

Graduate Texts in Mathematics

250

Editorial Board

S. Axler

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Graduate Texts in Mathematics

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(continued after index)

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Modern Fourier Analysis

Second Edition

 Springer

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*Για την Ιωάννα, την Κωνσταντίνα,
και την Θεοδώρα*

Preface

The great response to the publication of the book *Classical and Modern Fourier Analysis* has been very gratifying. I am delighted that Springer has offered to publish the second edition of this book in two volumes: *Classical Fourier Analysis, 2nd Edition*, and *Modern Fourier Analysis, 2nd Edition*.

These volumes are mainly addressed to graduate students who wish to study Fourier analysis. This second volume is intended to serve as a text for a second-semester course in the subject. It is designed to be a continuation of the first volume. Chapters 1–5 in the first volume contain Lebesgue spaces, Lorentz spaces and interpolation, maximal functions, Fourier transforms and distributions, an introduction to Fourier analysis on the n -torus, singular integrals of convolution type, and Littlewood–Paley theory.

Armed with the knowledge of this material, in this volume, the reader encounters more advanced topics in Fourier analysis whose development has led to important theorems. These theorems are proved in great detail and their proofs are organized to present the flow of ideas. The exercises at the end of each section enrich the material of the corresponding section and provide an opportunity to develop additional intuition and deeper comprehension. The historical notes in each chapter are intended to provide an account of past research but also to suggest directions for further investigation. The auxiliary results referred to the appendix can be located in the first volume.

A web site for the book is maintained at

<http://math.missouri.edu/~loukas/FourierAnalysis.html>

I am solely responsible for any misprints, mistakes, and historical omissions in this book. Please contact me directly (loukas@math.missouri.edu) if you have corrections, comments, suggestions for improvements, or questions.

Columbia Missouri,
June 2008

Loukas Grafakos

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