Appendix A
Topological Preliminaries

Abstract This appendix collects together the definitions and major results needed from point-set topology, algebraic topology, Euclidean space topology and the classification of compact surfaces with or without boundary. Only in a few cases are proofs given because they are generally widely available in a range of textbooks. Examples where proofs are given include a stronger version of Tychonoff’s Embedding Theorem than that normally encountered, Hodel’s result limiting the number of topologically distinct Čech-complete spaces and the Rudin-Zenor result that any separable, closed subset of a perfectly normal manifold is hereditarily separable.

A.1 Point Set Topology

In this Section we recall the definitions of metric and topological spaces along with some important topological properties. We also recall some useful results connecting various properties but shall present few proofs of these results. There are many good introductory books, including [2, 5, 10, 15].

Definition A.1 A metric space is a pair \((X, d)\), where \(X\) is a set and \(d : X \times X \to \mathbb{R}\) a (distance) function satisfying, for each \(x, y, z \in X\):

- \(d(x, y) \geq 0\) and \(d(x, y) = 0\) if and only if \(x = y\);
- \(d(x, y) = d(y, x)\);
- \(d(x, z) \leq d(x, y) + d(y, z)\).

Frequently we talk of a metric space \(X\) without specifying the metric.

Example A.2 The most familiar example of a metric space is Euclidean space \((\mathbb{R}^n, d)\), consisting of all ordered \(n\)-tuples of real numbers, with the Pythagorean metric, i.e., that defined by

\[
d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.
\]

The discrete metric on \(X\) assigns to each distinct pair of points the distance 1.
Recall that a subset $U \subset X$ of $X$ is open in $(X, d)$ provided that for each $x \in U$ there is $r > 0$ so that $y \in U$ whenever $d(x, y) < r$. The complement of an open set is called closed. Basic properties of open sets are axiomatised to give us a topological space.

**Definition A.3** A topological space is a pair $(X, \mathcal{T})$, where $X$ is a set and $\mathcal{T}$ a collection of subsets of $X$ satisfying:

- $\emptyset \in \mathcal{T}$;
- $X \in \mathcal{T}$;
- $U \cap V \in \mathcal{T}$ whenever $U, V \in \mathcal{T}$;
- $\bigcup \mathcal{I} \in \mathcal{T}$ whenever $\mathcal{I} \subset \mathcal{T}$.

The family $\mathcal{T}$ is called a topology (on $X$).

Frequently we talk of a topological space $X$ without specifying the topology. When $(X, \mathcal{T})$ is a topological space and $Y \subset X$ the family $\{Y \cap U / U \in \mathcal{T}\}$ is a topology on $Y$: with this topology $Y$ is called a subspace of $X$.

A family $\mathcal{B} \subset \mathcal{T}$ of open sets in a topological space $(X, \mathcal{T})$ is called a basis for the topology provided each open set is a union of members of $\mathcal{B}$. Often it is more convenient to specify a topology by specifying a basis. The following criterion for a family to be a basis for a topology is readily verified.

**Proposition A.4** A family $\mathcal{B}$ of subsets of a set $X$ is a basis for a topology on $X$ if and only if

- The union of all members of $\mathcal{B}$ is $X$;
- For each $x \in X$ and each $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

**Example A.5** Any metric space becomes a topological space by appending the family of open sets: this space is said to be induced by the metric. The topology induced by the discrete metric consists of all subsets of the set: it is the largest topology on the set and is called the discrete topology.

The smallest topology on a set consists of the empty set and the entire set: it is the indiscrete topology.

There are two important sets associated with any given set in a topological space.

**Definition A.6** Let $X$ be a topological space and $A \subset X$. The closure of $A$ is the set $\overline{A}$ which is the intersection of all closed subsets of $X$ containing $A$. The interior of $A$ is the set $\mathcal{A}$ which is the union of all open subsets of $A$. The set $\overline{A} \cap X \setminus \mathcal{A}$ is called the frontier of $A$ and is denoted by $\text{fr}(A)$.

As an arbitrary union of open sets is open, the interior of a set is open. Dually an arbitrary intersection of closed sets is closed, so the closure is always closed. For practical reasons, especially when we are taking the interior of a complicated set, $\mathcal{A}$ is often denoted by $\text{Int}(A)$. 
Definition A.7 For the following definition, recall that a $G_δ$-subset of a space is a countable intersection of open subsets. Dually, an $F_σ$-subset is a countable union of closed subsets.

Definition A.8 Given a totally ordered set $\langle X, < \rangle$, (see Definition B.13) by an interval in $X$ we mean a subset $I \subset X$ such that whenever $x, y, z \in X$ with $x, z \in I$ and $x < y < z$ then $y \in I$. Any interval of one of the forms

$$(a, b) = \{x \in X/a < x < b\}, \quad (\neg \infty, b) = \{x \in X/x < b\},$$

$$(a, \infty) = \{x \in X/a < x\}$$

is called an open interval. The set of all open intervals forms a basis for a topology on $X$ called the order topology.

The usual order on $\mathbb{R}$ is a total order and the order topology determined by this order is the same topology as that induced by the Pythagorean metric. Unless otherwise specified we will assume that $\mathbb{R}^n$ and its subspaces are endowed with the topology induced by the Pythagorean metric defined above. We may refer to this topology as the usual topology on $\mathbb{R}^n$.

Definition A.9 A function $f : X \to Y$ between topological spaces is continuous provided that $f^{-1}(V)$ is an open subset of $X$ whenever $V$ is an open subset of $Y$. A function $h : X \to Y$ is called a homeomorphism provided that $h$ is bijective and both $h$ and its inverse $h^{-1} : Y \to X$ are continuous.

We now describe some important topological properties.

Definition A.10 A topological space $X$ is Hausdorff provided that for any two distinct points $x, y \in X$ there are disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$. The space $X$ is regular provided that for each point $x \in X$ and each closed set $C \subset X$ with $x \notin C$ there are disjoint open sets $U, V \subset X$ such that $x \in U$ and $C \subset V$. The space $X$ is normal provided that for each pair of closed sets $C, D \subset X$ with $C \cap D = \emptyset$ there are disjoint open sets $U, V \subset X$ such that $C \subset U$ and $D \subset V$. A regular, Hausdorff space is called $T_3$ while a normal, Hausdorff space is called $T_4$.

Definition A.11 Let $X$ be a topological space. Then $X$ is said to be metrisable if its topology is induced by a metric on $X$.

Metrisable spaces are always Hausdorff but there are plenty of Hausdorff spaces which are not metrisable. Indiscrete spaces of more than one point are not Hausdorff.

Definition A.12 A set $N \subset X$ for which there is an open set $U \subset X$ with $x \in U \subset N$ is called a neighbourhood of $x$. A collection $\mathcal{N}$ of neighbourhoods of a point $x$ in a topological space is called a neighbourhood basis (at $x$) provided for each neighbourhood $M$ of $x$ there is $N \in \mathcal{N}$ so that $N \subset M$. A space $X$, each point of which has a countable neighbourhood basis, is called first countable. The space $X$ is second countable provided that it has a countable basis. Suppose that $D \subset S \subset X$. Then $D$ is dense in $S$ provided $S \subset \overline{D}$. If $X$ itself has a countable dense subset then we say that $X$ is separable.
The collection of all open subsets containing \( x \) forms a neighbourhood basis. In a metric space \((X, d)\) the collection of open balls at \( x \) of radius \( \frac{1}{n} \) for \( n \) a natural number forms a neighbourhood basis at \( x \): thus metrisable spaces are first countable. Second countable spaces are first countable: given a countable basis \( \mathcal{B} \), for any point \( x \), the collection of members of \( \mathcal{B} \) containing \( x \) is a countable neighbourhood basis at \( x \). The real line \( \mathbb{R} \), indeed each euclidean space \( \mathbb{R}^n \), is second countable. Consequently by Proposition A.15 below every subset of \( \mathbb{R}^n \) is separable. Manifolds are also first countable.

The following theorem gives an important condition for a space to be metrisable.

**Theorem A.13** (Urysohn’s Metrisation Theorem) Every \( T_3 \) second countable space is metrisable.

**Definition A.14** Let \( X \) be a topological space. A family \( \mathcal{U} \) of subsets is a cover of \( X \) provided that the union of the members of \( \mathcal{U} \) is \( X \). A subfamily \( \mathcal{V} \subset \mathcal{U} \) is called a subcover provided that \( \mathcal{V} \) is also a cover of \( X \). If \( \mathcal{U} \) is a cover of \( X \) and each member of \( \mathcal{U} \) is open in \( X \) then \( \mathcal{U} \) is called an open cover of \( X \). The space \( X \) is compact provided every open cover of \( X \) has a finite subcover. The space \( X \) is locally compact provided each point of \( X \) has a compact neighbourhood. The space \( X \) is Lindelöf provided every open cover of \( X \) has a countable subcover.

Compact spaces are locally compact. On the other hand Euclidean space is locally compact but not compact. The following proposition implies that every subspace of \( \mathbb{R}^n \) is Lindelöf (as well as separable). Of course if \( \omega \) is a topological property then a space \( X \) is hereditarily \( \omega \) provided that every subspace of \( X \) has property \( \omega \).

**Proposition A.15** In a metrisable space the following six properties are equivalent.

- \( X \) is second countable;
- every subset of \( X \) is second countable;
- \( X \) is separable;
- every subset of \( X \) is separable;
- \( X \) is Lindelöf;
- every subset of \( X \) is Lindelöf.

**Definition A.16** A sequence \( (x_n) \) in a topological space \( X \) converges to a point \( x \in X \) provided that each neighbourhood of \( x \) contains all but finitely many members of \( (x_n) \). The space \( X \) is sequentially compact provided that each sequence in \( X \) has a convergent subsequence.

In a metrisable space the two properties compact and sequentially compact are the same but neither implies the other in a general topological space.

**Definition A.17** Let \( X \) be a topological space. A cover \( \mathcal{U} \) of \( X \) said to be locally finite provided that each point of \( X \) has a neighbourhood which meets only finitely many members of \( \mathcal{U} \). Given a cover \( \mathcal{U} \) of \( X \), another cover \( \mathcal{V} \) is called a refinement of \( \mathcal{U} \) provided each member of \( \mathcal{V} \) is a subset of some member of \( \mathcal{U} \). An open refinement is a refinement all of whose members are open. The space \( X \) is paracompact provided that every open cover of \( X \) has a locally finite open refinement.
Obviously every compact space is paracompact but there are many examples of paracompact spaces which are not compact. In fact the following result is found in most topology books, such as those already referred to in this appendix, so we shall not present its proof.

**Proposition A.18** Every metrisable space is paracompact.

**Definition A.19** Suppose that $A$ is some set and to each $\alpha \in A$ is assigned a topological space $(X_\alpha, T_\alpha)$. As a set the product of the sets $X_\alpha$ is

$$\prod_{\alpha \in A} X_\alpha = \{x : A \to \bigcup_{\alpha \in A} X_\alpha / x(\alpha) \in X_\alpha \text{ for each } \alpha \in A\}.$$ 

The *product of the topological spaces* $(X_\alpha, T_\alpha)$ is the topological space $(X, T)$, where $X = \prod_{\alpha \in A} X_\alpha$ and $T$ is the Tychonoff topology having as basis the collection

$$\left\{ \prod_{\alpha \in A} U_\alpha / U_\alpha \in T_\alpha \text{ for each } \alpha \in A \text{ with } U_\alpha = X_\alpha \text{ except for finitely many } \alpha \right\}.$$

When $A = \{1, \ldots, n\}$ we may write $X_1 \times \cdots \times X_n$ instead of $\prod_{i=1}^n X_i$. Thus we can identify $\mathbb{R}^n$ with the usual topology as the product of $n$ copies of $\mathbb{R}$ each with the usual topology.

As hinted in the previous paragraph there is no expectation that we must have $X_\alpha \neq X_\beta$ when $\alpha \neq \beta$. If in fact $X_\alpha = X$ for all $\alpha \in X$, then we call the corresponding product space the *topological power* and denote it by $X^A$. As examples, giving $2 = \{0, 1\}$ and $\mathbb{N}$ the respective discrete topologies then $2^{\mathbb{N}}$ is homeomorphic to the familiar Cantor set while $\mathbb{N}^\mathbb{N}$ is homeomorphic to the irrational numbers with the usual topology.

**Definition A.20** Suppose that $X$ is a topological space $X$ and $\sim$ is an equivalence relation on $X$. The *quotient* of $X$ by $\sim$ is the topological space $X/ \sim$ whose points are the $\sim$-equivalence classes and whose topology is $\{ V/\pi^{-1}(V) \text{ is open in } X \}$, where $\pi : X \to X/ \sim$ is the natural projection.

**Definition A.21** A topological space $X$ is *Tychonoff* provided for each point $x \in X$ and each closed subset $C \subset X$ with $x \notin C$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.

Every metrisable space is Tychonoff.

The following result is found in almost all books on topology so we will omit the proof.

**Proposition A.22** Every locally compact, Hausdorff space is Tychonoff.

The following result is stronger than the usual version of Tychonoff’s Embedding Theorem appearing in most text-books as the latter version does not show that the exponent $\kappa$ may be as low as the cardinality of a base and in place of $\mathcal{F}$ uses the set of *all* continuous functions $X \to [0, 1]$, which may have higher cardinality.
**Proposition A.23** (Tychonoff’s Embedding Theorem) Suppose that $X$ is a Tychonoff space having a base $\mathcal{B}$ of cardinality $\kappa \geq \omega$. Then there is an embedding $e : X \to [0, 1]^\kappa$.

**Proof** We assume that $\emptyset, X \notin \mathcal{B}$. For each pair $(U_1, U_2) \in \mathcal{B}^2$ for which it is possible we choose a continuous function $f_{(U_1, U_2)} : X \to [0, 1]$ such that $f(U_1) \subseteq [0, \frac{1}{2})$ and $f(X \setminus U_2) = \{1\}$. Let $\mathcal{F}$ consist of the collection of all such functions $f_{(U_1, U_2)}$. Then $\mathcal{F}$ has cardinality at most the cardinality of $\mathcal{B}^2$, which is $\kappa$.

For each $x \in X$ and each $U_2 \in \mathcal{B}$ for which $x \in U_2$ there is $U_1 \in \mathcal{B}$ such that $f_{(U_1, U_2)} \in \mathcal{F}$, indeed, given such $x$ and $U_2$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus U_2) = \{1\}$. Then $f^{-1}([0, \frac{1}{2}))$ is an open set containing $x$ so contains a member $U_1 \in \mathcal{B}$ which also contains $x$.

Define $e' : X \to [0, 1]^{\mathcal{F}}$ by declaring the $f_{(U_1, U_2)}$-coordinate of $e'(x)$ to be $f_{(U_1, U_2)}(x)$. It is routine to show that $e'$ is injective and continuous and has continuous inverse defined on $e'(X)$, cf. the standard way of proving Tychonoff’s Embedding Theorem in any textbook.

Since $\mathcal{F}$ has cardinality at most $\kappa$ there is an injection of $\mathcal{F}$ to $\kappa$ which gives rise to an embedding of $[0, 1]^{\mathcal{F}}$ into $[0, 1]^\kappa$ and hence, by composition with $e'$, the required embedding $e$. \hfill \Box

Incidentally, if $X$ is second countable and Tychonoff then Tychonoff’s Embedding Theorem implies that $X$ embeds in a countable power of the interval $[0, 1]$. It is a standard result that a countable product of metrisable spaces is metrisable, so Urysohn’s Metrisation Theorem A.13 is a corollary of Tychonoff’s Embedding Theorem.

**Definition A.24** A topological space is Čech-complete if it is Tychonoff and is a $G_\delta$-subset of any one (and hence all) of its compactifications.

The next result is due to Hodel [11, Proposition 5.2]. Recall that the weight of a topological space is the minimum size of a base for the space.

**Lemma A.25** Suppose that $\kappa$ is an infinite ordinal. Then there are at most $2^\kappa$ many topologically distinct Čech-complete spaces of weight $\kappa$.

**Proof** Let $X$ be a Čech-complete space of weight $\kappa$. By Proposition A.23 there is an embedding $e : X \to [0, 1]^\kappa$. Then the closure $e(X)$ of $e(X)$ in $[0, 1]^\kappa$ is a compactification of $X$ so by the definition of Čech-complete, $e(X)$ is a $G_\delta$-set in $e(X)$, equivalently, $e(X) \setminus e(X)$ is an $F_\sigma$-set in $e(X)$ and hence in $[0, 1]^\kappa$. It follows that the complement of $e(X)$ in $[0, 1]^\kappa$ is the union of an open set and an $F_\sigma$-set in $[0, 1]^\kappa$.

Let $\mathcal{S} = \{ S \subseteq [0, 1]^\kappa / S \text{ is the union of an open set and an } F_\sigma \text{-set} \}$. Because $[0, 1]$ is second countable, it follows that $[0, 1]^\kappa$ has weight at most $\kappa$ and hence $[0, 1]^\kappa$ has at most $2^\kappa$ many open sets and at most $2^\kappa$ many $F_\sigma$-sets. Thus $|\mathcal{S}| \leq 2^\kappa$. Thus there are at most $2^\kappa$ many topologically distinct Čech-complete spaces of weight $\kappa$. \hfill \Box
We denote by $2$ the discrete metric space whose underlying set is $\{0, 1\}$.

**Definition A.26** A topological space $X$ is *connected* provided every continuous function $X \to 2$ is constant.

With the exception of the discrete spaces, the spaces we have already considered are connected.

Most authors present other ways to define when a space is connected but we prefer this definition, which is close to our intuitive idea of connectedness. In any case it is equivalent to other definitions. Perhaps the most important equivalent condition is that a space $X$ is connected if and only if the only subsets of $X$ which are both open and closed are $\emptyset$ and $X$ itself.

**Definition A.27** A topological space $X$ is *path connected* provided that for each pair of points $x, y \in X$ there is a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) = x$ and $f(1) = y$. The function $f$ is called a *path* while the points $f(0)$ and $f(1)$ are called the *end points*.

Note that path connected spaces are connected but not vice versa. The standard example of a space which is connected but not path connected is the subspace of $\mathbb{R}^2$ given by

$$\left\{ (x, y) \in \mathbb{R}^2 / y = \sin \frac{1}{x} \text{ and } x > 0 \right\} \cup \{0\} \times [-1, 1].$$

In most topology books it is shown that a subspace of $\mathbb{R}$ is connected if and only if it is an interval. Consequently the two parts making up the set above are both connected as they are homeomorphic to an interval. From this fact it is not too difficult to show that any continuous function from the space above to $2$ must be constant. It is a good challenge to the beginner to show that there is no path with its end points in different parts of the space.

The following result, known as the Customs Passage Theorem, states that in order to move from outside a country, $X \setminus A$, to inside the country, $A$, a smuggler must cross the frontier. This is particularly evident when $C$ is the image of some path in $X$.

**Proposition A.28** Suppose that $C \subset X$ is connected and $A \subset X$ is such that

$$C \cap (X \setminus A) \neq \emptyset \neq C \cap A.$$  

Then $C \cap \text{fr}(A) \neq \emptyset$.

**Definition A.29** A topological space $X$ is *perfectly normal* provided that for each pair $A, B \subset X$ of disjoint closed subsets there is a continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. The subset $A$ is a *regular $G_{\delta}$ subset* provided there is a sequence $\langle U_n \rangle$ of open subsets of $X$ such that $A = \cap_{n=1}^{\infty} U_n = \cap_{n=1}^{\infty} \overline{U_n}$. 

Theorem A.30  For a topological space $X$ the following conditions are equivalent:

(a) $X$ is perfectly normal;
(b) For each closed subset $A \subset X$ there is a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$;
(c) $X$ is normal and every closed subset of $X$ is a $G_δ$ subset of $X$;
(d) Every closed subset of $X$ is a regular $G_δ$ subset of $X$.

Proof (a)$\Rightarrow$(d). Given closed $A \subset X$, apply the definition to the disjoint closed subsets $A$ and $\emptyset$ of $X$. Declare $U_n = f^{-1}\left(\left[0, \frac{1}{n}\right]\right)$. Then the sequence $\langle U_n \rangle$ exhibits that $A$ is a regular $G_δ$ subset of $X$.

(d)$\Rightarrow$(c). We need only show that $X$ is normal. Suppose that $A, B \subset X$ are disjoint and closed in $X$. Let $\langle U_n \rangle$ and $\langle V_n \rangle$ be respective sequences exhibiting the regular $G_δ$ property for $A$ and $B$. We may assume that $U_{n+1} \subset U_n$ and $V_{n+1} \subset V_n$ for each $n$. For each $n$ let $S_n = U_n \setminus V_n$ and $T_n = V_n \setminus U_n$. Then each set $S_n$ and $T_n$ is open in $X$. Hence $O = \bigcup_{n=1}^{\infty} S_n$ and $P = \bigcup_{n=1}^{\infty} T_n$ are also open subsets of $X$.

We show that $A \subset O$ and $B \subset P$. Suppose that $x \in A$: then $x \notin B$ so there is $n$ such that $x \notin V_n$; since $x \in U_n$ it follows that $x \in S_n \subset O$ so $A \subset O$. Similarly $B \subset P$.

We finally show that $O \cap P = \emptyset$. Indeed, suppose that $O \cap P \neq \emptyset$, say $x \in O \cap P$. Then there are $m, n$ so that $x \in S_m \cap T_n$. Suppose that $m \geq n$. As $x \in S_m$ it follows that $x \in U_m$. On the other hand, as $x \in T_n$ it follows that $x \notin U_n$ and hence $x \notin U_m$, giving a contradiction. Thus $O \cap P = \emptyset$.

(c)$\Rightarrow$(b). Given closed $A \subset X$, let $\langle U_n \rangle$ be open subsets with $U_{n+1} \subset U_n$ and $A = \bigcap_{n=1}^{\infty} U_n$. For each $n$ use normality of $X$ and Urysohn’s characterisation of normality to find continuous $f_n : X \rightarrow \left[0, \frac{1}{2^{n+1}}\right]$ so that $f_n(A) = 0$ and $f_n(X \setminus U_n) = \frac{1}{2^{n+1}}$. Define $f : X \rightarrow [0, 1]$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then $f$ is continuous and $f^{-1}(0) = A$ as required.

(b)$\Rightarrow$(a). Given disjoint closed sets $A, B \subset X$, suppose that $g, h : X \rightarrow [0, 1]$ are continuous functions such that $g^{-1}(0) = A$ and $h^{-1}(0) = B$. Define $f : X \rightarrow [0, 1]$ by $f(x) = \frac{g(x)}{g(x) + h(x)}$. Then $f$ is continuous and $f^{-1}(0) = A$ while $f^{-1}(1) = B$.

The following result is a slight generalisation of [16, Theorem 2], the latter being the case $C = M$. The proof follows that of Rudin and Zenor.

Lemma A.31  Let $C$ be a separable, closed subset of a perfectly normal manifold $M$. Then $C$ is hereditarily separable.

Proof  Suppose instead that $S \subset C$ is a non-separable subset of $C$. Define a sequence $\langle x_\alpha \rangle_{\alpha < \omega_1}$ of points of $S$ as follows. Let $x_0 \in S$ be arbitrary. Then $S \not\subseteq \{x_0\}$. Suppose given $\alpha < \omega_1$ such that $x_\beta \in S$ has been defined for all $\beta < \alpha$ so that $S \not\subseteq \{x_\beta / \beta < \alpha\}$. Choose any $x_\alpha \in S \setminus \{x_\beta / \beta < \alpha\}$. As $\{x_\beta / \beta \leq \alpha\}$ is countable,
it follows that $S \not\subseteq \{x_\beta / \beta \leq \alpha\}$. Replacing $S$ by $\{x_\alpha / \alpha < \omega_1\}$ if necessary, we may assume that $S = \{x_\alpha / \alpha < \omega_1\}$ and $x_\alpha \notin \{x_\beta / \beta < \alpha\}$ for all $\alpha < \omega_1$.

For each $\alpha < \omega_1$ choose a euclidean neighbourhood $N_\alpha$ of $x_\alpha$ in $M$. Then by Proposition A.15 $N_\alpha$ is hereditarily separable so $O_\alpha = C \cap N_\alpha \setminus \{x_\beta / \beta < \alpha\}$ is a hereditarily separable subset of $C$ containing $x_\alpha$. Note that $O_\alpha$ is open in the subspace $C$. Henceforth we work in the subspace $C$.

The set $S$ is not separable so cannot be covered by any countable subcollection of the collection $\{O_\alpha / \alpha < \omega_1\}$ of separable sets. Thus we may find by induction an uncountable subset $A \subseteq \omega_1$ such that if $\alpha, \beta \in A$ satisfy $\beta < \alpha$ then $x_\alpha \notin O_\beta$.

Let $T = \{x_\alpha / \alpha \in A\}$. Then $T \cap O_\alpha = \{x_\alpha\}$ for each $\alpha \in A$.

Since $M$ is perfectly normal, so is $C$ and hence by Theorem A.30 the closed subset $C \setminus (\cup_{\alpha \in A} O_\alpha)$ is a $G_\delta$ set. Thus there is a sequence $\{U_n\}$ of open subsets of $C$ such that $U_n \supseteq U_{n+1}$ for each $n$ and $\cap_{n=1}^\infty U_n = C \setminus (\cup_{\alpha \in A} O_\alpha)$. Since we have $T \subseteq \cup_{\alpha \in A} O_\alpha \Rightarrow \cap_{n=1}^\infty (C \setminus U_n)$ it follows that for some $n$ the set $A_n = \{\alpha \in A / x_\alpha \in C \setminus U_n\}$ is uncountable.

Set $S_n = \{x_\alpha / \alpha \in A_n\} \subseteq T$. We claim that $S_n$ is closed. Indeed, suppose that $x \in C \setminus S_n$. Since $U_n \cup (\cup_{\alpha \in A} O_\alpha) = C$, it follows that either $x \in U_n$ or there is some $\alpha \in A$ such that $x \in O_\alpha$. If $x \in U_n$ then $U_n$ is an open set containing $x$ but disjoint from $S_n$, so $C \setminus S_n$ is a neighbourhood of $x$. If instead $x \in O_\alpha$ then either $O_\alpha$ is an open set containing $x$ and disjoint from $S_n$ or $\alpha \in A_n$ and $O_\alpha \setminus \{x_\alpha\}$ is an open set containing $x$ and disjoint from $S_n$; in either case $C \setminus S_n$ is a neighbourhood of $x$. As a neighbourhood of each of its points, $C \setminus S_n$ is open and hence $S_n$ is closed. As $C$ is perfectly normal, by Theorem A.30 there is a continuous function $f : C \to [0, 1]$ such that $f^{-1}(0) = S_n$.

For each $\alpha \in A_n$ choose a compact set $K_\alpha \subseteq O_\alpha$ such that the component of $C \setminus K_\alpha$ containing $x_\alpha$ lies in $O_\alpha$. For example, using local compactness one might find a compact neighbourhood of $x_\alpha$ lying in $O_\alpha$ and then let $K_\alpha$ be the frontier of this compact neighbourhood. By the connected version of Proposition A.28 this choice of $K_\alpha$ will satisfy our requirements. Note that $K_\alpha \cap S_n = \emptyset$. The set $f(K_\alpha)$ is compact so there is a positive integer $m_\alpha$ such that $f(K_\alpha) \subseteq \left[\frac{1}{m_\alpha}, 1\right]$. Thus there is a positive integer $m$ and an uncountable subset $B \subseteq A_n$ such that $f(K_\alpha) \subseteq \left[\frac{1}{m}, 1\right]$ for all $\alpha \in B$.

For each $\alpha \in B$, let $V_\alpha$ be the component of $f^{-1}\left(\left[0, \frac{1}{m}\right]\right)$ containing $x_\alpha$. Then the collection $\{V_\alpha / \alpha \in B\}$ is an uncountable collection of mutually disjoint open subsets of $C$. This contradicts our assumption that $C$ is separable. Thus $C$ is hereditarily separable. \qed

There are various criteria for a function to be continuous. In the restricted environment of first countable spaces it is enough for convergent sequences to be mapped to convergent sequences.

**Lemma A.32** Suppose that $f : X \to Y$ is a function. If $f$ is continuous at $x \in X$ then whenever $(x_n)$ is a sequence in $X$ converging to $x$ the sequence $(f(x_n))$ converges to $f(x)$. The converse holds when $X$ is first countable.
A useful result in point-set topology is the fact that a continuous bijection from a compact space to a Hausdorff space has a continuous inverse and hence is a homeomorphism. Since we are often dealing with $L$ or its powers and $L^n$, and hence $L^n$ for any positive integer $n$, is sequentially compact we find the following modification of this result useful.

**Lemma A.33** Let $f : X \rightarrow Y$ be a continuous bijection from a sequentially compact space $X$ to a first countable, Hausdorff space $Y$. Then $f$ has a continuous inverse and hence is a homeomorphism.

**Proof** Suppose that $C \subset X$ is closed: it suffices to show that $f(C)$ is closed in $Y$. It is a standard fact that a subset of a first countable space is closed if and only if every convergent sequence of points of the subspace has its limit in the subspace. So suppose that $(y_n)$ is a sequence of points of $f(C)$ which converges in $Y$. For each $n$, choose $x_n \in C$ such that $f(x_n) = y_n$. As a closed subset of a sequentially compact space $C$ is also sequentially compact so $(x_n)$ has a subsequence, say $(x_{n_k})$, which converges in $C$, say to $x$. By Lemma A.32 the sequence $(f(x_{n_k}))$ converges to $f(x)$. □

The product topology leads to a natural topology on the set $C(X, Y)$ of all continuous functions from a topological space $X$ to a topological space $Y$ because $C(X, Y) \subset Y^X$ and the power may be given the product topology.

**Definition A.34** Suppose that $X$ and $Y$ are two topological spaces. The subspace topology that $C(X, Y)$ inherits from the product space $Y^X$ is called the *pointwise topology*. To indicate that $C(X, Y)$ carries this topology we write $C_p(X, Y)$.

Although useful, as seen in Theorem 2.1, the pointwise topology is unsatisfactory in the sense that the only way in which it makes use of the topology on $X$ is that not all functions $X \rightarrow Y$ need be continuous. Another important topology, which does use the topology on $X$ more directly is the following.

**Definition A.35** Suppose that $X$ and $Y$ are two topological spaces. The *compact-open topology* topology on $C(X, Y)$ is the topology for which the following sets form a sub-basis.

$$\{ (K, O) \mid K, O \subset X \text{ with } K \text{ compact and } O \text{ open} \},$$

where $(K, O) = \{ f \in C(X, Y) \mid f(K) \subset O \}$. To indicate that $C(X, Y)$ carries this topology we write $C_k(X, Y)$.

### A.2 Algebraic Topology

We will not say much about algebraic topology in this section and instead refer the interested reader to any of numerous books which introduce the topic. Some good books include [4, 9, 12, 13].
Definition A.36 Given a topological space $X$ and a specified point $a \in X$, the fundamental group of $X$ based at $a$ is the set of equivalence classes of paths $\sigma : [0, 1] \to X$ with $\sigma(0) = \sigma(1) = a$ by the equivalence relation $\sigma \sim \tau$ if there exists a continuous function $H : [0, 1] \times [0, 1] \to X$ such that $H(s, 0) = \sigma(s)$, $H(s, 1) = \tau(s)$ and $H(0, t) = H(1, t) = a$ for each $s, t \in [0, 1]$, and furnished with the binary operation $[\sigma][\tau] = [\sigma \ast \tau]$ where $\sigma \ast \tau(s) = \begin{cases} \sigma(2s) & \text{if } s \leq 1/2 \\ \tau(2s - 1) & \text{if } s \geq 1/2 \end{cases}$ where $[\sigma]$ denotes the class of $\sigma$ under the equivalence relation $\sim$.

The fundamental group of $X$ based at $a$ is denoted by $\pi(X, a)$, sometimes $\pi(X)$. A continuous function $\sigma : [0, 1] \to X$ with $\sigma(0) = \sigma(1) = a$ is called a loop (based at $a$).

One can check that the axioms for a group are satisfied, where the identity element is the equivalence class of the constant loop based at $a$ and the inverse of the class of the loop $\sigma$ is the class of the loop $\bar{\sigma}$ defined by $\bar{\sigma}(s) = \sigma(1 - s)$.

Given two spaces $X$ and $Y$ and $a \in X$ and $b \in Y$ any continuous function $f : X \to Y$ which satisfies $f(a) = b$ induces a homomorphism $f_* : \pi(X, a) \to \pi(Y, b)$.

Some standard facts about the fundamental group include the following:

- if there is a path in $X$ from $a \in X$ to $b \in X$ then $\pi(X, a)$ is isomorphic to $\pi(X, b)$ but the isomorphism is not canonical;
- if $X$ is contractible then $\pi(X)$ is trivial;
- $\pi(S^1)$ is isomorphic to $\mathbb{Z}$;
- if $X$ is in the form of the figure 8, for example $X = \{(r, \theta) / r^2 = \cos 2\theta \}$ where we are using polar coordinates in $\mathbb{R}^2$, then $\pi(X)$ is a free group on two generators.

Definition A.37 A topological pair is a pair $(X, A)$ consisting of a topological space $X$ and a subspace $A$. A map of pairs $f : (X, A) \to (Y, B)$ is a continuous function $f : X \to Y$ such that $f(A) \subset B$. A homotopy (of a map of pairs) is a continuous function of the form $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subset B$.

The topological pair $(X, \varnothing)$ will be abbreviated to $X$. The identity map is the map $1_X : (X, A) \to (X, A)$ given by $1_X(x) = x$; where no confusion will arise we will denote $1_X$ by $1$. We use $i : A \to X$ and $j : X \to (X, A)$ to denote the inclusion maps.

We will consider a class of topological pairs and maps of these pairs (technically a category of topological pairs).

Definition A.38 A homology theory assigns

- an abelian group $H_q(X, A)$ to each topological pair $(X, A)$ and each $q \in \mathbb{Z}$;
- a homomorphism $f_* : H_q(X, A) \to H_q(Y, B)$ to each map $f : (X, A) \to (Y, B)$ of pairs and each $q \in \mathbb{Z}$; and
- a homomorphism $\partial : H_q(X, A) \to H_{q-1}(A)$, called the boundary, to each topological pair $(X, A)$ and each $q \in \mathbb{Z}$

such that the following seven axioms are satisfied:
Axiom 1 (Identity) \(1_* = 1\);

Axiom 2 (Composition) \((gf)_* = g_* f_*\) whenever \(gf\) is defined;

Axiom 3 (Commutativity) \(\partial f_* = (f|A)_* \partial\) when \(f : (X, A) \to (Y, B)\);

Axiom 4 (Exactness) The sequence

\[
\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots
\]

of homomorphisms is exact, that is, the image of any one homomorphism is the kernel of the next;

Axiom 5 (Homotopy) If \(f\) is homotopic to \(g\) then \(f_* = g_*\);

Axiom 6 (Excision) If \(U \subset X\) is open with \(\overline{U} \subset A\) and \(e : (X \setminus U, A \setminus U) \to (X, A)\) is the inclusion then \(e_* : H_q(X \setminus U, A \setminus U) \to H_q(X, A)\) is an isomorphism for each \(q\) (the map \(e\) is called an excision);

Axiom 7 (Dimension) The group \(H_q([0])\) is trivial for each \(q \neq 0\).

The group \(H_0([0])\) is called the coefficient group. In our discussion below as well as our applications we assume the coefficient group is \(\mathbb{Z}\).

A standard example of a homology theory is singular homology theory which we describe briefly. For more details see such texts as [9] and [13].

Let \(\Delta^q = \{(x_0, \ldots, x_n) \in \mathbb{R}^{q+1} / \sum_{i=0}^q x_i = 1 \text{ and } x_i \geq 0 \text{ for each } i\}\). \(\Delta^q\) is called the q-simplex. Consider the points \(v_0, \ldots, v_q \in \mathbb{R}^{q+1}\), where all coordinates of \(v_i\) are 0 except the \(i + 1^\text{st}\), which is 1. Then \(\Delta^q\) is the convex hull of \(v_0, \ldots, v_q\), so we may write \(\Delta^q = \{\sum_{i=0}^q t_i v_i / t_i \geq 0 \text{ for each } i \text{ and } \sum_{i=0}^q t_i = 1\}\).

The \(i\)th face of \(\Delta^q\) is the \((q - 1)\)-simplex \(\Delta^q_{i-1} = \{\sum_{i=0}^q t_j v_j \in \Delta^q / t_i = 0\}\).

Define the face map \(F^q_i : \Delta^q_{i-1} \to \Delta^q\) by

\[
F^q_i(v_j) = \begin{cases} v_j & \text{if } j < i \\ v_{j+1} & \text{if } j \geq i \end{cases}
\]

and extend \(F^q_i\) linearly. Observe that for \(i < j\) we have \(F_j F_i = F_{j-1} F_i : \Delta^q_{i-1} \to \Delta^q_{j-1}\).

Let \(X\) be a topological space. For each \(q \geq 0\) denote by \(S_q(X)\) the free abelian group generated by \(\{\sigma : \Delta^q \to X / \sigma\text{ is continuous}\}\). A continuous function \(\sigma : \Delta^q \to X\) is called a singular q-simplex. A typical element of \(S_q(X)\) may be thought of as a formal sum \(\sum a_i \sigma_i\), where \(a_i \in \mathbb{Z}\) and \(\sigma_i\) is a singular q-simplex: such a sum is called a q-chain. Note that when \(q < 0\) there are no singular q-simplexes so \(S_q(X)\) is the trivial group in that case.

Define the face homomorphism \(V^q_i : S_q(X) \to S_{q-1}(X)\) by \(V^q_i(\sigma) = \sigma F^q_i\). For this to have meaning when \(q = 0\) we require \(V^q_i\) to be the trivial homomorphism. Observe that \(V^q_i V^q_{j+1} = V^q_j V^q_{i+1} : S_{q+1}(X) \to S_{q-1}(X)\), again
with $i < j$. The boundary homomorphism $\partial_q : S_q(X) \to S_{q-1}(X)$ is defined by $\partial_q = \sum_{i=0}^{j} (-1)^i V_i^q$.

It may be shown that $\partial_q \partial_{q+1} = 0$ for each $q \in \mathbb{Z}$, so if we set $Z_q(X) = \text{Ker}(\partial_q)$ and $B_q(X) = \text{Im}(\partial_{q+1})$ then $B_q(X) \subseteq Z_q(X)$. The elements of $Z_q(X)$ are called cycles while those of $B_q(X)$ are called boundaries. As each is an abelian group we may take the quotient $H_q(X) = Z_q(X)/B_q(X)$. The abelian group $H_q(X)$ is called the $q$th homology group of $X$.

If $X$ and $Y$ are two spaces and $f : X \to Y$ is continuous then there is induced a homomorphism $f_* : S_q(X) \to S_q(Y)$ defined by $f_*(\sigma) = f(\sigma)$. This homomorphism in turn induces a homomorphism $f_* : H_q(X) \to H_q(Y)$.

Now suppose that $(X, A)$ is a topological pair. Then the homomorphism $i_* : S_q(A) \to S_q(X)$ induced by the inclusion $i : A \to X$ is a monomorphism. Declare $S_q(X, A) = S_q(X)/i_*(S_q(A))$. When $A = \emptyset$ this reduces to $S_q(X)$. We may define a homomorphism $\partial_q : S_q(X, A) \to S_{q-1}(X, A)$ by $\partial_q([c]) = [\partial_q(c)]$. As $\partial_q \partial_{q+1} = 0$, we may define $Z_q(X, A) = \text{Ker}(\partial_q)$, $B_q(X, A) = \text{Im}(\partial_{q+1})$ and $H_q(X, A) = Z_q(X, A)/B_q(X, A)$, much as before.

If $f : (X, A) \to (Y, B)$ is a map of pairs then $f$ induces a homomorphism $f_* : H_q(X, A) \to H_q(Y, B)$.

We leave it to the reader to consult other texts to see how the boundary $\partial : H_q(X, A) \to H_{q-1}(A)$ is defined and also that the axioms are satisfied.

Some standard facts about homology groups include the following:

- in singular homology $H_0(X)$ is a free abelian group with one generator for each path component of $X$;
- if $X$ is contractible then $H_q(X)$ is trivial for all $q \neq 0$;

\[
H_q(S^n) \approx \begin{cases} 0 & \text{if } n \neq q \neq 0 \\
\mathbb{Z} & \text{if } n \neq q = 0 \text{ or } n = q \neq 0 \\
\mathbb{Z} \oplus \mathbb{Z} & \text{if } n = q = 0 
\end{cases}
\]

- if $X$ is in the form of the figure 8, for example $X = \{(r, \theta) / r^2 = \cos 2\theta\}$ where we are using polar coordinates in $\mathbb{R}^2$, then $H_1(X)$ is a free abelian group on two generators while $H_q(X)$ is trivial for $q > 1$.

Not surprisingly the homology groups are often easier to handle than the fundamental group because always homology groups are abelian. In fact for many spaces $X$ the homology group $H_1(X)$ is the quotient of the fundamental group $\pi(X)$ by its commutator subgroup.

### A.3 Some Topology of Euclidean Space

This Section presents some important results from Euclidean space topology. Frequently, except in low dimensions, it is necessary (or at least preferable) to resort to algebraic topology to prove the results. Again we do not present the proofs and
instead refer the interested reader to any of a good choice of books on algebraic
topology, for example [9] and [10].

We use the following notation: for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we let
\[
\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}; \quad \mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}; \quad \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}.
\]

**Proposition A.39** Suppose that \( e : \mathbb{B}^n \to \mathbb{S}^n \) is an embedding. Then \( \mathbb{S}^n \setminus e(\mathbb{B}^n) \) is
path connected.

Our next result is an extension of the Jordan Curve Theorem to higher dimensions. The proof of the first part of it follows from calculations of the homology groups of
\( \mathbb{R}^n \setminus e(\mathbb{S}^m) \) for an embedding \( e : \mathbb{S}^m \to \mathbb{R}^n \); in particular \( H_0 \left( \mathbb{R}^n \setminus e \left( \mathbb{S}^{n-1} \right) \right) \approx \mathbb{Z} \oplus \mathbb{Z} \) for \( n > 0 \).

**Theorem A.40** (Jordan-Brouwer Separation Theorem) Let \( e : \mathbb{S}^{n-1} \to \mathbb{R}^n \) be an
embedding. Then \( \mathbb{R}^n \setminus e \left( \mathbb{S}^{n-1} \right) \) consists of two components, each having \( e \left( \mathbb{S}^{n-1} \right) \) as
its boundary.

As in the case of the Jordan Curve Theorem, one component of \( \mathbb{R}^n \setminus e \left( \mathbb{S}^{n-1} \right) \) is
bounded in \( \mathbb{R}^n \) whereas the other is unbounded: the former is called the interior
while the latter is the exterior. We denote the interior component by \( I \left( e \left( \mathbb{S}^{n-1} \right) \right) \)
and the exterior component by \( E \left( e \left( \mathbb{S}^{n-1} \right) \right) \). It is common to replace the range \( \mathbb{R}^n \)
in the Jordan-Brouwer Separation Theorem by its one point compactification \( \mathbb{S}^n \): of
course in that case we cannot distinguish the complementary components as interior
and exterior.

The following result follows from the previous two.

**Corollary A.41** Suppose that \( e, f : \mathbb{B}^n \to \mathbb{S}^n \) are two embeddings such that
\( e \left( \mathbb{S}^{n-1} \right) = f \left( \mathbb{S}^{n-1} \right) \). Then either \( e(\mathbb{B}^n) = f(\mathbb{B}^n) \) or \( e(\mathbb{B}^n) \cap f(\mathbb{B}^n) = \emptyset \).

The following result is usually deduced from the Jordan-Brouwer Separation
Theorem.

**Theorem A.42** (Invariance of Domain) Suppose that \( U \subset \mathbb{R}^n \) is open and \( f : U \to \mathbb{R}^n \) a continuous injection. Then \( f(U) \) is open and \( f \) is an embedding.

**Corollary A.43** There is no neighbourhood of \((0, \ldots, 0)\) in \( \mathbb{R}^n_+ \) which is homeo-
morphic to \( \mathbb{R}^n \).

**Proof** Indeed, if there were then there would be a continuous injection \( f : \mathbb{R}^n \to \mathbb{R}^n_+ \) such that \( f(\mathbb{R}^n) \) is a neighbourhood of \((0, \ldots, 0)\) in \( \mathbb{R}^n_+ \). By Theorem A.42 \( f(\mathbb{R}^n) \) is
open in \( \mathbb{R}^n \) and hence is also a neighbourhood of \((0, \ldots, 0)\) in \( \mathbb{R}^n \). However this con-
tradicts the fact that \((0, \ldots, 0)\) is on the frontier of \( \mathbb{R}^n_+ \) and hence any neighbourhood
in \( \mathbb{R}^n \) meets \( \mathbb{R}^n_+ \). \( \square \)

In dimension 2 the Jordan Curve Theorem may be strengthened to the Schoenflies
Theorem: see, for example, [14, Theorem 6, p. 68]. We note in passing that, unlike
the Jordan-Brouwer Separation Theorem, the Schoenflies Theorem does not extend
to higher dimensions, Alexander’s Horned Sphere [1], providing a counterexample
in dimension 3.
Theorem A.44 (Schoenflies Theorem) Let \( e : \mathbb{S}^1 \rightarrow \mathbb{R}^2 \) be an embedding. Then \( e \) extends to a homeomorphism \( \hat{e} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

A.4 Compact Surfaces

We recount important features pertaining to compact surfaces, including their classification and algebraic topology. Compact, connected surfaces were classified gradually over the decades 1860–1920. There are two sequences of compact, connected surfaces: orientable and non-orientable. The sequence of orientable surfaces starts with the sphere \( \mathbb{S}^2 \) and the torus \( \mathbb{T}^2 \) while the sequence of non-orientable surfaces starts with the projective plane and the Klein bottle. Compact, connected surfaces with boundary are obtained from compact, connected surfaces by taking a finite collection of disjoint closed discs and removing their interiors. Although there is a complete classification available, finding a complete proof is more of a challenge and the interested reader is referred to [7]. There are many other sources for at least partial proofs, for example [6], [8, p. 204 and 226] and [14, p. 161]. Appendix D of [7] gives a brief history from Möbius in 1861 to Brahma in 1921. A major step is showing that compact surfaces are able to be given a combinatorial or a differential structure either of which is then used to analyse the possibilities. We will just list the possibilities. We shall also list the fundamental and homology groups of these surfaces.

Definition A.45 A surface is a connected, Hausdorff space each point of which has a neighbourhood homeomorphic to \( \mathbb{R}^2 \).

A surface with boundary is a connected, Hausdorff space each point of which has a neighbourhood homeomorphic either to \( \mathbb{R}^2 \) or to \( \mathbb{R}^2_+ \).

Example A.46 For each non-negative integer \( n \) let \( \Sigma_n \) be obtained from the 2-sphere \( \mathbb{S}^2 \) as follows. Choose \( n \) pairs of mutually disjoint closed discs embedded in \( \mathbb{S}^2 \). Assume that they are nicely embedded in the sense that their boundaries are round circles of small radius and any two in a pair are close to each other. For any pair of discs, remove the interiors from \( \mathbb{S}^2 \) and then join the remaining boundaries by a cylinder which lies outside \( \mathbb{S}^2 \) except for the end circles of the cylinder which are identified with the boundaries of the discs which have been removed. The resulting surface is \( \Sigma_n \) and is called a sphere with \( n \) handles. See Fig. A.1 for a depiction of \( \Sigma_5 \). The surface \( \Sigma_n \) is orientable in the sense that it is two-sided: if we start on the outer surface we can never get to the inner surface by moving around on the surface. \( \Sigma_1 \) is homeomorphic to the torus \( \mathbb{S}^1 \times \mathbb{S}^1 \).

Example A.47 For each positive integer \( n \) let \( \Gamma_n \) be obtained from the 2-sphere \( \mathbb{S}^2 \) as follows. Choose \( n \) mutually disjoint closed discs embedded in \( \mathbb{S}^2 \). Assume that they are nicely embedded in the sense that their boundaries are round circles of small radius. For any disc, remove the interiors from \( \mathbb{S}^2 \) and then identify each point on the
remaining boundary with the antipodal point on the boundary. The resulting surface is $\Gamma_n$ and is called a sphere with $n$ crosscaps. The surface $\Gamma_n$ is non-orientable in the sense that it is one-sided: if we start on the outer surface we can get to the inner surface by moving around on the surface as each cross cap twists the surface much as a Möbius band does: in fact if we removed two discs as above but only identified antipodal points on the boundary of one then the resulting surface with boundary would be a Möbius band. $\Gamma_1$ is homeomorphic to the projective plane while $\Gamma_2$ is homeomorphic to the Klein bottle.

Example A.48 If we take either $\Sigma_n$ of Example A.46 or $\Gamma_n$ of Example A.47 and remove a further $m$ small mutually disjoint discs we obtain a surface with boundary, which we denote respectively by $\Sigma_{n,m}$ and $\Gamma_{n,m}$. The boundaries of $\Sigma_{n,m}$ and $\Gamma_{n,m}$ consist of $m$ disjoint circles.

In fact we have already described all possible compact surfaces with boundary. As already noted [7] is a thorough source of a proof. On the other hand [6] contains a brief, well-illustrated proof but like most proofs it does glide over some important details.

Theorem A.49 Every compact surface with boundary is of exactly one of the following two types:

- $\Sigma_{n,m}$ for $n, m \in \omega$;
- $\Gamma_{n,m}$ for $n \in \mathbb{N}$ and $m \in \omega$.

Note that we exclude $n = 0$ in the second case because $\Gamma_0$ is the same as $\Sigma_0 = \mathbb{S}^2$.

It is usual to call $n$ in $\Sigma_{n,m}$ or $\Gamma_{n,m}$ the genus of the surface (with boundary).

Next we discuss briefly the fundamental group of these surfaces. The basic ideas are the following. We can generate the fundamental group of $\Sigma_n$ by $2n$ loops as follows. Not surprisingly each handle supplies us with two loops, one going along the handle and the other around the handle; of course each loop begins at and returns to some base point. If we denote the homotopy class of the loop going along (around)
the \( i \)th handle by \( a_i \) (\( b_i \)) then there is a relation as well and the fundamental group has the presentation \( \langle a_1, b_1, \ldots, a_n, b_n \mid \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \rangle \): it is an interesting exercise to see how the geometry works and the reader is invited to try it on \( \Sigma_1 \) and \( \Sigma_2 \). In the case of \( \Sigma_1 \), if we cut the surface along the two loops \( a_1 \) and \( b_1 \) what remains is a disc and if we trace a curve near the boundary of this disc we see that this curve is not only homotopically trivial but also is of the form \( a_1 b_1 a_1^{-1} b_1^{-1} \) or some cyclic permutation. In the case of \( \Sigma_2 \) we may do the same, including cutting along \( a_2 \) and \( b_2 \), but need also to cut along a curve from the intersection of the two curves \( a_1 \) and \( b_1 \) to the intersection of the curves \( a_2 \) and \( b_2 \) via the base point.

On the other hand for \( \Gamma_n \) we need a single generator \( c_i \) for each crosscap. The generator \( c_i \) corresponds to a loop running from the base point to the boundary of the \( i \)th disc into which the crosscap was inserted then returning to the base point along the same route as we came to the base point. Notice that because antipodal points of the boundary of the \( i \)th disc were identified to form the crosscap, once we have travelled half way around the boundary we are back at the beginning point. If we travel the whole way around the boundary of the disc in the case of \( \Gamma_1 \) then, much as in the case of \( \Sigma_1 \), the loop is homotopically trivial. On the other hand this is equivalent to travelling twice around \( c_1 \), so \( c_1^2 = 1 \). For \( \Gamma_n \), again much as for \( \Sigma_n \), we traverse each of the crosscap loops twice in order to obtain \( c_1^2 \ldots c_n^2 = 1 \). The fundamental group of \( \Gamma_n \) has the presentation \( \langle c_1, \ldots, c_n \mid \prod_{i=1}^{n} c_i^2 \prod_{i=1}^{m} d_i \rangle \).

Whether we are dealing with \( \Sigma_{n,m} \) or \( \Gamma_{n,m} \), the extra complication of having boundary components is dealt with in much the same way: a loop following around the handles or crosscaps as above then around each of the \( m \) holes will represent the sole relation. This leads to the following theorem.

**Theorem A.50** The fundamental groups of compact surfaces with boundary have the following presentations:

- \( \pi(\Sigma_{n,m}) \approx \langle a_1, b_1, \ldots, a_n, b_n, d_1, \ldots, d_m \mid \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \prod_{i=1}^{m} d_i \rangle; \)
- \( \pi(\Gamma_{n,m}) \approx \langle c_1, \ldots, c_n, d_1, \ldots, d_m \mid \prod_{i=1}^{n} c_i^2 \prod_{i=1}^{m} d_i \rangle. \)

Of course we have not presented anywhere near a complete proof, especially that there need be no further generators or relations. The interested reader is referred to [3, Theorem 3.1.5] and [7, Sect. 6.6], for example.

In the case where \( m > 0 \), i.e., there really is a boundary component, we may solve either \( \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1} \prod_{i=1}^{m} d_i = 1 \) or \( \prod_{i=1}^{n} c_i^2 \prod_{i=1}^{m} d_i = 1 \) for \( d_m \) and thereby delete the generator \( d_m \) and the single generator to get the following.

**Corollary A.51** The fundamental group of a compact surface with non-empty boundary is free.

Using standard results from homology theory (for example the first homology group is obtained by abelianising the fundamental group), we obtain the following.

**Theorem A.52** The homology groups with integer coefficients of compact surfaces with boundary are as follows:
• $H_q(Σ_{n,m}) ≈ \begin{cases} 
\mathbb{Z} & \text{if } q = 0 \text{ or } q = 2 \text{ and } m = 0 \\
\mathbb{Z}^2 & \text{if } q = 1 \text{ and } m = 0 \\
\mathbb{Z}^2 \oplus \mathbb{Z}^{n-1} & \text{if } q = 1 \text{ and } m > 0 \\
0 & \text{otherwise}; 
\end{cases}

• $H_q(Γ_{n,m}) ≈ \begin{cases} 
\mathbb{Z} & \text{if } q = 0 \\
\mathbb{Z}_2 \oplus \mathbb{Z}^{n-1} & \text{if } q = 1 \text{ and } m = 0 \\
\mathbb{Z}^n \oplus \mathbb{Z}^{m-1} & \text{if } q = 1 \text{ and } m > 0 \\
0 & \text{otherwise.} 
\end{cases}$

References

1. Alexander, J.W.: An example of a simply connected surface bounding a region which is not simply connected. Proc. NAS 10, 8–10 (1924)
Appendix B
Set Theory Preliminaries

In this appendix we introduce some of the more relevant notions from Set Theory, beginning with cardinality, ordered sets and trees. We then discuss the countable ordinals and their properties, especially Fodor’s Lemma, closed unbounded sets and stationary sets. We finish the Appendix with a brief mention of the Continuum Hypothesis CH and Martins Axiom MA. The reader may refer to such books as [2, 4, 5].

B.1 Cardinality, Ordered Sets and Trees

In this section we discuss briefly cardinality of sets, a generalisation of the number of elements in a finite set. We then consider ordered sets of various kinds and finish with a brief mention of trees. More details of these topics may be found in [2], [4] and [5] or any other book on introductory Set Theory.

Firstly we should mention that we work in ZFC, assuming the axioms of Zermelo-Fraenkel, ZF, which we shall not state here. The C part of ZFC refers to the Axiom of Choice which we do state and, like the ZF axioms, use unashamedly.

**Axiom B.1** (Axiom of Choice) A non-empty product of non-empty sets is non-empty. More precisely, if \( A \neq \emptyset \) and for each \( \alpha \in A \), \( X_\alpha \) is a non-empty set then \( \prod_{\alpha \in A} X_\alpha \) is non-empty. An element of \( \prod_{\alpha \in A} X_\alpha \) is called a choice function.

**Definition B.2** Suppose that \( S \) and \( T \) are two sets. Write \( |S| \leq |T| \), read the cardinality of \( S \) is at most that of \( T \), when there is an injective function \( S \to T \). Write \( |S| = |T| \), read \( S \) and \( T \) have the same cardinality, when there is a bijection \( S \to T \). A set \( S \) is countable if and only if \( |S| \leq |N| \); otherwise it is uncountable.

The Schröder-Bernstein Theorem ensures antisymmetry of the relation \( \leq \). We do not present the proof here but recommend the interested reader seek out the rather clever yet simple proof.
Theorem B.3 (Schröder-Bernstein Theorem) Suppose that S and T are two sets such that $|S| \leq |T|$ and $|T| \leq |S|$. Then $|S| = |T|$.

Here are some more standard facts concerning cardinality. Again we do not present the proofs.

Lemma B.4 There is a bijection $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$.

Corollary B.5 The set $\mathbb{Q}$ of rational numbers is countable.

Corollary B.6 A countable union of countable sets is countable.

In contrast with Corollary B.5, Cantor showed that $\mathbb{R}$ is uncountable. In fact it may be shown that $|\mathbb{R}| = |2^\mathbb{N}|$, where $2^S$ denotes the set of all functions from $S$ to a two-point set, so that uncountability of $\mathbb{R}$ may also be deduced from the next result.

Without going into the details we note in passing that there is a systematic way of labelling infinite cardinalities (more properly, cardinals). The least infinite cardinal is denoted by $\aleph_0$: so $|\mathbb{N}| = \aleph_0$. Then one may inductively define $\aleph_\alpha$ for any ordinal $\alpha$: in particular $\aleph_1$ is the least uncountable cardinal. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$: thus $\mathfrak{c} > \aleph_0$.

Proposition B.7 For any set $S$ we have $|S| < |2^S|$.

Identifying a function $f : S \rightarrow \{0, 1\}$ with the subset $\{s \in S / f(s) = 1\}$ determines a natural bijection between $2^S$ and the power set of $S$, so they have the same cardinality and frequently we may label the power set by $2^S$.

Definition B.8 Suppose that $S$ and $T$ are two sets. Then the sum, product and power of their cardinalities are defined as follows:

- $|S| + |T| = |(S \times \{0\}) \cup (T \times \{1\})|$;
- $|S| \cdot |T| = |S \times T|$;
- $|T|^{|S|} = |T^S|$, where $T^S$ denotes the set of all functions $S \rightarrow T$.

For the following theorem we adopt the usual convention of denoting cardinals by Greek letters $\kappa$, $\lambda$, etc.

Theorem B.9 For any cardinals $\kappa$, $\lambda$ and $\mu$, with $\lambda$ and $\mu$ infinite, we have the following:

(i) $\kappa + \lambda = \sup\{\kappa, \lambda\}$;
(ii) $\kappa \cdot \lambda = \sup\{\kappa, \lambda\}$;
(iii) if $2 \leq \kappa \leq \lambda$ then $\kappa^\lambda = \lambda^\lambda$;
(iv) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

Lemma B.10 Let $S$ be a set with $|S| = \kappa$ for some infinite cardinal. Then the cardinality of the collection of subsets of $S$ having cardinality less than $\kappa$ is also $\kappa$. 

The cardinality of closed uncountable subsets of \( \mathbb{R} \) is interesting: looking at the sets individually we find that they each have cardinality \( c \); looking at them collectively we find that there are \( c \) many of them.

**Lemma B.11** Let \( C \subset \mathbb{R} \) be closed and uncountable. Then \( |C| = c \).

**Proof** The proof is in two stages. Firstly we show that every uncountable closed subset of \( \mathbb{R} \) contains an uncountable perfect set, i.e., one which is closed and each of its points is a limit point of the set.

Suppose that \( C \subset \mathbb{R} \) is closed and uncountable. Let

\[
D = \{ x \in C \mid (x - \varepsilon, x + \varepsilon) \cap C \text{ is uncountable for each } \varepsilon > 0 \}.
\]

Note that \( D \subset C \) and that \( D \) is perfect. Furthermore \( C \setminus D \) is countable because if \( x \in C \setminus D \) then there is an open interval \( (a_x, b_x) \subset \mathbb{R} \) containing \( x \) so that \( a_x, b_x \in \mathbb{Q} \) and \((a_x, b_x) \cap C \) is countable. Hence \( C \setminus D \subset \bigcup_{x \in C \setminus D} (a_x, b_x) \cap C \). By Corollary B.5 and Lemma B.4 there are at most countably many sets making up the union because there are only countably many possibilities for the points \( a_x \) and \( b_x \). Since each of the sets \((a_x, b_x) \cap C \) is countable it follows from Corollary B.6 that \( C \setminus D \) is countable. Then \( D \) is uncountable.

The second stage of the proof is to construct an injection of \( 2^\mathbb{N} \) into the perfect set \( D \). From this it will follow from Theorem B.3 that \( |C| = c \). By induction on \( n \in \mathbb{N} \) we construct points \( x_\sigma \) and positive real numbers \( r_\sigma \) for each finite sequence \( \sigma : \{1, \ldots, n\} \rightarrow \{0, 1\} \) so that for each such \( \sigma \),

- \( x_\sigma \in D \);
- \( r_\sigma \leq \frac{1}{2^n} \);
- \( [x_\sigma - r_\sigma, x_\sigma + r_\sigma] \subset (x_\tau - r_\tau, x_\tau + r_\tau) \) whenever there is \( m < n \) such that \( \tau = \sigma \restriction \{1, \ldots, m\} \);
- \( (x_\sigma - r_\sigma, x_\sigma + r_\sigma) \cap (x_\tau - r_\tau, x_\tau + r_\tau) = \emptyset \) whenever neither of \( \sigma \) and \( \tau \) is a restriction of the other.

Induction begins at \( n = 0 \) (so there is only one choice for \( \sigma \), viz the empty sequence \( \emptyset \)) by choosing \( x_\emptyset \in D \) arbitrarily and \( r_\emptyset = 1 \).

Suppose that \( x_\sigma \) and \( r_\sigma \) have been chosen whenever the sequence \( \sigma \) has domain a set of the form \( \{1, \ldots, m\} \) for \( m \leq n \). Let \( \sigma : \{1, \ldots, n\} \rightarrow \{0, 1\} \) be arbitrary. Then \( \sigma \) has two possible extensions to \( \{1, \ldots, n + 1\} \), viz \( \sigma_0, \sigma_1 \) defined by setting \( \sigma_i(n + 1) = i \). We need to choose \( x_{\sigma_0} \) and \( r_{\sigma_i} \) for \( i = 0, 1 \). Because \( D \) is perfect the set \( (x_\sigma - r_\sigma, x_\sigma + r_\sigma) \cap D \) is infinite so we may choose two distinct points \( x_{\sigma_0}, x_{\sigma_1} \in (x_\sigma - r_\sigma, x_\sigma + r_\sigma) \cap D \) so that \( x_{\sigma_0} \neq x_{\sigma} \neq x_{\sigma_1} \). Now we may choose \( r_{\sigma_i} \leq \frac{1}{2^{n+1}} \) such that \( [x_{\sigma_i} - r_{\sigma_i}, x_{\sigma_i} + r_{\sigma_i}] \subset (x_\sigma - r_\sigma, x_\sigma + r_\sigma) \) for \( i = 0, 1 \) and \( (x_{\sigma_0} - r_{\sigma_0}, x_{\sigma_0} + r_{\sigma_0}) \cap (x_{\sigma_1} - r_{\sigma_1}, x_{\sigma_1} + r_{\sigma_1}) = \emptyset \). Then the conditions are satisfied for \( \sigma_0 \) and \( \sigma_1 \) and, since \( \sigma \restriction \{1, \ldots, n\} \rightarrow \{0, 1\} \) was arbitrary, the induction continues.

The injection \( 2^\mathbb{N} \rightarrow D \) may be constructed as follows. Given \( \sigma : \mathbb{N} \rightarrow \{0, 1\} \), for each \( n \in \mathbb{N} \) let \( \sigma_n = \sigma \restriction \{1, \ldots, n\} \). Then the sequence \( \langle x_{\sigma_n} \rangle \) of points of \( D \) converges
to a point \( x_\sigma \in D \). It is readily verified that the function \( 2^\mathbb{N} \to D \) defined by \( \sigma \mapsto x_\sigma \) is injective.

**Lemma B.12** Let \( \mathcal{C} \) be the collection of all closed and uncountable subsets of \( \mathbb{R} \). Then \(|\mathcal{C}| = c\).

**Proof** Since the usual topology on \( \mathbb{R} \) is second countable, for example the collection of open intervals with rational end points is a basis for the topology and is countable by Lemma B.4, it follows that this topology has at most \( 2^{\aleph_0} = c \) many open sets. So \(|\mathcal{C}| \leq c\). On the other hand the collection \( \{[0, r] / r \in \mathbb{R}\} \) consists of uncountable closed subsets of \( \mathbb{R} \) and is clearly of cardinality \( c \) so \( c \leq |\mathcal{C}| \). □

Next we look at ordered sets.

**Definition B.13** A partially ordered set is a pair \((X, \leq)\) consisting of a set \( X \) and a partial order \( \leq \) on \( X \), i.e., a binary relation satisfying the properties:

- for each \( a \in X \), \( a \leq a \);
- for each \( a, b \in X \), if \( a \leq b \) and \( b \leq a \) then \( a = b \);
- for each \( a, b, c \in X \), if \( a \leq b \) and \( b \leq c \) then \( a \leq c \).

Let \((X, \leq)\) be a partially ordered set and let \( x, y \in X \). Write \( x < y \) to mean \( x \leq y \) and \( x \neq y \). Say that \( x \) is a predecessor of \( y \) and \( y \) a successor of \( x \) provided that \( x \leq y \). If \( x \in X \) is such that there is no \( y \in X \) with \( x < y \) (respectively \( y < x \)) then \( x \) is a maximal element (respectively minimal element).

A totally ordered set is a partially ordered set \((X, \leq)\) satisfying the further condition:

- for each \( a, b \in X \) either \( a \leq b \) or \( b \leq a \).

A well-ordered set is a totally ordered set \((X, \leq)\) satisfying the further condition:

- for each non-empty \( Y \subset X \) there is \( \alpha \in Y \) such that for each \( y \in Y \) we have \( \alpha \leq y \). (In this case \( \alpha \) is the least member of \( Y \).)

A subset \( S \subset X \) of a well-ordered set is said to be bounded provided there is \( \alpha \in X \) such that \( y \leq \alpha \) for all \( y \in S \). In such a case call \( \alpha \) an upper bound for \( S \). If \( S \) is bounded above then \( S \) has a least upper bound.

Note that a non-empty well-ordered set has a least member, usually denoted by \( 0 \). Because of this we do not need to follow ‘bounded’ in the definition above by the word ‘above.’

The set of non-negative integers with the usual order forms a well-ordered set. This set is denoted by \( \omega \). On the other hand, while the usual order on the real line \( \mathbb{R} \) is a total order it is not a well-order, for example the set of positive reals has no least element. However the **Well-Ordering Principle** asserts that every set may be well-ordered; in particular there is some total order on \( \mathbb{R} \) with respect to which \( \mathbb{R} \) is well-ordered.

While the Well-Ordering Principle might seem rather strong, it is equivalent in \( ZF \) to the Axiom of Choice, Axiom B.1, and to Zorn’s Lemma.
Lemma B.14 (Zorn’s Lemma) Let \((X, \leq)\) be a non-empty partially ordered set such that each subset \(Y \subset X\) for which \((Y, \leq)\) is a totally ordered set has an upper bound in \(X\). Then \(X\) has a maximal element.

Definition B.15 A subset \(B \subset \mathbb{R}\) is called a Bernstein set provided that every uncountable closed subset of \(\mathbb{R}\) meets both \(B\) and \(\mathbb{R} \setminus B\).

Proposition B.16 There exist Bernstein subsets of \(\mathbb{R}\).

Proof Let \(\mathcal{C}\) be the collection of all closed and uncountable subsets of \(\mathbb{R}\). By Lemma B.12 and the Well-Ordering Principle we may well-order \(\mathcal{C}\) by the ordinals less than \(c\): i.e., we may write \(\mathcal{C} = \{C_\alpha / \alpha < c\}\). We also require a well-order on \(\mathbb{R}\) which we denote by \(<\). Inductively we will construct two sequences \(\langle x_\alpha \rangle_{\alpha < c}\) and \(\langle y_\alpha \rangle_{\alpha < c}\) so that \(x_\alpha, y_\alpha \in C_\alpha\) and no two members of either sequence are the same.

Begin the induction by selecting \(x_0\) to be the first and \(y_0\) the second elements of \(C_0\) under the order \(<\). Now suppose that \(\alpha < c\) and \(x_\beta\) and \(y_\beta\) have been chosen for each \(\beta < \alpha\). The set \(\{x_\beta / \beta < \alpha\} \cup \{y_\beta / \beta < \alpha\}\) has cardinality less than \(c\) so by Lemma B.11 \(C_\alpha \setminus (\{x_\beta / \beta < \alpha\} \cup \{y_\beta / \beta < \alpha\})\) is non-empty: let \(x_\alpha\) and \(y_\alpha\) be the first and second elements of \(C_\alpha \setminus (\{x_\beta / \beta < \alpha\} \cup \{y_\beta / \beta < \alpha\})\) under the order \(<\). This completes the induction. Note that for each \(\alpha, \beta < c\) with \(\alpha \neq \beta\), no two of \(x_\alpha, x_\beta, y_\alpha\) and \(y_\beta\) are the same.

Set \(B = \{x_\alpha / \alpha < c\}\). Then \(B\) is a Bernstein set. \(\square\)

The final topic in this section is trees.

Definition B.17 A tree is a partially ordered set \((T, \leq)\) in which the set of predecessors of each element is well-ordered by \(\leq\). A branch of a tree \((T, \leq)\) is a maximal totally ordered subset of \(T\). If \(x \in T\) then any branch containing \(x\) which also contains successors of \(x\) contain a least successor, called an immediate successor. Every non-empty(!) tree has a unique least element, called the root.

Given a tree \((T, \leq)\) and \(x \in T\) one may think of \(T\) sending out a branch to each of its immediate successors. Note, though, that a branch not only includes elements above a given element but also the trunk below.

Example B.18 Suppose that \((X, \leq)\) is a well-ordered set and \(Y\) is any set. For \(x \in X\) let \(\overset{x}{\sim} = \{y \in X / y < x\}\). Let \(T = \{f : \overset{x}{\sim} \to Y / x \in X\}\), and declare \(f \leq g\), for \(f, g \in T\), provided that the domain of \(g\) contains the domain of \(f\) and that \(f\) and \(g\) agree on the domain of \(f\). Then \((T, \leq)\) is a tree.

The height of elements of a tree and of the tree itself is defined by reference to ordinals, which are touched on in the next section. The root of a tree is at height 0 and inductively given an element at known height \(\alpha\) its immediate successors all have height \(\alpha + 1\). As we shall see in the next section transfinite induction requires also
consideration of each limit when all predecessors of that limit have been inductively assumed to satisfy the requirement sought: essentially it is a matter of taking the smallest ordinal compatible with the requirement that if \( x < y \) then the height of \( x \) should be strictly less than that of \( y \). The best way to handle this is using ordinal types, as described in [5, p. 117]. Order type also gives a tidy way to define the height of a tree.

In the case where the well-ordered set \((X, \leq)\) of Example B.18 contains no element having \(|X|\) many predecessors, the tree \((T, \leq)\) has height \(|X|\) when \(X\) is infinite. A special case of this tree is when \(Y = \{0, 1\}\) is a two-point set, in which case we have a binary tree of height \(|X|\): the function \(\emptyset : \emptyset \to \{0, 1\}\) is the least element of \(T\) and, like other elements of \(T\) except those whose domains consist of all of the predecessors of a maximal element of \(X\), has precisely two immediate successors.

In certain cases one can estimate the cardinality of the number of branches of a tree. As a simple example, take the binary tree \((T, \leq)\) of height \(n\) where \(X = \{0, 1, \ldots, n\}\): it has \(2^n\) branches. If \(X = \mathbb{N}\), with the usual order, then \((T, \leq)\) has \(|2^{\aleph_0}| = \mathfrak{c}\) many branches. In Proposition 7.13 we consider the \(\mathfrak{c}\)-branching tree of height \(\aleph_1\): it has \(\mathfrak{c}^{\aleph_1}\) many branches and, by Theorem B.9(iv) and (ii), we have \(\mathfrak{c}^{\aleph_1} = (2^{\aleph_0})^{\aleph_1} = 2^{\aleph_1}\).

### B.2 The Countable Ordinals \(\omega_1\)

This section introduces the set of countable ordinals, \(\omega_1\), and some of its properties. The references [2, 4, 5] provide a more formal introduction to the ordinals than we give here and the interested reader is strongly urged to refer to those, or other, books for more details. The way in which set theorists construct sets of ordinals means that it is natural to denote the set of countable ordinals by \(\omega_1\) but also denote the first uncountable ordinal by the same symbol \(\omega_1\): so \(\omega_1\) denotes an uncountable set of ordinals and also a single ordinal at the same time. We follow this practice and hope that it causes the reader no confusion.

Each element of \(\omega\) has only finitely many predecessors but \(\omega\) itself is infinite. Going up a step, we let \(\omega_1\) denote the set of countable ordinals: each member of \(\omega_1\) has only countably many predecessors but \(\omega_1\) itself is uncountable.

If you do not want to go through the formalities of constructing \(\omega_1\) but are happy with the Well-Ordering Principle then you may think of it in the following way. Take any uncountable set \(X\). The Well-Ordering Principle assures us that there is a well-order on \(X\); denote it by \(\leq\). If there is a member of \(X\) having uncountably many predecessors then the last property for a well-order tells us that there is a least such element of \(X\); in that case replace \(X\) by the predecessors of this least element. If there is no member of \(X\) having uncountably many predecessors then make no replacement. In either case we obtain an uncountable well-ordered set each element of which has only countably many predecessors. However we may have well-ordered \(X\) the resulting sets will all be order isomorphic, i.e., there will be an order-preserving
bijection between the resulting sets. We may think of $\omega_1$ as being this well-ordered set.

Notice that if an element of a well-ordered set has any successors then it has a least successor, called the immediate successor: the immediate successor of $x$ is denoted by $x + 1$. An element of a well-ordered set which is neither an immediate successor nor 0 is called a limit. Some authors allow 0 to be called a limit ordinal too. Sometimes it will be convenient to denote the successor of the successor of $\alpha$ by $\alpha + 2$ and so on. This allows us to write any ordinal in the form $\lambda + n$ where $\lambda$ is a limit ordinal or 0 and $n \in \omega$.

We can construct $\aleph_1$ many limit ordinals in $\omega_1$ as follows. Given $\lambda \in \omega_1$ such that $\lambda$ is either a limit ordinal or 0, we note that $\lambda + n$ for any $n \in \omega$, has only countably many predecessors. Hence by Corollary B.6 the set $\bigcup_{n \in \omega} \{ \alpha \in \omega_1 / \alpha < \lambda + n \}$ is countable so has a non-empty complement. The least element of this complement is a limit ordinal larger than $\lambda$.

The Principle of Mathematical Induction exploits the well-ordering on $\omega$ in the following sense: a subset $S \subset \omega$ which contains 0 and also satisfies the condition $n \in S \Rightarrow n + 1 \in S$ for each $n \in \omega$ must be all of $\omega$. Indeed, if $S \neq \omega$ then $\omega \setminus S$ contains a least member, say $n$. We cannot have $n = 0$, so there is $m \in \omega$ such that $n = m + 1$. Then $m \in S$ so $m + 1 \in S$ yields a contradiction.

The Principle extends to $\omega_1$ but there is an extra requirement because if $S \neq \omega_1$ then the least member of $\omega_1 \setminus S$ may be a limit ordinal. Thus the Principle of Induction for $\omega_1$ states that if $S \subset \omega_1$ satisfies the three conditions

- $0 \in S$;
- if $\alpha \in S$ then $\alpha + 1 \in S$;
- if $\lambda$ is a limit ordinal and $\alpha \in S$ for every $\alpha < \lambda$ then $\lambda \in S$,

then $S = \omega_1$. Often the second and third conditions are combined into a slightly stronger single condition:

- if $\alpha \in \omega_1$ is such that $\beta \in S$ for every $\beta < \alpha$ then $\alpha \in S$.

The same idea applies to construction by induction: see Proposition B.31 for a simple example.

Now give $\omega_1$ the order topology defined in Definition A.8. Note that if $\beta \in \omega_1$ is a successor ordinal or 0 then $\{ \beta \}$ is an open subset while if $\beta$ is a limit ordinal then any open subset containing $\beta$ must also contain all elements in some interval of the form $(\alpha, \beta] = \{ \gamma \in \omega_1 / \alpha < \gamma \leq \beta \}$.

**Lemma B.19** In $\omega_1$ with the order topology every increasing sequence converges.

**Proof** Suppose that $\langle \alpha_n \rangle$ is an increasing sequence in $\omega_1$: we may assume that $\langle \alpha_n \rangle$ is strictly increasing. For each $n$ the element $\alpha_n$ has countably many predecessors. By Corollary B.6 the set $S = \cup_{n \in \omega} \{ \beta \in \omega_1 / \beta \leq \alpha_n \}$ is bounded above and hence has a least upper bound, say $\alpha \in \omega_1$. The element $\alpha$ cannot be a successor ordinal because if $\alpha = \beta + 1$, for some $\beta \in \omega_1$, then $\alpha_n < \beta$ for each $n$ (recall that $\langle \alpha_n \rangle$ is strictly increasing) so $\beta$ is also an upper bound for $S$, contradicting $\alpha$ being the least upper bound. Then $\alpha = \lim_{n \to \infty} \alpha_n$. □
Lemma B.20  In \( \omega_1 \) with the order topology every decreasing sequence is eventually constant.

Proof  Suppose that \( \langle \alpha_n \rangle \) is a decreasing sequence in \( \omega_1 \) with \( \alpha_0 = \alpha \); we prove the statement by induction on \( \alpha \). If \( \alpha = 0 \) then the statement is obviously true as then \( \alpha_n = 0 \) for all \( n \). Suppose \( \alpha > 0 \). Then either \( \alpha_n = \alpha \) for all \( n \) (in which case the statement is true) or there is some \( n \) such that \( \alpha_n < \alpha \). In the latter case we may apply the inductive hypothesis to the subsequence obtained by deleting from \( \langle \alpha_n \rangle \) the terms before the \( n \)th to obtain the conclusion as the latter sequence starts at \( \alpha_n < \alpha \). \( \square \)

Corollary B.21  \( \omega_1 \) is sequentially compact.

Proof  Suppose that \( \langle \alpha_n \rangle \) is a sequence in \( \omega_1 \). If \( \langle \alpha_n \rangle \) has an increasing subsequence then the result follows from Lemma B.19. If \( \langle \alpha_n \rangle \) has no increasing subsequence then it must have a decreasing subsequence which, by Lemma B.20 must eventually be constant and hence convergent. \( \square \)

Lemma B.22  Every limit ordinal in \( \omega_1 \) is the limit of some increasing sequence of ordinals.

Proof  Suppose that \( \lambda \in \omega_1 \) is a limit ordinal. Then \([0, \lambda)\) is countable so we may choose a bijection \( f : \omega \to [0, \lambda) \). By induction we define a sequence \( \langle \lambda_n \rangle \) which is increasing and converges to \( \lambda \). Set \( \lambda_0 = f(0) \). Suppose given \( \lambda_n \). Choose the least integer \( m > f^{-1}(\lambda_n) \) for which \( f(m) > \lambda_n \), and set \( \lambda_{n+1} = f(m) \). This construction proceeds because \( \lambda \) is a limit ordinal and hence for any \( \alpha < \lambda \) the set \( (\alpha, \lambda] \) is infinite. It is not too difficult to show that \( \lim_{n \to \infty} \lambda_n = \lambda \). \( \square \)

Proposition B.23  (Fodor’s Lemma or the Pressing Down Lemma) Suppose that \( f : \omega_1 \to \omega_1 \) is a function satisfying \( f(\alpha) < \alpha \) whenever \( \alpha > 0 \). Then there is \( \beta \in \omega_1 \) such that \( f^{-1}(\beta) \) is unbounded in \( \omega_1 \).

Proof  Suppose that for all \( \beta \in \omega_1 \) the set \( f^{-1}(\beta) \) is bounded. We construct an increasing sequence \( \langle \alpha_n \rangle \). Set \( \alpha_0 = 0 \) and suppose now that \( \alpha_n \) has been constructed. For each \( \beta \leq \alpha_n \) the set \( f^{-1}(\beta) \) is bounded and hence countable. Since \( \alpha_n \) has countably many predecessors it follows from Corollary B.6 that \( \bigcup_{\beta \leq \alpha_n} f^{-1}(\beta) \) is countable. Let \( \alpha_{n+1} \) be the least upper bound of \( \bigcup_{\beta \leq \alpha_n} f^{-1}(\beta) \).

By Lemma B.19 the sequence \( \langle \alpha_n \rangle \) converges, say to \( \alpha \). Because \( \alpha > \alpha_{n+1} \) for any \( n \) it follows that \( f(\alpha) > \alpha_n \). Taking the limit as \( n \to \infty \) it follows that \( f(\alpha) \geq \alpha \), contradicting the assumption about \( f \). \( \square \)

Note that if in addition the function \( f \) in Proposition B.23 is continuous then \( f^{-1}(\beta) \) is also closed. Subsets of \( \omega_1 \) which are closed and unbounded are important so we study some of their properties. Obviously \( \omega_1 \) itself is closed and unbounded. The set of limit ordinals is also closed and unbounded.

Lemma B.24  Suppose that \( \langle C_n \rangle \) is a sequence of closed, unbounded subsets of \( \omega_1 \). Then \( \bigcap_{n=0}^{\infty} C_n \) is also closed and unbounded.
Appendix B: Set Theory Preliminaries

Proof An arbitrary intersection of closed subsets of a topological space is closed so we need show that \( \bigcap_{n=0}^{\infty} C_n \) is unbounded.

Suppose \( \beta \in \omega_1 \); we exhibit a member \( \alpha \in \bigcap_{n=0}^{\infty} C_n \) with \( \alpha > \beta \) by constructing an increasing sequence \( \langle \alpha_n \rangle \). Set \( \alpha_0 = \beta \). Let \( f : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) be a bijection as given by Lemma B.4 and write \( f(n) = (f_1(n), f_2(n)) \). Suppose that \( \alpha_n \) has been constructed. Because \( C_{f_1(n+1)} \) is unbounded there is \( \alpha_{n+1} \in C_{f_1(n+1)} \) such that \( \alpha_{n+1} > \alpha_n \). The important points to note about the sequence \( \langle \alpha_n \rangle \) are that it is increasing and for each non-negative integer \( m \) there is an infinite subsequence which is in \( C_m \).

As an increasing sequence \( \langle \alpha_n \rangle \) converges to some \( \alpha \in \omega_1 \) by Lemma B.19. Furthermore \( \alpha > \beta \). Finally as each \( C_m \) is closed and contains a subsequence of \( \langle \alpha_n \rangle \) we have \( \alpha \in C_m \). \( \square \)

The following result is a simple version of the \( \Delta \)-system lemma. A more general setting as well as its proof appears in [3, Lemma 2.4].

Lemma B.25 Suppose that \( \mathcal{F} \) is an uncountable family of finite sets. Then there is an uncountable subfamily \( \mathcal{G} \subset \mathcal{F} \) and a finite set \( F \) such that \( G \cap H = F \) for each pair \( G, H \in \mathcal{G} \) of distinct members of \( \mathcal{G} \).

Proof By Corollary B.6 there is an integer \( n \) and an uncountable subfamily \( \mathcal{F}' \subset \mathcal{F} \) such that each member of \( \mathcal{F}' \) has exactly \( n \) elements. Replacing \( \mathcal{F} \) by \( \mathcal{F}' \) we see that it suffices to verify the lemma for the case where all members of \( \mathcal{F} \) contain exactly \( n \) members for some \( n \in \mathbb{N} \). This is done by induction on \( n \).

If \( n = 1 \) then all members of \( \mathcal{F} \) are mutually disjoint so we may take \( F = \emptyset \).

Assume that the lemma is true when each member of the family has \( n - 1 \) elements and suppose that \( \mathcal{F} \) is an uncountable family of sets each member of which contains \( n \) elements. Using Zorn’s Lemma B.14, let \( \hat{\mathcal{F}} \) be a subfamily of \( \mathcal{F} \) which is maximal with respect to the property that members of \( \hat{\mathcal{F}} \) are mutually disjoint. We consider two cases.

- If \( \hat{\mathcal{F}} \) is uncountable then we may set \( \mathcal{G} = \hat{\mathcal{F}} \) and \( F = \emptyset \).
- If \( \hat{\mathcal{F}} \) is countable then by Corollary B.6 \( \cup \hat{\mathcal{F}} \) is countable so there is \( x \in \cup \hat{\mathcal{F}} \) which belongs to uncountably many members of \( \mathcal{F} \). Let \( \tilde{\mathcal{F}} \subset \hat{\mathcal{F}} \) be an uncountable subfamily each member of which contains \( x \). The family \( \{ F \setminus \{ x \} / F \in \tilde{\mathcal{F}} \text{ and } x \in F \} \) is uncountable and each member has \( n - 1 \) elements. Hence by the induction hypothesis there is an uncountable subfamily \( \mathcal{G} \subset \tilde{\mathcal{F}} \) and a finite set \( F' \) such that \( (G \setminus \{ x \}) \cap (H \setminus \{ x \}) = F' \) for each pair \( G, H \in \mathcal{G} \). Set \( F = F' \cup \{ x \} \). \( \square \)

Definition B.26 A subset \( S \subset \omega_1 \) is stationary provided \( S \cap C \neq \emptyset \) for each closed, unbounded \( C \subset \omega_1 \).

Lemma B.27 Suppose that \( \mathcal{C} \) is a collection of subsets of \( \omega_1 \) satisfying the conditions
(i) every countable subset of \( \omega_1 \) is a member of \( \mathcal{C} \);
(ii) if \( \mathcal{D} \subset \mathcal{C} \) is countable then \( \bigcup \mathcal{D} \in \mathcal{C} \);
(iii) \( \omega_1 \notin \mathcal{C} \).

Then there is an uncountable set \( S \subset \omega_1 \) and a collection \( \{ B_\alpha / \alpha \in S \} \) of subsets of \( \omega_1 \) such that \( B_\alpha \notin \mathcal{C} \) and \( B_\alpha \cap B_\beta = \emptyset \) whenever \( \alpha, \beta \in S \) with \( \alpha \neq \beta \).

**Proof** For each \( \alpha \in \omega_1 \) choose an injection \( f_\alpha : [0, \alpha) \to \omega \). Define
\[
A : \omega \times \omega_1 \to 2^{\omega_1} \text{ by } A(n, \alpha) = \{ \beta < \omega_1 \mid \alpha < \beta \text{ and } f_\beta(\alpha) = n \}. \quad (i)
\]
Because each \( f_\gamma \) is injective it follows that for each \( n \in \omega \) and each \( \alpha, \beta \in \omega_1 \), if \( \alpha \neq \beta \) then
\[
A(n, \alpha) \cap A(n, \beta) = \emptyset.
\]
Note that \( \bigcup_{n \in \omega} A(n, \alpha) = (\alpha, \omega_1) \), because if \( \beta \in (\alpha, \omega_1) \) then \( \alpha \in [0, \beta) \), the domain of \( f_\beta \), so \( \beta \in A(f_\beta(\alpha), \alpha) \subset \bigcup_{n \in \omega} A(n, \alpha) \) so we have shown that \( (\alpha, \omega_1) \subset \bigcup_{n \in \omega} A(n, \alpha) \). The converse is obvious.

Thus for each \( \alpha \in \omega_1 \) we have \( \omega_1 = [0, \alpha) \cup (\bigcup_{n \in \omega} A(n, \alpha)) \), a countable union. It now follows from (ii) and (iii) that for each \( \alpha \in \omega_1 \) at least one of the sets \( [0, \alpha) \) and \( A(n, \alpha) \) (for \( n \in \omega \)) cannot be in \( \mathcal{C} \); as \( [0, \alpha) \) is countable we have \( [0, \alpha) \in \mathcal{C} \), so one of the sets \( A(n, \alpha) \) cannot be in \( \mathcal{C} \). Thus there is \( \nu(\alpha) \in \omega \) such that \( A(\nu(\alpha), \alpha) \notin \mathcal{C} \).

Consider the function \( \nu : \omega_1 \to \omega \). There must be \( n \in \omega \) such that \( S = \nu^{-1}(n) \) is uncountable. The collection \( \{ A(n, \alpha) / \alpha \in S \} \) satisfies the requirements. \( \square \)

**Theorem B.28** \( \omega_1 \) can be partitioned into \( \aleph_1 \) many mutually disjoint stationary sets.

**Proof** Let \( \mathcal{C} \) denote the collection of all non-stationary subsets of \( \omega_1 \). Clearly \( \mathcal{C} \) satisfies conditions (i) and (iii) of Lemma B.27. \( \mathcal{C} \) also satisfies condition (ii). Indeed, suppose that \( S_n \in \mathcal{C} \) for each \( n \in \omega \). For each \( n \) there is a closed, unbounded \( C_n \subset \omega_1 \) such that \( C_n \cap S_n = \emptyset \). By Lemma B.24 the set \( \bigcap_{n \in \omega} C_n \) is also closed and unbounded. Furthermore \( (\bigcap_{n \in \omega} C_n) \cap (\bigcup_{n \in \omega} S_n) = \emptyset \). Hence \( \bigcup_{n \in \omega} S_n \notin \mathcal{C} \) as claimed.

Lemma B.27 now yields an uncountable collection of mutually disjoint subsets of \( \omega_1 \) none of which lies in \( \mathcal{C} \); hence all of which are stationary. \( \square \)

Readers might find it surprising that every continuous function \( f : \omega_1 \to \mathbb{R} \) is eventually constant, i.e., there are \( \alpha \in \omega_1 \) and \( c \in \mathbb{R} \) such that whenever \( \beta > \alpha \) then \( f(\beta) = c \). In fact much more is true. First we give a result which describes some of the structure of closed unbounded subsets of \( \omega_1 \times X \) for a small space \( X \).

**Lemma B.29** Suppose that \( X \) is a first countable, Lindelöf space and \( C \subset \omega_1 \times X \) is closed and has points with arbitrarily large \( \omega_1 \)-coordinate. Then there is \( c \in X \) such that \( C \cap (\omega_1 \times \{ c \}) \) also has arbitrarily large \( \omega_1 \)-coordinate.

**Proof** Suppose to the contrary that for each \( x \in X \) the \( \omega_1 \)-coordinate of \( C \cap (\omega_1 \times \{ x \}) \) is bounded: then for each \( x \in X \) there is \( \alpha_x \in \omega_1 \) such that \( C \cap (\{ \alpha_x \} \times X) = \emptyset \). It is claimed that there is a open neighbourhood \( V_x \) of \( x \) in \( X \) such that \( C \cap (\{ \alpha_x \} \times V_x) = \emptyset \). If, to the contrary, there is no such neighbourhood then, using first countability, we may find a sequence \( \{ x_n \} \) converging

---

1 The matrix \( A \) is called the Ulam matrix [6].
to \( x \) such that \( C \cap (\{\alpha_x, \omega_1\} \times \{x_n\}) \neq \emptyset \) for each \( n \), say \( \beta_n \in [\alpha_x, \omega_1) \) is such that \((\beta_n, x_n) \in C\). By Lemma B.21 and taking a subsequence if necessary we may assume that \((\beta_n)\) also converges, say to \( \beta \in [\alpha_x, \omega_1) \). Thus \((\beta_n, x_n)\) converges to \((\beta, x)\). Because \( C \) is closed it then follows that \((\beta, x) \in C \cap (\{\alpha_x, \omega_1\} \times \{x\})\), which contradicts the assumption that \( C \cap (\{\alpha_x, \omega_1\} \times \{x\}) = \emptyset \). Thus we have now shown that there is an open neighbourhood \( V_x \) of \( x \) in \( X \) such that \( C \cap (\{\alpha_x, \omega_1\} \times V_x) = \emptyset \).

The collection \( \{V_x / x \in X\} \) is an open cover of the Lindelöf space \( X \) so has a countable subcover, say \( \{V_x / x \in Y\} \), where \( Y \) is a countable subset of \( X \). Let \( \alpha = \sup\{\alpha_x / x \in Y\} \). Then no point of \( C \) can have \( \omega_1\)-coordinate greater than \( \alpha \), which contradicts our assumption. \( \square \)

**Lemma B.30** ([1, Lemma 2.2]) Suppose that \( X \) is a first countable, Lindelöf, Hausdorff space. Then every continuous function \( f : \omega_1 \to X \) is eventually constant.

**Proof** Suppose that \( f : \omega_1 \to X \) is a continuous function.

Because \( X \) is Hausdorff, the graph of \( f \), \( \Gamma_f = \{(t, f(t)) / t \in \omega_1\} \), is closed in \( \omega_1 \times X \), so what we have just proved applies to \( \Gamma_f \). Since \( \Gamma_f \) contains points with arbitrarily large \( \omega_1\)-coordinate then by Lemma B.29 there is \( c \in X \) such that \( \Gamma_f \cap (\omega_1 \times \{c\}) \) also has arbitrarily large \( \omega_1\)-coordinate.

Use first countability of \( X \) to find a countable basis of open neighbourhoods of \( c \), say \( \{V_n\} \). For each \( n \) the two sets \( f^{-1}(c) \) and \( f^{-1}(X \setminus V_n) \) are closed and disjoint, and \( f^{-1}(c) \) is unbounded. By Lemma B.24 any two closed, unbounded subsets of \( \omega_1 \) intersect, from which it follows that \( f^{-1}(X \setminus V_n) \) must be bounded. Then

\[
 f^{-1}(X \setminus \{c\}) = \bigcup_{n=1}^{\infty} f^{-1}(X \setminus V_n)
\]

must also be bounded, say by \( \beta \). Hence for any \( t \in [\beta, \omega_1) \) we have \( f(t) = c \) as required. \( \square \)

In the following proposition we are considering strictly increasing functions \( \omega \to \omega \). Given two such functions \( f, g \), we declare \( f <^* g \) provided \( f(n) < g(n) \) for all but finitely many \( n \in \omega \).

**Proposition B.31** There is an \( \omega_1\)-sequence \( \langle f_\alpha : \omega \to \omega \rangle_{\alpha < \omega_1} \) of strictly increasing functions such that \( f_{\alpha+1}(n) = f_\alpha(n) + 1 \) for all \( \alpha < \omega_1 \) and \( n \in \omega \) and \( f_\alpha <^* f_\beta \) whenever \( \alpha < \beta \).

**Proof** Start the sequence by defining \( f_0(n) = n \) for all \( n \in \omega \). The statement of the proposition already tells us how to define \( f_{\alpha+1} \) given \( f_\alpha \).

Now suppose that \( \alpha > 0 \) is a limit ordinal and \( f_\beta \) has been defined for \( \beta < \alpha \). Use Lemma B.22 to choose an increasing sequence \( \langle f_n \rangle \) converging to \( \alpha \). Define \( f_\alpha \) inductively on \( m \in \omega \) by \( f_\alpha(m+1) = \max\{f_\alpha(n(p)) / n, p \leq m\} + 1 \). Then one may show that \( \langle f_\alpha \rangle \) satisfies the requirements. \( \square \)
B.3 The Continuum Hypothesis and Martin’s Axiom

We now discuss the Continuum Hypothesis, CH, and Martin’s Axiom, MA, briefly. We refer the reader to [2, 4] and [5] or any other book on introductory Set Theory for a more precise discussion of and further details on these hypotheses as well as many others which, like CH and MA, are independent of the usual ZFC axioms of set theory. Recall that the cardinality of \(\mathbb{R}\) is denoted by \(c\), so \(c \geq \aleph_1\).

For many years after Cantor discovered that the cardinality of the real numbers was strictly greater than that of the natural numbers mathematicians wondered whether there exists a set \(S\) such that \(|\mathbb{N}| < |S| < |\mathbb{R}|\); equivalently, whether \(c > \aleph_1\). The statement that no such set exists came to be known as the Continuum Hypothesis, CH.

Axiom B.32 \(CH : c = \aleph_1\).

It took decades before the question whether \(CH\) is true or not was settled and the answer probably would have surprised the earlier mathematicians. There are models of Set Theory in which \(CH\) is true and models in which its negation is true. The negation of \(CH\) is denoted by \(\neg CH\).

One really useful consequence of assuming \(CH\) is that \(\mathbb{R}\) can be well-ordered by use of \(\omega_1\). More precisely we can write \(\mathbb{R} = \{x_\alpha / \alpha < \omega_1\}\). This sometimes allows an inductive proof based on the countable ordinals \(\omega_1\); see Example 6.5, for example.

Martin’s Axiom, MA, has two forms, one topological in nature and the other more set theoretic. See [7, Sect. 1] for a discussion and proof of the equivalence of the two versions.

Definition B.33 A topological space has the countable chain condition (abbreviated ccc) provided that every pairwise disjoint family of open sets is countable.

Axiom B.34 MA: In every non-empty, compact, ccc, Hausdorff space the intersection of fewer than \(c\) many dense open sets is non-empty.

Theorem B.35 \(CH \Rightarrow MA\).

Proof Suppose that \(X\) is a non-empty, compact, ccc, Hausdorff space. One version of the Baire category theorem tells us that every locally compact, Hausdorff space is a Baire space. When \(CH\) holds, a collection of fewer than \(c\) many dense open subsets of \(X\) is countable, hence by the Baire category theorem it has dense intersection. \(\Box\)

Definition B.36 Let \((X, \leq)\) be a partially ordered set. Elements \(x, y \in X\) are compatible if there is some \(z \in X\) such that \(z \leq x\) and \(z \leq y\). Elements which are not compatible are said to be incompatible. \((X, \leq)\) has the countable chain condition, abbreviated ccc, provided that there is no uncountable pairwise incompatible subset. A subset \(D \subset X\) is dense if for each \(x \in X\) there is \(d \in D\) such that \(d \leq x\).

Axiom B.37 MA: Let \((X, \leq)\) be a non-empty partially ordered set which has the ccc and let \(\emptyset\) be a collection of fewer than \(c\) dense subsets of \(X\). Then there is a subset \(Y \subset X\) such that:
• for each \( x, y \in Y \) there is \( z \in Y \) such that \( z \leq x \) and \( z \leq y \);
• for each \( x \in X \) and \( y \in Y \), if \( y \leq x \) then \( x \in Y \);
• for each \( D \in \mathcal{D} \) we have \( D \cap Y \neq \emptyset \).

Because \( \text{CH} \Rightarrow \text{MA} \), there are three mutually exclusive possibilities involving \( \text{CH} \) and \( \text{MA} \) and their negations: \( \text{CH} \), \( \text{MA} \wedge \neg \text{CH} \) and \( \neg \text{MA} \). Each of them is consistent with the usual axioms of Set Theory.

References

Index

A
Adding a $k$-handle, 46
$\aleph$-space, 22
$\aleph_0$-space, 22
Almost metaLindelöf, 22
$\alpha$-favourable, strongly, 24
$\alpha$-favourable, weakly, 24
Analytic, 24
Arc, 5
Arc connected, 5
Arc connected, manifold is, 5
Atlas, 2
Attaching map, 46
Axiom of Choice, 185

B
Baire space, 24
Baire space, strongly, 24
Basis, 168
Basis, criterion for, 168
Bernstein set, 189
Boundary of a manifold with boundary, 2
Boundary of a manifold with boundary is a manifold, 2
Bounded set, 188
Branched 1-manifold, complete feather, 156
Branched 1-manifold, looped, 154
Branched 1-manifold, single, 154
Branched 1-manifold, triple, 154
Choice function, 185
Closed, 168
Closed long ray, 3
Closure, 168
Coefficient group, 178
Collared, 42
Compact, 170
Compact-open topology, 176
Compatible, 196
Connected, 173
Continuous, 169
Continuous flow, 63
Continuum Hypothesis, CH, 198
Converges, 170
Coordinate transformation, 101
Cosmic, 22
Countable chain condition, 196
Countable ordinals, 190
Countable set, 185
Countably compact space, 51
Countably fan tight, 24
Countably strongly fan tight, 24
Countably tight, 24
Cover, 170
Cover, $\gamma$, 25
Cover, $\omega$, 25
Cover, $k$, 22
Cover, large, 25
Crosscaps, sphere with, 182
Customs Passage Theorem, 173

C
Cardinal arithmetic, 186
Cardinality of a set, 185
Čech-complete, 172
Chart, 2

D
$\Delta$-system lemma, 193
Dense, 169
Development, 23
Diffeomorphism, 103
Differentiable function, 103
Differentiable manifold, 101
Differentiable submanifold, 103
Differential structure, 101
Differential structure, exotic, 115
Differential structure, product, 104
Differential structure, usual on $\mathbb{R}^n$, 102
Differential structure, usual on $S^n$, 102
Differential structures on long line, $2^{\aleph_1}$ many distinct, 114
Differential structures on real line, f rack many, 113
Differential structures, equivalent, 103
Dimension, 1
Dimension of a foliation, 130
Direction matrix for continuous $\mathbb{L}_{\geq 0}^n \to \mathbb{L}_{\geq 0}^m$, 69
Direction matrix, reduced, 72
Discrete flow, 65
Discrete metric, 167
Discrete open expansion, 23
Double of a manifold with boundary, 12
Driving Creek Railway, 155
Driving Creek Railway, location, 148

E
End points, 173
Euclidean space, 167
Eventually constant, 8
Excision, 178
Exotic differential structure, 115
Extremely normal, 23

F
Family, generating*, 26
Fell topology, 26
Finitistic, 22
Finitistic, star, 21
Finitistic, strongly, 22
First countable, 169
Flow, continuous, 63
Flow, discrete, 65
Fodor’s Lemma, 192
Foliated chart, 130
Foliation, 130
Foliation of $L^2$ less a compactum, 147
Foliation of long plane; only two, 150
Foliation tube lemma, 137
Foliation with a single leaf, 133
Foliation with only sequentially compact leaves has $c$ many, 135
Foliation, codimension, 130

G
Game $\mathcal{G}_S(D)$, 25
Game, Babinkostova’s $G^k_s(A, B)$, 25
Game, Banach-Mazur, 24
Game, Choquet, 24
Game, Gruenhage’s, 25
Game, stationary strategy for, 25
Game, strategy for, 25
Game, topological, 24
Game, winning strategy for, 25
Gartside-Good-Knight-Mohamad manifold, 17
$G^*_s$-diagonal, 23
$G^*_s$-diagonal, quasi-, 23
$G^*_s$-diagonal, quasi-regular, 23
$G^*_s$-diagonal, regular, 23
Genus of a surface, 182
Group, semi-direct product, 82
$G^*_s$-subset, 169
$G^*_s$-subset, regular, 173

H
Handle, adding a $k$-, 46
Handlebody, 46
Handlebody decomposition, 46
Handle, sphere with, 181
Hausdorff, 169
Hausdorff, locally, 157
Hemicompact, 22
Hereditary topological property, 170
Homeomorphism, 169
Homogeneous, 41
Homology groups of a surface, 183
Homology theory, 177
Hurewicz, 22
Index

I
I-diagonals at height $c$, 66
Incompatible, 196
Interior, 168
Interval, 169
Interval, open, 169
Invariance of Domain, 180
Isotopy, 65

J
JCM-space, 23
Jointly metrisable on compacta, 23
Jordan-Brouwer Separation Theorem, 180

K
Kneser foliation, 131
$k$-space, 24

L
$L$ space, 95
Lašnev, 22
Leaf of a foliation, 130
Leaf space, 130
Leaf topology, 130
Least member in a well-ordered set, 188
Lexicographic order, 3
Lindelöf, 170
$k$-Lindelöf, 22
Lindelöf, $\omega_1$-, 22
Lindelöf, linearly, 22
Lindelöf, linearly $\omega_1$-, 22
Lindelöf, strongly hereditarily, 22
Line with double origin, 154
Linearly $\omega_1$-Lindelöf, 22
Linearly metaLindelöf, 22
$L^n$, $I$-diagonals at height $c$, 66
$L^n$, semidiagonals, 66
Locally collared, 42
Locally compact, 170
Locally finite, 170
Locally Hausdorff, 157
Long line, 3
Long line has $2^{\aleph_1}$ distinct differential structures, 114
Long line has differential structure, 105
Long line has many differential structures, 106
Long line is sequentially compact, 6
Long line power, classification of homeomorphisms up to isotopy, 80
Long line, continuous flow, 64

M
Manifold, 1
Manifold has cardinality $c$, 4
Manifold has finitely presented homology when $\omega$-bounded, 53
Manifold is arc connected, 5
Manifold is Tychonoff, 4
Manifold with boundary, 1
Manifold with boundary, double, 12
Manifold with boundary, interior, 2
Manifold, $2^\omega$ many, 61
Manifold, $2^{\aleph_1}$ many $\omega$-bounded, 59
Manifold, $\omega$-bounded, 52
Manifold, boundary, 2
Manifold, dimension, 1
Manifold, non-Hausdorff, 153
Manifold, non-Hausdorff is $T_1$, 153
Manifold, perfectly normal but not metrisable, 91
Manifold, perfectly normal implies metrisable, 95
Mapping class group, 81
Mapping class group, uncountable, 85
Martin’s Axiom, MA, 196
Maximal element, 188
Metacompact, 21
MetaLindelöf, almost, 22
MetaLindelöf, linearly, 22
MetaLindelöf, nearly linearly $\omega_1$-, 22
MetaLindelöf, 21
Metric space, 167
Metric, discrete, 167
Metric, Pythagorean, 167
Metrizability, equivalent conditions for a manifold, 27
Metrizable, 169
Microbundle, 26
Microbundle, tangent, 27
Minimal element, 188
Monotonically normal, 23
Moore space, 23
Mooreising, 14
Moving Off Property, 23
M₁ space, 22
M₃ space, 22

N
Nearly linearly ω₁-meta-Lindelöf, 22
Neighbourhood, 169
Neighbourhood basis, 169
k-network, 22
Non-orientable surface, 182
Normal, 169
Normal, extremely, 23
Normal, monotonically, 23
Normal, perfectly, 23
Normal, weakly, 23
Nyikosising, 17

O
ω₁-Meta-Lindelöf, 22
ω₁ contains ℵ₁ many mutually disjoint stationary sets, 194
ω₁, continuous function to R is eventually constant, 192
ω₁, countable intersection of closed, unbounded sets, 194
ω₁, decreasing sequences eventually constant, 192
ω₁, increasing sequences converge, 191
ω₁ sequentially compact, 192
Open, 168
Open cell, countable union, 39
Open cover, 170
Open long ray, 3
Open refinement, 170
Order topology, 169
Ordinal, immediate successor, 191
Ordinal, limit, 191
Orientable surface, 181
Oriented foliation, 161

P
Paracompact, 21, 170
Paracompact, strongly, 21
Para-Lindelöf, 21

Q
q-space, 23
Quotient of a topological space, 171

R
Radial, 24
Rationals are countable, 186
Real line has c many differential structures, 113
Realcompact, 15
Reeb foliation, 132
Refinement, 170
Regular, 169
Rigid foliation, 162
Rigid second countable non-Hausdorff manifold, 159
Rigid space, 159

S
Schoenflies Theorem, 181
Schroder-Bernstein Theorem, 186
Screenable, 21
Screenable, selectively, 22
Second countable, 169
Selection principle \( S_1(\mathfrak{A}, \mathfrak{B}) \), 25
Selection principle \( S_{fin}(\mathfrak{A}, \mathfrak{B}) \), 25
Selection principles, 25
Selectively screenable, 22
Semidiagonals of \( \mathbb{I}^n \), 66
Separable, 169
Separable manifold contains a dense open cell, 41
Sequential, 24
Sequentially compact, 51, 170
Set of operations, 26
Set saturated by leaves of a foliation, 138
Sierpinski topology, 26
\( \sigma \)-closure preserving base, 22
\( \sigma \)-Metacompact, 21
\( \sigma \)-ParaLindelöf, 21
 Singular homology theory, 178
Sphere with crosscaps, 182
Sphere with handles, 181
Squat, 8
Squat, first countable, Lindelöf T2 space is, 8
Squat, metrisable manifolds are always, 8
Squat, non-metrisable manifolds may be, 8
S space, 95
Star, 24
Star finitistic, 21
Star-finite open refinement, 21
Stationary set, 193
Strategy, 25
Strategy, stationary, 25
Strategy, winning, 25
Stratifiable, 22
Strongly finitistic, 22
Strongly hereditarily Lindelöf, 22
Strongly hereditarily separable, 94
Strongly paracompact, 21
Subcover, 170
Submanifold, 2
Submanifold, differentiable, 103
Submetrisable, 23
Subparacompact, 23
Subspace, 168
Successor, 188
Support of a singular chain, 53
Surface, 181
Surface with boundary, 181
Surface, fundamental group, 183
Surface, genus, 182
Surface, homology groups, 183
Surface, non-orientable, 182
Surface, orientable, 181
Surfaces, classification of compact, 182

T
T3, 169
T4, 169
Tangent microbundle, 27
\( \theta \)-refinable, 23
Tightness, 95
Topological game, 24
Topological power, 171
Topological space, 168
Topology, 168
Topology, cocompact, 26
Topology, compact-open, 176
Topology, discrete, 168
Topology, indiscrete, 168
Topology, pointwise, 176
Topology, Sierpinski, 26
Topology, Tychonoff, 171
Topology, upper Fell, 26
Topology, upper Kuratowski, 26
Totally ordered set, 188
Tree, 189
Tree, branch, 189
Tree, height, 189
Tree, root, 189
Tychonoff, 171
Tychonoff topology, 171
Tychonoff’s Embedding Theorem, 172
Type I space, 49
Type I space, canonical sequence, 49

U
Ulam matrix, 194
Uncountable set, 185
Upper bound, 188
Upper bound, least, 188
Upper Fell topology, 26
Upper Kuratowski topology, 26
Urysohn’s Metrisation Theorem, 26
Usual topology on \( \mathbb{R}^n \), 169

V
Volterra space, 24

W
Weakly normal, 23
Weight, 172
Well-ordered set, 188
Well-Ordering Principle, 188

Z
Zorn’s Lemma, 189