

Complex General Relativity

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by

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PREFACE

This book is written for theoretical and mathematical physicists and mathematicians interested in recent developments in complex general relativity and their application to classical and quantum gravity. Calculations are presented by paying attention to those details normally omitted in research papers, for pedagogical reasons. Familiarity with fibre-bundle theory is certainly helpful, but in many cases I only rely on two-spinor calculus and conformally invariant concepts in gravitational physics. The key concepts the book is devoted to are complex manifolds, spinor techniques, conformal gravity, α -planes, α -surfaces, Penrose transform, complex space-time models with non-vanishing torsion, spin- $\frac{1}{2}$ fields and spin- $\frac{3}{2}$ potentials. Problems have been inserted at the end, to help the reader to check his understanding of these topics.

Thus, I can find at least four reasons for writing yet another book on spinor and twistor methods in general relativity: (i) to write a textbook useful to beginning graduate students and research workers, where two-component spinor calculus is the unifying mathematical language. This enables one to use elegant and powerful techniques, while avoiding a part of mathematics that would put off physics-oriented readers; (ii) to make it possible to a wide audience to understand the key concepts about complex space-time, twistor space and Penrose transform for gravitation; (iii) to present a self-consistent mathematical theory of complex space-times with non-vanishing torsion; (iv) to present the first application to boundary-value problems in cosmology of the Penrose formalism for spin- $\frac{3}{2}$ potentials. The self-contained form and the length have been chosen to make the monograph especially suitable for a series of graduate lectures.

Section 7.2 is based on work in collaboration with Hugo A. Morales-Técotl and Giuseppe Pollifrone. It has been a pleasure and a privilege, for me, to work with both of them. Sections 8.2-8.9 are based on work in collaboration with Giuseppe Pollifrone and, more recently, with Gabriele Gionti, Alexander Kamenshchik and

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Igor Mishakov. Only collaborative work of which I was the first author has been included.

A constant source of inspiration have been the original papers by Roger Penrose and Richard Ward, as well as the more comprehensive monographs by the same authors. The years spent in Cambridge as a graduate student, and the many conversations on geometrical methods with Giuseppe Marmo, also played a key role.

Financial support by the Ministero per l'Università e la Ricerca Scientifica e Tecnologica to attend the Twistor Conference in Seale-Hayne is also gratefully acknowledged. I very much enjoyed such a beautiful and stimulating Conference, and its friendly atmosphere. I am indebted to Stephen Huggett for hospitality in Seale-Hayne and for encouraging my research, and to Mauro Carfora for inviting me to give a series of graduate lectures at SISSA on the theory of the Dirac operator. Partial support by the European Union under the Human Capital and Mobility Program was also obtained.

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