Appendix A

Order relations and groups

A.1 Order relations, posets, Zorn’s lemma

A relation $\geq$ on an arbitrary set $X$ is called a **partial order (relation)** if it is reflexive ($x \geq x$, $\forall x \in X$), transitive ($x \geq y \geq z \Rightarrow x \geq z$, $\forall x,y,z \in X$) and skew-symmetric ($x \geq y \geq x \Rightarrow x = y$, $\forall x,y \in X$). The pair $(X, \geq)$ is then said a **partially ordered set** (shortened to **poset**).

An equivalent writing of $a \geq b$ is $b \leq a$.

The partial order $\geq$ is a **total order** if, further, either $x \geq y$ or $y \geq x$ for any $x,y \in X$.

If $(X, \geq)$ is a partially ordered set:

(i) $Y \subset X$ is **upper bounded** (resp. **lower bounded**) if it admits an **upper bound** (lower bound), i.e. $x \in X$ such that $x \geq y$ ($y \geq x$) for any $y \in Y$;

(ii) an element $x_0 \in X$ for which there exists no element $x \neq x_0$ in $X$ such that $x \geq x_0$ is **maximal** in $X$. (Note that for us a maximal element in $X$ may not be an upper bound in $X$).

If $(X, \geq)$ is a poset, a subset $Y \subset X$ is (totally) **ordered** if the relation $\geq$, restricted to $Y \times Y$, is a total order.

Recall that Zorn’s lemma is an equivalent statement to the Axiom of Choice (also known as Zermelo’s axiom).

**Theorem A.1 (“Zorn’s lemma”).** If any ordered subset in a poset $(X, \geq)$ is upper bounded, $X$ admits a maximal element.

Useful among the various notions on posets $(X, \geq)$ are those of supremum and infimum:

(i) $a$ is called **least upper bound** (or **supremum**, or just **sup**) of the set $A \subset X$, written $a = \sup A$, if $a$ is an upper bound of $A$ and any other upper bound $a'$ of $A$ satisfies $a \leq a'$;

(ii) $a$ is called **greatest lower bound**, (or **infimum** or **inf**) of $A \subset X$, written $a = \inf A$, if $a$ is a lower bound for $A$ and any other lower bound $a'$ satisfies $a' \leq a$;

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It is immediate to see that any subset \( A \subset X \) has at most one least upper bound and one greatest lower bound.

A.2 Round-up on group theory

A group is an algebraic structure \((G, \circ)\) consisting in a set \(G\) and an operation \(\circ : G \times G \to G\) (the composition law, often called \textit{product}) satisfying three properties:

1. \(\circ\) is \textit{associative}
   \[
   g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3, \quad \text{for any } g_1, g_2, g_3 \in G;
   \]

2. there exists an element \(e \in G\), called \textit{identity} or \textit{neutral} element, such that
   \[
   e \circ g = g \circ e = g, \quad \text{for any } g \in G;
   \]

3. each element \(g \in G\) admits an \textit{inverse}, i.e.
   \[
   \text{for any } g \in G \text{ there exists } g^{-1} \in G \text{ such that } g \circ g^{-1} = g^{-1} \circ g = e.
   \]

The identity and the inverse to a given element are easily seen to be unique.

A group \((G, \circ)\) is \textit{commutative} or \textit{Abelian} if \(g \circ g' = g' \circ g\) for any \(g, g' \in G\); otherwise it is \textit{noncommutative} or \textit{non-Abelian}.

A subset \(G' \subset G\) in a group is a \textit{subgroup} if it becomes a group with the product of \(G\) restricted to \(G' \times G'\). A subgroup \(N\) in a group \(G\) is \textit{normal} if it is invariant under \textit{conjugation}, i.e. for any \(n \in N\) and \(g \in G\) the conjugate element \(g \circ n \circ g^{-1}\) belongs to \(N\).

If \(N\) is a normal subgroup in \(G\), then \(G/N\) denotes the quotient, i.e. the set of equivalence classes in \(G\) with respect to the equivalence relation \(g \sim g' \leftrightarrow g = ng'\) for some \(n \in N\). It is easy to prove that \(G/N\) inherits a natural group structure from \(G\).

The \textit{centre} \(Z\) of \(G\) is the commutative subgroup of \(G\) made by elements \(z\) that commute with every element of \(G\). In other words, \(z \in Z \leftrightarrow z \circ g = g \circ z\) for any \(g \in G\).

If \((G_1, \circ_1)\) and \((G_2, \circ_2)\) are two groups, a \textit{(group) homomorphism} from \(G_1\) to \(G_2\) is a map \(h : G_1 \to G_2\) that \textit{preserves the groups' structures}, i.e.:

\[
h(g \circ_1 g') = h(g) \circ_2 h(g') \quad \text{for any } g, g' \in G_1.
\]

With the obvious notation it is clear that \(h(e_1) = e_2\) and \(h(g^{-1}_1) = (h(g))^{-1}_2\) for any \(g \in G_1\).

The \textit{kernel} \(\text{Ker}(h) \subset G\) of a homomorphism \(h : G \to G'\) is the pre-image under \(h\) of the identity \(e'\) of \(G'\), i.e. the set of elements \(g\) such that \(h(g) = e'\). Notice \(\text{Ker}(h)\) is a normal subgroup. Clearly \(h\) is one-to-one if and only if its kernel contains the identity of \(G\) only. It turns out that the image \(h(G)\) of a homomorphism \(h : G \to G'\) is a subgroup of \(G'\) isomorphic to \(G/\text{Ker}(h)\).
A group isomorphism is a bijective group homomorphism. An isomorphism \( h : G \to G \) is an automorphism of \( G \). The set \( \text{Aut}(G) \) of automorphisms of \( G \) is itself a group under composition of maps.

If \( G_1, G_2 \) are groups, the direct product \( G_1 \otimes G_2 \) is a group with the following structure. The elements of \( G_1 \otimes G_2 \) are pairs \((g_1, g_2)\) of the Cartesian product of the sets \( G_1, G_2 \). The composition law is

\[
(g_1, g_2) \circ (f_1, f_2) := (g_1 \circ f_1, g_2 \circ f_2) \quad \forall (g_1, g_2), (f_1, f_2) \in G_1 \times G_2.
\]

The neutral element is obviously \((e_1, e_2)\), where \( e_1, e_2 \) are the identities of \( G_1, G_2 \). Moreover, \( G_1 \) and \( G_2 \) can be identified with normal subgroups of \( G_1 \otimes G_2 \).

The ensuing generalisation of the notion of product plays a big role in physical applications. Let \((G_1, \circ_1), (G_2, \circ_2)\) be groups and suppose that for any \( g_1 \in G_1 \) there is a group isomorphism \( \psi_{g_1} : G_2 \to G_2 \) such that:

(i) \( \psi_{g_1} \circ \psi_{g'_1} = \psi_{g_1 \circ g'_1} \);
(ii) \( \psi_{e_1} = \text{id}_{G_2} \),

where \( \circ \) is the composition of functions and \( e_1 \) the neutral element in \( G_1 \). (Equivalently, \( \psi_g \in \text{Aut}(G_2) \) for any \( g \in G_1 \), and the map \( G_1 \ni g \mapsto \psi_g \) is a group homomorphism from \( G_1 \) to \( \text{Aut}(G_2) \).) We can endow the Cartesian product \( G_1 \times G_2 \) with a group structure simply by defining the composite of \((g_1, g_2), (f_1, f_2) \in G_1 \times G_2\) as

\[
(g_1, g_2) \circ \psi (f_1, f_2) := (g_1 \circ_1 f_1, g_2 \circ_2 \psi_{g_1}(f_2)).
\]

The operation is well defined, so \((G_1 \otimes_\psi G_2, \circ_\psi)\) is a group called the semidirect product of \( G_1 \) and \( G_2 \) by \( \psi \). The order of the factors in the product is clearly relevant.

Looking at the semidirect product \((G \otimes_\psi N, \circ_\psi)\) we could prove \( N \) is a normal subgroup of \( G \otimes_\psi N \), and

\[
\psi_g(n) = g \circ_\psi n \circ_\psi g^{-1} \quad \text{for any } g \in G, n \in N.
\]

There is also a converse of sorts. Consider a group \((H, \circ)\), let \( G \) be a subgroup of \( H \) and \( N \) a normal subgroup. Assume \( N \cap G = \{e\} \), \( e \) being the identity of \( H \). Suppose also \( H = GN \), meaning that for any \( h \in H \) there exist \( g \in G \) and \( n \in N \) such that \( h = gn \). Then one can prove that the pair \((g, n)\) is uniquely determined by \( h \), and \( H \) is isomorphic to the semidirect product \( G \otimes_\psi N \) with

\[
\psi_g(n) := g \circ h \circ g^{-1} \quad \text{for any } g \in G, n \in N.
\]

If now \( V \) is a vector space (real or complex), \( GL(V) \) denotes the group of bijective linear maps \( f : V \to V \) with the usual composition law. \( GL(V) \) is called the (general) linear group of \( V \).

If \( V := \mathbb{R}^n \) or \( \mathbb{C}^n \) then \( GL(V) \) is denoted by \( GL(n, \mathbb{C}) \) or \( GL(n, \mathbb{R}) \), respectively.

Let us define linear representations of a group. Take \((G, \circ)\) a group and \( V \) a vector space. A (linear) representation of \( G \) on \( V \) is a homomorphism \( \rho : G \to GL(V) \).
A representation $\rho : G \to GL(V)$ is called:

1) **faithful** if it is injective;
2) **free** if the subgroup made of elements $h_v$ such that $\rho(h_v)v = v$ is trivial for any $v \in V \setminus \{0\}$, i.e. it contains only the neutral element of $G$;
3) **transitive** if, for any $v, v' \in V \setminus \{0\}$ there exists $g \in G$ with $v' = \rho(g)v$;
4) **irreducible** if no proper subspace $S \subset V$ exists that is **invariant** under the action of $\rho(G)$, i.e. $\rho(g)S \subset S$ for any $g \in G$.

In case $V$ is a Hilbert or Banach space and $\rho$ defines *bounded operators on the entire $V$*, the representation is said irreducible if there are no *closed* $\rho(G)$-invariant subspaces in $V$. 
Appendix B

Elements of differential geometry

Let $n, m = 1, 2, \ldots, k = 0, 1, \ldots$ be fixed integers and $\Omega \subset \mathbb{R}^n$ an open non-empty set. A map $f : \Omega \rightarrow \mathbb{R}^m$ is of class $C^k$ (or simply $C^k$), written $f \in C^k(\Omega; \mathbb{R}^m)$, if all partial derivatives of the components of $f$ are continuous up to order $k$ included. Conventionally, $C^k(\Omega) := C^k(\Omega; \mathbb{R})$.

A function $f : \Omega \rightarrow \mathbb{R}^m$ is (of class) $C^\infty$, or smooth, if it is $C^k$ for any $k = 0, 1, \ldots$, so one defines

$$C^\infty(\Omega; \mathbb{R}^n) := \bigcap_{k=0,1,\ldots} C^k(\Omega; \mathbb{R}^n).$$

Again, $C^\omega(\Omega) := C^\omega(\Omega; \mathbb{R})$. Eventually, $f : \Omega \rightarrow \mathbb{R}^m$ is $C^\omega$ or real-analytic if it is $C^\omega$ and it admits a Taylor expansion (in several real variables) at any $p \in \Omega$, on some open ball around $p$ of finite radius, contained in $\Omega$. Usually, when the order $k$ of differentiability is not mentioned explicitly it means that $k = \infty$.

**Notation B.1.** In this section upper indices denote coordinates of $\mathbb{R}^n$ and components of (contravariant) vectors. Thus the standard coordinates on $\mathbb{R}^n$ will be denoted by $x^1, \ldots, x^n$, instead of $x_1, \ldots, x_n$. ■

### B.1 Smooth manifolds, product manifolds, smooth functions

The most general and powerful tool apt to describe the features of spacetime, three-dimensional physical space, and the abstract space of physical systems in classical theories, is the notion of smooth manifold. In practice a smooth manifold is a collection of objects, generally called points, that admits local coordinates identifying points with $n$-tuples of $\mathbb{R}^n$. 
Definition B.2. Let \( n = 1, 2, 3, \ldots \) and \( k = 1, 2, \ldots, \infty \) be fixed numbers. A \( C^k \) manifold of dimension \( n \) is a set \( M \), whose elements are called points, equipped with the geometric structure defined below.

1. \( M \) has a differentiable structure \( \mathcal{A} = \{(U_i, \phi_i)\}_{i \in I} \) of class \( C^k \), that is a collection of pairs \((U_i, \phi_i)\), called local charts, where \( U_i \) is a subset in \( M \) and \( \phi_i \) a map from \( U_i \) to \( \mathbb{R}^n \) (the local coordinate system or local frame) such that:

   i. \( \bigcup_{i \in I} U_i = M \), any \( \phi_i \) is injective and \( \phi_i(U_i) \) is open in \( \mathbb{R}^n \) (so \( M \) is called an \( n \)-dimensional manifold, or just \( n \)-manifold);

   ii. local charts in \( \mathcal{A} \) must be pairwise \( C^k \)-compatible. Two injective maps \( \phi: U \to \mathbb{R}^n, \psi: V \to \mathbb{R}^n \) with \( U, V \subset M \) are \( C^k \)-compatible if either \( U \cap V = \emptyset \), or \( U \cap V \neq \emptyset \) and the maps \( \phi \circ \psi^{-1}: \psi(U \cap V) \to \phi(U \cap V), \psi \circ \phi^{-1}: \phi(U \cap V) \to \psi(U \cap V) \) are both \( C^k \);

   iii. \( \mathcal{A} \) is maximal, i.e.: if \( U \subset M \) is open and \( \phi: U \to \mathbb{R}^n \) compatible with every local chart of \( \mathcal{A} \), then \( (U, \phi) \in \mathcal{A} \).

2. Topological requirements:

   i. \( M \) is a second-countable Hausdorff space;

   ii. \( M \) is, by way of \( \mathcal{A} \), locally homeomorphic to \( \mathbb{R}^n \). In other terms, if \( (U, \phi) \in \mathcal{A} \) then \( U \) is open and \( \phi: U \to \phi(U) \) is a homeomorphism.

A smooth \( C^\infty \) manifold is more often called real-analytic manifold.

Remark B.3. (1) Every local chart \((U, \phi)\) enables us to assign \( n \) real numbers \((x_1^p, \ldots, x_n^p) = \phi(p)\) bijectively to every point \( p \) of \( U \). The entries of the \( n \)-tuple are the coordinates of \( p \) in the local chart \((U, \phi)\). Points in \( U \) are thus in one-to-one correspondence with \( n \)-tuples of \( \phi(U) \subset \mathbb{R}^n \).

(2) If \( U \cap V \neq \emptyset \), the compatibility of local charts \((U, \phi), (V, \psi)\) implies that the Jacobian matrix of \( \phi \circ \psi^{-1} \) is invertible and so has everywhere non-zero determinant. Conversely, if \( \phi \circ \psi^{-1}: \psi(U \cap V) \to \phi(U \cap V) \) is bijective, of class \( C^k \), and with non-vanishing Jacobian determinant on \( \psi(U \cap V) \), then also \( \psi \circ \phi^{-1}: \phi(U \cap V) \to \psi(U \cap V) \) is \( C^k \) and the local charts are compatible. The proof can be found in the renowned [CoFr98II].

Theorem B.4 (Implicit function theorem). Let \( D \subset \mathbb{R}^n \) be open, non-empty, and \( f: D \to \mathbb{R}^n \) a \( C^k \) function for some \( k = 1, 2, \ldots, \infty \). If the Jacobian of \( f \) at \( p \in D \) has non-zero determinant there exist open neighbourhoods \( U \subset D \) of \( p \) and \( V \) of \( f(p) \) such that: (i) \( f \upharpoonright U: U \to V \) is bijective, (ii) the inverse \( f^{-1} \upharpoonright V: V \to U \) is \( C^k \).

(3) The topological requirements in (2)(ii) (valid for the standard topology of \( \mathbb{R}^n \)) are technical and guarantee unique solutions to differential equations on \( M \) (necessary in physics when the equations describe the evolution of physical systems) and the existence of integrals on \( M \). Condition (2)(ii) intuitively says that \( M \) is, around any point, “continuous” like \( \mathbb{R}^n \). Standard counterexamples show that the Hausdorff property of \( \mathbb{R}^n \) is not carried over to \( M \) by local homeomorphisms, so it must be imposed explicitly.
(4) Let $M$ be a second-countable Hausdorff space. A collection of local charts $\mathcal{A}$ on $M$ satisfying (i) and (ii) in (1), but not necessarily (iii), plus (ii) in (2) is called a $C^k$ atlas on the $n$-manifold $M$. It is not hard to see that any atlas $\mathcal{A}$ on $M$ is contained in some maximal atlas. Two atlases on $M$ such that every chart of one is compatible with any chart of the other induce the same differentiable structure on $M$. Thus to assign a differentiable structure it suffices to prescribe a non-maximal atlas, one of the many that determine it. The unique differentiable structure associated to a given atlas is said to be induced by the atlas.

(5) If $1 \leq k < \infty$ there might be superfluous charts in the differentiable structure (only a finite number!), eliminating which gives a $C^\infty$ atlas. ■

**Examples B.5.** (1) The simplest examples of differentiable manifolds, of class $C^\infty$ and dimension $n$, are non-empty open subsets of $\mathbb{R}^n$ (including $\mathbb{R}^n$ itself) with standard differentiable structure determined by the identity map (the inclusion, alone, defines an atlas).

(2) Consider the unit sphere $S^2$ in $\mathbb{R}^3$ (with topology inherited from $\mathbb{R}^3$) centred at the origin:

$$S^2 := \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

in canonical coordinates $x^1, x^2, x^3$ of $\mathbb{R}^3$. It has dimension 2 and a smooth structure induced by $\mathbb{R}^3$ by defining an atlas with 6 local charts $(S^2_{(i)\pm}, \phi_{(i)\pm})$ ($i = 1, 2, 3$) as follows. Take the axis $x^i$ ($i = 1, 2, 3$) and the pair of open hemispheres $S^2_{(i)\pm}$ with south-north direction given by $x^i$, and consider local charts $\phi_{(i)\pm} : S^2_{(i)\pm} \to \mathbb{R}^2$ that map $p \in S^2_{(i)\pm}$ to its coordinates on the plane $x^i = 0$. It can be proved (see below) that $S^2$ cannot be covered by a single (global) chart, in contrast to $\mathbb{R}^3$ (or any open subspace). This proves that the class of smooth manifolds does not reduce to open non-empty subsets of $\mathbb{R}^n$, and hence is quite interesting. A similar example is the circle in $\mathbb{R}^2$. ■

Given $C^k$ manifolds $M$ and $N$ of respective dimensions $m, n$, we can construct a third $C^k$ manifold of dimension $m + n$ over the topological product $M \times N$. (The resulting space will be Hausdorff and second-countable.) This is called product manifold of $M$ and $N$, and denoted simply by $M \times N$. The structure described herebelow is called product structure. Given local charts $(U, \phi)$ on $M$ and $(V, \psi)$ on $N$ it is immediate to see

$$U \times V \ni (p, q) \mapsto (\phi(p), \psi(q)) =: \phi \oplus \psi(p, q) \in \mathbb{R}^{m+n} \quad (B.1)$$

is a local homeomorphism. If $(U', \phi')$ and $(V', \psi')$ are other charts, compatible with the previous ones, the charts $(U \times V, \phi \oplus \psi)$ and $(U' \times V', \phi' \oplus \psi')$ are obviously compatible. As $(U, \phi)$ and $(V, \psi)$ vary on $M$ and $N$ the charts $(U \times V, \phi \oplus \psi)$ define an atlas on $M \times N$. The structure this atlas generates is, by definition, the product structure.

**Definition B.6.** Given $C^k$ manifolds $M, N$ of dimension $m, n$, the product manifold is the set $M \times N$ equipped with product topology and $C^k$ structure induced by the local charts $(U \times V, \phi \oplus \psi)$ as of (B.1), when $(U, \phi), (V, \psi)$ vary on $M, N$. 

Since a manifold is locally indistinguishable from $\mathbb{R}^n$, the differentiable structure allows to make sense of differentiable functions defined on a manifold other than $\mathbb{R}^n$ or subsets. The idea is simple: reduce locally to the standard notion on $\mathbb{R}^n$ using the local charts that cover the manifold.

**Definition B.7.** Let $M, N$ be manifolds of dimensions $m, n$ and class $C^p, C^q$ respectively ($p, q \geq 1$). A continuous map $f : M \to N$ is said $C^k$ ($0 \leq k \leq p, q$, possibly $k = \infty$ or $\omega$) if $\psi \circ f \circ \phi^{-1}$ is a $C^k$ map from $\mathbb{R}^m$ to $\mathbb{R}^n$, for any choice of local charts $(U, \phi)$ on $N$ and $(V, \psi)$ on $M$.

The collection of $C^k$ functions from $M$ to $N$, $k = 0, 1, 2, \ldots, \infty, \omega$ is denoted $C^k(M; N)$; if $N = \mathbb{R}$ one just writes $C^k(M)$.

A $C^k$ diffeomorphism $f : M \to N$ is a bijective $C^k$ map with $C^k$ inverse. If there is a $C^k$ diffeomorphism $f$ mapping $M$ to $N$, the two manifolds are called diffeomorphic (under $f$).

**Remark B.8.** (1) Notice how we allowed for differentiable maps of class $C^0$, which are actually just continuous maps (like $C^0$ diffeomorphisms are just homeomorphisms). Every $C^k$ diffeomorphism is clearly a homeomorphism, which explains why there cannot exist any diffeomorphism between $S^2$ and (a subset of) $\mathbb{R}^2$, for the former is compact, the latter not. Consequently, the sphere $S^2$ does not admit global charts.

(2) For $f : M \to N$ to be $C^p$ it is enough that $\psi \circ f \circ \phi^{-1}$ is $C^k$ for any local charts $(U, \phi), (V, \psi)$ in the given atlases, without having to check the condition for every possible local charts on the manifolds.

A useful notion is that of embedded submanifold. $\mathbb{R}^n$ is an embedded submanifold in $\mathbb{R}^m$ if $m > n$. In the canonical coordinates $x_1, \ldots, x^m$ on $\mathbb{R}^m$, $\mathbb{R}^n$ is identified with the subspace given by equations $x^{n+1} = \cdots = x^m = 0$, while the first $n$ coordinates of $\mathbb{R}^m$, $x_1, \ldots, x^n$, are identified with the standard coordinates on $\mathbb{R}^n$. Now the idea is to replace $\mathbb{R}^n, \mathbb{R}^m$ using local frames, and generalise to manifolds $N, M$.

**Definition B.9.** Let $M$ be a $C^k$ ($k \geq 1$) manifold of dimension $m > n$. An embedded $C^k$ submanifold of $M$ of dimension $n$ is the following $n$-manifold $N$ of class $C^k$.

(a) $N$ is a subset in $M$ with induced topology.

(b) The differentiable structure di $N$ is given by the atlas $\{ (U_i, \phi_i) \}_{i \in I}$ where:

(i) $U_i = V_i \cap N, \phi_i = \psi \mid_{V_i \cap N}$ for a suitable local chart $(V_i, \phi_i)$ on $M$;

(ii) in the frame $x_1, \ldots, x^m$ associated to $(V_i, \phi_i)$, the set $V_i \cap N$ is determined by $x^{n+1} = \cdots = x^m = 0$, and the remaining coordinates $x_1, \ldots, x^n$ are the local framing associated to $\phi_i$.

To finish we state an important result (see [doC92, Wes78] for example) to decide when a subset in a manifold is an embedded submanifold. The proof is straightforward from Dini’s theorem [CoFr98II].

**Theorem B.10 (On regular values).** Let $M$ be a $C^k$ manifold of dimension $m$. Consider the set

$$N := \{ p \in M \mid f_j(p) = v_j, \ j = 1, \ldots, c \}$$
determined by \( c(< m) \) constants \( v_j \) and \( c \) functions \( f_j : M \to \mathbb{R} \) of class \( C^k \). Suppose that around each point \( p \in N \) there exists a local chart \((U, \phi)\) on \( M \) such that the Jacobian matrix \( \partial (f_j \circ \phi^{-1})/\partial x^i|_{\phi(p)} \) has rank \( r \). Then \( N \) is an embedded \( C^k \) submanifold in \( M \) of dimension \( n := m - c \).

In particular, if the square \( c \times c \) matrix

\[
\frac{\partial f_j \circ \phi^{-1}}{\partial x^k}, \quad j = 1, \ldots, c, \quad k = m - c + 1, m - c + 2, \ldots, m
\]

is non-singular at \( \phi(p) \), \( p \in N \), then the first \( n \) coordinates \( x^1, \ldots, x^n \) define a frame system around \( p \) in \( N \).

**B.2 Tangent and cotangent spaces. Covariant and contravariant vector fields**

Let \( M \) be \( C^k \) manifold of dimension \( n \) \((k \geq 1)\). Consider the space \( C^k(M) \) as an \( \mathbb{R} \)-vector space with linear combinations

\[
(af + bg)(p) := af(p) + bg(p), \quad \text{for any } p \in M
\]

where \( a, b \in \mathbb{R}, f, g \in C^k(M) \). Given a point \( p \in M \), a **derivation** at \( p \) is an \( \mathbb{R} \)-linear map \( L_p : C^k(M) \to \mathbb{R} \) satisfying the Leibniz rule:

\[
L_p(fg) = f(p)L_p(g) + g(p)L_p(f), \quad f, g \in C^k(M).
\]  

(B.2)

A linear combination \( aL_p + bL'_p \) of derivations at \( p \) \((a, b \in \mathbb{R})\),

\[
(aL_p + bL'_p)(f) := aL_p(f) + bL'_p(f), \quad f, g \in C^k(M),
\]

is still a derivation. Hence derivations at \( p \) form a vector space over \( \mathbb{R} \), which we denote \( \mathcal{D}^k_p \). Every local chart \((U, \phi)\) with \( U \ni p \) automatically gives \( n \) derivations at \( p \), as follows. If \( x^1, \ldots, x^n \) are coordinates associated to \( \phi \), define the \( k \)th derivation to be

\[
\frac{\partial}{\partial x^k} \bigg|_p : f \mapsto \frac{\partial f \circ \phi^{-1}}{\partial x^k} \bigg|_{\phi(p)}, \quad f, g \in C^1(M).
\]  

(B.3)

If \( 0 \) is the null derivation and \( c^1, c^2, \ldots, c^n \in \mathbb{R} \) satisfy \( \sum_{k=1}^n c^k \frac{\partial}{\partial x^k} \bigg|_p = 0 \), we choose a differentiable function conciding with the coordinate map \( x^l \) on an open neighbourhood of \( p \) (whose closure is in \( U \)) and vanishing outside. Then the \( n \) derivations \( \frac{\partial}{\partial x^k} \bigg|_p \) at \( p \) are **linearly independent**: \( \sum_{k=1}^n c^k \frac{\partial}{\partial x^k} \bigg|_p f = 0 \) implies \( c^l = 0 \). Since we are free to choose \( l \) arbitrarily, every coefficient \( c^r \) is zero for \( r = 1, 2, \ldots, n \). Hence the \( n \) derivations \( \frac{\partial}{\partial x^k} \bigg|_p \) form a basis for an \( n \)-dimensional subspace of \( \mathcal{D}^k_p \) (actually if \( k = \infty \) the
The proof is direct from the definitions. Because the Jacobian
\[ \frac{\partial}{\partial y^i} \bigg|_p = \sum_{k=1}^{n} \frac{\partial x^k}{\partial y^i} \bigg|_{\psi(p)} \frac{\partial}{\partial x^k} \bigg|_p. \]

(B.4)

The proof is direct from the definitions. Because the Jacobian \( \frac{\partial x^k}{\partial y^i} \bigg|_{\psi(p)} \) is invertible by definition of chart, the subspace of \( \mathcal{D}_p^\infty \) spanned by the \( \frac{\partial}{\partial x^k} \bigg|_p \) coincides with the span of the \( \frac{\partial}{\partial x^k} \bigg|_p \). The subspace is thus intrinsically defined.

**Definition B.11.** Let \( M \) be an \( n \)-dimensional \( C^k \) manifold \( (k \geq 1) \), and fix a point \( p \in M \). The vector subspace of derivations at \( p \) generated by the \( n \) derivations \( \frac{\partial}{\partial x^i} \bigg|_p \), \( k = 1, 2, \ldots, n \), in any local coordinate system \( (U, \phi) \) with \( U \ni p \), is called tangent space of \( M \) at \( p \) and is written \( T_p M \). The elements of the tangent space at \( p \) are the tangent vectors at \( p \) to \( M \). Tangent vectors are examples of contravariant vectors.

We recall that the space \( V^* \) of linear maps from a real vector space \( V \) to \( \mathbb{R} \) is called dual space to \( V \). If the dimension of \( V \) is finite, so is the dimension of \( V^* \), for they coincide. In particular, if \( \{e_i\}_{i=1}^n \) is a basis of \( V \), the dual basis in \( V^* \) is the basis \( \{e^j\}_{j=1}^n \) defined via: \( e^j(e_i) = \delta^j_i \), \( i, j = 1, \ldots, n \), by linearity. With \( f \in V^* \), \( v \in V \), one uses the notation \( \langle v, f \rangle := f(v) \).

**Definition B.12.** Let \( M \) be an \( n \)-dimensional \( C^k \) manifold \( (k \geq 1) \), \( p \in M \) a given point. The dual space to \( T_p M \) is called cotangent space of \( M \) at \( p \), written \( T^*_p M \). Points of the cotangent space at \( p \) are called cotangent vectors at \( p \) or 1-forms at \( p \), and are instances of covariant vectors (covectors). For any basis \( \frac{\partial}{\partial x^i} \bigg|_p \) of \( T_p M \), the \( n \) elements of the dual basis are indicated by \( dx^i|_p \). By definition

\[ \left\langle \frac{\partial}{\partial x^j}|_p, dx^i|_p \right\rangle = \delta^i_j. \]

Let us move on to vector fields on a manifold \( M \).

Suppose \( M \) is an \( n \)-dimensional \( C^k \) manifold (including \( k = \infty \) and \( k = \omega \)). A contravariant \( C^r \) vector field, \( r \geq 0, 1, \ldots, k \), is a map assigning a vector \( v(p) \in T_p M \) to any \( p \in M \), so that for any local chart \( (U, \phi) \) with coordinates \( x^1, \ldots, x^n \) where

\[ v(q) = \sum_{i=1}^{n} v^i(x_{q}^1, \ldots, x_{q}^n) \frac{\partial}{\partial x^i} \bigg|_q, \]

the \( n \) functions \( v^i = v^i(x^1, \ldots, x^n) \) are \( C^r \) on \( \phi(U) \). Similarly, a covariant \( C^r \) vector field, \( r \geq 0, 1, \ldots, k \) is a map sending \( p \in M \) to a covector \( \omega(p) \in T^*_p M \), so that for
any local chart \((U, \phi)\) with coordinates \(x^1, \ldots, x^n\) where
\[
\omega(q) = \sum_{i=1}^{n} v_i(x^1_{q}, \ldots, x^n_{q}) \, dx^i|_q ,
\]
the \(n\) functions \(\omega_i = \omega_i(x^1, \ldots, x^n)\) are \(C^r\) on \(\phi(U)\).

**Remarks B.13.** Take \(v \in T_pM\) and two local charts \((U, \phi), (V, \psi)\) with \(U \cap V \ni p\) and respective coordinates \(x^1, \ldots, x^n, \, x'^1, \ldots, x'^n\). Then \(v = \sum_{i=1}^{n} v^j \frac{\partial}{\partial x^j}\)|\(_p = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x'^j}\)|\(_p\). Hence \(\sum_{i=1}^{n} v^j \frac{\partial}{\partial x^j}\)|\(_p = \sum_{j=1}^{n} v^j \frac{\partial x'^j}{\partial x^j}\)|\(_{\psi(p)}\) \(\frac{\partial}{\partial x^i}\)|\(_p\), so \(\sum_{i=1}^{n} \left(v^j - \sum_{j=1}^{n} \frac{\partial x'^j}{\partial x^j}\)|\(_{\psi(p)}\) \(v^j\) \(\right) \frac{\partial}{\partial x^i}\)|\(_p\) = 0. Since the derivations \(\frac{\partial}{\partial x^i}\)|\(_p\) are linearly independent, we conclude that the components of a tangent vector in \(T_pM\) transform, under coordinate change, as
\[
v^j = \sum_{j=1}^{n} \frac{\partial x^i}{\partial x'^j}\)|\(_{\psi(p)}\) \(v^j\). \hspace{1cm} \text{(B.5)}
\]
The same argument gives the formula for covariant vectors \(\omega = \sum_{i=1}^{n} \omega_i \, dx^i|_p = \sum_{j=1}^{n} \omega'_j \, dx'^j|_p\), namely
\[
\omega_i = \sum_{j=1}^{n} \frac{\partial x'^j}{\partial x^i}\)|\(_{\psi(p)}\) \(\omega'_j\). \hspace{1cm} \text{(B.6)}

### B.3 Differentials, curves and tangent vectors

Let \(f : M \to \mathbb{R}\) be a \(C^r\) scalar field on the \(C^k\) \(n\)-manifold \(M\), and assume \(k \geq r > 1\). The **differential** \(df\) of \(f\) is the covariant vector field of class \(C^{r-1}\)
\[
df|_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\)|\(_{\psi(p)}\) \(dx^i|_p\)
in any local chart \((U, \psi)\).

Consider a \(C^r\) curve inside the \(C^k\) manifold \(M\) \((r = 0, 1, \ldots, k)\), i.e. a \(C^r\) function \(\gamma : I \to M\) where \(I \subset \mathbb{R}\) is an open interval thought of as a submanifold in \(\mathbb{R}\). Assume explicitly that \(r > 1\). We can define the **tangent vector** to \(\gamma\) at \(p \in \gamma(I)\) by
\[
\dot{\gamma}(p) := \sum_{i=1}^{n} \frac{dx^i}{dt}\)|\(_{t_p}\) \(\frac{\partial}{\partial x^i}\)|\(_p\) ,
\]
where \(\gamma(t_p) = p\), in any local chart around \(p\). The definition does **not** depend on the chart. Had we defined
\[
\dot{\gamma}(p) := \sum_{j=1}^{n} \frac{dx'^j}{dt}\)|\(_{t_p}\) \(\frac{\partial}{\partial x'^j}\)|\(_p\)
in another frame system around $p$, using (B.5) would have given

$$\dot{\gamma}(p) = \dot{\gamma}'(p).$$

So we have this definition.

**Definition B.14.** A $C^r$ curve, $r = 0, 1, \ldots, k$, in the $n$-dimensional $C^k$ manifold $M$ is a $C^r$ map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an open interval (embedded in $\mathbb{R}$). When $r > 1$, the tangent vector to $\gamma$ at $p = \gamma(t_p)$, $t_p \in I$, is the vector $\dot{\gamma}(p) \in T_p M$ given by

$$\dot{\gamma}(p) := \sum_{i=1}^{n} \left. \frac{d}{dt} \left|_{t_p} \frac{\partial}{\partial x^i} \right|_p \right.,$$

in any local framing around $p$.

### B.4 Pushforward and pullback

Let $M$ and $N$ be manifolds of dimensions $m$ and $n$, and $f : N \to M$ a function (all at least $C^1$). Given a point $p \in N$ consider local charts $(U, \phi)$ around $p$ in $N$ and $(V, \psi)$ around $f(p)$ in $M$. Indicate by $(y^1, \ldots, y^n)$ the coordinates on $U$, by $(x^1, \ldots, x^m)$ those on $V$ and introduce maps $f^k(y^1, \ldots, y^n) = y^k(f \circ \phi^{-1})$, $k = 1, \ldots, m$. Now define:

(i) the **pushforward** $df_p : T_p N \to T_{f(p)} M$, in coordinates:

$$df_p : T_p N \ni \sum_{i=1}^{n} u^i \frac{\partial}{\partial y^i} \bigg|_p \mapsto \sum_{j=1}^{m} \left( \sum_{i=1}^{n} \frac{\partial f^j}{\partial y^i} \bigg|_{\phi(p)} \right) u^i \frac{\partial}{\partial x^j} \bigg|_p; \quad (B.8)$$

(ii) the **pullback** $f^* : T^*_{f(p)} M \to T^*_p N$, in coordinates:

$$f^* : T^*_p N \ni \sum_{j=1}^{m} \omega_j dx^j|_{f(p)} \mapsto \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \frac{\partial f^j}{\partial y^i} \bigg|_{\phi(p)} \right) \omega_j dy^i|_p. \quad (B.9)$$

It is not hard to see they do not depend on local frame systems. The pushforward is also written $f_p : T_p N \to T_{f(p)} M$. 

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