A.1 Taylor Series Expansion for Function of One Variable

If a function \( f(x) \) is analytic at all points inside and on a circle \( C \) of radius (say, \( R \)) with its center at \( x_0 \), then the value of the function for all points \( x \) in the circle can be determined by the Taylor’s theorem i.e.,

\[
f(x) = f(x_0) + \left. \frac{\partial}{\partial x} f(x) \right|_{x=x_0} (x-x_0) + \frac{(x-x_0)^2}{2!} \left. \frac{\partial^2}{\partial x^2} f(x) \right|_{x=x_0} + \cdots + \frac{(x-x_0)^n}{n!} \left. \frac{\partial^n}{\partial x^n} f(x) \right|_{x=x_0} + \cdots \quad (A.1)
\]

The Taylor series expansion for two variables \((x, y)\) is given by,

\[
f(x, y) = f(x_0, y_0) + \left. \frac{\partial}{\partial x} f(x, y) \right|_{x=x_0} (x-x_0) + \left. \frac{\partial}{\partial y} f(x, y) \right|_{y=y_0} (y-y_0)
+ \frac{1}{2!} \left[ (x-x_0)^2 \left. \frac{\partial^2}{\partial x^2} f(x, y) \right|_{x=x_0} + 2(x-x_0)(y-y_0) \left. \frac{\partial^2}{\partial x \partial y} f(x, y) \right|_{x=x_0, y=y_0} + (y-y_0)^2 \left. \frac{\partial^2}{\partial y^2} f(x, y) \right|_{x=x_0, y=y_0} \right] + \cdots \quad (A.2)
\]

A closely related expression can be obtained by expanding the function about the origin \((x_0 = 0)\), known as, Maclaurin’s series,

\[
f(x) = f(0) + \left. \frac{\partial}{\partial x} f(x) \right|_{x=0} x + \frac{x^2}{2!} \left. \frac{\partial^2}{\partial x^2} f(x) \right|_{x=0} + \cdots + \frac{x^n}{n!} \left. \frac{\partial^n}{\partial x^n} f(x) \right|_{x=0} + \cdots \quad (A.3)
\]

A.2 Gradient and Laplacian Operators

Cartesian Coordinate \((x, y, z)\)

This is the simplest coordinate system that consists of three mutually orthogonal axis. Any arbitrary vector \(\hat{r}\) can be expressed as,

\[
\hat{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad (A.4)
\]

The gradient is defined as,

\[
\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \quad (A.5)
\]

The Laplacian operator is given by,

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (A.6)
\]
Cylindrical Coordinate \((\rho, \phi, z)\)

Any arbitrary vector (see, Fig. A.1) is given by,

\[
\vec{r} = x \hat{i} + y \hat{j} + z \hat{k} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}
\]  
(A.7)

where, \(\rho \geq 0\), \(0 \leq \phi \leq 2\pi\) and \(-\infty < z < \infty\).

The Gradient: \(\nabla = \frac{\partial}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial}{\partial \phi} \hat{\phi} + \frac{\partial}{\partial z} \hat{z}\)  
(A.8)

The Laplacian: \(\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \phi} \left( \sin \theta \frac{\partial}{\partial \phi} \right) + \frac{\partial^2}{\partial z^2}\)  
(A.9)

Spherical Coordinate \((\rho, \theta, \phi)\)

Any arbitrary vector can be written as (see, Fig. A.2),

\[
\vec{r} = x \hat{i} + y \hat{j} + z \hat{k} = \rho \sin \theta \cos \phi \hat{i} + \rho \sin \theta \sin \phi \hat{j} + \rho \cos \theta \hat{k}
\]  
(A.10)

where, \(\rho \geq 0\), \(0 \leq \theta \leq \pi\), \(0 \leq \phi \leq 2\pi\).

The gradient and Laplacian operator are given by,

\[
\nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi}
\]  
(A.11)

and

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.
\]  
(A.12)
Fig. A.2 Description of a spherical coordinate system
B.1 Wave Impedance

Let us assume that, a plane wave is travelling in an unbounded lossless medium \( (\epsilon, \mu) \) along the \( z \)-direction and the electric field has only \( x \)-component. So, the corresponding scalar wave equation is given by,

\[
\nabla^2 E_x(x, y, z) + k^2 E_x(x, y, z) = 0 \tag{B.1}
\]

Since, the wave is a uniform plane wave that travels in the \( z \)-direction, its solution is not a function of \( x \) & \( y \), so the solution reduces to, \( E_x(x, y, z) = w(z) \). Realizing that the wave in hand is a travelling wave, the solution in rectangular coordinates is given by,

\[
E_x(z) = w(z) = C_1 e^{-ikz} + C_2 e^{ikz} = E_0^+ e^{-ikz} + E_0^- e^{ikz} \quad \text{(say)} \tag{B.2}
\]

where, \( E_0^+ \) and \( E_0^- \) represents the amplitude of the positive and negative travelling waves.

There must be a component of magnetic field associated with electric field, but as of now, we do not know which component of magnetic field coexist. We determine the corresponding magnetic field by using one of the Maxwell’s equations, i.e.,

\[
\vec{\nabla} \times \vec{E} = -i \omega \mu \vec{H}
\]

\[
\Rightarrow \vec{H} = -\frac{1}{i \omega \mu} \vec{\nabla} \times \vec{E} = -\frac{1}{i \omega \mu} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ E_x & 0 & 0 \end{vmatrix} \tag{B.3}
\]

Substituting the electric field expression we get,

\[
\vec{H} = -\frac{1}{i \omega \mu} \frac{\partial E_x}{\partial z} \hat{y} = \frac{k}{\omega \mu} \left[ E_0^+ e^{-ikz} - E_0^- e^{ikz} \right] \hat{y} = \sqrt{\frac{\epsilon}{\mu}} \left[ E_0^+ - E_0^- \right] \hat{y} = [H_y^+ - H_y^-] \hat{y} \tag{B.4}
\]

where, \( E_0^+ = E_0^+ e^{-ikz} \) and \( E_0^- = E_0^- e^{ikz} \). Sorting out the positive and negative parts of \( y \)-component of magnetic and electric field gives,

\[
H_y^+ = \sqrt{\frac{\epsilon}{\mu}} E_x^+ \tag{B.5}
\]

\[
H_y^- = -\sqrt{\frac{\epsilon}{\mu}} E_x^- \tag{B.6}
\]
Note that, each term of magnetic field (A/m) is related to the corresponding term which is in V/m, the factor of $\sqrt{\frac{\mu}{\varepsilon}}$ must have units of mhos (A/V). Therefore, $\sqrt{\frac{\mu}{\varepsilon}}$ is known as wave impedance of the medium, which is the ration of electric to magnetic field i.e.,

$$Z_{\text{wave}} = \frac{E_{x}}{H_{y}} = \sqrt{\frac{\mu}{\varepsilon}}$$  \hspace{1cm} (B.7)

### B.2 Fourier Transform

A brief and limited introduction to Fourier transform is provided below:

#### B.2.1 1-D Fourier Transform

Harmonic function is the building block of wave and Fourier optics. The principle of superposition can be employed to represent any waveform in terms of the real (cosine) and imaginary (sine) parts of the harmonic function. Fourier transform allows the decomposition of a complex valued function $f(t)$ into harmonic (sine and cosine) function of different amplitude and frequencies provided some conditions are satisfied,

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi \nu t} d\nu.$$  \hspace{1cm} (B.8)

The inverse of the above process is called inverse Fourier transform,

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi \nu t} dt.$$  \hspace{1cm} (B.9)

This gives the important information about the frequency content in the function $f(t)$. $f(t)$ and $F(\nu)$ form a Fourier transform pair.

#### B.2.2 2-D Fourier Transform

Following 1-D Fourier transform, we consider Fourier transform of a function of two variables, $f(x, y)$. This function can be considered as a spatial pattern or optical field in a plane. As for 1-D case, the building blocks that can be used to construct any arbitrary function $f(x, y)$ are the 2-D harmonic function, $e^{-i2\pi(v_x x + v_y y)}$, where $v_x$ and $v_y$ are the spatial frequencies in $x$ and $y$ directions respectively. Mathematically, this can be expressed as,

$$f(x, y) = \iint_{-\infty}^{\infty} F(v_x, v_y)e^{-i2\pi(v_x x + v_y y)} dv_x dv_y$$  \hspace{1cm} (B.10)

The inverse function also exists and is given by,

$$F(v_x, v_y) = \iint_{-\infty}^{\infty} f(x, y)e^{i2\pi(v_x x + v_y y)} dx dy$$  \hspace{1cm} (B.11)

$F(v_x, v_y)$ is the inverse 2-D Fourier transform which gives the important information about the spatial frequencies present in the function $f(x, y)$. The modulus of the function i.e., $|F(v_x, v_y)|^2$ is also called the 2D spectral density. If $|F(v_x, v_y)|^2$ extends over a wide spatial frequency range, the function $f(x, y)$ has a wide bandwidth. The same is true for 1-D signal.
B.2.3 Properties of 1-D and 2-D Fourier Transform

For ready reference and easy derivation, it is better to know some of the properties of Fourier transform (F.T.). These transform can be directly verified from the definition itself. If, \( F.T. \{ f(x) \} = F(v) \) and \( F.T. \{ g(x) \} = G(v) \), then,

**Linearity:** If, \( g(x) = a_1 f_1(x) + a_2 f_2(x) \), then, \( G(v) = a_1 F_1(v) + a_2 F_2(v) \).

**Translation:** If, \( g(x) = f(x - x_0) \), then \( G(v) = e^{-i2\pi v x_0} F(v) \).

**Modulation:** If, \( g(x) = e^{i2\pi v_x x} f(x) \), then \( G(v) = F(v - v_0) \).

**Scaling:** If, \( g(x) = f(ax) \), then, \( G(v) = \frac{1}{|a|} F(\frac{v}{a}) \).

**Convolution:**

1-D: If, \( g(x) = \int_{-\infty}^{\infty} f_1(x)f_2(x - x_0) \, dx \), then, \( G(v) = F_1(v)F_2(v) \).

2-D: If, \( g(x, y) = \int_{-\infty}^{\infty} f_1(x, y)f_2(x - x_0, y - y_0) \, dx \, dy \), then, \( G(v_x, v_y) = F_1(v_x, v_y)F_2(v_x, v_y) \).

**Parseval’s Theorem:** \( \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(v)|^2 \, dv \).

B.3 Higher Order Modes of Light

Gaussian beam is the lowest mode \((l = 0, m = 0)\) of transverse electromagnetic field. Like Gaussian mode, higher order modes are also solution of paraxial Helmholtz equation. Particularly, those solutions are interesting that exhibit non-Gaussian behavior. They have diverse application in beam shaping, optical tweezers and even in super-resolution STED microscopy.

**Hermite-Gaussian Beam**

The optical intensity of Hermite-Gaussian beam of order \((l, m)\) is given by,

\[
I_{l,m}(x, y, z) = |A_{l,m}|^2 \left[ \frac{W_0}{W(z)} \right]^2 G_l^2 \left[ \frac{\sqrt{2}x}{W(z)} \right]^2 G_m^2 \left[ \frac{\sqrt{2}y}{W(z)} \right]^2,
\]

(B.12)

where, \( G_l(x') = H_l(x')e^{-x'^2/2} \), \( l = 0, 1, 2, \ldots \) is the Hermite-Gaussian function of order \( l \), and \( A_{l,m} \) is a constant. Specifically, \( G_1(x') = 2x'e^{-x'^2/2} \) and \( G_2(x') = 2(2x'^2 - 1)e^{-x'^2/2} \).

**Laguerre-Gaussian Beams**

LG beam is the solution of paraxial Helmholtz equation in cylindrical coordinates \((\rho, \phi, z)\),

\[
I_{l,m}(\rho, \phi, z) = |A_{l,m}|^2 \left[ \frac{W_0}{W(z)} \right]^2 \left( \frac{\rho}{W(z)} \right)^2 \left[ L_m^l \left( \frac{2\rho^2}{W^2(z)} \right) \right]^2 e^{-2\rho^2/W^2(z)},
\]

(B.13)

where, \( L_m^l \) is the generalized Laguerre polynomial function.

---

### Table C.1

The following table lists first few positive roots (rounded-off) of Bessel Function of order upto 4

<table>
<thead>
<tr>
<th>Order</th>
<th>(n=0)</th>
<th>(n=1)</th>
<th>(n=2)</th>
<th>(n=3)</th>
<th>(n=4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(J_n(x) = 0)</td>
<td>2.405</td>
<td>3.831</td>
<td>5.135</td>
<td>6.380</td>
<td>7.588</td>
</tr>
<tr>
<td></td>
<td>5.520</td>
<td>7.015</td>
<td>8.417</td>
<td>9.761</td>
<td>11.065</td>
</tr>
</tbody>
</table>

### Table C.2

Some of the Fluorescent Dyes for STED Microscopy

<table>
<thead>
<tr>
<th>Name of the Dye</th>
<th>(\lambda_{\text{exc}})</th>
<th>(\lambda_{\text{STED}})</th>
<th>Resolution</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mn doped ZnSe Qdots</td>
<td>440</td>
<td>676</td>
<td>45 nm</td>
<td>S. Irvine et al., Angew. Chem. 47,2685 (2008)</td>
</tr>
<tr>
<td>ATTO-532</td>
<td>470</td>
<td>603</td>
<td>&lt;25 nm</td>
<td>G. Donnert et al., PNAS USA 103, 11440 (2006)</td>
</tr>
<tr>
<td>FITC</td>
<td>488</td>
<td>592</td>
<td>&lt;60 nm</td>
<td>G. Moneron et al., Opt. Expr. 18, 1302 (2010)</td>
</tr>
<tr>
<td>ATTO-532</td>
<td>488</td>
<td>600</td>
<td>&lt;40 nm</td>
<td>L. Meyer et al., Small 4, 1095 (2008)</td>
</tr>
<tr>
<td>GFP</td>
<td>490</td>
<td>575</td>
<td>70 nm</td>
<td>K. Willig et al., Nat. Meth. 3, 721 (2006)</td>
</tr>
<tr>
<td>NK51</td>
<td>532</td>
<td>647</td>
<td>40–45 nm</td>
<td>(xyz) R. Schmidt et al., Nat. Meth. 5, 539 (2008)</td>
</tr>
<tr>
<td>ATTO-590</td>
<td>570</td>
<td>690</td>
<td>20 nm</td>
<td>D. Wildanger et al., J. Microsc. 236, 35 (2009)</td>
</tr>
<tr>
<td>ATTO-647N</td>
<td>635</td>
<td>750</td>
<td>50 nm</td>
<td>K. Willig et al., Nat. Meth. 4, 915 (2007)</td>
</tr>
<tr>
<td>Name of the Dye</td>
<td>λ_{exc} (nm)</td>
<td>λ_{emi} (nm)</td>
<td>EC (×10⁻³)</td>
<td>QY</td>
</tr>
<tr>
<td>----------------</td>
<td>--------------</td>
<td>--------------</td>
<td>-------------</td>
<td>----</td>
</tr>
<tr>
<td><strong>Photoactivable fluorescent proteins</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PA-GFP(G)</td>
<td>504</td>
<td>517</td>
<td>17.4</td>
<td>0.79</td>
</tr>
<tr>
<td>PS-CFP2(G)</td>
<td>490</td>
<td>511</td>
<td>47.0</td>
<td>0.23</td>
</tr>
<tr>
<td>PA-mCherry1(R)</td>
<td>564</td>
<td>595</td>
<td>18.0</td>
<td>0.46</td>
</tr>
<tr>
<td>PA-TagRFP(R)</td>
<td>562</td>
<td>595</td>
<td>66.0</td>
<td>0.38</td>
</tr>
<tr>
<td><strong>Photoconvertible fluorescent proteins</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tdEos(R)</td>
<td>569</td>
<td>581</td>
<td>33.0</td>
<td>0.60</td>
</tr>
<tr>
<td>Dendra2(R)</td>
<td>553</td>
<td>573</td>
<td>35.0</td>
<td>0.55 ND</td>
</tr>
<tr>
<td>mEos2(R)</td>
<td>573</td>
<td>584</td>
<td>46.0</td>
<td>0.66</td>
</tr>
<tr>
<td><strong>Photoswitchable fluorescent proteins</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dronpa</td>
<td>503</td>
<td>517</td>
<td>95.0</td>
<td>0.85</td>
</tr>
<tr>
<td>rsFastLime</td>
<td>496</td>
<td>518</td>
<td>39.1</td>
<td>0.77</td>
</tr>
<tr>
<td>E2GFP</td>
<td>515</td>
<td>523</td>
<td>29.3</td>
<td>0.91</td>
</tr>
<tr>
<td>rsCherry</td>
<td>572</td>
<td>610</td>
<td>80.0</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Synthetic fluorophores</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cy5</td>
<td>649</td>
<td>664</td>
<td>250.0</td>
<td>0.28</td>
</tr>
<tr>
<td>Alexa Fluor-647</td>
<td>650</td>
<td>665</td>
<td>240.0</td>
<td>0.33</td>
</tr>
<tr>
<td>Rhodamine B</td>
<td>530</td>
<td>620</td>
<td>105.0</td>
<td>0.65</td>
</tr>
<tr>
<td>C-Rhodamine</td>
<td>545</td>
<td>575</td>
<td>90.0</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Table C.3  Fluorescent Dyes for Localization Microscopy (Data obtained from, ZEISS Microscopy Online Campus (http://www.zeiss-campus.magnet.fsu.edu/articles/superresolution/palm/introduction.html))