN 978-0-387-21526-6 (eBook)
DOI 10.1007/978-0-387-21526-6

SPIN 10780644

Springer-Verlag New York Berlin Heidelberg
A member of BertelsmannSpringer Science+ Business Media GmbH

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DAISY AND CLYDE ATKINSON
HAZEL AND WRAY FLEMING

and

DAQING HAN, SUZHEN QIN
HUIDI TANG, ELIZABETH JING HAN

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research level monographs.

California Institute of Technology — J.E. Marsden
Brown University — L. Sirovich
University of Houston — M. Golubitsky

Preface

This textbook has grown out of a course that we teach periodically at the University of Iowa. We have beginning graduate students in mathematics who wish to work in numerical analysis from a theoretical perspective, and they need a background in those "tools of the trade" that we cover in this text. Ordinarily, such students would begin with a one-year course in *real and complex analysis*, followed by a one- or two-semester course in *functional analysis* and possibly a graduate level course in *ordinary differential equations, partial differential equations*, or *integral equations.* We still expect our students to take most of these standard courses, but we also want to move them more rapidly into a research program. The course based on this book is designed to facilitate this goal.

The textbook covers basic results of functional analysis and also some additional topics that are needed in theoretical numerical analysis. Applications of this functional analysis are given by considering, at length, numerical methods for solving partial differential equations and integral equations.

The material in the text is presented in a mixed manner. Some topics are treated with complete rigor, whereas others are simply presented without proof and perhaps illustrated (e.g., the principle of uniform boundedness). We have chosen to avoid introducing a formalized framework for *Lebesgue measure and integration* and also for *distribution theory.* Instead we use standard results on the completion of normed spaces and the unique extension of densely defined bounded linear operators. This permits us to introduce the Lebesgue spaces formally and without their concrete realization using measure theory. The weak derivative can be introduced similarly using the unique extension of densely defined linear operators, avoiding the need for a formal development of distribution theory. We describe some of the standard material on measure theory and distribution theory in an intuitive manner, believing this is sufficient for much of subsequent mathematical development. In addition, we give a number of deeper results without proof, citing the existing literature. Examples of this are the *open mapping theorem*, the *Hahn-Banach theorem*, the *principle of uniform boundedness*, and a number of the results on *Sobolev spaces.*

The choice of topics has been shaped by our research program and interests at the University of Iowa. These topics are important elsewhere, and we believe this text will be useful to students at other universities as well.

The book is divided into chapters, sections, and subsections where appropriate. Mathematical relations (equalities and inequalities) are numbered by chapter, section, and their order of occurrence. For example, (1.2.3) is the third-numbered mathematical relation in Section 1.2 of Chapter 1. Definitions, examples, theorems, lemmas, propositions, corollaries, and remarks are numbered consecutively within each section, by chapter and section. For example, in Section 1.1, Definition 1.1.1 is followed by Example 1.1.2.

The first three chapters cover basic results of functional analysis and approximation theory that are needed in theoretical numerical analysis. Early on, in Chapter 4, we introduce methods of nonlinear analysis, as students should begin early to think about both linear and nonlinear problems. Chapter 5 is a short introduction to finite difference methods for solving time-dependent problems. Chapter 6 is an introduction to Sobolev spaces, giving different perspectives of them. Chapters 7 through 10 cover material related to elliptic boundary value problems and variational inequalities. Chapter 11 is a general introduction to numerical methods for solving integral equations of the second kind, and Chapter 12 gives an introduction to boundary integral equations for planar regions with a smooth boundary curve.

We give exercises at the end of most sections. The exercises are numbered consecutively by chapter and section. At the end of each chapter, we provide some short discussions of the literature, including recommendations for additional reading.

During the preparation of the book, we received helpful suggestions from numerous colleagues and friends. We particularly thank P.G. Ciarlet, William A. Kirk, Wenbin Liu, and David Stewart. We also thank the anonymous referees whose suggestions led to an improvement of the book.

Kendall Atkinson
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Iowa City, IA

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