Springer Monographs in Mathematics

For other titles published in this series, go to
http://www.springer.com/series/3733

John Rhodes · Benjamin Steinberg

The q-theory of Finite Semigroups

Springer

John Rhodes
University of California
Department of Mathematics
1000 Centennial Dr.
Berkeley CA 94720-3840
USA
blvdbastille@aol.com

Benjamin Steinberg
Cereleton University
School of Mathematics and Statistics
1125 Colonel by Drive
Ottawa ON K1S 5B6
Canada
bsteinbg@math.carleton.ca

ISSN: 1439-7382
ISBN: 978-0-387-09780-0 e-ISBN: 978-0-387-09781-7
DOI 10.1007/978-0-387-09781-7

Library of Congress Control Number: 2008939929

Mathematics Subject Classification (2000): 20M07, 20M35, 20F20, 20F32, 20A15, 20A26, 16Y60, 06B35, 06E15, 06A15

Printed on acid-free paper

springer.com

In memory of Bret Tilson

Preface

When people are trying to learn mathematics for the purpose of research, they usually start at the forefront and then go backwards as needed in order to understand the results more fully. Yet with a mathematics book, it is not uncommon for people to start on page one and read onwards. This book is a research manuscript, and we heartily encourage the reader to delve in, read what is of interest, and go back as necessary. We hope that the material at the end of the book — the charts, tables and indices — will make this easier going.

Not many books have appeared in recent years dedicated to state-of-the-art Finite Semigroup Theory. There was the famous "Arbib" book in the 1960s [171] containing the lectures of Kenneth Krohn, John Rhodes and Bret Tilson. Samuel Eilenberg's treatise [85], with two chapters by Tilson [362,363], appeared more than 30 years ago. It revolutionized semigroup theory with the introduction of pseudovarieties of semigroups and varieties of languages. However, the most recent book on the subject is that of Jorge Almeida [7], which was originally published in 1992! Almeida's book made profinite methods in semigroup theory accessible. Howard Straubing's 1994 book [346] does touch on semigroup theory, but it is more concerned with applications to Computer Science than with semigroups themselves.

This volume is intended both to introduce a quantized version of Eilenberg's theory, by going to operators on pseudovarieties and relational morphisms, and to fill in some of the vacuum that has accrued from years that have passed with no new books on finite semigroups. Also, we have tried to include a number of classical results that profit from a recasting in a modern language to increase accessibility.

The philosophy of the book and an explanation of the individual chapters is covered in the Introduction, which we strongly recommend reading before entering into the body of the text. Here we try to provide a brief guide to the structure of the current volume.

Let us begin with what this book does not do. We do not intend in any way to give a basic introduction to Semigroup Theory. We have included for the reader's convenience an appendix on the basic structure theory, such as Green's relations and Rees's Theorem, but it is by no means complete. The classic treatise of A. H. Clifford and G. B. Preston [68] still serves admirably for learning this material, as does the "Arbib" book [171]. One can also turn to the books of J. M. Howie [139] and P. M. Higgins [133]. For the current text, it is handy to be familiar with a little bit of category theory [185], basic topology [158], and perhaps some combinatorial group theory [184].

This is a book about *finite* semigroups. Except for free semigroups, free groups and profinite semigroups, the reader should not expect to encounter infinite semigroups. Readers interested in the Prime Decomposition Theorem for infinite semigroups should consult the relevant papers on the topic [3, 50–52, 88, 125, 214, 215, 276, 277, 279]. Outside of a brief foray to prove Schützenberger's Theorem, we do not enter into Formal Language Theory and Automata Theory. Formal Language Theory is an important aspect of Finite Semigroup Theory, and much of the motivation for problems in the field derive from it. But there is already a sizable literature of excellent books devoted to this facet (for instance, [85, 177, 224, 229, 346]). Nonetheless, it would be an important task in the future to reinterpret the results of this book from the language theoretic point of view. Another important aspect of Finite Semigroup Theory that we touch upon only slightly is the finite basis problem. Mark Sapir, Marcel Jackson and Mikhail Volkov, among others, have done important work on this subject [77, 142–144, 146, 159, 246, 306–309, 372, 374].

The book is divided into four parts. The first part, entitled ***The $\mathfrak{q}$-operator and Pseudovarieties of Relational Morphisms***, is the heart of $\mathfrak{q}$-theory. Chapter 1 is foundational material, and much of it may be familiar to the reader. However, at the end of the chapter the important notions of division of relational morphisms and relational morphism between relational morphisms are introduced. Chapter 2 introduces the classes of relational morphisms that are of central importance in this volume: continuously closed classes and pseudovarieties of relational morphisms. We introduce the operator $\mathfrak{q}$ and its image is characterized. Maximal and minimal models, meaning maximal and minimum classes defining a given operator, are studied, and we give examples showing that all the commonly occurring operators in Finite Semigroup Theory fit into our framework. The Derived and Kernel Semigroupoids are presented and their basic properties explored in order to introduce the pseudovarieties of relational morphisms $\mathbf{V}_D$ and $\mathbf{V}_K$, whose images under $\mathfrak{q}$ are the operators $\mathbf{V} * (-)$ and $\mathbf{V} ** (-)$, thanks to the Derived Semigroupoid and Kernel Semigroupoid Theorems.

The last chapter of Part I, Chapter 3, develops the equational theory for pseudovarieties of relational morphisms. Because the development parallels Reiterman's Theorem and uses heavily profinite semigroups, we have provided a brief introduction to the classical theory of pseudoidentities and profinite and pro-**V** semigroups. Then we establish the generalization of Reiterman's Theorem to pseudovarieties of relational morphisms and give examples of pseudoidentity bases for the pseudovarieties of relational morphisms associated to the most commonly studied operators: semidirect products, Mal'cev products and joins. Inevitable substitutions are introduced and studied, leading to a basis theorem for the composition of two pseudovarieties of relational morphisms. Classical basis theorems from the literature are deduced as consequences. We discuss when the basis theorem for the semidirect product [27] applies and provide a new basis theorem covering the general case.

The second part of the book is called ***Complexity in Finite Semigroup Theory***. Chapter 4 provides a comprehensive look at group complexity starting from the very beginning with a proof of the Prime Decomposition Theorem and ending with such advanced topics as Ash's Theorem [33], the Ribes and Zalesskii Theorem [300], Rhodes's Presentation Lemma [43,334], Tilson's $2\mathscr{J}$-class Theorem [360] and Henckell's Aperiodic Pointlikes Theorem [121, 129]. Also, some important and hard-to-find results from the literature are featured, including Graham's Theorem [107] on the idempotent-generated subsemigroup of a Rees matrix semigroup. Using the language and viewpoint of $\mathfrak{q}$-theory, a simplified presentation of the computability of complexity for semigroups in **DS** is presented [170, 269, 368]. A large part of this chapter is dedicated to an updated presentation of the results in the "Arbib" book [171] concerning Mal'cev products and subdirectly indecomposable semigroups. We do not make any attempt to duplicate the material covered in Tilson's chapters in Eilenberg's book [362, 363], and so as a consequence the Fundamental Lemma of Complexity is not proved here. Few of the results in Chapter 4 are new, although some of the proofs are novel.

Chapter 5 introduces our general scheme for defining the complexity hierarchy associated to a single operator or a pair of operators on the lattice of pseudovarieties. We discuss many examples from the literature and present some new ones as well. The focus of the chapter is on two-sided complexity. The decomposition theory for maximal proper surmorphisms [293, 296] is included for the convenience of the reader and because the language of pseudovarieties of relational morphisms sheds some new light on these results. The remainder of the chapter takes the first steps in generalizing results from group complexity to two-sided complexity. The Ideal Theorem is proved and the two-sided complexity of the full transformation monoid is computed.

Part III of the book, ***The Algebraic Lattice of Semigroup Pseudovarieties***, provides a study of lattice theoretic aspects of the algebraic lattice of pseudovarieties of semigroups. Chapter 6 is essentially a condensed survey of algebraic and continuous lattices [97]. It is intended to establish notation and to introduce ideas that may not be familiar to a semigroup audience. Some examples related to $\mathfrak{q}$-theory are provided. For instance, it is shown that any continuous lattice of countable weight is the fixed-point lattice of an idempotent continuous operator on the lattice of pseudovarieties of semigroups. Chapter 7 is dedicated to the abstract spectral theory of the lattice of pseudovarieties of semigroups (and to a much lesser degree the lattices of pseudovarieties of relational morphisms and continuous operators). In particular, a multitude of new results about join irreducibility are obtained. Semigroups S with the property that $S \in \mathbf{V} \vee \mathbf{W}$ implies $S \in \mathbf{V}$ or $S \in \mathbf{W}$ are studied, as well as their exclusion pseudovarieties. We introduce several novel families of such semigroups, leading to the following application: if $\mathbf{H}$ is a pseudovariety of groups containing a non-nilpotent group, then the pseudovariety $\overline{\mathbf{H}}$ of semigroups whose subgroups belong to $\mathbf{H}$ is finite join irreducible. This generalizes results of Margolis, Sapir and Weil [195]. Also, the classification of Kovács-Newman semigroups that we began in an earlier paper [287] is completed.

The final part of the book, entitled ***Quantales, Idempotent Semirings, Matrix Algebras and the Triangular Product***, consists of two chapters. Chapter 8 introduces our slight weakening of the notion of a quantale and develops the general algebraic theory of these objects. We also study the equivalence between Boolean bialgebras and profinite semigroups, leading to a bialgebra structure on the Boolean algebra of regular languages over a finite alphabet. Chapter 9 contains almost entirely new material. Viewing finite quantales as idempotent semirings, we develop a decomposition theory for finite semirings in general via the triangular product of Plotkin [240], following the model of group complexity. The centerpiece of the chapter is formed by a pair of results we refer to as the Triangular Decomposition Theorem and the Ideal Decomposition Theorem. They basically give semiring analogues of the results of Munn and Ponizovskiĭ on semigroup algebras over fields [210,245] by embedding the semigroup algebra (over a semiring) of a finite semigroup inside an iterated triangular product of matrix algebras over the group algebras of its Schützenberger subgroups. Applying these theorems in the context of idempotent semirings yields the decomposition half of the Prime Decomposition Theorem for idempotent semirings. A large segment of the chapter is devoted to proving that matrix algebras over the power semigroup of a group are irreducible with respect to the triangular product. The moral of the story is that much more of ring theory works for semirings than one might expect. Finally, Chapter 9 ends with applications of the decomposition theory of idempotent semirings to computing the group complexity of power semigroups.

There is not a strict logical sequence for reading the various parts of the book (hopefully, the dependency graph is at least acyclic!). All readers should be familiar with the material in Chapter 1 to proceed. The basic definitions and terminology introduced in Chapter 2, at least as far as to the end of Section 2.3, are needed for Chapters 3–8, although specific results are rarely required. The new material in Chapter 3 is highly technical and is not required for most of the rest of the book, except for a brief reappearance in Chapter 7; the reader already familiar with pseudoidentities and profinite semigroups may skip it entirely on a first reading. Much of Part II requires only Chapter 1 and the language of Chapter 2, not the results. The key exceptions are the Derived and Kernel Semigroupoid Theorems, which are used repeatedly.

Part III requires Part I, but Chapter 6 can be read entirely independently of Part II. Chapter 7 occasionally appeals to results from Chapters 4 and 5. Chapter 8 depends on Part I and Chapter 6, whereas Chapter 9 depends only on a basic knowledge of definitions from Chapters 1 and 4, familiarity with the Fundamental Lemma of Complexity and on a small part of Chapter 8, namely quantic nuclei and some examples.

We have interspersed throughout the text a large number of exercises, some routine and others more difficult. The exercises form an integral part of the subject matter, and the reader is encouraged to solve as many of them as possible.

The current volume has greatly benefitted from the comments and criticisms of our colleagues and students. The following mathematicians deserve special thanks for their hard work and thoughtfulness: Karl Auinger, Bridget Brimacombe, Attila Egri-Nagy, Karl Hofmann, Gabor Horvath, Mark Kambites, Jimmie Lawson, Stuart Margolis, Chrystopher Nehaniv, Boris and Eugene Plotkin, Luis Ribes, Kimmo Rosenthal and Mikhail Volkov. The anonymous reviewers should also be acknowledged for their careful reading of the text. All errors and inaccuracies that remain are the sole responsibility of the authors.

The first author would like to thank U.C. Berkeley's generous retirement funds. The second author has been funded by an NSERC Discovery Grant during the course of this research. Both authors gratefully acknowledge support from FCT through the *Centro de Matemática da Universidade do Porto* and both have received partial support from the FCT and POCTI approved project POCTI/32817/MAT/2000 in participation with the European Community Fund FEDER. Some of this work was done while the second author was at the University of Porto, Portugal. The second author would also like to thank his colleagues at the University of Porto, in particular Jorge Almeida, Manuel Delgado and Pedro Silva, for their friendship and collaboration throughout the years and for making Porto an enjoyable environment to work in.

The first author would like to thank his wonderful wife, Laura Morland, for everything, especially her editorial skills. The second author is grateful to Ariane Masuda for her love and support through the long process of bringing this book to fruition, as well as for her careful reading of the appendix. *Obrigado por tudo.*

The inspirational spark for $\mathfrak{q}$-theory itself might never have been ignited, were it not for the superb mathematics library at "Chevaleret" (Paris VI–CNRS–Paris VII), and we thank its librarians for their *disponibilité*. However, none of this work would have been possible if not for the many excellent cafés that graciously allowed us to occupy their tables for overly excessive time periods. Our special appreciation goes to the Paris cafés: Les Monts

d'Auvergne, Café Lowrider, Au Père Tranquille, Aux Cadrans; the Porto cafés: Café Aviz, Café Majestic; the Berkeley café: Au Coquelet; and the Ottawa café: The Wild Oat.

This book is dedicated to Bret Tilson in gratitude to his contributions to Semigroup Theory, as well as his early participation in the research that led to this book. He was its first reader. Bret has had a great impact on all our lives and is sorely missed.

Paris, August 2008 *John Rhodes*
Ottawa, August 2008 *Benjamin Steinberg*

Contents

Part III The Algebraic Lattice of Semigroup Pseudovarieties

Part IV Quantales, Idempotent Semirings, Matrix Algebras and the Triangular Product

List of Tables

List of Figures

Introduction

What is $\mathfrak{q}$-theory? To explain the title of the current volume, we first need to put the whole field of Finite Semigroup Theory into a historical context. So let us begin with a theorem-based history of *finite* semigroups (biased, of course, by the authors' own viewpoint, and certainly by no means complete). Early theorems in the subject followed in the footsteps of other branches of algebra by aiming to classify semigroups up to isomorphism. An early success in this direction, and arguably the first theorem about finite semigroups, was the Rees-Suschkewitsch Theorem characterizing, up to isomorphism, simple [350] and 0-simple semigroups [258] as certain matrix semigroups over groups, a Wedderburn Theorem for semigroups, if you like. The sequel to this work was J. A. Green's fundamental paper [108] where the relations that now bear his name were introduced. This set up the framework to understand finite semigroups in terms of local coordinates at each (regular) $\mathscr{J}$-class: locally speaking, finite semigroups are Rees matrices over groups. Out of Green's paper came the famous eggbox pictures of Alfred Clifford and Gordon Preston [68]. However, a crucial missing ingredient was to understand in coordinates how the elements above a regular $\mathscr{J}$-class J act on it. This gap in our knowledge was filled in by Marcel-Paul Schützenberger via his famous representation by monomial matrices, now called the Schützenberger representation [311, 312]. From a more global viewpoint, his representation gives wreath product coordinates to the action of a semigroup on the left or right of a $\mathscr{J}$-class, and thus the wreath product entered into the subject quite early on.

Another way in which early semigroup theory trod the well-beaten paths blazed by other areas of algebra was via the representation theory of finite semigroups. The pioneering work of Clifford [66,67], W. D. Munn [210,211] and I. S. Ponizovskiĭ [245] completely determined the irreducible representations of a finite semigroup, modulo group theory, as well as characterized when the semigroup algebra is semisimple. A more global viewpoint of these results that takes advantage of the Schützenberger representation can be found in John Rhodes and Yechezkel Zalcstein's paper in *Monoids and semigroups with applications* [297]. Donald McAlister, in his 1971 survey paper [198], pointed out that at the time, the sole significant application of representation theory to finite semigroups was Rhodes's [270], where the congruence on a finite semigroup induced by the Jacobson radical of its semigroup algebra was shown to coincide with (in modern terminology) the largest $\mathbb{L}\mathbf{1}$-congruence, and a character theoretic interpretation was given to the computation of the complexity of completely regular semigroups [171, Chapter 9]. (This theme has been reprised only quite recently in the paper [18], where representation theory was used to simplify some key results in Formal Language Theory.) In recent years, Mohan Putcha [251–255] has modernized semigroup representation theory, exploring relations with algebraic groups, algebraic monoids [250,262], quasi-hereditary algebras [70] and P. Gabriel's theory of quivers [92–94]. (See also [1,57,58,335,336] for recent connections of finite semigroup representation theory with Probability Theory, random walks, hyperplane arrangements and Coxeter groups.) This brings to a close the first chapter of our history, which takes us up to the early 1960s. Most of the results that we have discussed thus far can be found in the treatise of Clifford and Preston [68].

The direction initiated by the Rees-Suschkewitsch Theorem eventually had to be abandoned, for the program of classifying finite semigroups up to isomorphism is a hopeless one: there are simply too many of them. From an asymptotic viewpoint, the class of 3-nilpotent semigroups, i.e., semigroups satisfying the identity $xyz = 0$, covers almost all finite semigroups up to isomorphism [162]. Clearly it serves no purpose to try to classify such semigroups. More precisely, if the semigroups of order n are distributed uniformly, then the probability that a labeled random semigroup of order n is 3-nilpotent goes to 1 as n goes to infinity. Intuitively, this happens because any multiplication table where the product of three elements is zero is automatically associative, and therefore 3-nilpotent semigroups are easy to construct. In fact, there are purported to be $1,843,120,128$ semigroups of order 8 up to isomorphism and anti-isomorphism, and approximately 99% are nilpotent according to S. Satoh, K. Yama, and M. Tokizawa [310]. So, whereas groups are gems, all of them precious, the garden of semigroups is filled with weeds. One needs to yank out these weeds to find the interesting semigroups. (On the other hand, semigroups have wider applicability than do groups, especially in Computer Science.)

Thus by the mid-1960s, it was time for a revolution in thinking about finite semigroups, and it would have to start from outside the world of algebra. Automata and sequential machines [161,209] had already made their appearance on the stage in the late 1950s, and the idea that there should be an algebraic theory of automata and machines was very much in the air. In particular, the work of Stephen Kleene [161], reinterpreted via the syntactic monoid, stated that regular languages (studied in Logic and Computer Science) are precisely subsets of the free monoid saturated by a finite index congruence. But it was not until Kenneth Krohn and Rhodes introduced the fundamental notion of division that a successful algebraic decomposition theory of machines and semigroups could be achieved [169]. Other attempts [118,119] at an algebraic decomposition theorem for sequential machines failed precisely because they did not use division: the full transformation monoid T_n cannot embed in a semidirect product without embedding in one of the factors.

The statement of the Krohn-Rhodes Prime Decomposition Theorem can be formulated to any student who has taken a basic course in Group Theory: *every* finite semigroup divides (is a quotient of a subsemigroup of) an iterated wreath product of its simple group divisors and the three element monoid of transformations of the set $\{0, 1\}$ consisting of the constant maps and the identity map. This monoid is the semigroup analogue to the *flip-flop* sequential machine from Computer Science and Electrical Engineering [171]. The Prime Decomposition Theorem is an example of a "mature" theorem. Many early theorems in semigroup theory (and too many recent ones!) involved inventing a class of semigroups and then studying it. Not so for the Prime Decomposition Theorem: all the notions in the statement already existed, at least implicitly: wreath/semidirect products, simple groups (we like the terminology SNAGS for simple non-abelian groups) and division (the composition of the $\mathbb{H}$ and $\mathbb{S}$ operators from Garrett Birkhoff's Universal Algebra).

Let us briefly remind the reader of how the Jordan-Hölder program for Finite Group Theory goes. If G is a finite group, there is a composition series

$$\{1\} = N_m < N_{m-1} < \cdots < N_1 = G$$

for G where $N_{i+1} \lhd N_i$ and N_i/N_{i+1} is simple, all i. These simple groups, called the composition factors of G, are unique, although the order in which they appear is not. The "monomial map" [113, 141, 381], going back to the work of Frobenius on induced representations, then places G inside the iterated wreath product of these simple group divisors. Therefore, if you understand simple groups (and according to the Classification of Finite Simple Groups [101–106], we supposedly do) and you understand wreath products and sequential coordinates, then you understand all finite groups.

The Prime Decomposition Theorem transported the whole Jordan-Hölder program to the realm of semigroups [283]! This led naturally to the notion of group complexity of a finite semigroup: when decomposing a finite semigroup S into a wreath product of groups and aperiodic (i.e., group-free) semigroups, how many groups do you need? The minimal number of groups is deemed the complexity of the semigroup. This notion was formalized by Krohn and Rhodes in their 1968 Annals paper [170], where they computed the complexity of union of groups semigroups (also called completely regular semigroups). Since that time, the major driving open problem in Finite Semigroup Theory has been to find an algorithm to compute group complexity. The problem has been open for more than 40 years! The literature on the subject — which comprises decomposition theorems, partial results, lower bounds and upper bounds — is much too vast to do it justice in this introduction. Chapter 4 of the current volume, together with Bret Tilson's chapters [362, 363] in Volume B of Samuel Eilenberg's *Automata, languages, and machines* [85], provides the most complete treatment of the subject to date.

Contemporary with the Prime Decomposition Theorem was Schützenberger's celebrated 1965 theorem on star-free languages [313]. It characterized star-free languages as the languages recognized by aperiodic semigroups (semigroups of complexity 0). In fact, the difficult direction of this theorem, decomposing the language accepted by an aperiodic semigroup, can be achieved quite easily as a consequence of the Prime Decomposition Theorem [207] and the relationship between sequential machines, wreath products and languages developed in [169], [171, Chapter 5] and summarized in [85, Chapter VI]. Moreover, the Krohn-Rhodes prime decomposition/sequential machine approach has been used successfully to characterize many other classes of languages, in particular by Howard Straubing [343, 344, 346]. Nowadays, the name "the wreath product principle" is often attached to this approach. This should not diminish the importance of Schützenberger's Theorem, which indicated that there is a relationship between classes of languages and classes of finite semigroups. Other theorems appeared soon thereafter that expressed that naturally occurring classes of regular languages corresponded via their syntactic monoids to naturally occurring classes of semigroups. These include the characterization of locally testable languages as being recognized by local semilattices [60, 204, 383, 384] (all of which appeared in the early 1970s) and Simon's characterization of piecewise testable languages as those recognized by $\mathscr{J}$-trivial monoids (1975) [318].

Once isomorphism was off the table, what was sorely lacking was a framework for classification in finite semigroup theory. The Prime Decomposition Theorem and the language results would seem to suggest that Universal Algebra could provide a possible framework. But classical Universal Algebra, with its reliance on free objects and equations and the requirement for closure under infinite products, wasn't quite the answer.

A fruitful tendency in modern mathematics is to turn theorems into definitions — after which the original theorem becomes a verification that some object satisfies the definition. For instance, the Heine-Borel Theorem originally stated that any covering of a closed interval by open intervals has a finite subcover. This led to the modern definition of a compact space, which eventuated in the Heine-Borel Theorem becoming the statement that closed intervals are compact. Another example is Rees's Theorem, which originally characterized 0-simple semigroups as Rees matrix semigroups over groups. This then led to the definition of Rees matrix semigroups over arbitrary semigroups, which in turn led to infinite iterated Rees matrix semigroups and the Synthesis Theorem [3, 50–52, 276, 277].

Eilenberg turned the language characterization theorems into a definition [85]. Together with Schützenberger [85, 86] he defined a pseudovariety of finite semigroups to be a class of finite semigroups closed under forming *finite* products and taking divisors. He defined the companion notion of a variety of formal languages and provided a correspondence between semigroup pseudovarieties and varieties of languages. From the point of view of this book, Eilenberg proved that the algebraic lattice **PV** of pseudovarieties of finite semigroups is isomorphic to the algebraic lattice of varieties of languages. Schützenberger's Theorem then became the theorem that the variety of star-free languages corresponds under the Eilenberg Correspondence to the pseudovariety of aperiodic semigroups. The Prime Decomposition Theorem turned into the statement that a finite semigroup belongs to the smallest pseudovariety of semigroups closed under semidirect product containing its simple group divisors and the flip-flop.

At this point (around 1976), Finite Semigroup Theory essentially became the study and classification of pseudovarieties of finite semigroups. Notice that the pseudovariety notion nearly wipes out the 3-nilpotent weeds: whereas almost all semigroups are, probabilistically speaking, 3-nilpotent, there are only a small finite number of pseudovarieties of 3-nilpotent semigroups. The starring role in the new theory, created in large part under the impetus of attacking the problem of group complexity, was then taken by the semidirect product of pseudovarieties of semigroups, an associative multiplication on the lattice $\mathbf{PV}$. However, other non-associative products, such as the Mal'cev product [59], the two-sided semidirect product [293] and the Schützenberger product [85], have also garnered quite a bit of attention in the literature, not to mention the join operation on $\mathbf{PV}$, which corresponds to direct products of semigroups. This was the state of play at the end of the 1970s.

In the 1980s, it became evident that there were two deficiencies in Eilenberg's book [85]. The resolution of these issues led to Finite Semigroup Theory as we know it today. The first deficiency was foundational, or one of scope. In Eilenberg [85] there is a result called the Tilson Trace-Delay Theorem. This result was used to prove $\mathbf{V} * \mathbf{D} = \mathbb{L}\mathbf{V}$ for various pseudovarieties $\mathbf{V}$ generated by monoids. Key tools used in these results were the so-called "graph congruences" and the derived transformation semigroup. Also in his Chapter XII of Eilenberg's book [362], Tilson introduced the derived semigroup, obtaining a one-way connection between the semidirect product and the derived semigroup. The derived semigroup was a salient feature of Tilson's simplified proof of Rhodes's Fundamental Lemma of Complexity [268], although it was only in combination with the Rhodes expansion [54] that the full power of the technique was revealed. However, graph congruences and semigroups were not the proper setting for these results and did not allow for a complete understanding of what was really behind them.

In a seminal paper in 1987 [364], Tilson, influenced by personal conversations with Stuart Margolis, as well as by the work of Margolis and Jean-Eric Pin [192–194], Denis Thérien and A. Weiss [358], Straubing [345] and Robert Knast [163], realized that categories would place the aforementioned results in their proper context. He replaced the derived semigroup with the derived category and proved the all-important Derived Category Theorem. If $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of monoids and $\mathbf{V} * \mathbf{W}$ is their semidirect product, then Tilson established $S \in \mathbf{V} * \mathbf{W}$ if and only if there is a relational morphism $\varphi\colon S \to T$ with $T \in \mathbf{W}$ such that the derived category D_φ of φ divides a monoid in $\mathbf{V}$ [364]. This led to a theory of pseudovarieties of categories and the fundamental global versus local problem: when is category membership in the pseudovariety generated by a collection of monoids determined by looking at the local monoids of a category?

The Derived Category Theorem created a paradigm that was followed by much of later work, and which in the current text we turn into a definition. Tilson's 1987 "Categories as algebra" paper [364] was followed two years later by its two-sided analogue in his joint paper with Rhodes [293], which introduced the kernel category of a relational morphism and showed that a monoid S belongs to the two-sided semidirect product $\mathbf{V} ** \mathbf{W}$ of pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ if and only if there is a relational morphism $\varphi\colon S \to T$ with $T \in \mathbf{W}$ and the kernel category K_φ of φ dividing a monoid in $\mathbf{V}$. Combined with Rhodes's classification of maximal proper surmorphisms [267], this led to a decomposition theory [293, 296] that has had sweeping applications to Formal Language Theory [55, 234, 346, 349, 378, 379]. The advent of categories has removed much of the necessity for ad hoc wreath product decompositions and machine equations [169] from the theory.

Whereas the work of Tilson, Margolis and Pin brought the derived category to the attention of finite semigroup theorists, it should be mentioned that the derived category was already a well-known construct to category theorists; for instance, Daniel Quillen [257] used it to formulate a criterion for when the geometric realization of a functor between categories induces a homotopy equivalence between their nerves; the derived category of a functor $F\colon C \to D$ (pre-identifications) is precisely the category of elements [186] of the composition of F with the Yoneda embedding $Y\colon D \to \mathbf{Set}$ (the category of elements was also put to good effect by P. J. Higgins [132] to construct groupoid coverings); and it is well-known to stand in an adjoint relationship with the Grothendieck construction [186], called by some the semidirect product of categories. William Nico also studied the relationship between the derived category and wreath products [219] before Tilson did, in fact at the categorical level, but his theory lacked the crucial ingredient of division. M. Loganathan's little known paper [183] on the cohomology of inverse semigroups gave a proof of McAlister's P-theorem [199] — and its generalization by L. O'Carroll [223] — using the derived category of a functor between categories even earlier than the paper of Margolis and Pin on the same subject [193].

The second deficiency in the theory of pseudovarieties espoused by Eilenberg and Schützenberger was the equational theory. They established [86] that pseudovarieties have ultimate equational descriptions, but in practice this approach is useless. An ultimate equational description of a pseudovariety $\mathbf{V}$ is a sequence of identities such that a semigroup belongs to $\mathbf{V}$ if and only if it satisfies all but finitely many of the identities in the sequence. One would like to be able to say what it means for a pseudovariety of semigroups to be defined by a finite number of "identities," that is, to be finitely based; the ultimate equational descriptions do not allow for this. Jan Reiterman in 1982 came up with the correct solution to the problem: pseudoidentities [261]. A usual identity is a formal equality between elements of a free semigroup. Reiterman's idea was that the role of a free semigroup in the finite world can be taken by a free profinite semigroup. A pseudoidentity is then a formal equality between elements of a free profinite semigroup.

Reiterman's Theorem shows that pseudovarieties are exactly the classes defined by pseudoidentities. However, it was only under the impetus of Jorge Almeida that profinite semigroups and the syntactic approach became a fundamental tool in Semigroup Theory, particularly in the 1990s. Much of Almeida's early work on the subject is encapsulated in his volume *Finite semigroups and universal algebra* [7]; further references can be found in the bibliography of the current text. The profinite approach was generalized to pseudovarieties of categories [27, 149], although there are a number of subtleties in this context. One goal then became to try and find a basis of pseudoidentities for $\mathbf{V} * \mathbf{W}$ given a basis of pseudoidentities for the categories dividing elements of $\mathbf{V}$ and knowledge of $\mathbf{W}$ [27] and similarly for other products [235]. This led to the notions of hyperdecidability [9] and tameness [19,20]. Nowadays, profinite semigroups are an indispensable tool in semigroup theory, in particular for studying pointlikes [9, 235, 322, 327, 330], stabilizer pairs (here one should compare the profinite argument in [130] with the argument in [124]) and related notions [9, 322, 330]. Many of these latter notions had their roots in the early work of Rhodes and Tilson on Type I/Type II semigroups [295] and Rhodes and Karsten Henckell on pointlike sets [121], but it was mostly Almeida who pushed these notions, especially with regard to the profinite approach [9, 19, 20].

The explosion of techniques in the 1980s resulted in profound work such as Ash's Theorem [33], solving the Rhodes Type II conjecture. The conjecture was proved independently by Luis Ribes and Pavel Zalesskii [300], using the method of profinite groups acting on profinite trees, via a translation of the problem into the profinite topology on a free group by Pin and Christophe Reutenauer [232]. The Type II Theorem describes which elements of a finite semigroup always relate to 1 under a relational morphism to a group. A review of the innumerable consequences of the resolution of the Rhodes conjecture appears in the *IJAC* survey paper of Henckell, Margolis, Pin and Rhodes [126]. Some highlights include characterizations of the pseudovarieties of semigroups generated by inverse semigroups, orthodox semigroups and power semigroups of groups. Actually, the first two cases were handled by special cases of the conjecture established earlier by Christopher Ash [32] and by Jean-Camille Birget, Margolis and Rhodes [53].

Ash, most likely due to his background as a logician, injected an important new technique into Finite Semigroup Theory: Ramsey Theory. The basic idea is that if one takes a long product of generators of a finite semigroup, then there must be a lot more repetition of idempotents than you might expect. Nowadays, there is a non-profinite proof of Ash's Theorem that does not rely on Ramsey Theory [34], but the technique remains invaluable. Karl Auinger and Benjamin Steinberg managed to synthesize the techniques of Ash and Ribes and Zalesskii to study related problems over other pseudovarieties of groups [37,39–41,328,329,331], in particular establishing intimate connections between semidirect product decompositions of semigroups and the geometry of profinite groups. This is based on earlier work of Margolis, Mark Sapir and Pascal Weil [196] amalgamating combinatorial group theory, via Stallings Folding [321] and M. Hall's Theorem [112], with inverse semigroup theory.

We are now close to being able to state the thesis of this book. Tilson's Derived Category Theorem [364] showed how to define the semidirect product operator $\mathbf{V} * (-)$ in terms of relational morphisms. Similarly, its sequel, the Kernel Category Theorem [293], showed how to define the two-sided semidirect product operator $\mathbf{V} ** (-)$ in terms of relational morphisms. Various authors [27, 293, 296] used this idea to define operators $\mathbf{V} * (-)$ where $\mathbf{V}$ is a pseudovariety of categories and even to define a semidirect product of pseudovarieties of categories [150]. Steinberg, in his Ph.D. thesis [322, 330], provided a necessary and sufficient condition on a relational morphism so that it could be factored as a division followed by the projection from a direct product. This resulted in a relational morphism description of the operator $\mathbf{V} \vee (-)$, as well as a number of new decidability results for joins of pseudovarieties. The Mal'cev product $\mathbf{V} ⓜ \mathbf{W}$ is defined by declaring that $S \in \mathbf{V} ⓜ \mathbf{W}$ if and only if there is a relational morphism $\varphi\colon S \to T$ with $T \in \mathbf{W}$ and so that, for each idempotent $e \in T$, the semigroup $e\varphi^{-1}$ belongs to $\mathbf{V}$. This product is then, by virtue of its construction, defined in terms of a class of relational morphisms.

In "Categories as algebra. II," Steinberg and Tilson "beefed up" the entire theory of the derived category of a monoid relational morphism to the level of categories [339]. One of the principal results of that paper states the following: Let $\varphi\colon S \to T$ be a relational morphism of monoids. Then the derived category D_φ of φ divides a monoid in $\mathbf{V} * \mathbf{W}$ if and only if $\varphi = \varphi_1\varphi_2$ where D_{φ_1} divides a monoid in $\mathbf{V}$ and D_{φ_2} divides a monoid in $\mathbf{W}$. This result was dubbed the Composition Theorem. To express it in a more compact and elegant manner, they defined the class $\mathbf{V}_D$ of all relational morphisms whose derived category belongs to $\mathbf{V}$. One can compose classes of relational morphisms in an obvious way, and the Composition Theorem then admits the following reformulation:

$$(\mathbf{V} * \mathbf{W})_D = \mathbf{V}_D \mathbf{W}_D.$$

In the course of their research, Steinberg and Tilson tossed around the idea of defining pseudovarieties of relational morphisms. In particular, Steinberg had a notion of division of relational morphisms and pseudoidentities for relational morphisms. Pseudovarieties of relational morphisms could then be used to define operators on the lattice of semigroup pseudovarieties. But Tilson argued that without further evidence, it was better to delay bestowing the name pseudovariety on a class of relational morphisms that later on might not prove to be worthy of the name. In fact, Tilson had obviously flirted with the idea of defining pseudovarieties of relational morphisms in the past, as evidenced by his notion of a weakly closed class and the complexity of a relational morphism in [362]. Tilson also was attached to the idea that a semigroup

S should be identified with a unique relational morphism: its collapsing map $S \to 1$. He identified a pseudovariety of semigroups $\mathbf{V}$ with its set of collapsing morphisms, and his viewpoint was that the action of a set R of relational morphisms on $\mathbf{V}$ should be the composition $\mathsf{R}\mathbf{V}$, which is a collection of collapsing morphisms corresponding to a pseudovariety of semigroups. The disadvantage to this approach is that the axioms that were being considered for classes of relational morphisms were *not* satisfied by the set of collapsing morphisms of a pseudovariety and so one could not identify a pseudovariety of semigroups with a pseudovariety of relational morphisms in this way. In the end Tilson and Steinberg abandoned this line of research.

On June 9, 2000, Rhodes and Steinberg met to discuss semigroups at *Les Monts d'Auvergne*, a Parisian café in the neighborhood of Chevaleret (at the rue Maurice et Louis de Broglie). Rhodes had been reading "Categories as algebra. II" [339] and under its influence had also arrived at the idea of defining operators via relational morphisms. However, he was more interested in the operators themselves than in classes of relational morphisms. Inspired by the Composition Theorem, he believed that the *whole* of semigroup theory should be viewed as studying the composition of continuous operators on the lattice $\mathbf{PV}$ of pseudovarieties. The reasoning is as follows: the two-sided semidirect product and the Mal'cev product are non-associative, but the composition of operators is associative. How one chooses to bracket, say, iterated two-sided semidirect products is actually forced by associativity once one decides which factor is the operator and which factor is the variable. For instance, if your operators are $\alpha = \mathbf{A} ** (-)$ and $\beta = \mathbf{G} ** (-)$, then one can perfectly well write down nice associative expressions like $\alpha\beta\alpha$, or even $(\alpha\beta)^\omega\alpha$. Only after choosing a pseudovariety $\mathbf{V}$ on which to operate (think of this as making an observation in quantum physics) does a choice of bracketing and non-associativity appear. Thus taking $\mathbf{V}$ to be the trivial pseudovariety $\mathbf{1}$ yields

$$\alpha\beta\alpha(\mathbf{1}) = \mathbf{A} ** (\mathbf{G} ** (\mathbf{A} ** \mathbf{1})).$$

Suppose now that $\alpha' = (-) ** \mathbf{A}$ and $\beta' = (-) ** \mathbf{G}$. Then we can again form pleasant associative expressions like $\alpha'\beta'\alpha'$, only this time performing the experiment of evaluating at $\mathbf{1}$ results in

$$\alpha'\beta'\alpha'(\mathbf{1}) = ((\mathbf{1} ** \mathbf{A}) ** \mathbf{G}) ** \mathbf{A}.$$

One could also "mix" variables and consider $\alpha'' = \mathbf{A} \textcircled{m} (-)$ and $\beta'' = (-) \textcircled{m} \mathbf{G}$.

Getting ahead of ourselves for the moment, if we stick with the operators $\alpha = \mathbf{A} ** (-)$ and $\beta = \mathbf{G} ** (-)$, then the two-sided complexity hierarchy is obtained by taking the operator $(\alpha^\omega\beta^\omega)^n\alpha^\omega$ and sampling it at the trivial pseudovariety $\mathbf{1}$. After the quantum effect of applying the operator to the trivial pseudovariety, we arrive at the level n two-sided complexity pseudovariety given by

$$\mathbf{K}_n = (\alpha^\omega\beta^\omega)^n\alpha^\omega(\mathbf{1}) = \mathbf{A} **^\omega (\mathbf{G} **^\omega (\cdots(\mathbf{A} **^\omega (\mathbf{G} **^\omega \mathbf{A}))\cdots))$$

where **G** appears n times. This quantum idea of replacing points by operators and then only getting the points back by performing an observation (or an experiment) by evaluating at a point is why this book is called $\mathfrak{q}$-theory: $\mathfrak{q}$ as in quantum! In fact, this book is very much a *quantized* version of Eilenberg [85]; pseudovarieties of semigroups are replaced by operators on the lattice of pseudovarieties; relational morphism and division of semigroups is replaced by relational morphism and division of relational morphisms. Tilson does identify (in his Chapter XII of Eilenberg [362]) a semigroup with its collapsing morphism, but this is not a true quantization; in this book, we quantize a semigroup to all relational morphisms having it as the domain.

Returning to our conversation at *Les Monts d'Auvergne*, Rhodes told Steinberg that he wanted to define a homomorphism from classes of relational morphisms to operators and characterize which operators were in the image. The viewpoint was that both **PV** and the lattice of relational morphism pseudovarieties were complete lattices, so they have the least upper bound property like the real numbers, and that one could define natural topologies for which our homomorphism was continuous. This begins to functor us over to soft analysis and the abstract spectral theory of continuous lattices [97]. Steinberg, who had already been influenced away from the idea by Tilson, spoke about the issues involved in deciding between the various candidates for the title of pseudovariety of relational morphisms and why he and Tilson had dropped the idea. It was fair to say he was at that moment against developing the notion. The following day the authors again met at *Les Monts d'Auvergne*, only this time their positions had switched under the influence of each other's arguments of the previous day: Steinberg thought the idea was great and wanted to develop it; Rhodes believed it should be dropped. Nonetheless, from this meeting the current volume was born. Shortly thereafter Tilson became a coauthor, but several months later creative differences arose, and he divorced himself from the project. However, Tilson was certainly a major influence on the early development of this book, and he followed its progress with interest up until his death.

Our original program went something like this: First we defined division of relational morphisms and pseudovarieties of relational morphisms. Pseudovarieties could be composed and a homomorphism $\mathfrak{q}$ taking pseudovarieties to operators was defined. Then we characterized which operators on the lattice **PV** of pseudovarieties arose from pseudovarieties of relational morphisms. These turned out to be Scott continuous functions (in the sense of [97]) satisfying an additional condition that we termed the global Mal'cev condition. For instance, the Schützenberger product satisfies the global Mal'cev condition and hence can be defined by a pseudovariety of relational morphisms. Next our goal was to take Steinberg's notion of pseudoidentities for relational morphisms and prove a Reiterman's Theorem in this context. This was all completed by Fall 2000. This notion of a pseudoidentity led to a general notion of inevitable substitutions that encompassed the ideas used previously to study pointlikes [121, 322, 330], idempotent-pointlikes [235] and inevitable graphs [9, 33]. Since application of operators is a special instance of composition (where a pseudovariety is identified with a constant map), the basis theorems of Almeida and Weil [27] and Pin and Weil [235] became instances of a basis theorem for composition of pseudovarieties of relational morphisms. Complexity hierarchies associated to iteration of operators were to be defined, with group complexity as the model, and encompassing group complexity [170], dot-depth [71], p-length [368] and two-sided complexity. The goal was then to generalize Almeida and Steinberg's notion of tameness [19, 20] to obtain decidability results for arbitrary hierarchies arising from iteration of operators. For instance, if we could find a good basis of pseudoidentities for the pseudovariety of relational morphisms defining the Schützenberger product operator, this would provide a program to attack the dot-depth problem. Unfortunately, we realized in early 2001 that some additional hypotheses were needed on the factors for the basis theorem to work and that, in particular, there were missing hypotheses in [27], thereby invalidating many of the results of [19, 20].

To understand the issues underlying the instances when the basis theorem holds and when it doesn't, we were led to begin a systematic study of lattice theoretic considerations with regard to **PV** and continuous operators on **PV**. In particular, we introduced a new class of relational morphisms, called a *continuously closed class*, and showed that all continuous operators are the $\mathfrak{q}$-image of a continuously closed class. We were also led to study order-theoretic properties of the map $\mathfrak{q}$ itself. As a map of partially ordered sets it turns out to have both a left and right adjoint, and hence, given any continuous operator satisfying the global Mal'cev property, there is a unique maximal and a unique minimal pseudovariety of relational morphism defining it: each such operator is the $\mathfrak{q}$-image of a closed interval in the lattice of pseudovarieties of relational morphisms. This leads to many interesting questions, such as whether $\mathbf{V}_D$ is the minimal pseudovariety of relational morphisms defining $\mathbf{V} * (-)$ (we term this the "Tilson Problem"). Trying to resolve the case of the trivial pseudovariety $\mathbf{1}$, we were led to the fascinating question of whether every finite semigroup embeds in a relatively free finite semigroup. This was answered positively by George Bergman, who established that $\mathbf{1}_D$, the pseudovariety of divisions, is indeed the unique minimal pseudovariety of relational morphisms defining the identity operator [47]. Imagine what techniques will be needed to address the general case!

We then entered into the so-called "abstract spectral theory" of lattices [97], a sweeping generalization of Stone's duality between Boolean algebras and profinite spaces [61, 97, 117, 147, 341, 342]. This in turn brought us to quantales [303], the so-called quantum locales. These are complete lattices with a semigroup multiplication preserving all sups. They generalize the well-known locales of pointless topology [97, 147, 186] and play a role in the search for a non-commutative Gelfand space for C^*-algebras [172, 173, 303]. The monoid of pseudovarieties of relational morphisms and the monoid of continuous operators on **PV** are quantales in a slightly weakened sense. Finite quantales (in the classical sense) are nothing more than finite idempotent semirings, which have already been considered by Libor Polák in the context of Formal Language Theory [241–244]. It seemed natural to start applying the complexity of operators program in this context. To achieve this, a product was needed. Neither the usual wreath product nor the wreath product of ordered semigroups [237] works in this context. It turns out that it is the triangular product of Boris Plotkin [239, 240, 376] that does the job. Plotkin's triangular product is an axiomatization of the block triangular form obtained for a matrix representation by taking a Jordan-Hölder composition series. Our decision to use this product was influenced very much by the viewpoint of Almeida, Margolis, Steinberg, and Mikhail Volkov [18].

The establishment of a Prime Decomposition Theorem for Idempotent Semirings leads to a large number of open questions, including computing the complexity of a finite idempotent semiring and completing the classification of irreducible idempotent semirings. Also, this theorem opens up a new avenue of attack on the dot-depth problem as the Schützenberger product is a special case of the triangular product of semigroups. Our decomposition theorem works in this context as well and can be used to obtain lower triangular and block lower triangular Boolean matrix representations. The problem of dot-depth two is equivalent to deciding whether a finite semigroup divides a semigroup of lower triangular Boolean matrices [233]. The power set of a finite semigroup is an idempotent semiring; the Prime Decomposition Theorem for Idempotent Semirings works as a powerful tool for studying the complexity of power semigroups. In particular, it leads to an improved version of results of Cary Fox and Rhodes [91], allowing us to compute the exact group complexity of the power semigroup of a finite inverse semigroup, as well as an asymptotically tight bound on the complexity of the power semigroup of the full transformation of degree n.

The contents of the current volume are summarized in the Preface. Let us add that at the end of the text we compile a list of 74 problems generated by the results of this book. We invite our readers to solve them all!