Appendix

Non-exceptional Branch Points; The Vanishing of the $L^{th}$ Derivative, $L$ Even

We discuss how to demonstrate that a minimal surface $X$ with a non-exceptional branch point at $w = 0$ of order $n$, $L$ even cannot be a minimum, if we consider the generator $\tau := c \in w^{-(n+1)} + \bar{c} w^{n-1} + \delta c w^{-r} + \bar{\delta} \bar{c} w^r$. As we have observed, the $\epsilon^{L-1}$ term of the $L^{th}$ derivative is zero.

Here we need a trick. Going to the next highest derivative gives us additional parameters to work with, allowing us to show that, with appropriate choices, the leading $\epsilon^{L-1}$ term of the $(L + 1)^{st}$ derivative is negative. This implies

$$E^{(L)}(0) = O(\epsilon^L),$$

(A.1)

and this remains true if we change the choice for $\tau$ to

$$\tau := c \epsilon w^{-n-1} + \bar{c} \epsilon w^{n+1} + \delta c w^{-r} + \bar{\delta} \bar{c} w^r + \rho, \quad c \in \mathbb{C}.$$  

(A.2)

We infer

$$2(m + 1) = L(n + 1) - (n + 1 - r), \quad L = \text{even},$$

and so $n + 1 - r$ is even, which implies that

$$1 \leq r \leq n - 1.$$  

(A.3)

Next we define a meromorphic $\phi_t(0)$, real on $S^1$, such that

$$\phi_t(0) := -i \mu c^2 \epsilon^2 w^{-2n-1} - i \delta c^2 \epsilon (n + 1 - r) w^{-n-1-r} + \gamma c^2 w^{-r} + \cdots,$$

(A.4)

with an arbitrary $\lambda \in \mathbb{C}$. Then it follows

$$w \hat{Z}_{tw}(0) \phi(0) + 2w \hat{Z}_t w(0) = 2c^3 \epsilon (i \lambda A_1 w^{-r} + i \rho \delta (2n + 2 - r) A, w^{-r} \cdots, \cdots)$$

$$+ 2c^3 \epsilon^3 (\mu^2 A_1 w^{-2n} + \cdots, \cdots).$$

For simplicity, we shall assume, for the moment, that $\delta = 0$. This does not alter (A.1). This leads to the definition.

$$\phi_{tt}(0) := -2c^3 \epsilon^3 \mu^2 w^{-3n-1} - i \lambda c^3 w^{-n-1-r} + \cdots,$$

(A.5)
and inductively to
\[
D_t^\beta \phi(0) := c^{\beta+1} \left[ \text{const } \epsilon^{\beta+1} w^{-(\beta+1)n-1} + \text{const } \epsilon^{\beta-1} w^{-(\beta-1)n-r-1} + \ldots \right]
\]
for \(1 \leq \beta \leq L/2 - 1\). \hspace{1cm} (A.6)

Now we write the formula for \(E^{(L+1)}(0)\) in a different order as
\[
E^{(L+1)}(0) = I_0 + I_1 + I_2 + I_4 + I_5 + I_6
\]
with
\[
I_0 := 4 \text{Re} \int_{S^1} w[D_t^L \hat{Z}(0)]_w \tau \, dw;
\]
\[
I_1 := \sum_{M=s+2}^{L-1} \frac{4L!}{M!(L-M)!} \text{Re} \int_{S^1} w[D_t^M \hat{Z}(0)]_w g_{L-M} \, dw, \quad s := L/2;
\]
\[
I_2 := \frac{4L!}{(s+1)!(L-s-1)!} \text{Re} \int_{S^1} w[D_t^{s+1} \hat{Z}(0)]_w g_{L-s-1} \, dw;
\]
\[
I_3 := \frac{2L!}{s!} \text{Re} \int_{S^1} w[D_t^s \hat{Z}(0)]_w \cdot h_s \, dw, \quad \alpha + \beta = s;
\]
\[
I_4 := \frac{2L!}{\sigma!} \text{Re} \int_{S^1} w[D_t^\sigma \hat{Z}(0)]_w \cdot h_\sigma \, dw, \quad \sigma = s-1 = L/2 - 1, \quad \beta = L - \sigma - \alpha;
\]
\[
I_5 := \sum_{M=2}^{s-2} \frac{2L!}{M!M!} \text{Re} \int_{S^1} w[D_t^M \hat{Z}(0)]_w \cdot h_M \, dw, \quad \beta = L - M - \alpha;
\]
\[
I_6 := \frac{4L!}{(L-1)!} \text{Re} \int_{S^1} w \hat{Z}_{tw}(0) \cdot \hat{X}_w D_t^{L-1} \phi(0) \, dw
\]
\[+ 2 \text{Re} \int_{S^1} w \hat{X}_w \cdot \hat{X}_w D_t^L \phi(0) \, dw.
\]

The standard reasoning yields \(I_0 = 0\), \(I_6 = 0\), and the pole-removal process yields that \(g_{L-M}\) is holomorphic; thus also \(I_1 = 0\).

**Lemma A.1** We have \(I_5 = O(\epsilon^L)\). This leaves us with
\[
E^{(L+1)}(0) = I_2 + I_3 + I_4 + O(\epsilon^L). \hspace{1cm} (A.7)
\]

**Proof** We begin by considering the contribution of the last complex component to \(I_5\). Since \(\alpha \leq M\), we have in
\[
w[D_t^M \hat{Z}(0)]_w \cdot h_M = \sum_{\alpha=0}^{M} c_{\alpha M}^\alpha \psi(M, \alpha)[D_t^\alpha \hat{Z}(0)]_w D_t^\beta \phi(0), \hspace{1cm} (A.8)
\]
that \(\beta = L - M - \alpha \geq L - 2(s-2) = L - 2(L/2 - 2) \geq 4\).

We will show that there is no pole associated with a term that has \(\epsilon^\gamma, \gamma \leq L - 1\), as a coefficient. We have
\[
w[D_t^M \hat{Z}(0)]_w \cdot [D_t^\alpha \hat{Z}(0)]_w = \text{const } \epsilon^{M+\alpha} w^{1+2m-(\alpha+M)(n+1)} + \ldots.
\]
In order to achieve a coefficient of order $\epsilon^{\gamma}$, $\gamma \leq L - 1$, we must consider the contribution from the second term of (A.6),

$$\text{const} \epsilon^{\beta - 1} w^{-(\beta - 1)n - r - 1}, \quad M + \alpha + \beta = L.$$ 

The order of the $w$-term will then be

$$1 + 2m - (M + \alpha)(n + 1) - (\beta - 1)n - r - 1$$

$$= (2m + 2) - (M + \alpha + \beta - 1)(n + 1) + (\beta - 1) - r - 2$$

$$= r + (\beta - 3) - r \geq 1;$$

thus there is no pole.

The lowest $w$-powers associated to $\epsilon^{M - 2}$ in $(D_t^M \hat{Z}(0))_w$ are of the order $m - (M - 2)(n + 1)$.

Considering the order of the largest pole in $D_t^\beta \phi(0)$ with coefficient $\epsilon^{\beta + 1}$ and looking at the total contribution to a pole of order $\epsilon^{L - 1}$ in (A.8), we obtain a term of the form

$$\text{const} \epsilon^{M - 2 + \alpha + (\beta + 1)} w^{1 + 2m - (M - 2)(n + 1) - \alpha(n + 1) - (\beta + 1)n - 1} = \text{const} \epsilon^{L - 1} w^r + \beta - 1;$$

so again there is no pole.

What about the first two complex components? In the first case from above we get terms of the form $(j \leq 2m - 2n)$:

$$\text{const} \epsilon^{M + \alpha + \beta - 1} (A_j \cdot A_{2m - 2n + 1}) w^{1 + 2m - n - \alpha(n + 1) - (\beta - 1)n - r - 1},$$

and

$$1 + 2m - n - \alpha(n + 1) - (\beta - 1)n - r - 1$$

$$= 2 + 2m - n - \alpha(n + 1) - \beta n + n - r - 2$$

$$= 2 + 2m - \alpha(n + 1) - \beta(n + 1) + \beta - r - 2$$

$$= 2 + 2m - (L - M)(n + 1) + \beta - r - 2$$

$$= 2 + 2m - (L - 1)(n + 1) + (M - 1)(n + 1) + \beta - r - 2$$

$$= r + (M - 1)(n + 1) + \beta - r - 2$$

$$= (M - 1)(n + 1) + \beta - 2 \geq (n + 1) + 2 = n + 3$$

since $M \geq 2$ and $k \geq 4$.

Again, there is no pole, and similarly for the second case from above. This completes the proof of Lemma A.1. \[\square\]

**Lemma A.2** We have

$$I_3 = O(\epsilon^{L}) \quad (A.9)$$

with a real number $T \geq 0$. 

**Proof** We have \( n \geq 2 \) because, by (A.3),
\[
n - 1 - r \geq 0, \quad n - 1 - r \text{ even, } r \geq 1.
\]
We begin by considering what comes from the first two complex components in the products
\[
w[D^s_t \hat{Z}(0)]_w \cdot [D^s_t \hat{Z}(0)]_w \tau, \quad w[D^s_t \hat{Z}(0)]_w \cdot [D^{s-1}_t \hat{Z}(0)]_w \phi_t(0), \quad \ldots \quad (A.10)
\]
\( s = L/2 \).

(i) First we have to understand \([D^s_t \hat{Z}(0)]_w\):
\[
[D^s_t \hat{Z}(0)]_w = \left\{ 2H \text{Re} \left[ i \sum_{\alpha + \beta = s-1} \frac{(s-1)!}{\alpha!\beta!} w[D^s_t \hat{Z}(0)]_w D^\beta_t \phi(0) \right] \right\}_w
\]
\[
= \left\{ 2H \text{Re} \left[ i w(D^{s-1}_t \hat{Z}(0))_w \tau + i (s-1)(D^{s-2}_t \hat{Z}(0))_w \phi_t(0) + \ldots \right] \right\}_w
\]
\[
= \left\{ 2H \text{Re} \left[ c^s e^s i^s T_1 R_m w^{\gamma_1} + c^s i^{s-1} e^{s-2} T_2 R_m w^{\gamma_2} + \ldots \right] \right\}_w
\]
where \( T_1, T_2 \) are real constants with \( T_2 > 0 \), and
\[
\gamma_1 := -\frac{1}{2}(n + 1 - r) < 0, \quad \gamma_2 := \frac{1}{2}(3n + 1 - r) > 0.
\]
Recall that
\[
{2H[\text{Re}(aw^{-\nu})]}_w = v \bar{a} w^{\nu - 1}.
\]
Hence,
\[
\left\{ 2H \left[ \text{Re}(c^s e^s i^s T_1 R_m w^{\gamma_1} + \ldots + c^s i^{s-1} e^{s-2} T_2 R_m w^{\gamma_2} + \ldots) \right] \right\}_w
\]
\[
= \bar{c}^s e^s (1)^s i^s T_1 R_m (-\gamma_1) w^{-\gamma_1 - 1} + \ldots + (1)^s \bar{c}^s e^s i^s T_4 + \ldots
\]
\[
+ c^s i^{s-1} e^{s-2} T_2 R_m \gamma_2 w^{\gamma_2 - 1} + \ldots.
\]
Renaming \((-1)^s+1\gamma_1\) as \( T_1 \), \((-1)^s T_4\) as \( T_4 \), and \( \gamma_2 T_2 > 0 \) as \( T_2 \), we obtain
\[
[D^s_t \hat{Z}(0)]_w = c^s e^s i^s T_1 R_m w^{-\gamma_1 - 1} + \ldots + c^s e^s i^s T_4
\]
\[
+ \ldots + c^s i^{s-1} e^{s-2} T_2 R_m w^{\gamma_2 - 1} + \ldots \quad (A.11)
\]
whence
\[
w[D^s_t \hat{Z}(0)]_w \cdot [D^s_t \hat{Z}(0)]_w = (iL^{-1}|c|^L e^{L-2} T_1 T_2 |R_m|^2 w^{\nu} + \ldots
\]
\[
+ 2iL^{-1}|c|^L e^{L} T_2 T_4 |R_m|^2 w^{\nu_2} + \ldots) + O(e^{L-1})
\]
with \( \nu := -\gamma_1 - 1 + \gamma_2 = \frac{1}{2}(n + 1 - r) + \frac{1}{2}(3n + 1 - r) - 1 = 2n - r. \)
Multiplication by \( \tau \) yields

\[
w[D_t^s \hat{Z}(0)]_w \cdot [D_t^s \hat{Z}(0)]_w \tau = (c|c|^{L-1}i^{L-1}T_1 T_2 |R_m|^2 w^{\gamma-(n+1)} + \ldots \\
+ 2c|c|^{L-1}T_2 T_4 R_m w^\gamma-(n+1) + \ldots) \\
+ O(\epsilon^L)
\]

where \( \gamma - (n + 1) = (2n - r) - (n + 1) = (n - 1) - r \geq 0 \) and \( \gamma_2 - (n + 1) = \frac{1}{2}(n - 1 - r) \geq 0 \).

Thus, we obtain

\[
\int_{S^1} w[D_t^s \hat{Z}(0)]_w \cdot [D_t^s \hat{Z}(0)]_w \tau dw = O(\epsilon^L). \tag{A.12}
\]

(ii) Next we claim that there is no contribution of order \( \epsilon^{L-1} \) or lower which comes from any of the complex components of the terms in (A.10) which are indicated by \( \ldots \), that is, from

\[
w[D_t^s \hat{Z}(0)]_w [D_t^{s-\beta} \hat{Z}(0)]_w D_t^\beta \phi(0) \quad \text{for } \beta > 1. \tag{A.13}
\]

Recall that \( \phi_{tt}(0) \) is defined so that

\[
\hat{\phi}_{ttw}(0) = -ic^3 \epsilon (n + 1 - r) \lambda w^{-n-1-r} - 2c^3 \mu^2 \epsilon^3 w^{-3n-1} \\
+ \text{ terms with lower order poles.} \tag{A.14}
\]

Using this we see that

\[
w[D_t^s \hat{Z}(0)]_w \cdot [D_t^{s-2} \hat{Z}(0)]_w D_t^\beta \phi(0)
\]

has no pole associated with coefficients of order \( \epsilon^L \) or lower, and similarly for all \( \beta > 2 \).

(iii) Now we investigate in the second term of (A.10) what contribution comes from the third complex component of the terms involved. This contribution, \( \tilde{C} \), is

\[
\tilde{C} = \int w \cdot [c^s \epsilon^s i^s T_1 \tilde{R}_m w^{-\gamma_1-1} + \ldots + c^s \epsilon^s i^s T_4 + \ldots \\
+ c^s \epsilon^{s-2} i^{s-1} T_2 R_m w^{\gamma_2-1} + \ldots] \\
\cdot [i^s \epsilon^{s-1} \epsilon^{s-1} R_m T_3 w^{\gamma_3} + \ldots] \cdot (-i \epsilon^2 \mu c^2 w^{-2n-1} + \gamma c^2 w^{-r} + \ldots)
\]

where \( \gamma_3 := \frac{1}{2}(r + n + 1) \) and \( T_3 \geq 0 \).

This leads to

\[
\tilde{C} = \int [\gamma |c|^{L} c i^{L-1} \epsilon^{L-1} T_1 T_3 |R_m|^2 w^{\gamma_3-\gamma_1-1} + \ldots \\
+ \gamma c |c|^{L} i^{L-1} \epsilon^{L-1} T_3 R_m T_3 w^{\gamma_3-\gamma_1} + \ldots \\
- c^{L+1} \epsilon^{L-1} i^{L-1} T_2 T_3 \mu \tilde{R}_m^2 w^{\gamma_3-2n-1} + \ldots + O(\epsilon^L)
\]
and
\[
\gamma_3 - \gamma_1 - r = \frac{1}{2}(r + n + 1) + \frac{1}{2}(n + 1 - r) - r = n - r + 1 > 0,
\]
\[
\gamma_2 + \gamma_3 - 2n - 1 = \frac{1}{2}(3n + 1 - r) - 1 + \frac{1}{2}(r + n + 1) - 2n - 1 = 0,
\]
and \(\gamma_3 - r > 0\). Thus we obtain
\[
\tilde{C} = O(\epsilon L).
\]

(A.15)

Now we have to study the contributions coming from the first complex components.

(iv) The first term of (A.8) will have a lowest \(w\)-power of the form
\[
\epsilon^s \cdot \epsilon^s (A_j \cdot A_{2m-2n+1}) w^\gamma
\]
for some \(j \leq 2m - 2n\) and
\[
\gamma := 1 + 2m + n - s(n + 1) - (n + 1)
\]
\[
= \frac{1}{2} \{ (2m + 2) - 2n - L(n + 1) + 2n - 4 \}
\]
\[
= \frac{1}{2} \{ [(2m + 2) - (L - 1)(n + 1)] + [(2m + 2) - (n + 1) - 4] \}
\]
\[
= \frac{1}{2} \{ r + (L - 2)(n + 1) + r - 4 \}.
\]
Since \(L \geq 6\) it follows that
\[
\gamma \geq \frac{1}{2} \{ 2r + 4(n + 1) - 4 \} = r + 2n > 0.
\]

(v) Similarly the second term of (A.10) is harmless, and what we have seen in (ii) also applies to the other terms (A.13) of (A.10).

Inspecting (i)–(v) we obtain the assertion of Lemma A.2. \(\square\)

We now need to investigate \(I_4\), which is defined as
\[
I_4 = \frac{2L!}{\sigma! \sigma!} \text{Re} \int_{S^1} w[D_t^\sigma \hat{Z}(0)]_w \left\{ \sum_{\sigma = 0}^{\sigma} \frac{\sigma!}{\alpha! \beta!} \psi(\sigma, \alpha)[D_t^\sigma \hat{Z}(0)]_w D_t^\beta \phi(0) \right\} dw
\]
with \(\sigma = s - 1 = L/2 - 1\) and \(\alpha + \beta + \sigma = L\) whence \(2 \leq \beta \leq L/2 + 1\).

Lemma A.3 The terms in (A.16) with \(\beta \geq 3\) are of the order \(O(\epsilon L)\).

Proof We need to show that the terms with \(\beta \geq 3\) and coefficients \(\epsilon^{L-1}\) have no poles.
(i) For example, if we consider the lowest order zero of the last complex component of \( w[D_*^\sigma \hat{Z}(0)]_w \) with coefficient \( \epsilon^\sigma \), this is a term of the form
\[
\text{const } \epsilon^\sigma w^{1+\sigma(n+1)}.
\]
Multiplication by \( \text{const } \epsilon^\sigma w^{m-\sigma(n+1)} \) and then by \( \text{const } \epsilon^{\beta-1}w^{-(\beta-1)n-r-1} \) yields
\[
\text{const } \epsilon^{L-1}w^\gamma
\]
with
\[
\gamma = 1 + 2m - \sigma(n+1) - \alpha(n+1) - (\beta - 1)n - (r + 1)
\]
\[
= \beta - 3 \geq 0,
\]
i.e. there is no pole.

If, on the other hand, we consider the contribution of \( \epsilon^{\beta+1}w^{-(\beta+1)n-1} \), and consider also the term with coefficient \( \epsilon^{\alpha-2} \) in \( [D_*^\alpha \hat{Z}(0)]_w \), that is
\[
\text{const } \epsilon^{\alpha-2}w^{m-(\alpha-2)(n+1)}
\]
the total product will again be of the form \( \text{const } \epsilon^{L-1}w^\gamma' \) with
\[
\gamma' = 1 + 2m - \sigma(n+1) - (\alpha - 2)(n+1) - (r + 1) - (\beta + 1)n - 1
\]
\[
= \beta - 2 > 0,
\]
again there is no pole.

(ii) What about the first complex components? The worst terms are of the form
\[
\text{const}(A_j \cdot A_{2m-2n+1})w^{1+2m+n-\alpha(n+1)-(\beta+1)n-1}
\]
and
\[
1 + 2m + n - \alpha(n+1) - (\beta + 1)n - 1
\]
\[
= 1 + 2m + n - (n+1) - \beta n - n - 1
\]
\[
= 1 + 2m - \alpha(n+1) - \beta(n+1) + \beta - 1
\]
\[
= (2 + 2m) - (\alpha + \beta)(n+1) + \beta - 2
\]
\[
= (2 + 2m) - (L/2 + 1)(n+1) + \beta - 2
\]
\[
\geq (L - 1)(n+1) - (L/2 + 1)(n+1) + \beta - 2
\]
\[
= (L/2 - 2)(n+1) + \beta - 2 > \beta - 2 \geq 0
\]
since \( \alpha + \beta = L/2 + 1 \).

This completes the proof of Lemma A.3. \( \square \)

From (A.5) and Lemma A.3 we infer
\[
I_4 = \frac{2L!}{\sigma!\sigma!} \text{Re} \int_{S^1} w[D_*^\sigma \hat{Z}(0)]_w \cdot [D_*^\sigma \hat{Z}(0)]_w \phi_{tt}(0) \, dw + O(\epsilon^L). \quad (A.17)
\]
In the product $w[D^\sigma_t \hat{Z}(0)] \cdot [D^\sigma_t \hat{Z}(0)]_w$ we can ignore the contributions from the first complex components, and from the last one we obtain (as in the case $2m + 2 = L(n + 1)$, $L$ odd) the contribution

$$c^{2\sigma} i^{2\sigma} \epsilon^{2\sigma} k^2 R^2_m w^\gamma + \cdots + O(\epsilon^{2\sigma + 1}), \quad \sigma = \frac{L}{2} - 1, \quad 2\sigma = L - 2,$$

with

$$k := (m - n)(m + 1 - 2(n + 1)) \cdots (m + 1 - \sigma(n + 1)) > 0$$

and

$$\gamma := 1 + 2m - (L - 2)(n + 1) = n + r.$$

We obtain for the integrand on the right-hand side of (A.17) the expansion

$$-2c^{L+1}\epsilon^{L-1}i^{L-1}k^2\lambda R^2_m w^{-1} + \cdots + O(\epsilon^L).$$

Thus we infer that

$$I_4 = -\frac{2(L - 1)!}{\sigma!\sigma!} \Re \int_{S^1} c^{L+1}\epsilon^{L-1}i^{L-1}k^2\lambda R^2_m \frac{dw}{w} + O(\epsilon^L). \quad (A.18)$$

We must now investigate $I_2$. The first term of $I_2$ is (omitting constants)

$$\int_{S^1} w[D^{L/2+1}_t \hat{Z}(0)]_w \cdot D^{L/2-1}_t \hat{Z}(0)\phi(0) \, dw.$$

The term $wD^{L/2+1}_t \hat{Z}(0)$ has a highest order pole of the form $k_1\epsilon^{L/2}w^{-\frac{1}{2}(n+1-r)}$, $k_1 \neq 0$, arising from the generator $ce/w^{n+1}$.

The term $w[D^{L/2+1}_t \hat{Z}(0)]_w$ is of the form $k_2\epsilon^{L/2-1}w^{\frac{1}{2}(n+1-r)-1}$ and therefore has a zero of order $\frac{1}{2}(n - 1 - r)$ yielding a contribution to $I_2$ of the form

$$\epsilon^{L-1} \int_{S^1} k_1k_2R^2_m/w \, dw$$

but we know nothing about $k_2$. However, the other terms in $I_2$ do not contribute, as the pole terms in $D^\beta_t \phi$, $\beta > 0$, arising from $ce/w^{n+1}$ have orders too low to contribute. The lack of information about $k_2$ means that we also have no information about the sum $I_2 + I_4$, which could have a zero $\epsilon^{L-1}$ term, yielding absolutely nothing.

The trick, in this case, is to choose $\delta \neq 0$, $\rho \neq 0$. Then we see that the $(L + 1)^{st}$ derivative is of the form $I_2 + I_4 + O(\epsilon^L)$. Suppose that in this sum the $\epsilon^{L-1}\rho\delta$ term is zero for all choices of $\rho, \delta$. Then, if we choose $\lambda := -i\rho\delta(2n + 2 - r)$, the $\lambda$ linear terms in the higher order derivatives $D^\beta_t \phi$, $\beta \geq 2$, no longer contribute to the $\epsilon^{L-1}$ terms in $I_2 + I_4$. However, this means that there is no cancellation due to these derivatives, implying that $k_2 \neq 0$ and also that $I_4 = O(\epsilon^L)$ as well. Hence, if $\kappa = k_2k_2 \neq 0$, it follows that

$$E^{(L+1)}(0) = -\frac{2 \cdot (L - 1)!}{s!s!} \Re \int_{S^1} \epsilon^{L-1}c^{L+1}i^{L-1}\kappa R^2_m \frac{dw}{w} + O(\epsilon^L) \quad (A.19)$$

where $\kappa \neq 0$. 

Then by an appropriate choice of $c$ and $\epsilon > 0$ we can make $E^{(L+1)}(0)$ negative. Hence there is a real $v > 0$ such that

$$E^{(L+1)}(0) = -2 \cdot (L + 1)!v\epsilon^{L-1} + O(\epsilon^L), \quad (A.20)$$

whereas

$$E^{(L)}(0) = O(\epsilon^L). \quad (A.21)$$

Now we want to prove Proposition 4.2 of Chap. 4, using (A.20) and (A.21). In addition we need the following auxiliary result to be verified later.

**Lemma A.4** For $\alpha = 2, 3, \ldots, L$ there are constants $b_{\alpha} \in \mathbb{R}$ such that

$$E^{(L+\alpha)}(0) = b_{\alpha}\epsilon^{L-\alpha+1} + O(\epsilon^{L-\alpha+2}). \quad (A.22)$$

**Proof of Proposition 4.2** Let us write $\hat{Z}(t, \epsilon)$ instead of $\hat{Z}(t)$ in order to express the dependence of $\hat{Z}$ on $t$ and $\epsilon$, and set

$$E(t, \epsilon) := D(\hat{Z}(t, \epsilon)) \quad (A.23)$$

and so $E(0, 0) = D(\hat{X})$. Applying Taylor’s theorem with respect to $t$ and recalling that $E^{(j)}(0, \epsilon) = 0$ for $1 \leq j \leq L - 1$, $0 < \epsilon \leq \epsilon_0$ and some $\epsilon_0 > 0$ we obtain

$$E(t, \epsilon) = D(\hat{X}) + \sum_{\alpha=0}^{L} \frac{1}{(L+\alpha)!}E^{(L+\alpha)}(0, \epsilon)t^{L+\alpha} + R(t, \epsilon) \quad (A.24)$$

where the remainder $R(t, \epsilon)$ can be estimated by

$$|R(t, \epsilon)| \leq M|t|^{2L+1}$$

if $|t| \leq t_0$ and $0 < \epsilon < \epsilon_0$, \quad (A.25)

for some constant $M > 0$ and some sufficiently small $t_0 > 0$. Choosing $\epsilon_0 > 0$ sufficiently small, we may assume the following, taking (A.20)–(A.22) into account: There are positive numbers $v, a, c_2, \ldots, c_L$ such that for $0 < \epsilon \leq \epsilon_0$ we have

$$\frac{1}{L!}E^{(L)}(0, \epsilon) \leq a\epsilon^L, \quad \frac{1}{(L+1)!}E^{(L+1)}(0, \epsilon) \leq -\nu\epsilon^{L-1}, \quad (A.26)$$

$$\frac{1}{(L+\alpha)!}E^{(L+\alpha)}(0, \epsilon) \leq c_{\alpha}\epsilon^{L-\alpha+1}.$$  

From (A.24)–(A.26) we infer for $0 < t < t_0$ and $0 < \epsilon < \epsilon_0$ that

$$E(t, \epsilon) \leq D(\hat{X}) + (\epsilon a - v t)\epsilon^{L-1}t^L + \sum_{\alpha=2}^{L} c_{\alpha}\epsilon^{L-\alpha+1}t^{L+\alpha} + Mt^{2L+1}.$$  

Setting $t := 2a\epsilon v^{-1}$ and choosing $\epsilon^* := \min\{\epsilon_0, (2a)^{-1}v_{t0}\}$ we obtain for $0 < \epsilon < \epsilon^*$ that

$$(\epsilon a - v t)\epsilon^{L-1}t^L = -a\epsilon^{L+1}v^{-L}e^{2L}$$

$$= -be^{2L} \quad \text{with } b = 2^{L}\nu^{-L}a^{L+1} > 0$$

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and
\[ \sum_{\alpha=2}^{L} c_{\alpha} \varepsilon^{L-\alpha+1} t^{L+\alpha} + M t^{2L+1} \leq M^* \varepsilon^{2L+1} \]

with
\[ M^* := \sum_{\alpha=2}^{L} c_{\alpha} \left( \frac{2a}{v} \right)^{L+\alpha} + M \left( \frac{2a}{v} \right)^{2L+1}. \]

This yields
\[ E(t, \varepsilon) \leq D(\hat{X}) + (M^* \varepsilon - b) \varepsilon^2 \]

for \( t = \frac{2a \varepsilon}{v} \) and \( 0 < \varepsilon < \varepsilon^* \).

Choosing \( \varepsilon_\ell := \min\{\varepsilon^*, (2M^* \ell)^{-1} b\} \) and \( t_\ell := 2a \varepsilon_\ell v^{-1} \) we obtain
\[ \varepsilon_\ell \to +0, \quad t_\ell \to +0 \quad \text{and} \quad E(t_\ell, \varepsilon_\ell) < D(\hat{X}). \]

Thus Proposition 4.2 of Chap. 4 is proved. \(\square\)

Therefore the proof of Theorem 4.1 in Chap. 4 is complete as soon as we have verified Lemma A.4.

Before we do that let us mention that Proposition 4.2 of Chap. 4 can certainly not be derived from (A.20) and (A.21) alone as one sees by the following

**Example** The function
\[ f(t) := t^2 (t - \varepsilon)^2 = \varepsilon^2 t^2 - 2\varepsilon t^3 + t^4 \]

satisfies \( f''(0) = 2\varepsilon^2 \) and \( f'''(0) = -12\varepsilon \), but still \( t = 0 \) is even a global minimizer for \( f \). This shows the need of further information, e.g. on the higher order Taylor coefficients, in order to ensure that the minimal surface \( \hat{X} \) is not a local minimizer.

Instead of Lemma A.4 we state a somewhat stronger result which immediately yields the desired result.

**Lemma A.5** Let \( L \) be even and \( Q := L + 2k + 2, k = 0, 1, \ldots, L - 1 \). Then
\[ E^{(Q-1)}(0) = O(\varepsilon^\mu) \quad \text{and} \quad E^{(Q)}(0) = O(\varepsilon^\mu) \quad \text{for} \ \mu := L - (k + 1). \] (A.27)

**Proof** We argue only for those terms of the integrands that come from the last complex components, as the reasoning for the first is similar (as usual). Furthermore we prove the statement only for \( E^{(Q)}(0) \) since the reasoning for \( E^{(Q-1)}(0) \) is the same.

Recall that
\[ E^{(Q)}(0) = \sum c_{\alpha \beta \gamma} \Re \int_{S^1} w[D_\ell^\alpha \hat{Z}(0)] w \cdot [D_\ell^\gamma \hat{Z}(0)] w D_\ell^\beta \phi(0) \, dw, \quad \gamma \leq \alpha. \]

Consider only those terms of \( D_\ell^\beta \phi(0) \) arising from the term \( \varepsilon^2 w^{-r} \) in the definition (A.4) of \( \phi_l(0) \) since these will contribute the lowest order \( \varepsilon \)-terms.
For $\gamma + \beta < L/2 - 1$ the sum
\[
\sum c_{\alpha\beta\gamma} w[D_t^{\gamma} \hat{Z}(0)] w D_t^{\beta} \phi(0)
\] (A.28)
has no pole. We will only consider the terms with $\beta = 0, 1$ since for $\beta > 1$ the pole orders decrease.

Let us investigate the term
\[
w[D_t^{Q/2-1} \hat{Z}(0)] w \cdot [D_t^{Q/2-1} \hat{Z}(0)] w \phi_t(0).
\] (A.29)
The term $[D_t^{Q/2-1} \hat{Z}(0)] w$ has no pole, and terms of order $\epsilon^\nu, \nu < L/2$, contribute only holomorphic terms to the third complex component. Thus, if we can show that any term in $[D_t^{Q/2-1} \hat{Z}(0)] w \phi_t(0)$ of order $\epsilon^\nu$ with $\nu < L/2 - (k + 1)$ contributes no poles, then the minimal order $\mu$ of any $\epsilon$-coefficient will be
\[
\mu = L/2 + L/2 - (k + 1) = L - (k + 1),
\]
as claimed. Now
\[
[D_t^{L/2+k} \hat{Z}(0)] w = \left[2H \text{Re} \left[(D_t^{L/2+k-1} \hat{Z}(0)) w \tau + (D_t^{L/2+k-2} \hat{Z}(0)) w \phi_t(0) + \cdots\right]\right] w.
\]
The key fact is that in order to obtain one additional contribution from $\phi_t(0)$ (no $\epsilon$) we need to go down two derivatives. Thus, in general, our construction shows that in the formulation of $[D_t^{L/2+k} \hat{Z}(0)] w$ dropping down $2\rho$ orders in the derivatives of $\hat{Z}$ yields a contribution of $\rho$ additional $\phi_t(0)$-terms. Choose $\rho$ so that $v + 2\rho = L/2 + k + 2$, implying that
\[
\rho \geq k + 1.
\]
Then the contribution of the last complex component to (A.29) is
\[
\epsilon^v R_m w^{1+m-[L/2+(k-2\rho)](n+1)-\rho r-r}, \quad v \leq L/2 - (k + 2).
\]
The $w$-exponent is equal to
\[
\gamma := \frac{1}{2} \{(2m + 2) - (L - 1)(n + 1) + [2(2\rho - k) - 1](n + 1) - 2\rho r - 2r\}
\]
\[
= \frac{1}{2} \{r + (4\rho - 2k - 1)(n + 1) - 2(\rho + 1)r\}.
\]
But $2\rho \geq 2k + 2$; therefore
\[
\gamma \geq \frac{1}{2} \{r + (2\rho + 1)(n + 1) - 2(\rho + 1)r\} = \frac{1}{2} \{(2\rho + 1)(n - r + 1)\},
\]
and so $\gamma > 0$.

For $\gamma = L/2 + s, 0 < s < k$, the same argument shows that for $\mu \leq L/2 - (s + 2)$ no pole-term with a coefficient $\epsilon^\mu$ arises in (A.28). This completes the proof of the lemma.
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