Appendix A
Markov Processes

For the convenience of the reader, in this appendix we give a summary of some of
the concepts and results for general stochastic processes and Markov processes that
are used in the main text. Many of them can be found in Sharpe (1988); see also

A.1 Measurable Spaces

Given a class $\mathcal{F}$ of functions on a non-empty set $E$, we define $b\mathcal{F} = \{f \in \mathcal{F} : f$ is bounded$\}$ and $p\mathcal{F} = \{f \in \mathcal{F} : f$ is positive$\}$. We say $\mathcal{F}$ separates points if for every $x \neq y \in E$ there exists $f \in \mathcal{F}$ so that $f(x) \neq f(y)$. For a class $\mathcal{G}$ of functions on or subsets of $E$, we use $\sigma(\mathcal{G})$ to denote the $\sigma$-algebra on $E$ generated by $\mathcal{G}$, that is, $\sigma(\mathcal{G}) = \cap\{\mathcal{F} : \mathcal{F}$ is a $\sigma$-algebra on $E$ and all elements of $\mathcal{G}$ are $\mathcal{F}$-measurable$\}$. If $(E, \mathcal{E})$ is a measurable space, we also use $\mathcal{E}$ to denote the class of real $\mathcal{E}$-measurable functions on $E$. We write $\mu(f)$ for the integral of a function $f \in \mathcal{E}$ with respect to a measure $\mu$ on $(E, \mathcal{E})$ if the integral exists. Let $\mathbb{R}$ denote the one-dimensional Euclidean space.

Let $\| \cdot \|$ denote the supremum/uniform norm of functions. We say a sequence $\{f_n\}$ of functions on $E$ converges uniformly to a function $f$ on $E$ if $\|f_n - f\| \to 0$ as $n \to \infty$. We say $\{f_n\}$ converges boundedly and pointwise to $f$ if there is a constant $C \geq 0$ such that $\|f_n\| \leq C$ for all $n \geq 1$ and $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$.

A monotone vector space $\mathcal{L}$ on the set $E$ is defined to be a collection of bounded real functions on $E$ satisfying the conditions: (i) $\mathcal{L}$ is a vector space over $\mathbb{R}$; (ii) $\mathcal{L}$ contains the constant function $1_E$; (iii) if $\{f_n\} \subset p\mathcal{L}$ and $f_n \to f$ increasingly for a bounded function $f$, then $f \in \mathcal{L}$.

**Proposition A.1** (Monotone Class Theorem; Sharpe, 1988, p.364) Let $\mathcal{K}$ be a collection of bounded real functions on the set $E$ which is closed under multiplication. If $\mathcal{L}$ is a monotone vector space containing $\mathcal{K}$, then $\mathcal{L} \supset \sigma(\mathcal{K})$. 
Proposition A.2 (Modified Monotone Class Theorem) Let $\mathcal{H}$ be a vector space of bounded real functions on the set $E$ which contains $1_E$ and is closed under multiplication. If another collection of bounded real functions $\mathcal{I}$ contains $\mathcal{H}$ and is closed under bounded pointwise convergence, then $\mathcal{I} \supset b\sigma(\mathcal{H})$.

Proof. Let $\mathcal{L}$ be the intersection of all classes of bounded real functions that contain $\mathcal{H}$ and are closed under bounded pointwise convergence. Then $\mathcal{L}$ is closed under bounded pointwise convergence and $\mathcal{H} \subset \mathcal{L} \subset \mathcal{I}$. For $f \in \mathcal{L}$ let

$$\mathcal{L}_f = \{ g \in \mathcal{L} : af + bg \in \mathcal{L} \text{ for all } a, b \in \mathbb{R} \}.$$ 

It is easy to see that $\mathcal{L}_f \subset \mathcal{L}$. For $f \in \mathcal{H}$ we have $\mathcal{H} \subset \mathcal{L}_f$ and so $\mathcal{L}_f = \mathcal{L}$. For $g \in \mathcal{L}$, for every $g \in \mathcal{H}$ we have $f \in \mathcal{L}_g$ and so $g \in \mathcal{L}_f$. It follows that $\mathcal{H} \subset \mathcal{L}_f$, yielding $\mathcal{L}_f = \mathcal{L}$. Therefore $\mathcal{L}$ is a vector space. By the monotone class theorem we have $\mathcal{L} \supset b\sigma(\mathcal{H})$, which implies the desired result.

Let us consider a measurable space $(E, \mathcal{E})$. Suppose that $\mu$ is a $\sigma$-finite measure on $(E, \mathcal{E})$. A set $N \subset E$ is called a $\mu$-null set if there is $N_0 \in \mathcal{E}$ so that $N \subset N_0$ and $\mu(N_0) = 0$. For $A, B \subset E$ we define the symmetric difference

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

It is easy to show that

$$\mathcal{E}^\mu := \{ A \subset E : A \triangle B \text{ is a } \mu\text{-null set for some } B \in \mathcal{E} \}$$

is a $\sigma$-algebra, which is called the $\mu$-completion of $\mathcal{E}$. We can let $\mu(A) = \mu(B)$ for $B \in \mathcal{E}$ such that $A \triangle B$ is a $\mu$-null set to extend $\mu$ uniquely to a $\sigma$-finite measure on $(E, \mathcal{E}^\mu)$. The measure space $(E, \mathcal{E}, \mu)$ is said to be complete if $\mathcal{E} = \mathcal{E}^\mu$. The universal completion of $\mathcal{E}$ is the $\sigma$-algebra $\mathcal{E}^u$ defined to be the intersection of the $\mu$-completions of $\mathcal{E}$ as $\mu$ runs over all finite measures on $(E, \mathcal{E})$.

Proposition A.3 If $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\sigma$-algebras on the set $E$ such that $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_1^u$, then $\mathcal{E}_2^u = \mathcal{E}_1^u$.

Proof. Let $A \in \mathcal{E}_1^u$ and let $\mu$ be a finite measure on $\mathcal{E}_2$. Since $A \in \mathcal{E}_1^\mu$, it is easy to find $A_1, A_2 \in \mathcal{E}_1 \subset \mathcal{E}_2$ so that $A_1 \subset A \subset A_2$ and $\mu(A_1) = \mu(A_2)$. Then $A \in \mathcal{E}_2^\mu$, implying $\mathcal{E}_1^u \subset \mathcal{E}_2^u$. To show the reverse inclusion, let $A \in \mathcal{E}_2^u$ and let $\mu$ be a finite measure on $\mathcal{E}_1$. Then $\mu$ extends uniquely to $\mathcal{E}_2 \subset \mathcal{E}_2^u$ and $A \in \mathcal{E}_2^\mu$. Consequently, there are $A_1, A_2 \in \mathcal{E}_2 \subset \mathcal{E}_1^u \subset \mathcal{E}_1^\mu$ such that $A_1 \subset A \subset A_2$ and $\mu(A_1) = \mu(A_2)$. This yields the existence of $B_1, B_2 \in \mathcal{E}_1$ such that $B_1 \subset A_1, A_2 \subset B_2$ and $\mu(B_1) = \mu(B_2)$. Then $A \in \mathcal{E}_1^\mu$, which implies $\mathcal{E}_2^u \subset \mathcal{E}_1^u$. $\square$

Let $(E, \mathcal{E})$ be a measurable space. The trace or restriction of $\mathcal{E}$ on a subset $A \subset E$ is defined to be the $\sigma$-algebra $\mathcal{E}_A := \{ B \cap A : B \in \mathcal{E} \}$. For a measure $\mu$ on $(E, \mathcal{E})$, the (outer) trace or restriction $\mu_A$ of $\mu$ on $(A, \mathcal{E}_A)$ is defined by $\mu_A(C) = \inf \{ \mu(B) : C = B \cap A, B \in \mathcal{E} \}$. The trace $\mu_A$ can be realized as follows.
A_0 \in \mathcal{E} \text{ with } A_0 \supset A \text{ having minimal } \mu\text{-measure. Then for } C \in \mathcal{E}_A \text{ of the form } C = B \cap A \text{ with } B \in \mathcal{E} \text{ we have } \mu_A(C) = \mu(B \cap A_0); \text{ see Sharpe (1988, p.367).}

**Proposition A.4** (Sharpe, 1988, p.368) Let \( A \subset E \) and let \( \mathcal{E}_A \) be the trace of \( \mathcal{E} \) on \( A \). Then we have:

1. **given a finite measure** \( \mu \) on \( (A, \mathcal{E}_A) \), the formula \( \mu(B) := \mu(B \cap A) \) for \( B \in \mathcal{E} \)
2. **defines a finite measure** \( \mu \) on \( (E, \mathcal{E}) \) whose trace on \( A \) is \( \mu \);

Let \( (E, \mathcal{E}) \) and \( (F, \mathcal{F}) \) be measurable spaces. A \( \sigma \)-finite kernel from \( (E, \mathcal{E}) \) to \( (F, \mathcal{F}) \) is a function \( K = K(\cdot, \cdot) \) on \( E \times F \) having values in \([0, \infty]\) such that:

1. for each \( A \in \mathcal{F} \) the mapping \( x \mapsto K(x, A) \) is \( \mathcal{E} \)-measurable;
2. for each \( x \in E \) the mapping \( A \mapsto K(x, A) \) is a \( \sigma \)-finite measure on \( (F, \mathcal{F}) \).

A kernel \( K \) is said to be **finite or bounded** if \( x \mapsto K(x, F) \) is a finite or bounded, respectively, function on \( E \). The kernel \( K \) is called **Markov** or **sub-Markov** if \( K(x, F) = 1 \) or \( K(x, F) \leq 1 \), respectively, for each \( x \in E \). A kernel from \( (E, \mathcal{E}) \) to \( (E, \mathcal{E}) \) is simply called a kernel on \( (E, \mathcal{E}) \). Given a bounded kernel \( K \) from \( (E, \mathcal{E}) \) to \( (F, \mathcal{F}) \), for any \( f \in b\mathcal{F} \) we can define \( Kf \in b\mathcal{E} \) by

\[
Kf(x) = K(x, f) = \int_F f(y)K(x, dy), \quad x \in E,
\]

and for any finite measure \( \mu \) on \( (E, \mathcal{E}) \) we can define a finite measure \( \mu K \) on \( (F, \mathcal{F}) \) by

\[
\mu K(B) = \int_E K(x, B)\mu(dx), \quad B \in \mathcal{F}.
\]

**Proposition A.5** (Sharpe, 1988, p.376) A bounded kernel \( K \) from \( (E, \mathcal{E}) \) to \( (F, \mathcal{F}) \) extends in a unique way to a bounded kernel \( K \) from \( (E, \mathcal{E}^u) \) to \( (F, \mathcal{F}^u) \).

For a metrizable topological space \( E \) with a metric \( d \) compatible with its topology, let \( \mathcal{C}(E) := \mathcal{C}(E, d) \) denote the space of \( d \)-continuous real functions on \( (E, d) \) and let \( \mathcal{C}_u(E) := \mathcal{C}_u(E, d) \) denote the space of \( d \)-uniformly continuous real functions on \( E \). The advantage of \( \mathcal{C}_u(E) \) is that if \( (E, d) \) is separable and totally bounded, then \( b\mathcal{C}_u(E) \) with the supremum norm is separable, whereas \( b\mathcal{C}(E) \) is not. The Borel \( \sigma \)-algebra \( \mathcal{B}(E) = \mathcal{B}(E, d) \) on \( E \) is defined to be the \( \sigma \)-algebra generated by \( b\mathcal{C}(E) \) or, equivalently, by all open subsets of \( E \). If \( E \) is locally compact, we let \( C_0(E) \) denote the space of continuous real functions on \( E \) vanishing at infinity. A topological space is called a **Radon topological space** or **Lusin topological space** if it is homeomorphic to a universally measurable subset or a Borel subset, respectively, of a compact metric space. A measurable space \( (F, \mathcal{F}) \) is called a **Radon measurable space** or **Lusin measurable space** if it is measurably isomorphic to \( (E, \mathcal{B}(E)) \) with \( E \) being a Radon or Lusin topological space, respectively.
A.2 Stochastic Processes

Let \((\Omega, \mathcal{G}, P)\) be a probability space. We shall use either \(E(X)\) or \(P(X)\) to denote the expectation of a random variable \(X\) defined on this space. A collection \((\mathcal{G}_t)_{t \in I}\) of sub-\(\sigma\)-algebras of \(\mathcal{G}\) indexed by an interval \(I \subset \mathbb{R}\) is called a filtration of \((\Omega, \mathcal{G})\) if \(\mathcal{G}_r \subset \mathcal{G}_t\) for every \(r \leq t \in I\). If a filtration \((\mathcal{G}_t)_{t \in I}\) is defined on \((\Omega, \mathcal{G}, P)\), we call \((\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in I})\) a filtered probability space.

Suppose that \((\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in I})\) is a filtered probability space. A random variable \(T\) taking values in \(I \cup \{\infty\}\) is called a stopping time or an optional time over the filtration \((\mathcal{G}_t)_{t \in I}\), in case \(\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{G}_t\) for all \(t \in I\). Given a stopping time \(T\) over \((\mathcal{G}_t)_{t \in I}\), we can define a \(\sigma\)-algebra

\[
\mathcal{G}_T := \{A \in \mathcal{G}(I) : A \cap \{T \leq t\} \in \mathcal{G}_t \text{ for every } t \in I\},
\]

where \(\mathcal{G}(I) = \sigma(\bigcup_{t \in I} \mathcal{G}_t)\). Let \(\tau = \sup(I)\) and let \(\mathcal{G}_{\tau+} = \cap\{\mathcal{G}_s : t < s \in I\}\) for \(t \in I \setminus \{\tau\}\). We say \((\mathcal{G}_t)_{t \in I}\) is right continuous if \(\mathcal{G}_{t+} = \mathcal{G}_t\) for every \(t \in I \setminus \{\tau\}\). Let \(\mathcal{G}_{\tau+} = \mathcal{G}_\tau\) in case \(\tau \in I\). If \(T\) is a stopping time over \((\mathcal{G}_{t+})_{t \in I}\), we define \(\mathcal{G}_{T+}\) by (A.3) with \(\mathcal{G}_{t+}\) replaced by \(\mathcal{G}_{t+}\).

The special case \(I = [0, \infty)\) is often considered. Suppose that \((\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) is a filtered probability space. Let \(\mathcal{B}\) be the \(\mathcal{P}\)-completion of \(\mathcal{G}\) and let \(\mathcal{N} = \{A \in \mathcal{B} : P(A) = 0\}\). Let \(\mathcal{G}_t = \sigma(\mathcal{G}_t \cup \mathcal{N})\) for \(t \geq 0\). We call \((\mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) the augmentation of \((\mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) by the probability \(P\). If \(\mathcal{G} = \mathcal{G}\) and \(\mathcal{G}_t = \mathcal{G}_t\) for every \(t \geq 0\), we say \((\mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) are augmented. We say a filtered probability space \((\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) satisfies the usual hypotheses if \((\mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) are augmented and \((\mathcal{G}_t)_{t \geq 0}\) is right continuous.

**Proposition A.6** Suppose that \((\mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) are augmented. If \(S\) and \(T\) are stopping times over \((\mathcal{G}_t)_{t \geq 0}\) such that \(P\{S \neq T\} = 0\), then \(\mathcal{G}_S = \mathcal{G}_T\).

**Proof.** For any \(A \in \mathcal{G}_S\) we have \(A \in \mathcal{G}_\infty\) and \(A \cap \{S \leq t\} \in \mathcal{G}_t\) for \(t \geq 0\). Since \((\mathcal{G}, (\mathcal{G}_t)_{t \geq 0})\) are augmented and \(P\{S \neq T\} = 0\), we have \(A \cap \{T \leq t\} \in \mathcal{G}_t\) for \(t \geq 0\). Then \(A \in \mathcal{G}_T\). That proves \(\mathcal{G}_S \subset \mathcal{G}_T\). Similarly we have \(\mathcal{G}_T \subset \mathcal{G}_S\). \(\Box\)

We say the filtration \((\mathcal{G}_t)_{t \geq 0}\) is quasi-left continuous if for every increasing sequence of stopping times \(\{T_n\}\) with limit \(T\) we have \(\mathcal{G}_T = \sigma(\bigcup_{n=1}^{\infty} \mathcal{G}_{T_n})\). A stopping time \(T\) is called a predictable time if there is an announcing sequence of stopping times \(\{T_n\}\) such that \(\lim_{n \to \infty} T_n = T\) and \(T_n < T\) on \(\{T < \infty\}\) for each \(n \geq 1\). A stopping time \(T\) is said to be totally inaccessible if for every predictable time \(S\) we have \(S \neq T\) a.s. on \(\{T < \infty\}\).

Suppose that \(E\) is a metrizable topological space. For clarity we sometimes write \(\mathcal{B}(E)\) for the Borel \(\sigma\)-algebra of \(E\). Let \(\mathcal{B}^u(E)\) denote the universal completion of \(\mathcal{B}(E)\). Let \(\mathcal{B}^\bullet(E)\) be a \(\sigma\)-algebra on \(E\) such that \(\mathcal{B}(E) \subset \mathcal{B}^\bullet(E) \subset \mathcal{B}^u(E)\). Then Proposition A.3 implies that \(\mathcal{B}^u(E)\) is also the universal completion of \(\mathcal{B}^\bullet(E)\). Let \(I \subset \mathbb{R}\) be an interval and let \((\Omega, \mathcal{G}, P)\) be a probability space. A collection \((X_t)_{t \in I}\) of measurable maps of \((\Omega, \mathcal{G})\) into \((E, \mathcal{B}^\bullet(E))\) is called a stochastic process. For fixed \(\omega \in \Omega\), the map \(t \mapsto X_t(\omega)\) from \(I\) to \(E\) is called a sample path of \((X_t)_{t \in I}\). The natural \(\sigma\)-algebras \(\mathcal{F}^\bullet\) and \((\mathcal{F}_t^\bullet)_{t \in I}\) of \((X_t)_{t \in I}\) are defined by
\[ \mathcal{F}^* = \sigma(\{f(X_s) : s \in I, f \in \mathcal{B}^*(E)\}) \]

and

\[ \mathcal{F}^*_t = \sigma(\{f(X_s) : s \in I_t, f \in \mathcal{B}^*(E)\}), \]

where \( I_t = (-\infty, t] \cap I \). The process \((X_t)_{t \in I}\) is \(\mathcal{B}^*(E)\)-adapted relative to a filtration \((\mathcal{F}_t)_{t \in I}\) in case \(\mathcal{F}^* \subset \mathcal{F}_t\) for every \( t \in I \). It is \(\mathcal{B}^*(E)\)-progressive relative to \((\mathcal{F}_t)_{t \in I}\) if the mapping \((\omega, s) \mapsto X_s(\omega)\) restricted to \(\Omega \times I_t\) is \((\mathcal{F}_t \times \mathcal{B}(I_t))\)-measurable for every \( t \in I \). Clearly, a \(\mathcal{B}^*(E)\)-progressive process is \(\mathcal{B}^*(E)\)-adapted. We simply say \((X_t)_{t \in I}\) is adapted or progressive if it is \(\mathcal{B}^0(E)\)-adapted or \(\mathcal{B}^0(E)\)-progressive, respectively.

Let \((X_t)_{t \in I}\) be a stochastic process taking values in \((E, \mathcal{B}^*(E))\). For any \( t_1 < \cdots < t_n \in I \) let \(P_{t_1, \ldots, t_n}\) be the probability measure on \((E^n, \mathcal{B}^*(E)^n)\) induced by the mapping \(\omega \mapsto (X_{t_1}(\omega), \ldots, X_{t_n}(\omega))\). We call

\[ \{P_{t_1, \ldots, t_n} : t_1 < \cdots < t_n \in I, n = 1, 2, \ldots\} \]

the family of finite-dimensional distributions of \((X_t)_{t \in I}\). If another process \((Y_t)_{t \in I}\) has identical finite-dimensional distributions as \((X_t)_{t \in I}\), we say it is a realization of \((X_t)_{t \in I}\). If the processes \((X_t)_{t \in I}\) and \((Y_t)_{t \in I}\) are defined on the same probability space and if \(P\{X_t = Y_t\} = 1\) for every \( t \in I \), we say \((Y_t)_{t \in I}\) is a modification of \((X_t)_{t \in I}\). We say a process \((X_t)_{t \in I}\) is continuous or right continuous if all its sample paths \(t \mapsto X_t(\omega)\) are continuous or right continuous on \(I\), respectively. A path or process \((X_t)_{t \geq 0}\) is said to be càdlàg (continu à droite avec limites à gauche) if it is right continuous at every \( t \geq 0 \) and possesses left limit at every \( t > 0 \).

Suppose that \(F\) is a non-empty set and \((t, x) \mapsto f(t, x)\) is a real or complex function defined on the product space \([0, \infty) \times F\). We say \((t, x) \mapsto f(t, x)\) is locally bounded provided

\[ \sup_{0 \leq s \leq t} \sup_{x \in F} |f(s, x)| < \infty, \quad t \geq 0. \]

A real or complex stochastic process \((X_t)_{t \geq 0}\) is said to be locally bounded if \((t, \omega) \mapsto X_t(\omega)\) is a locally bounded function on \([0, \infty) \times \Omega\).

Now let us consider a metric space \((E, d)\). Suppose that \(T\) is a subset of \([0, \infty)\) such that \(0 \in T\) and \(t \mapsto x(t)\) is a path from \(T\) to \(E\). For any \( \varepsilon > 0 \) the number of \(\varepsilon\)-oscillations of \(t \mapsto x(t)\) on \(T\) is defined as

\[ m(\varepsilon) := \sup\{n \geq 1 : \text{there are } 0 = t_0 < t_1 < \cdots < t_n \in T \text{ so that } d(x(t_{i-1}), x(t_i)) \geq \varepsilon \text{ for all } 1 \leq i \leq n\}. \]

An earlier version of the proof of the following proposition was suggested to the author by Tom Kurtz.
**Proposition A.7** Let \((E, d)\) be a complete separable metric space and let \((X_t)_{t \geq 0}\) be a stochastic process in \((E, \mathcal{B}^0(E))\). If \((X_t)_{t \geq 0}\) has a càdlàg realization, then it has a càdlàg modification.

**Proof.** Suppose that \((\xi_t)_{t \geq 0}\) is a càdlàg realization of \((X_t)_{t \geq 0}\). Let \((\mathcal{F}_t)_{t \geq 0}\) be the natural filtration of \((\xi_t)_{t \geq 0}\). Take a countable dense subset \(T = \{0, r_1, r_2, \ldots\}\) of \([0, \infty)\) and let \(T_n = \{0, r_1, \ldots, r_n\}\). For \(\varepsilon > 0\) and \(a > 0\) let \(m^a(\varepsilon)\) and \(m_n^a(\varepsilon)\) denote the numbers of \(\varepsilon\)-oscillations of \(t \mapsto X_t\) on \(T \cap [0, a]\) and \(T_n \cap [0, a]\), respectively. Let \(\mu^a(\varepsilon)\) and \(\mu_n^a(\varepsilon)\) denote respectively those numbers of \(t \mapsto \xi_t\). Then \(m_n^a(\varepsilon) \to m^a(\varepsilon)\) and \(\mu_n^a(\varepsilon) \to \mu^a(\varepsilon)\) increasingly as \(n \to \infty\). Let \(\tau_n^\varepsilon(0) = 0\) and for \(k \geq 0\) define

\[
\tau_n^\varepsilon(k+1) = \min\{t > \tau_n^\varepsilon(k) : t \in T_n, d(\tau_n^\varepsilon(k), \xi_t) \geq \varepsilon\}
\]

if \(\tau_n^\varepsilon(k) < \infty\) and \(\tau_n^\varepsilon(k+1) = \infty\) if \(\tau_n^\varepsilon(k) = \infty\). Since \(T_n\) is discrete, for any \(u \geq 0\) we have

\[
\{\tau_n^\varepsilon(k+1) \leq u\} = \bigcup_{s < t \in T_n \cap [0, u]} (\{\tau_n^\varepsilon(k) = s\} \cap \{d(\xi_s, \xi_t) \geq \varepsilon\}). \quad (A.4)
\]

By the separability of \((E, d)\) we have \(\mathcal{B}^0(E \times E) = \mathcal{B}^0(E) \times \mathcal{B}^0(E)\). Then \(\{d(\xi_s, \xi_t) \geq \varepsilon\} \in \mathcal{F}_u\) for \(s < t \leq u\). Using (A.4) one can show inductively that each \(\tau_n^\varepsilon(k)\) is a stopping time over \((\mathcal{F}_t)\). Since \(\{\mu_n^a(\varepsilon) \geq k\} = \{\tau_n^\varepsilon(k) \leq a\}\), each \(\mu_n^a(\varepsilon)\) is a random variable and hence so is \(\mu^a(\varepsilon) = \lim_{n \to \infty} \mu_n^a(\varepsilon)\). Similarly, \(m_n^a(\varepsilon) \to m^a(\varepsilon)\) and \(\mu_n^a(\varepsilon) \to \mu^a(\varepsilon)\) as \(n \to \infty\). Since \((\xi_t)_{t \geq 0}\) is a càdlàg realization of \((X_t)_{t \geq 0}\), we get

\[
P\{m^a(\varepsilon) < \infty\} = P\{\mu^a(\varepsilon) < \infty\} = 1.
\]

Let \(\Omega_1 = \bigcap_{j=1}^\infty \{m^j(1/j) < \infty\}\). Then \(P(\Omega_1) = 1\). It is simple to show that for \(\omega \in \Omega_1\) the limit \(Y_t(\omega) := \lim_{T \ni s \to t+} X_s(\omega)\) exists at \(t \geq 0\) and \(Z_t(\omega) := \lim_{T \ni s \to t-} X_s(\omega)\) exists at \(t > 0\). Fix \(x_0 \in E\) and let \(Y_t(\omega) = x_0\) for all \(t \geq 0\) and \(\omega \in \Omega \setminus \Omega_1\). Then \((Y_t)_{t \geq 0}\) is a càdlàg process. Since \((X_t)_{t \geq 0}\) is clearly right continuous in probability, we have \(Y_t = X_t\) a.s. for every \(t \geq 0\). Therefore \((Y_t)_{t \geq 0}\) is a càdlàg modification of \((X_t)_{t \geq 0}\).

\[\square\]

### A.3 Right Markov Processes

Let \(E\) be a Radon topological space and let \(\mathcal{B}^\bullet(E)\) be a \(\sigma\)-algebra such that \(\mathcal{B}^0(E) \subset \mathcal{B}^\bullet(E) \subset \mathcal{B}^u(E)\). A family of Markov or sub-Markov kernels \((P_t)_{t \geq 0}\) on \((E, \mathcal{B}^\bullet(E))\) is called a *transition semigroup* if it satisfies the following Chapman–Kolmogorov equation:

\[
P_{r+t}(x, B) = \int_E P_r(x, dy) P_t(y, B) \quad (A.5)
\]
for all $r, t \geq 0$, $x \in E$ and $B \in \mathcal{B}^*(E)$. By Proposition A.5, we can always regard $(P_t)_{t \geq 0}$ as kernels on $(E, \mathcal{B}^*(E))$. A Borel transition semigroup $(P_t)_{t \geq 0}$ is a transition semigroup on a Lusin topological space $E$ such that $P_t f \in b\mathcal{B}^0(E)$ for each $t \geq 0$ and $f \in b\mathcal{B}^0(E)$. We say the transition semigroup $(P_t)_{t \geq 0}$ is Markov or conservative if each $P_t$ is a Markov kernel. We say $(P_t)_{t \geq 0}$ is normal if $P_0(x, \cdot) = \delta_x$ for every $x \in E$.

Let us consider a transition semigroup $(P_t)_{t \geq 0}$ on $(E, \mathcal{B}^*(E))$. A family $(\mu_t)_{t \in \mathbb{R}}$ of $\sigma$-finite measures on $(E, \mathcal{B}^*(E))$ is called an entrance rule for $(P_t)_{t \geq 0}$ if $\mu_s P_{t-s} \rightarrow \mu_t$ increasingly as $s \rightarrow t \in \mathbb{R}$. By an entrance law at $\alpha \in [-\infty, \infty)$ for $(P_t)_{t \geq 0}$ we mean a family of $\sigma$-finite measures $(\mu_t)_{t \geq \alpha}$ such that $\mu_s P_{t-s} = \mu_t$ for $t \geq s > \alpha$. We say $(\mu_t)_{t \geq \alpha}$ is bounded if $t \mapsto \mu_t(E)$ is a bounded function on $(\alpha, \infty)$. A probability entrance law is an entrance law $(\mu_t)_{t \geq \alpha}$ where each $\mu_t$ is a probability measure. If there is a $\sigma$-finite measure $\mu_\alpha$ such that $\mu_t = \mu_\alpha P_{t-\alpha}$ for all $t > \alpha$, we say the entrance law $(\mu_t)_{t \geq \alpha}$ is excessive and call $(\mu_t)_{t \geq \alpha}$ a closed entrance law. We say an entrance law $(\mu_t)_{t \geq \alpha}$ is minimal or extremal if every entrance law dominated by $(\mu_t)_{t \geq \alpha}$ is proportional to it. Note that an entrance law $(\mu_t)_{t \geq \alpha}$ at $\alpha \in [-\infty, \infty)$ may be extended to an entrance rule $(\mu_t)_{t \in \mathbb{R}}$ by setting $\mu_t = 0$ for $t \leq \alpha$. In this sense, we can regard the entrance law as a special case of the entrance rule. The concepts of entrance rules and entrance laws can obviously be extended to semigroups of bounded kernels. We sometimes make use of those extensions.

Let $\mathcal{K}^1(P)$ denote the set of all probability entrance laws $(\mu_t)_{t \geq 0}$ at zero for $(P_t)_{t \geq 0}$ endowed with the $\sigma$-algebra generated by all mappings $\{\mu \mapsto \mu_t(f) : t \geq 0, f \in b\mathcal{B}^*(E)\}$. Let $\mathcal{K}^1_m(P)$ be the set of minimal probability entrance laws in $\mathcal{K}^1(P)$. From Dynkin (1978, Theorems 3.1 and 9.1) we know $\mathcal{K}^1(P)$ is a simplex, that is, $\mathcal{K}^1_m(P)$ is a measurable subset of $\mathcal{K}^1(P)$ and for each $\mu \in \mathcal{K}^1(P)$ there is a unique probability measure $Q_\mu$ on $\mathcal{K}^1_m(P)$ such that

$$
\mu_t(\cdot) = \int_{\mathcal{K}^1_m(P)} \nu_t(\cdot) Q_\mu(\mathrm{d}\nu), \quad t > 0.
$$

A $\sigma$-finite measure $m$ on $(E, \mathcal{B}^*(E))$ is called an excessive measure for $(P_t)_{t \geq 0}$ if $m P_t \leq m$ for every $t \geq 0$. The measure $m$ is called a purely excessive measure if $m P_t \leq m$ for every $t \geq 0$ and $m P_t \rightarrow 0$ as $t \rightarrow \infty$, and it is called an invariant measure if $m P_t = m$ for every $t \geq 0$. For $\alpha \geq 0$ we say a function $f \in \mathcal{P}^\alpha(E)$ is $\alpha$-super-mean-valued for $(P_t)_{t \geq 0}$ if $e^{-\alpha t} P_t f \leq f$ for all $t \geq 0$, and it is called an $\alpha$-excessive function for $(P_t)_{t \geq 0}$ if $e^{-\alpha t} P_t f \rightarrow f$ increasingly as $t \rightarrow 0$. In the special case with $\alpha = 0$, we simply say $f$ is super-mean-valued or excessive, respectively. Let $\mathcal{K}^\alpha$ denote the set of $\alpha$-excessive functions for $(P_t)_{t \geq 0}$.

A family of bounded kernels $(U_\alpha)_{\alpha > 0}$ on $(E, \mathcal{B}^*(E))$ is called a resolvent in case the resolvent equation

$$
U_\alpha f(x) - U_\beta f(x) = (\beta - \alpha)U_\alpha U_\beta f(x)
$$

(A.6)

is satisfied for all $\alpha, \beta > 0$, $x \in E$ and $f \in b\mathcal{B}^*(E)$. A resolvent $(U_\alpha)_{\alpha > 0}$ is called Markov or conservative if $\alpha U_\alpha$ is a Markov kernel for all $\alpha > 0$. A function
$f \in \mathcal{P}(E)$ is called $\alpha$-supermedian for the resolvent $(U^\alpha)_{\alpha>0}$ if $\beta U^\alpha + \beta f \leq f$ for all $\beta > 0$. Let $\mathcal{K}^\alpha$ denote the class of all $\alpha$-supermedian functions for $(U^\alpha)_{\alpha>0}$.

If $(P_t)_{t \geq 0}$ is a transition semigroup on $(E, \mathcal{B}^\bullet(E))$ such that $(t, x) \mapsto P_t f(x)$ is measurable with respect to $\mathcal{B}([0, \infty)) \times \mathcal{B}^\bullet(E)$ for every $f \in \mathcal{B}^\bullet(E)$, the operators $(U^\alpha)_{\alpha>0}$ defined by

$$U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt, \quad f \in \mathcal{B}^\bullet(E),$$

constitute a resolvent, which is called the resolvent of $(P_t)_{t \geq 0}$. We also call $U^\alpha$ the $\alpha$-potential operator of $(P_t)_{t \geq 0}$. The potential operator $U$ of $(P_t)_{t \geq 0}$ is defined by

$$U f(x) = \int_0^\infty P_t f(x) dt, \quad f \in \mathcal{P}(\mathcal{B}^\bullet(E)).$$

However, this kernel may not be $\sigma$-finite. It is easy to show that if $f$ is $\alpha$-super-mean-valued for $(P_t)_{t \geq 0}$, it is $\alpha$-supermedian for $(U^\alpha)_{\alpha>0}$.

A particularly important special case is where $E$ is a locally compact separable metric space. In this case, its one-point compactification is metrizable. A normal and conservative transition semigroup $(P_t)_{t \geq 0}$ on a locally compact separable metric space $E$ is called a Feller semigroup provided:

1. $P_t(C_0(E)) \subset C_0(E)$ for all $t \geq 0$;
2. $P_t f \rightarrow f$ pointwise as $t \rightarrow 0$ for all $f \in C_0(E)$.

If $(P_t)_{t \geq 0}$ is a Feller semigroup, then $P_t f \rightarrow f$ uniformly as $t \rightarrow 0$ for all $f \in C_0(E)$; see Sharpe (1988, p.50).

Suppose that $(P_t)_{t \geq 0}$ is a Markov transition semigroup on $(E, \mathcal{B}^\bullet(E))$ and $(\xi_t)_{t \in I}$ is a stochastic process in $(E, \mathcal{B}^\bullet(E))$ indexed by an interval $I \subset \mathbb{R}$. We assume that $(\xi_t)_{t \in I}$ is defined on $(\Omega, \mathcal{G}, \mathbf{P})$ and is $\mathcal{B}^\bullet(E)$-adapted to a filtration $(\mathcal{G}_t)_{t \in I}$ of $(\Omega, \mathcal{G})$. We say $\{(\xi_t, \mathcal{G}_t) : t \in I\}$ has the simple $\mathcal{B}^\bullet(E)$-Markov property with transition semigroup $(P_t)_{t \geq 0}$ if

$$\mathbf{P}[f(\xi_t) | \mathcal{G}_r] = P_{t-r} f(\xi_r), \quad r \leq t \in I, f \in \mathcal{B}^\bullet(E).$$

If $\{(\xi_t, \mathcal{G}_t) : t \geq 0\}$ satisfies the simple $\mathcal{B}^\bullet(E)$-Markov property with transition semigroup $(P_t)_{t \geq 0}$, the distribution $\mu_0$ of $\xi_0$ is called the initial law of $(\xi_t)_{t \geq 0}$. In this case, we necessarily have

$$\mathbf{P}[f_1(\xi_{t_1}) f_2(\xi_{t_2}) \cdots f_n(\xi_{t_n})] = \mu_0(P_{t_1} f_1 \cdots P_{t_{n-1}} f_{t_{n-1}} f_n)$$

for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ and $f_1, f_2, \ldots, f_n \in \mathcal{B}^\bullet(E)$, which is a simple consequence of (A.9) by an induction argument. Consequently, the restriction of $\mathbf{P}$ on the natural $\sigma$-algebra $\mathcal{F}$ is determined uniquely by (A.10).
Proposition A.8 Suppose that \( \{(\xi_t, G_t) : t \geq 0\} \) satisfies the simple \( \mathcal{B}^* (E) \)-Markov property (A.9). Let \( (\mathcal{G}^*, G_t^*) \) denote the augmentations of \( (\mathcal{G}, G_t) \) with respect to \( P \). Then \( \{(\xi_t, G_t^*) : t \geq 0\} \) satisfies the simple \( \mathcal{B}^u (E) \)-Markov property.

Proof. Let \( \mu_t \) denote the distribution of \( \xi_t \) on \( (E, \mathcal{B}^*(E)) \). For \( f \in b\mathcal{B}^u (E) \) we can choose \( f_1, f_2 \in b\mathcal{B}^*(E) \) so that \( f_1 \leq f \leq f_2 \) and \( \mu_t (f_2 - f_1) = 0 \). Then \( f_1 (\xi_t), f_2 (\xi_t) \in bG_t \) and

\[
P \left[ f_2 (\xi_t) - f_1 (\xi_t) \right] = \mu_t (f_2 - f_1) = 0. \tag{A.11}
\]

It follows that \( f (\xi_t) \in bG_t^* \), and so \( (\xi_t)_{t \geq 0} \) is \( \mathcal{B}^u (E) \)-adapted relative to \( (G_t^*)_{t \geq 0} \). Then to get the desired result it suffices to show

\[
P \left[ f (\xi_t) 1_A \right] = P \left[ P_{t-r} f (\xi_r) 1_A \right] \tag{A.12}
\]

for \( t \geq r \geq 0, A \in \mathcal{G}^* \) and \( f \in b\mathcal{B}^u (E) \). Let \( \mathcal{N} = \{N \in \mathcal{G}^* : P(N) = 0\} \). Then there is \( A_0 \in \mathcal{G} \) so that \( A \Delta A_0 \in \mathcal{N} \). By (A.9) we have (A.12) for \( f \in b\mathcal{B}^u (E) \). For \( f \in b\mathcal{B}^u (E) \) we can take \( f_1, f_2 \in b\mathcal{B}^*(E) \) so that \( f_1 \leq f \leq f_2 \) and (A.11) holds. Since (A.12) holds for both \( f_1 \) and \( f_2 \), it also holds for \( f \). \( \square \)

Corollary A.9 (Sharpe, 1988, p.6) Suppose that \( (P_t)_{t \geq 0} \) preserves \( \mathcal{B}^* (E) \) and \( \mathcal{B}^0 (E) \) with \( \mathcal{B}^0 (E) \subset \mathcal{B}^* (E) \subset \mathcal{B}^u (E) \subset \mathcal{B}^u (E) \). Let \( \{(\xi_t, G_t) : t \geq 0\} \) satisfy the simple \( \mathcal{B}^* (E) \)-Markov property (A.9). If \( (\xi_t)_{t \geq 0} \) is \( \mathcal{B}^0 (E) \)-adapted to \( (G_t)_{t \geq 0} \), then \( \{(\xi_t, G_t) : t \geq 0\} \) satisfies the simple \( \mathcal{B}^u (E) \)-Markov property.

Proof. By Proposition A.8 we infer \( \{(\xi_t, G_t^*) : t \geq 0\} \) satisfies the simple \( \mathcal{B}^u (E) \)-Markov property. Then \( \{(\xi_t, G_t) : t \geq 0\} \) satisfies the simple \( \mathcal{B}^u (E) \)-Markov property. \( \square \)

Definition A.10 (Sharpe, 1988, p.7) Suppose that \( (P_t)_{t \geq 0} \) is a normal Markov transition semigroup on \( (E, \mathcal{B}^* (E)) \). The collection \( \xi = (\Omega, \mathcal{G}, G_t, \xi_t, \theta_t, P_x) \) is called a \( \mathcal{B}^* (E) \)-Markov process with transition semigroup \( (P_t)_{t \geq 0} \) in case \( \xi \) satisfies the following conditions:

1. \( (\Omega, \mathcal{G}, G_t)_{t \geq 0} \) is a filtered measurable space, and \( (\xi_t)_{t \geq 0} \) is an \( E \)-valued process \( \mathcal{B}^* (E) \)-adapted to \( (G_t)_{t \geq 0} \).
2. \( (\theta_t)_{t \geq 0} \) is a collection of shift operators for \( \xi \), that is, maps of \( \Omega \) into itself satisfying \( \theta_s \circ \theta_t = \theta_{s+t} \) and \( \xi_s \circ \theta_t = \xi_{s+t} \) identically for \( t, s \geq 0 \).
3. For every \( x \in E \), \( P_x \) is a probability measure on \( (\Omega, \mathcal{G}) \) and \( x \mapsto P_x (H) \) is \( \mathcal{B}^* (E) \)-measurable for each \( H \in b\mathcal{G} \).
4. For every \( x \in E \), we have \( P_x \{x_0 = x \} = 1 \) and the process \( (\xi_t)_{t \geq 0} \) has the simple Markov property (A.9) relative to \( (G_t, P_x) \) with transition semigroup \( (P_t)_{t \geq 0} \).

We say \( \xi \) is right continuous if \( t \mapsto \xi_t (\omega) \) is right continuous for every \( \omega \in \Omega \).

If the above conditions (1)–(4) are satisfied, we also say that \( \xi \) is a realization of the semigroup \( (P_t)_{t \geq 0} \). In this case, for any finite measure \( \mu \) on \( (E, \mathcal{B}^* (E)) \) we may define the finite measure \( P_\mu \) on \( (\Omega, \mathcal{G}) \) by
\[ \mathbf{P}_\mu(H) = \int_E \mathbf{P}_x(H)\mu(dx), \quad H \in b\mathcal{G}. \tag{A.13} \]

In the sequel, we always assume \( \mu \) is a probability measure unless stated otherwise. It is easy to verify that \( (\xi_t)_{t \geq 0} \) has the simple \( \mathcal{B}^*(E) \)-Markov property relative to \( (\mathcal{G}_t, \mathbf{P}_\mu) \) with initial law \( \mu \). We mention that the measurability of \( x \mapsto \mathbf{P}_x(H) \) is used in the definition (A.13) of the measure \( \mathbf{P}_\mu \) on \( (\Omega, \mathcal{G}) \). Of course, this measurability follows automatically if \( (\mathcal{G}, \mathcal{G}_t) \) are the natural \( \sigma \)-algebras of \( \{\xi_t : t \geq 0\} \).

Consider a right continuous \( \mathcal{B}^*(E) \)-Markov process \( \xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, \mathbf{P}_x) \) with transition semigroup \( (P_t)_{t \geq 0} \) and resolvent \( (U^\alpha)_{\alpha > 0} \) on \( (E, \mathcal{B}^*(E)) \). Let \( \mathcal{G}^\mu \) denote the \( \mathbf{P}_\mu \)-completion of \( \mathcal{G} \) and let \( \mathcal{N}^\mu(\mathcal{G}) \) denote the family of \( \mathbf{P}_\mu \)-null sets in \( \mathcal{G}^\mu \). Then define:

\[
\mathcal{G} = \cap\{\mathcal{G}^\mu : \mu \text{ is an initial law on } E\}; \\
\mathcal{N}(\mathcal{G}) = \cap\{\mathcal{N}^\mu(\mathcal{G}) : \mu \text{ is an initial law on } E\}; \\
\mathcal{G}^\mu_t = \sigma(\mathcal{G}_t \cup \mathcal{N}^\mu(\mathcal{G})); \\
\mathcal{G}_t = \cap\{\mathcal{G}^\mu_t : \mu \text{ is an initial law on } E\}.
\]

Therefore \( (\mathcal{G}^\mu, \mathcal{G}^\mu_t) \) is the augmentation of \( (\mathcal{G}, \mathcal{G}_t) \) by the probability \( \mathbf{P}_\mu \). We call \( (\mathcal{G}, \mathcal{G}_t) \) the augmentation of \( (\mathcal{G}, \mathcal{G}_t) \) by the system of probabilities \( \{\mathbf{P}_\mu : \mu \text{ is a probability on } E\} \). It is easy to see that \( (\xi_t)_{t \geq 0} \) is \( \mathcal{B}^\mu(E) \)-adapted relative to \( (\mathcal{G}^\mu_t)_{t \geq 0} \) and each \( \mathbf{P}_\mu \) extends uniquely to \( \mathcal{G}^\mu_t \). Moreover, for any \( H \in b\mathcal{G} \) the mapping \( x \mapsto \mathbf{P}_x(H) \) is \( \mathcal{B}^\mu(E) \)-measurable and the equality in (A.13) remains true. Using Proposition A.8 and Corollary A.9 one can see \( (\xi_t)_{t \geq 0} \) has the simple \( \mathcal{B}^\mu(E) \)-Markov property relative to \( (\mathcal{G}^\mu_t, \mathbf{P}_\mu) \) and \( (\mathcal{G}_t, \mathbf{P}_\mu) \).

We say \( (\mathcal{G}, \mathcal{G}_t)_{t \geq 0} \) are augmented with respect to the system \( \{\mathbf{P}_\mu : \mu \text{ is a probability on } E\} \) provided \( \mathcal{G} = \mathcal{G}_0 \) and \( \mathcal{G}_t = \mathcal{G}_t \) for all \( t \geq 0 \). As observed in Sharpe (1988, p.25), further application of augmentation procedure to \( (\mathcal{G}_t)_{t \geq 0} \) is fruitless in the sense that \( [\mathcal{G}_t]^\mu = \mathcal{G}_t^\mu \) and \( [\mathcal{G}_t]^\mu = \mathcal{G}_t \). Similarly, letting \( \mathcal{G}^\mu_+ = \cap_{s \geq t} \mathcal{G}_s^\mu \) we have \( [\mathcal{G}^\mu_+]^+ = [\mathcal{G}_+]^\mu \), which will be denoted simply by \( \mathcal{G}^\mu_+ \). It is easy to see that for any initial law \( \mu \) on \( E \) the filtered space \( (\Omega, \mathcal{G}^\mu, \mathcal{G}^\mu_+, \mathbf{P}_\mu) \) satisfies the usual hypotheses. Let \( \mathcal{F}^\mu \) be the \( \mathcal{B}^\mu(E) \)-natural \( \sigma \)-algebra of \( \{\xi_t : t \geq 0\} \). Let \( \mathcal{F}^\mu \) denote the \( \mathbf{P}_\mu \)-completion of \( \mathcal{F}^\mu \) and let \( \mathcal{F} = \cap\{\mathcal{F}^\mu : \mu \text{ is an initial law on } E\} \). It was proved in Sharpe (1988, p.25) that

\[ \mathbf{P}_\mu[F \circ \theta_t|\mathcal{G}_t] = \mathbf{P}_\mu[F \circ \theta_t|\mathcal{G}^\mu_t] = \mathbf{P}_{\xi_t}(F) \tag{A.14} \]

for every \( t \geq 0 \), \( F \in b\mathcal{F} \) and initial law \( \mu \).

**Proposition A.11** Suppose that \( T \) is a stopping time over \( (\mathcal{G}_t)_+ \). Then \( \mathcal{G}^\mu_+ = \sigma(\mathcal{G}^\mu_+ \cup \mathcal{N}^\mu(\mathcal{G})) \).

**Proof.** Since \( \mathcal{G}^\mu_+ \supset \sigma(\mathcal{G}^\mu_+ \cup \mathcal{N}^\mu(\mathcal{G})) \) is obvious, we only need to verify the inclusion \( \mathcal{G}^\mu_+ \subset \sigma(\mathcal{G}^\mu_+ \cup \mathcal{N}^\mu(\mathcal{G})) \). It suffices to show for every \( A \in \mathcal{G}^\mu_+ \) there is \( B \in \mathcal{G}_T^\mu \) such that \( A \triangle B \in \mathcal{N}^\mu(\mathcal{G}) \), where “\( \triangle \)” denotes the symmetric difference defined by (A.1). For each \( n \geq 1 \) define the stopping time \( T_n \) over \( (\mathcal{G}_t)_+ \) by

\[ T_n(\omega) = \begin{cases} \\
\frac{k}{2^n} \text{ if } (k - 1)/2^n \leq T(\omega) < \frac{k}{2^n}, \\
\infty \text{ if } T(\omega) = \infty.
\end{cases} \tag{A.15} \]
Then $T_n \to T$ decreasingly as $n \to \infty$. In view of (A.15) we have $A \cap \{T_n = k/2^n\} \in \mathcal{G}_{2^n}^\mu$ for $1 \leq n < \infty$ and $1 \leq k = \infty$. Then there exists $A_{n,k} \in \mathcal{G}_{2^n}$ so that

$$(A \cap \{T_n = k/2^n\}) \Delta A_{n,k} \in \mathcal{N}^\mu(\mathcal{G}).$$

Since $\{T_n = k/2^n\} \in \mathcal{G}_{2^n}$, we have

$$B_{n,k} := A_{n,k} \cap \{T_n = k/2^n\} \in \mathcal{G}_{2^n}.$$

Observe also that

$$(A \cap \{T_n = k/2^n\}) \Delta B_{n,k} \in \mathcal{N}^\mu(\mathcal{G}).$$

Let $B_n = (\bigcup_{k=1}^\infty B_{n,k}) \cup B_{n,\infty}$. Then $A \Delta B_n \in \mathcal{N}^\mu(\mathcal{G})$ and

$$B_n \cap \{T_n = k/2^n\} = B_{n,k} \in \mathcal{G}_{2^n}.$$

It follows that $B_n \in \mathcal{G}_{T_n} \subset \mathcal{G}_{T_k}$ for $n \geq k$. By the right continuity of $(\mathcal{G}_{t+})$,

$$B := \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty B_n \in \bigcap_{k=1}^\infty \mathcal{G}_{T_k}^+ = \mathcal{G}_{T+}^+.$$

Moreover, we have $A \Delta B \in \mathcal{N}^\mu(\mathcal{G})$. That gives the desired result. \qed

**Corollary A.12** For any initial law $\mu$ on $E$ and any stopping time $T$ for $(\mathcal{G}_{t+}^\mu)$, there is a stopping time $S$ for $(\mathcal{G}_{t+})$ so that $\{T \neq S\} \in \mathcal{N}^\mu(\mathcal{G})$. In this case, we have

$$\mathcal{G}_{T+}^\mu = \mathcal{G}_{S+}^\mu = \sigma(\mathcal{G}_{S+}^\mu \cup \mathcal{N}^\mu(\mathcal{G})).$$

(A.16)

**Proof.** The first assertion was proved in Sharpe (1988, p.25). Then (A.16) follows by Propositions A.6 and A.11. \qed

**Proposition A.13** (Sharpe, 1988, p.26) Let $f \in \mathcal{B}^\mu(E)$ and let $\mu$ be an initial law on $E$. Then we have:

1. If $T$ is an stopping time over $(\mathcal{G}_{t+}^\mu)$, then $f(\xi_T)1_{\{T<\infty\}} \in \mathcal{G}_{T+}^\mu$.
2. If $T$ is an stopping time over $(\mathcal{G}_{T+})$, then $f(\xi_T)1_{\{T<\infty\}} \in \mathcal{G}_{T+}$.

**Definition A.14** (Sharpe, 1988, p.26) Suppose that $\xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, P_x)$ is a right continuous $\mathcal{B}^\mu(E)$-Markov process with transition semigroup $(P_t)_{t \geq 0}$ and $(\mathcal{G}, \mathcal{G}_t)$ have been augmented by $\{P_\mu : \mu \text{ is an initial law on } E\}$. We say $(\xi_t)_{t \geq 0}$ satisfies the strong Markov property relative to $(\mathcal{G}_{t+})$ provided

$$P_\mu[f(\xi_t) \circ \theta_T 1_{\{T<\infty\}} | \mathcal{G}_{T+}] = P_\mu[f(\xi_T) 1_{\{T<\infty\}}]$$

(A.17)

for every $t \geq 0$, stopping time $T$ over $(\mathcal{G}_{t+})$, initial law $\mu$ and $f \in \mathcal{B}^\mu(E)$. 

suppose that $A$ is a Markov process with transition semigroup $(P_t)_{t \geq 0}$. Let $(\mathcal{G}, \mathcal{G}_t)$ be the augmentations of $(\mathcal{G}^*, \mathcal{G}_t^*)$ by $\{P_\mu : \mu \text{ is an initial law on } E\}$. Then $(\xi_t)_{t \geq 0}$ has the strong Markov property relative to $(\mathcal{G}_t^*)$ if and only if

$$P_\mu [f(\xi_t) \circ \theta_{T_0} 1_{\{T_0 < \infty\}} | \mathcal{G}_{T_0^+}] = P_t f(\xi_t) 1_{\{T_0 < \infty\}}$$

(A.18)

for every $t \geq 0$, stopping time $T_0$ over $\mathcal{G}_t^*$, initial law $\mu$ and $f \in b\mathcal{B}(E)$.

**Proof.** Suppose that $(\xi_t)_{t \geq 0}$ has the strong Markov property relative to $\mathcal{G}_t^*$ and $T_0$ is a stopping time over $\mathcal{G}_t^*$. Since $t \mapsto \xi_t$ is clearly $\mathcal{B}(E)$-progressive over $\mathcal{G}_t^*$, for any $f \in b\mathcal{B}(E)$ the process $t \mapsto f(\xi_t)$ is progressive over $\mathcal{G}_t^*$. Then $f(\xi_t) 1_{\{T_0 < \infty\}} \in b\mathcal{B}_{T_0^+}$; see, e.g., Dellacherie and Meyer (1978, p.122) or Sharpe (1988, p.22). Consequently, we have $P_t f(\xi_{T_0}) 1_{\{T_0 < \infty\}} \in b\mathcal{B}_{T_0^+}$ for $t \geq 0$. By letting $T = T_0$ in (A.17) and taking the conditional expectation relative to $\mathcal{G}_{T_0^+}$ we obtain (A.18). For every stopping time $T_0$ over $\mathcal{G}_t^*$ and every $f \in b\mathcal{B}(E)$, let $T$ be a stopping time over $\mathcal{G}_t^*$ and let $A \in \mathcal{G}_t^*$. By Corollary A.12 there is a stopping time $T_0$ over $\mathcal{G}_t^*$ and an event $A_0 \in \mathcal{G}_{T_0^+}$ so that $\{T \neq T_0\} \in \mathcal{N}^\mu(\mathcal{G}_t^*)$ and $A \Delta A_0 \in \mathcal{N}^\mu(\mathcal{G}_t^*)$. By (A.18) for every $f \in b\mathcal{B}(E)$ we have

$$P_\mu [1_A f(\xi_t) \circ \theta_T 1_{\{T < \infty\}}] = P_\mu [1_A P_t f(\xi_T) 1_{\{T < \infty\}}].$$

(A.19)

As in the proof of Proposition A.8 it is easy to see the above equality also holds for $f \in b\mathcal{B}(E)$. That gives (A.17) for $f \in b\mathcal{B}(E)$.

If (A.18) is satisfied for every $t \geq 0$, stopping time $T_0$ over $\mathcal{G}_t^*$, initial law $\mu$ and $f \in b\mathcal{B}(E)$, we say the process $(\xi_t)_{t \geq 0}$ satisfies the strong Markov property relative to $\mathcal{G}_t^*$. Note that the condition that $(P_t)_{t \geq 0}$ sends $b\mathcal{B}(E)$ into itself is used to guarantee the measurability of the right-hand side relative to $\mathcal{G}_t^*$.

A real-valued process $(Z_t)_{t \geq 0}$ is called $\mathcal{P}_\mu(\mathcal{G})$-evanescent in case $\{\omega \in \Omega : Z_t(\omega) \neq 0 \text{ for some } t \geq 0\} \in \mathcal{N}^\mu(\mathcal{G})$. Let $\mathcal{N}^\mu(\mathcal{G})$ denote the class of $\mathcal{P}_\mu(\mathcal{G})$-evanescent processes and let $\mathcal{I} = \cap \{\mathcal{I}^\mu(\mathcal{G}) : \mu \text{ is an initial law on } E\}$.

Let $\mathcal{D}(\mathcal{G}_t)$ denote the class of bounded right continuous real processes adapted to $(\mathcal{G}_t)_{t \geq 0}$. Let $\mathcal{I}^\mu(\mathcal{G}_t)$ be the $\sigma$-algebra on $\Omega \times [0, \infty)$ generated by $\mathcal{D}(\mathcal{G}_t) \cup \mathcal{I}^\mu(\mathcal{G}_t)$ and let $\mathcal{I}^\mu(\mathcal{G}_t) = \cap \{\mathcal{I}^\mu(\mathcal{G}_t) : \mu \text{ is an initial law on } E\}$. We say an extended real function $f$ on $E$ is nearly optional relative to $\xi$ provided $(\omega, t) \mapsto f(\xi_t(\omega))$ is $\mathcal{I}^\mu(\mathcal{G}_t)$-measurable. Clearly, a continuous function on $E$ is nearly optional. By the monotone class theorem it is easy to see that a Borel function on $E$ is also nearly optional. The function $f$ is said to be nearly Borel relative to $\xi$ if for every initial law $\mu$ there are Borel functions $g$ and $h$ on $E$ so that $g \leq f \leq h$ and $\mathcal{P}_\mu [g(\xi_t) = h(\xi_t)]$ for all $t \geq 0 = 1$. Let $d$ be a metric on $E$ compatible with its topology. Recall that $\mathcal{C}_u(E) \equiv \mathcal{C}_u(E, d)$ denotes the set of real $d$-uniformly continuous functions on $E$ and $\mathcal{I}^\alpha$ is the class of all $\alpha$-excessive functions for $(P_t)_{t \geq 0}$.

**Theorem A.16** (Sharpe, 1988, p.31) Suppose that $\xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, P_x)$ is a right continuous $\mathcal{B}(E)$-Markov process with transition semigroup $(P_t)_{t \geq 0}$ and...
(\(\mathcal{G}, \mathcal{G}_t\)) have been augmented by \(\{P_\mu : \mu \text{ is an initial law on } E\}\). Then the following conditions are equivalent:

1. \(\{t \mapsto f(\xi_t) \text{ is not right continuous}\} \in \mathcal{N}(\mathcal{G})\) for every \(\alpha > 0\) and every \(f \in \mathcal{G}^\alpha\);
2. \(\{t \mapsto U^\alpha f(\xi_t) \text{ is not right continuous}\} \in \mathcal{N}(\mathcal{G})\) for every \(\alpha > 0\) and every \(f \in b\mathcal{C}_u(E)\);
3. \((\xi_t)_{t \geq 0}\) satisfies the strong Markov property relative to \((\mathcal{G}, \mathcal{G}_t)\), and \(U^\alpha f\) is nearly optional relative to \((\xi_t, \mathcal{G}_t)\) for every \(\alpha > 0\) and every \(f \in b\mathcal{C}_u(E)\);
4. \((\xi_t)_{t \geq 0}\) satisfies the strong Markov property relative to \((\mathcal{G}, \mathcal{G}_t)\), and \(P_s f\) is nearly optional relative to \((\xi_t, \mathcal{G}_t)\) for every \(s \geq 0\) and every \(f \in b\mathcal{C}_u(E)\);
5. \(\{t \mapsto P_s f(\xi_t) \text{ is not right continuous}\} \in \mathcal{N}(\mathcal{G})\) for every \(s \geq 0\) and every \(f \in b\mathcal{C}_u(E)\);
6. \(P_s f\) is nearly optional relative to \((\xi_t, \mathcal{G}_t)\) for every \(s \geq 0\) and every \(f \in b\mathcal{C}_u(E)\), and

\[
P_\mu \{f(\xi_t)1_{\{T < t\}} | \mathcal{G}^\mu_{T+}\} = P_{t-T} f(\xi_{T})1_{\{T < t\}}, \quad t \geq 0, f \in b\mathcal{C}_u(E),
\]

for every optional time \(T\) over \((\mathcal{G}, \mathcal{G}_t)\) and every initial law \(\mu\) on \(E\);
7. \(\{s \mapsto P_{t-s} f(\xi_s)1_{[0,t)}(s) \text{ is not right continuous}\} \in \mathcal{N}(\mathcal{G})\) for every \(t \geq 0\) and every \(f \in b\mathcal{C}_u(E)\);
8. \(\{s \mapsto P_{t-s} f(\xi_s)1_{[0,t)}(s) \text{ is not right continuous}\} \in \mathcal{N}(\mathcal{G})\) for every \(t \geq 0\) and every \(f \in b\mathcal{B}_u(E)\).

**Corollary A.17** (Sharpe, 1988, p.36) Let \((\mathcal{F}^*, \mathcal{F}_t^*)\) be the natural \(\sigma\)-algebras of \(\xi\) generated by \(\{\xi_t : t \geq 0\}\) and let \((\mathcal{F}, \mathcal{F}_t)\) be their augmentations. If \(\xi\) satisfies one of the conditions in Theorem A.16 relative to \((\mathcal{F}_t)\), then \((\mathcal{F}^*_t)\) and \((\mathcal{F}_t)\) are right continuous.

The properties in Theorem A.16 depend not only on the transition semigroup \((P_t)_{t \geq 0}\), but also on the particular realization \(\xi\). In particular, when \((P_t)_{t \geq 0}\) is a Borel semigroup, for every \(\alpha > 0\) and every \(f \in \mathcal{C}_u(E)\) the function \(U^\alpha f\) is nearly optional relative to \((\xi_t, \mathcal{G}_t)\), so the properties hold if and only if \((\xi_t)_{t \geq 0}\) satisfies the strong Markov property relative to \((\mathcal{G}_t)_{t \geq 0}\).

**Definition A.18** (Sharpe, 1988, p.38) The system \(\xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, P_x)\) is called a right Markov process or simply a right process with transition semigroup \((P_t)_{t \geq 0}\) provided:

1. \(\xi\) is a right continuous realization of \((P_t)_{t \geq 0}\);
2. \(\xi\) satisfies the conditions in Theorem A.16;
3. \((\mathcal{G}, \mathcal{G}_t)_{t \geq 0}\) are augmented and \((\mathcal{G}_t)_{t \geq 0}\) is right continuous.

We call \(\xi\) a Borel right process if it is a right process with Borel transition semigroup. A Markov transition semigroup \((P_t)_{t \geq 0}\) is called a right transition semigroup if it is the transition semigroup of a right process.

**Proposition A.19** (Sharpe, 1988, p.39) The minimum of two \(\alpha\)-excessive functions of a right semigroup is also \(\alpha\)-excessive.
The fine topology of a right process $\xi$ is the smallest topology on $E$ rendering continuous all functions in $\cup_{\alpha \geq 0} \mathcal{F}^\alpha$ as maps from $E$ to $[0, \infty]$; see Sharpe (1988, p.53 and p.232). A function $f \in b\mathcal{B}^0(E)$ is finely continuous relative to $\xi$ if and only if $t \mapsto f(\xi_t)$ is a.s. right continuous on $[0, \infty)$. More generally, we have:

**Theorem A.20** (Sharpe, 1988, p.53 and p.55) Let $f \in \mathcal{B}^u(E)$. If $t \mapsto f(\xi_t)$ is a.s. right continuous at $t = 0$, then $f$ is finely continuous relative to $\xi$. Conversely, if $f$ is finely continuous and nearly optional relative to $\xi$, then $t \mapsto f(\xi_t)$ is a.s. right continuous on $[0, \infty)$.

A right process $\xi$ is called a Hunt process if it is quasi-left continuous, that is, for every increasing sequence of stopping times $\{T_n\}$ with limit $T$ we have $\xi_{T_n} \to \xi_T$ a.s. on $\{T < \infty\}$. If $\xi$ is a Hunt process, then $t \mapsto \xi_t$ is a.s. càdlàg on $[0, \infty)$; see Sharpe (1988, p.221).

Let us consider two Radon topological spaces $E$ and $F$. Suppose that $\xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, P_x)$ is a right process in $E$ with transition semigroup $(P_t)_{t \geq 0}$ and $\psi$ is a map of $E$ to $F$. In addition, we assume:

1. $\psi$ is surjective and measurable relative to the $\sigma$-algebras $\mathcal{B}^u(E)$ and $\mathcal{B}^u(F)$;
2. for every $t \geq 0$ and every $f \in \mathcal{B}^u(F)$ there exists a function $Q_t f \in \mathcal{B}^u(F)$ so that $P_t(f \circ \psi) = (Q_t f) \circ \psi$;
3. the path $t \mapsto X_t := \psi(\xi_t)$ is a.s. right continuous in $F$.

Under the above conditions, the operator $f \mapsto Q_t f$ determines a probability kernel on $(F, \mathcal{B}^u(F))$ and $(Q_t)_{t \geq 0}$ form a Markov transition semigroup. Let $\Omega_1 = \{\omega \in \Omega : t \mapsto X_t(\omega)$ is right continuous$\}$. The above property (3) implies $P_x(\Omega_1) = 1$ for every $x \in E$, so we can replace $\Omega$ by $\Omega_1$ in the definition of $\xi$. Let $(\mathcal{F}^u, \mathcal{F}^u_t)$ be the $\mathcal{B}^u(F)$-natural $\sigma$-algebras of $\{X_t : t \geq 0\}$ on $\Omega_1$. A simple calculation shows that $P_{x_1}$ and $P_{x_2}$ coincide on $\mathcal{F}^u$ if $\psi(x_1) = \psi(x_2) = x$. We denote their common restriction on $\mathcal{F}^u$ by $Q_x$. Let $(\mathcal{F}, \mathcal{F}_t)$ be the augmentations of $(\mathcal{F}^u, \mathcal{F}^u_t)$ relative to the family of probability measures $\{Q_x : x \in F\}$.

**Theorem A.21** (Sharpe, 1988, p.75) The system $X = (\Omega_1, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, Q_x)$ is a right process in $F$ with transition semigroup $(Q_t)_{t \geq 0}$.

A general transition semigroup $(P_t)_{t \geq 0}$ on $(E, \mathcal{B}^*(E))$ may be extended to a conservative transition semigroup on a larger space. Simply take an abstract point $\partial \notin E$ and let $\tilde{E} = E \cup \{\partial\}$ be the Radon topological space obtained by adjoining $\partial$ to $E$ as an isolated point. Let $\mathcal{B}^*(\tilde{E}) = \sigma(\mathcal{B}^*(E) \cup \{\partial\})$ and define $(\tilde{P}_t)_{t \geq 0}$ on $(\tilde{E}, \mathcal{B}^*(\tilde{E}))$ by

$$
\tilde{P}_t(x, B) = \begin{cases}
P_t(x, B) & \text{if } x \in E \text{ and } B \in \mathcal{B}^*(E), \\
1 - P_t(x, E) & \text{if } x \in E \text{ and } B = \{\partial\}, \\
1_{B}(\partial) & \text{if } x = \partial.
\end{cases}
$$

(A.20)

It is trivial to see that $(\tilde{P}_t)_{t \geq 0}$ is a conservative transition semigroup on $(\tilde{E}, \mathcal{B}^*(\tilde{E}))$. We call $(P_t)_{t \geq 0}$ a right transition semigroup if $(\tilde{P}_t)_{t \geq 0}$ is a right semigroup in the sense of Definition A.18. In this case, suppose that $\hat{\xi} = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, \tilde{P}_x)$ is a
right process on $\hat{E}$ realizing $(\hat{P}_t)_{t \geq 0}$. It is obvious from (A.20) that $\partial$ is a trap for the process. That is, $\hat{P}_\partial \{ \xi_t = \partial \text{ for all } t \geq 0 \} = 1$. Let $\zeta = \inf \{ t \geq 0 : \xi_t = \partial \}$. By the strong Markov property, we have $\hat{P}_x \{ \xi_t = \partial \text{ for all } t \geq \zeta \} = 1$ and all $x \in \hat{E}$. In many respects, the process $\hat{\xi}$ is interesting only when it is in $E$. Indeed, if $(P_t)_{t \geq 0}$ is the object of interest, the adjunction of $\partial$ is quite artificial. In this situation, one may simplify the notation by making the convention that every function $f$ on $E$ is the object of interest, the adjunction of $\partial$ is quite artificial. In this situation, one may simplify the notation by making the convention that every function $f$ on $E$ is by definition a Markov resolvent such that every function $f$ on $E$ is automatically extended to $\hat{E}$ by setting $f(\partial) = 0$. Then $P_t f$ means exactly the same thing as $\hat{P}_t f$. Let $P_x = P_x^x$ for $x \in E$. The system $\xi = (\Omega, \mathcal{G}, \mathcal{F}_t, \xi_t, \theta_t, P_x)$ is called a right process on $E$ with lifetime $\zeta$ and transition semigroup $(P_t)_{t \geq 0}$. In the special case $P_x \{ \zeta = \infty \} = 1$ for all $x \in E$, the process $\xi$ is said to be conservative. We call $\xi$ a Hunt process if $\hat{\xi}$ is a Hunt process in $\hat{E}$.

**Theorem A.22** (Fitzsimmons, 1988, p.349 and p.350) Suppose that $\xi$ is a Borel right process with bounded potential operator $U$. Let $M(E)$ denote the space of finite Borel measures on $E$ endowed with the topology of weak convergence. Let $f \in L^0(E)$. Then we have:

1. $f$ is finely continuous relative to $\xi$ if and only if $\lim_{n \to \infty} \nu_n(f) = \nu(f)$ for all $\nu_n, \nu \in M(E)$ satisfying $\lim_{n \to \infty} \nu_n U = \nu U$;
2. $t \mapsto f(\xi_t)$ has left limits on $(0, \infty)$ a.s. if and only if $\lim_{n \to \infty} \nu_n(f)$ exists for all $\nu_n \in M(E)$ such that $\lim_{n \to \infty} \nu_n U$ exists in $M(E)$;
3. if $t \mapsto f(\xi_t)$ is quasi-left continuous, then $\lim_{n \to \infty} \nu_n(f) = \nu(f)$ for all $\nu_n, \nu \in M(E)$ satisfying $\lim_{n \to \infty} \nu_n U = \nu U$.

### A.4 Ray–Knight Completion

We first assume that $E$ is a compact metrizable space. A Ray resolvent $(U^\alpha)_{\alpha > 0}$ on $(E, \mathcal{B}(E))$ is by definition a Markov resolvent such that $U^\alpha \mathcal{C}(E) \subset \mathcal{C}(E)$ and $\cup_{\alpha > 0} U^\alpha \cap 2^E$ separates the points of $E$, where $\mathcal{F}^\alpha$ is the class of $\alpha$-supermedian functions for $(U^\alpha)_{\alpha > 0}$. It was proved in Getoor (1975, p.9) that to every Ray resolvent $(U^\alpha)_{\alpha > 0}$ there corresponds a unique Markov transition semigroup $(P_t)_{t \geq 0}$ on $(E, \mathcal{B}(E))$ such that $t \mapsto P_t f(x)$ is right continuous for $x \in E$ and $f \in \mathcal{C}(E)$, and

$$U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt, \quad \alpha > 0, x \in E, f \in \mathcal{C}(E).$$  \hspace{1cm} (A.21)

The Markov transition semigroup $(P_t)_{t \geq 0}$ defined by (A.21) is called the Ray semigroup associated with $(U^\alpha)_{\alpha > 0}$. A Ray semigroup is not necessarily normal. The set of branch points for $(P_t)_{t \geq 0}$ is $B := \{ x \in E : P_0(x, \cdot) \neq \delta_x(\cdot) \}$ and the set of non-branch points for $(P_t)_{t \geq 0}$ is $D := \{ x \in E : P_0(x, \cdot) = \delta_x(\cdot) \} = E \setminus B$.

**Proposition A.23** (Sharpe, 1988, p.44) Let $(P_t)_{t \geq 0}$ be a Ray semigroup on $E$ and let $B$ and $D$ be defined as above. Then:

1. for any $\{g_n\}$ uniformly dense in $\mathcal{C}(E) \cap \mathcal{F}^1$ we have $B = \cup_n \{ P_0 g_n < g_n \}$;
(2) $B$ is an $F_t$ set in $E$ and hence $B \in \mathcal{B}(E)$;
(3) for any $t \geq 0$ and $x \in E$ the probability measure $P_t(x, \cdot)$ is carried by $D$.

**Theorem A.24** (Sharpe, 1988, p.46) The restriction of $(P_t)_{t \geq 0}$ to $D$ is a right semigroup which may be realized on the space $\Omega$ of right continuous maps of $[0, \infty)$ into $D$ having left limits in $E$.

**Theorem A.25** (Sharpe, 1988, p.49) Suppose that $E$ is a locally compact, non-compact separable metric space and $(U^\alpha)_{\alpha>0}$ is a Markov resolvent on $(E, \mathcal{B}(E))$ such that $U^\alpha(C_0(E)) \subset C_0(E)$ for all $\alpha > 0$ and $\alpha U^\alpha f \to f$ pointwise as $\alpha \to \infty$ for all $f \in C_0(E)$. Then there is a right process $\xi$ with state space $E$ having resolvent $(U^\alpha)_{\alpha>0}$ such that:

1. $\xi$ is quasi-left continuous;
2. for all $t > 0$ the set $\{\xi_s(\omega) : 0 \leq s \leq t\}$ a.s. has compact closure in $E$;
3. a.s. the left limit $\xi_{t-} := \lim_{s \to t-} \xi_s$ exists in $E$ for all $t > 0$.

The conditions in Theorem A.25 are satisfied if $(U^\alpha)_{\alpha>0}$ is the resolvent generated by a Feller semigroup. Then a Feller semigroup has a Hunt realization.

Now suppose we are given a general Radon topological space $E$ with a totally bounded metric $d$ for its topology. Let $(U^\alpha)_{\alpha>0}$ be a Markov resolvent on $(E, \mathcal{B}(E))$ satisfying

$$\mathcal{B}(E) \subset \sigma(\{U^\alpha f : \alpha > 0, f \in \mathcal{C}_b(E,d)\}).$$

(A.22)

A set $U \subset \text{pb} \mathcal{B}^u(E)$ is called a rational cone if it is closed under positive rational linear combinations. For $D \subset \text{pb} \mathcal{B}^u(E)$, we denote by $q(D)$ the rational cone generated by $D$, that is, the smallest rational cone containing $D$. For a rational cone $U \subset \text{pb} \mathcal{B}^u(E)$, set $\lambda(U) = \{f_1 \wedge \cdots \wedge f_n : n \geq 1, f_i \in U\}$ and $u(U) = \{U^\alpha_1 f_1 + \cdots + U^\alpha_n f_n : n \geq 1, f_i \in U\}$ and strictly positive rationals $\alpha_i$. It is obvious that $u(U)$ is a rational cone contained in the cone $\cup_{\alpha>0} b \mathcal{R}^\alpha$. That $\lambda(U)$ is also a rational cone comes from the trivial identities $(\land_i a_i) + b = \land_i (a_i + b)$ and $(\land_i a_i) + (\land_j b_j) = \land_{i,j} (a_i + b_j)$.

For a given function class $D \subset \text{pb} \mathcal{B}^u(E)$, we set $\mathcal{R}_0 = u(q(D))$ and set $\mathcal{R}_n = \lambda(\mathcal{R}_{n-1} + u(\mathcal{R}_{n-1}))$ for $n \geq 1$ inductively, where $\mathcal{R}_{n-1} + u(\mathcal{R}_{n-1}) = \{f + g : f \in \mathcal{R}_{n-1}$ and $g \in u(\mathcal{R}_{n-1})\}$. The set $\mathcal{R}(D) := \cup_{n \geq 0} \mathcal{R}_n$ is called the rational Ray cone generated by $(U^\alpha)_{\alpha>0}$ and $D$; see Getoor (1975, p.58) and Sharpe (1988, p.90). The rational Ray cone $\mathcal{R} = \mathcal{R}(D)$ generated by $(U^\alpha)_{\alpha>0}$ and $D \subset \text{pb} \mathcal{B}^u(E)$ is the smallest rational cone contained in $\text{pb} \mathcal{B}^u(E)$ such that:

1. $U^\alpha(\mathcal{R}) \subset \mathcal{R}$ for all rationals $\alpha > 0$;
2. $f, g \in \mathcal{R}$ implies $f \wedge g \in \mathcal{R}$;
3. $\mathcal{R}$ contains $u(q(D))$.

Clearly, for each $f \in \mathcal{R}(D)$ there is a constant $\beta = \beta(f) > 0$ so that $f$ is a $\beta$-supermedian function for $(U^\alpha)_{\alpha>0}$. Furthermore, if $(U^\alpha)_{\alpha>0}$ is the resolvent associated with a conservative right semigroup $(P_t)_{t \geq 0}$ on $(E, d)$, for each $f \in \mathcal{R}(D)$ there exists $\beta = \beta(f) > 0$ so that $f$ is $\beta$-excessive for $(P_t)_{t \geq 0}$. 


Proposition A.26 (Sharpe, 1988, p.90) If $\mathcal{D}$ is a countable uniformly dense subset of $p\mathcal{C}_u(E,d)$ and contains the constant function $1_E$, then the rational Ray cone $\mathcal{R} = \mathcal{R}(\mathcal{D})$ is countable, contains the positive rational constant functions, and separates the points of $E$.

In the remainder of this section, we assume $\mathcal{D} \subset p\mathcal{C}_u(E,d)$ satisfies the conditions of Proposition A.26. Recall that $\|\cdot\|$ denotes the supremum norm. We give the rational Ray cone $\mathcal{R} = \mathcal{R}(\mathcal{D})$ an enumeration $\{g_0, g_1, g_2, \ldots\}$ with $g_0 = 0$. Clearly,

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{|g_n(x) - g_n(y)|}{2^n \|g_n\|}, \quad x, y \in E,$$

(A.23)

defines a metric $\rho$ on $E$, and each $g_n$ is $\rho$-uniformly continuous. Let $(\bar{E}, \bar{\rho})$ denote the completion of $(E, \rho)$. Observe that the map $x \mapsto (g_n(x))_{n \geq 1}$ of $E$ into $K := \prod_{n=1}^{\infty} [0, \|g_n\|]$ with the metric $q$ defined by

$$q(a, b) = \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{2^n \|g_n\|}, \quad a, b \in K,$$

is an isometry. It follows that the completion $(\bar{E}, \bar{\rho})$ is compact. The topology on $E$ induced by the metric $\rho$ is called the Ray topology of $(U^\alpha)_{\alpha > 0}$.

Proposition A.27 (Sharpe, 1988, p.91) Each function $f \in \mathcal{C}_u(E, \rho)$ extends to a unique $\bar{f} \in \mathcal{C}_u(\bar{E}, \bar{\rho})$. For each $\alpha > 0$, we have $U^\alpha(\mathcal{C}_u(E, \rho)) \subset \mathcal{C}_u(E, \rho)$ and $U^\alpha(\mathcal{C}_u(E, d)) \subset \mathcal{C}_u(E, \rho)$, and $\mathcal{C}_u(E, \rho)$ is the uniform closure of $\mathcal{R} - \mathcal{R} := \{f - g : f, g \in \mathcal{R}\}$.

Proposition A.28 (Sharpe, 1988, p.91) If $U^\alpha(\mathcal{C}_u(E, d)) \subset \mathcal{C}_u(E, d)$ for all $\alpha > 0$, then the Ray topology is coarser than the original topology.

Proposition A.29 (Sharpe, 1988, p.92) Let $\mathcal{B}^u(E)$ denote the $\sigma$-algebra on $E$ generated by the Ray topology. Then $\mathcal{B}(E) \subset \mathcal{B}^u(E) \subset \mathcal{B}^a(E)$ and $U^\alpha \mathcal{B}^u(E) \subset \mathcal{B}^u(E)$ for every $\alpha > 0$.

Proposition A.30 (Sharpe, 1988, pp.92–93) We have $E \in \mathcal{B}^u(\bar{E})$, so $(E, \rho)$ is a Radon space. If $(E, d)$ is Lusin and if $(U^\alpha)_{\alpha > 0}$ maps $\mathcal{B}(E)$ into itself, then $\mathcal{B}(E) = \mathcal{B}^u(E)$ and $E \in \mathcal{B}(\bar{E})$, so $(E, \rho)$ is a Lusin space.

For every $\bar{f} \in \mathcal{C}_u(\bar{E}, \bar{\rho})$ we clearly have $f := \bar{f}|_E \in \mathcal{C}_u(E, \rho)$. Then $U^\alpha f \in \mathcal{C}_u(E, \rho)$ by Proposition A.27 and so $U^\alpha f$ extends continuously to some $(U^\alpha f)^- \in \mathcal{C}_u(\bar{E}, \bar{\rho})$. Define the operators $(\bar{U}^\alpha)_{\alpha > 0}$ on $\mathcal{C}_u(\bar{E}, \bar{\rho})$ by

$$\bar{U}^\alpha \bar{f} = (U^\alpha f)^-, \quad \alpha > 0, \bar{f} \in \mathcal{C}_u(\bar{E}, \bar{\rho}).$$

Theorem A.31 (Sharpe, 1988, p.93) For $\alpha > 0$ and $x \in E$ the measure $\bar{U}^\alpha(x, \cdot)$ is carried by $E \in \mathcal{B}^a(\bar{E})$ and its restriction to $E$ is $U^\alpha(x, \cdot)$. Moreover, the family $(\bar{U}^\alpha)_{\alpha > 0}$ is a Ray resolvent on the space $\bar{E}$. 
We call \((\bar{U}^\alpha)_{\alpha > 0}\) the Ray extension of \((U^\alpha)_{\alpha > 0}\). The space \((\bar{E}, \bar{\rho})\) constructed above is called the Ray–Knight completion of \((E, \rho)\) with respect to \((U^\alpha)_{\alpha > 0}\). It depends not only on \(E, d\) and \((U^\alpha)_{\alpha > 0}\) but also on the choice of the family \(\mathcal{D} \subset p\mathcal{C}_u(E, d)\). If \((U^\alpha)_{\alpha > 0}\) is the resolvent associated with a conservative right semigroup \((P_t)_{t \geq 0}\), we also call \((\bar{E}, \bar{\rho})\) the Ray–Knight completion of \((E, \rho)\) with respect to \((P_t)_{t \geq 0}\). In this case, the Ray semigroup \((\bar{P}_t)_{t \geq 0}\) associated with \((\bar{U}^\alpha)_{\alpha > 0}\) is called the Ray extension of \((P_t)_{t \geq 0}\).

**Theorem A.32** (Sharpe, 1988, p.94) Let \((P_t)_{t \geq 0}\) be a conservative right semigroup on \(E\). Then there is a realization \(\xi = (\Omega, \mathcal{G}, \mathcal{H}, \xi_t, \theta_t, P_x)\) of \((P_t)_{t \geq 0}\) which is a right process in both \((E, d)\) and \((E, \rho)\) and the left limit \(\xi_{t-} := \lim_{s \to t-} \xi_s\) taken in the Ray topology exists in \(\bar{E}\) for all \(t > 0\).

**Theorem A.33** Suppose that \((P_t)_{t \geq 0}\) is a conservative Borel right semigroup on a Lusin topological space \(E\). Then every right continuous realization of \((P_t)_{t \geq 0}\) with the augmented natural \(\sigma\)-algebras is a right process. In particular, the semigroup can be realized canonically on the space of right continuous paths from \([0, \infty)\) to \(E\).

**Proof.** Let \(\xi\) be a right process with semigroup \((P_t)_{t \geq 0}\). Then each \(f \in \mathcal{S}^\alpha\) is a nearly Borel function of \(\xi\) relative to the Ray topology; see Sharpe (1988, p.95). By Proposition A.30, we have \(\mathcal{B}(E) = \mathcal{B}^\alpha(E)\), so each \(f \in \mathcal{S}^\alpha\) is nearly Borel in the original topology. Then the result follows by Sharpe (1988, p.98).

If \((P_t)_{t \geq 0}\) is a right semigroup on \((E, d)\) not necessarily conservative, the associated resolvent \((U^\alpha)_{\alpha > 0}\) is not necessarily Markov. In this case, we let \(\bar{E} = E \cup \{\partial\}\) for an abstract point \(\partial \notin E\). Let \((\tilde{E}, \tilde{\partial})\) be a topological extension of \((E, d)\) with \(\partial\) being an isolated point and let \((\tilde{P}_t)_{t \geq 0}\) denote the conservative extension of \((P_t)_{t \geq 0}\) on \(\tilde{E}\) with \(\partial\) being a cemetery. Let \((\tilde{U}^\alpha)_{\alpha > 0}\) denote the resolvent associated with \((\tilde{P}_t)_{t \geq 0}\). Let \(\tilde{\mathcal{D}}\) be the countable Ray cone for \((\tilde{U}^\alpha)_{\alpha > 0}\) constructed from \(\tilde{\mathcal{D}}\), which is a countable uniformly dense subset of \(p\mathcal{C}_u(\tilde{E}, d)\) and contains the constant function \(1_{\tilde{E}}\). Let \((\tilde{E}, \tilde{\rho}, \tilde{U}^\alpha, \tilde{P}_t)\) be the corresponding Ray–Knight completion of \((\tilde{E}, \tilde{\partial}, \tilde{U}^\alpha, \tilde{P}_t)\).

**Proposition A.34** In the situation described above, if there are constants \(\alpha > 0\) and \(\varepsilon > 0\) such that \(U^\alpha 1_E(x) \geq \varepsilon\) for all \(x \in E\), then \(\partial\) is an isolated point of \(\tilde{E}\).

**Proof.** Since \(\tilde{\mathcal{D}}\) is uniformly dense in \(p\mathcal{C}_u(\tilde{E}, d)\), there is a function \(\tilde{g} \in \tilde{\mathcal{D}}\) such that \(\tilde{g}(\partial) < \alpha \varepsilon / 2\) and \(\tilde{g}(x) \geq 1\) for every \(x \in E\). Fix a rational \(\beta \in (\alpha / 2, \alpha)\). Then \(\tilde{f} := U^\beta \tilde{g} \in \tilde{\mathcal{D}}\). Since \(\partial\) is a cemetery for \((\tilde{P}_t)_{t \geq 0}\), we have \(\tilde{f}(\partial) = \beta^{-1} \tilde{g}(\partial) < \varepsilon\). However, for every \(x \in E\) we have \(\tilde{f}(x) = \tilde{U}^\beta \tilde{g}(x) \geq U^\alpha 1_E(x) \geq \varepsilon\). Let \(\tilde{f}\) be the unique continuous extension of \(\tilde{f}\) to \(\tilde{E}\). It follows that \(\tilde{f}(x) \geq \varepsilon\) for every \(x \in \tilde{E} \setminus \{\partial\}\). Then the point \(\partial\) must be isolated in \(\tilde{E}\). □

By Proposition A.34, if \((P_t)_{t \geq 0}\) is a conservative right semigroup, then \(\partial\) is an isolated point of \(\tilde{E}\). In that case, the topology of \(E\) inherited from \(\tilde{E}\) coincides with its Ray topology defined directly by \((P_t)_{t \geq 0}\). In the general case, we also call the inherited topology of \(E\) the Ray topology of \((P_t)_{t \geq 0}\).
A.5 Entrance Space and Entrance Laws

Let $\xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \theta_t, \mathcal{P}_x)$ be a right process on the Radon topological space $E$ with transition semigroup $(P_t)_{t \geq 0}$ and resolvent $(U^\alpha)_{\alpha > 0}$. We first assume $(P_t)_{t \geq 0}$ is conservative. Let $d$ be a metric for the topology of $E$ such that the $d$-completion of $E$ is compact. Let $(\bar{E}, \bar{\rho}, \bar{U}^\alpha)$ be a Ray–Knight completion of $(E, d, U^\alpha)$ and let $(\bar{P}_t)_{t \geq 0}$ be the Ray extension of $(P_t)_{t \geq 0}$. Set $E_R = \{x \in \bar{E} : \bar{U}^1(x, \cdot) \text{ is carried by } E\}$, which is called the Ray space for $\xi$ or $(P_t)_{t \geq 0}$. It was proved in Sharpe (1988, p.191) that $E_R \in \mathcal{B}^u(\bar{E})$ is a Radon topological space and $E \subset E_R$. By the resolvent equation we have $E_R = \{x \in \bar{E} : \bar{U}^\alpha(x, \cdot) \text{ is carried by } E\}$ for each $\alpha > 0$.

**Theorem A.35** (Sharpe, 1988, p.191) Let $(\bar{E}_1, \bar{\rho}_1, \bar{U}^\alpha_1)$ and $(\bar{E}_2, \bar{\rho}_2, \bar{U}^\alpha_2)$ be Ray–Knight completions of $(E, d_1, U^\alpha)$ and $(E, d_2, U^\alpha)$ respectively, where $d_1$ and $d_2$ are totally bounded metrics for the original topology of $E$. Then the corresponding Ray spaces $E^1_R$ and $E^2_R$ are homeomorphic under a mapping $\psi : E^1_R \to E^2_R$ satisfying $\bar{U}^\alpha_1(x, B) = \bar{U}^\alpha_2(\psi(x), B)$ for all $\alpha > 0$ and $B \in \mathcal{B}(\bar{E})$.

Therefore, the Ray space $E_R$ together with the resolvent $(\bar{U}^\alpha)_{\alpha > 0}$ restricted to $E_R$ is uniquely determined, up to homeomorphism, by the original topology on $E$ and $(U^\alpha)_{\alpha > 0}$. This makes the Ray space a natural object. Let $D$ denote the set of non-branch points of $(P_t)_{t \geq 0}$ on $E$, and let $E_D = D \cap E_R = \{x \in E_R : P_0(x, \cdot) = \delta_x(\cdot)\}$ which is called the entrance space for $(P_t)_{t \geq 0}$. Since $D \in \mathcal{B}(\bar{E})$, we have $E_D \in \mathcal{B}(E_D, \bar{\rho})$.

**Proposition A.36** (Sharpe, 1988, pp.192–193) We have:

1. For $x \in E_R$ and $t > 0$, $\bar{P}_t(x, \cdot)$ is carried by $E$.
2. For $x \in E_R$, $\bar{P}_0(x, \cdot)$ is carried by $E_D$.
3. For $x \in B$, $\bar{P}_0(x, \cdot)$ is not concentrated at any point of $\bar{E}$.

The restriction $(Q_t)_{t \geq 0}$ of $(\bar{P}_t)_{t \geq 0}$ to $(E_D, \bar{\rho})$ is a right semigroup, and $E_D \setminus E$ is quasi-polar for any realization $Y$ of $(Q_t)_{t \geq 0}$ as a right process, that is, for every initial law $\mu$ on $(E_D, \bar{\rho})$ the $Q_\mu$-outer measure of $\{\omega : Y_t(\omega) \in E \text{ for all } t > 0\}$ is equal to one; see Sharpe (1988, p.193). The following theorem gives a complete characterization of probability entrance laws for a conservative right semigroup.

**Theorem A.37** (Sharpe, 1988, p.196) For every probability entrance law $(\eta_t)_{t \geq 0}$ for $(P_t)_{t \geq 0}$ on $E$, there is a unique probability measure $\eta_0$ on $\mathcal{B}^u(E_D, \bar{\rho})$ such that $\eta_t = \eta_0 \bar{P}_t$ for every $t > 0$.

**Corollary A.38** If $E$ is a locally compact separable metric space and $(P_t)_{t \geq 0}$ is a Feller semigroup on $E$, then all probability entrance laws for $(P_t)_{t \geq 0}$ are closable.

**Proof.** We assume $E$ is not compact, for otherwise the proof is easier. Let $\bar{E} = E \cup \{\partial\}$ be a one-point compactification of $E$. Then $\bar{E}$ is compact and separable, so it is metrizable. Let $\bar{d}$ be a metric on $\bar{E}$ compatible with its topology and let $d$ be the restriction of $\bar{d}$ to $E$. It is easy to see that the Ray–Knight completion of $E$ given by $d$ and $(P_t)_{t \geq 0}$ coincides with $\bar{E}$ and the entrance space is just $E$. Then the result follows from Theorem A.37. □
In the remainder of this section, let $E$ be a Lusin topological space and consider a Borel right semigroup $(P_t)_{t \geq 0}$ on $E$ which is not necessarily conservative. Let $E^0 = E \cup \{\partial\}$ be a topological extension of $E$ with $\partial$ being an isolated point. Let $\Omega$ denote the space of right continuous paths $w$ from $\mathbb{R}$ to $E^\partial$ such that there are constants $\alpha(w) < \beta(w)$ so that $w_t \in E$ for $t \in (\alpha(w), \beta(w))$ and $w_t = \partial$ for $t \in (\alpha(w), \beta(w))^c$. Let $(\mathcal{F}_t^0, \mathcal{F}_t^0)_{t \in \mathbb{R}}$ be the natural $\sigma$-algebras on $\Omega$ generated by the coordinate process. For $r \in [-\infty, \infty)$ let $\Omega_r$ denote the space of right continuous paths $w \in \Omega$ satisfying $\alpha(w) = r$.

**Theorem A.39** (Getoor and Glover, 1987, pp.57–58) Let $(\eta_t)_{t \in \mathbb{R}}$ be an entrance rule for $(P_t)_{t \geq 0}$. Then there exists a Radon measure $\rho(ds)$ on $\mathbb{R}$ and an entrance law $(\nu_t^r)_{t > r}$ for every $r \in [-\infty, \infty)$ so that

$$
\eta_t = \nu_t^{-\infty} + \int_{-\infty}^{\infty} \nu_t^s \rho(ds), \quad t \in \mathbb{R},
$$

(A.24)

where $\nu_t^s = 0$ for $t \leq s$ by convention.

**Theorem A.40** (Getoor and Glover, 1987, p.63) To each entrance rule $(\eta_t)_{t \in \mathbb{R}}$ for $(P_t)_{t \geq 0}$ there corresponds a unique $\sigma$-finite measure $Q_\eta$ on $(\Omega, \mathcal{F}_0^0)$ so that

$$
Q_\eta \{ w_{t_1} \in dx_1, w_{t_2} \in dx_2, \ldots, w_{t_n} \in dx_n \}
\quad = \eta_1(dx_1)P_{t_2-t_1}(x_1, dx_2)\cdots P_{t_n-t_{n-1}}(x_{n-1}, dx_n)
$$

(A.25)

for all $\{t_1 < \cdots < t_n\} \subset \mathbb{R}$ and $\{x_1, \ldots, x_n\} \subset E$. Moreover, if $(\eta_t)_{t \in \mathbb{R}}$ is an entrance law at $r \in \mathbb{R}$, then $Q_\eta$ is carried by $\Omega_r$.

The measure $Q_\eta$ defined by (A.25) is called the Kuznetsov measure corresponding to the entrance rule $(\eta_t)_{t \in \mathbb{R}}$; see Getoor and Glover (1987). This property roughly means that $\{w_t : t \in \mathbb{R}\}$ is a Markov process with transition semigroup $(P_t)_{t \geq 0}$ and one-dimensional distributions $(\eta_t)_{t \in \mathbb{R}}$. For the entrance rule $(\eta_t)_{t \in \mathbb{R}}$ given by (A.24), the measure $Q_\eta$ can be represented as

$$
Q_\eta(dw) = Q^{-\infty}(dw) + \int_{-\infty}^{\infty} Q^s(dw)\rho(ds),
$$

(A.26)

where $Q^s$ is the Kuznetsov measure corresponding to the entrance law $(\nu_t^s)_{t > s}$; see Getoor and Glover (1987, p.66). The theory of Kuznetsov measures was developed systematically in Dellacherie et al. (1992) and Getoor (1990).

**Example A.1** Let $(P_t)_{t \geq 0}$ be the transition semigroup of the absorbing-barrier Brownian motion in $(0, \infty)$. For any $t > 0$ the kernel $P_t(x, dy)$ has density

$$
p_t(x, y) = g_t(x - y) - g_t(x + y), \quad x, y > 0,
$$

(A.27)

where

$$
g_t(z) = \frac{1}{\sqrt{2\pi t}} \exp\{-z^2/2t\}, \quad t > 0, z \in \mathbb{R}.
$$

(A.28)
We can define an entrance law \((\kappa_t)_{t \geq 0}\) for \((P_t)_{t \geq 0}\) by

\[
\kappa_t(f) = \frac{2}{t} \int_0^\infty yg_t(y) f(y) dy = \frac{d}{dx} P_t f(0^+), \quad f \in b\mathcal{B}(0, \infty). \tag{A.29}
\]

The corresponding Kuznetsov measure \(\kappa(dw)\) is called Itô’s excursion law, which is carried by the set of positive continuous paths \(\{w_t : t > 0\}\) such that \(w_{0+} = w_t = 0\) for every \(t \geq \tau_0(w) := \inf\{s > 0 : w_s = 0\}\); see, e.g., Ikeda and Watanabe (1989, p.124).

### A.6 Concatenations and Weak Generators

Suppose that \(E\) is a Lusin topological space and \((P_t)_{t \geq 0}\) is a Borel right semigroup on this space. We consider a right process \(\xi = (\Omega, \mathcal{F}, \mathcal{G}_t, \xi_t, \theta_t, \mathbb{P}_x)\) with transition semigroup \((P_t)_{t \geq 0}\). Let \((\mathcal{F}, \mathcal{F}_t)\) be the augmentations of the \(\mathcal{B}(E)\)-natural \(\sigma\)-algebras \((\mathcal{F}^0_t, \mathcal{F}_t^0)\) generated by \(\{\xi_t : t \geq 0\}\). A right continuous \((\mathcal{F}_t)\)-adapted increasing process \(\{K(t) : t \geq 0\}\) is called an additive functional of \(\xi\) if \(K_0 = 0\) and for every bounded stopping time \(T\) we have a.s.

\[
K_{T+t} = K_T + K_t \circ \theta_T, \quad t \geq 0. \tag{A.30}
\]

Clearly, an additive functional \(\{K(t) : t \geq 0\}\) defines a \(\sigma\)-finite random measure \(K(ds)\) on \([0, \infty)\). For any \(\beta \in b\mathcal{B}(E)\) write

\[
K_t(\beta) = \int_{[0,t]} \beta(\xi_s) K(ds), \quad t \geq 0.
\]

Let \(b\mathcal{B}(K)\) denote the set of functions \(\beta \in b\mathcal{B}(E)\) so that \(t \mapsto e^{-K_t(\beta)}\) is a locally bounded stochastic process. Note that \(b\mathcal{B}(K) \supset pb\mathcal{B}(E)\). We say an additive functional \(\{K(t) : t \geq 0\}\) is admissible if each \(\omega \mapsto K_t(\omega)\) is measurable with respect to the natural \(\sigma\)-algebra \(\mathcal{F}_0^0\) and

\[
k(t) := \sup_{x \in E} \mathbb{P}_x[K(t)] \to 0, \quad t \to 0. \tag{A.31}
\]

In the sequel, we assume \(\{K(t) : t \geq 0\}\) is a continuous admissible additive functional of \(\xi\). Let \(b \in b\mathcal{B}(K)\) and let \(\gamma(x, dy)\) be a bounded Borel kernel on \(E\). For \(f \in b\mathcal{B}(E)\) we consider the linear evolution equation

\[
q_t(x) = \mathbb{P}_x\left[e^{-K_t(b)} f(\xi_t)\right] + \mathbb{P}_x \left[\int_0^t e^{-K_s(b)} \gamma(\xi_s, q_{t-s}) K(ds)\right], \tag{A.32}
\]

where \(t \geq 0\) and \(x \in E\). Recall that \(\| \cdot \|\) denotes the supremum norm.

**Proposition A.41** For every \(f \in b\mathcal{B}(E)\) there is a unique locally bounded Borel function \((t, x) \mapsto q_t(x)\) on \([0, \infty) \times E\) solving (A.32), which is given by
\[ q_t(x) = P_x\left[ e^{-K_t(b)} f(\xi_t) \right] + P_x \left\{ \int_0^t e^{-K_s(b)} K(ds_1) P_{\mu_{s_1}} f(\xi_{t-s_1}) \right\} + \sum_{i=2}^{\infty} P_x \left\{ \int_0^t e^{-K_{s_i}(b)} K(ds_1) P_{\mu_{s_i-1}} \left\{ \int_0^{t-s_i} e^{-K_{s_{i-1}}(b)} K(ds_2) \cdots \right\} \right\}, \]

where \( \sigma_i = \sum_{j=1}^i s_j \) and \( \mu_s = \gamma(\xi_s, \cdot) \). Moreover, the operators \( \pi_t : f \mapsto q_t \) form a locally bounded semigroup \((\pi_t)_{t \geq 0}\).

**Proof.** For \( r \geq 0 \) it is not hard to see that \((t, x) \mapsto q_t(x)\) satisfies (A.32) for \( t \geq 0 \) if and only if it satisfies the equation for \( 0 \leq t \leq r \) and \((t, x) \mapsto q_{r+t}(x)\) satisfies

\[ q_{r+t}(x) = P_x\left[ e^{-K_{r+t}(b)} f(\xi_t) \right] + P_x \left[ \int_0^t e^{-K_{r+s}(b)} \gamma(\xi_s, \xi_{r+t-s}) K(ds) \right] \quad \text{(A.33)} \]

for \( t \geq 0 \). Let \( t \mapsto l(t) \) be an increasing deterministic function so that \( e^{-K_t(b)} \leq l(t) \) for all \( t \geq 0 \). Fix a constant \( \delta > 0 \) so that \( k(\delta)(l(\delta)||\gamma(\cdot, 1)|| < 1 \). Observe that the \( i \)-th term of the series in the definition of \( q_t(x) \) is bounded by \( k(t)^i l(t)^i ||\gamma(\cdot, 1)||^i ||f|| \). Then the series converges uniformly on \([0, \delta] \times E\). Since each \( \omega \mapsto K_t(\omega) \) is measurable with respect to the natural \( \sigma \)-algebra, it is easy to see that \((t, x) \mapsto q_t(x)\) is jointly measurable and satisfies (A.32) on \([0, \delta] \times E\). By the relation of (A.32) and (A.33) we can extend \((t, x) \mapsto q_t(x)\) to a solution of (A.32) on \([0, \infty) \times E\). Moreover, the operator \( f \mapsto q_t \) determines a bounded Borel kernel \( \pi_t(x, dy) \) on \( E \) and \((\pi_t)_{t \geq 0}\) form a locally bounded semigroup. To show the uniqueness of the solution of (A.32), suppose that \((t, x) \mapsto v_t(x)\) is a locally bounded solution of (A.32) with \( v_0(x) \equiv 0 \). It is easily seen that

\[ ||v_t|| \leq l(t) ||\gamma(\cdot, 1)|| \sup_{x \in E} \left[ \int_0^t ||v_{t-s}|| K(ds) \right], \]

and hence

\[ \sup_{0 \leq s \leq t} ||v_s|| \leq k(t) l(t) ||\gamma(\cdot, 1)|| \sup_{0 \leq s \leq t} ||v_s|| \]

for every \( t \geq 0 \). Then we must have \( ||v_t|| = 0 \) for \( 0 \leq t \leq \delta \). Using the above procedure and the relation of (A.32) and (A.33) successively we get \( ||v_t|| = 0 \) for all \( t \geq 0 \). Since (A.32) is a linear equation, that gives the uniqueness of the solution. \( \square \)

**Proposition A.42** Let \( f \in \mathfrak{B}(E) \) and let \((t, x) \mapsto \pi_t f(x)\) be defined by (A.32). Then \( t \mapsto \pi_t f(x) \) is right continuous pointwise on \( E \) if and only if so is \( t \mapsto P_t f(x) \).

**Proof.** Clearly, the second term on the right-hand side of (A.32) tends to zero as \( t \to 0 \). Moreover, by (A.31) we have

\[ \lim_{t \to 0} \left| P_x\left[ (1 - e^{-K_t(b)}) f(\xi_t) \right] \right| \leq \lim_{t \to 0} \|f\| P_x\left[ |1 - e^{-K_t(b)}| \right] = 0. \]
strictly positive multiplicative functional

Let

Then Proposition A.30 implies

It follows that

which means if one of the limits exists, so do the other two and the equalities hold. Then we get the result by the semigroup properties of \((P_t)_{t \geq 0}\) and \((\pi_t)_{t \geq 0}\).

Now suppose that \(b(x) \geq \gamma(x, 1)\) for every \(x \in E\). Let \((\pi_t)_{t \geq 0}\) be defined by (A.32). Since \((P_t)_{t \geq 0}\) is not conservative in general, we can only understand \(\xi = (\Omega, \mathcal{G}, \mathcal{G}_t, \xi_t, \mathbb{P}_x)\) as a right process in the extended state space \(E \cup \{\partial\}\) with \(\partial\) being an isolated cemetery. Let \(\bar{E}\) be a Ray–Knight completion of \(E \cup \{\partial\}\) relative to \(\xi\). Then Proposition A.30 implies \(E \in \mathcal{B}(\bar{E})\). By Theorem A.32 we have \(\mathbb{P}_x\{\text{the left limit } \xi_{t-} := \lim_{s \rightarrow t-} \xi_s \text{ taken in the Ray topology exists in } \bar{E} \text{ for all } t > 0\} = 1\). Let \(\hat{\gamma}(x, dy)\) be a sub-Markov kernel on \(E\) satisfying \(\gamma(x, dy) = b(x)\hat{\gamma}(x, dy)\). We extend \(\hat{\gamma}(x, dy)\) to a Markov kernel from \(E\) to \(\bar{E}\) by setting \(\hat{\gamma}(x, \{\partial\}) = 1 - \hat{\gamma}(x, E)\). Fix \(x_0 \in E\) and let \(b(x) = b(x_0)\) and \(\hat{\gamma}(x, \cdot) = \hat{\gamma}(x_0, \cdot)\) for \(x \in \bar{E}\ \setminus E\). Let \(\hat{\xi} = (\Omega, \mathcal{G}, \mathcal{G}_t, \hat{\xi}_t, \mathbb{P}_x)\) be the subprocess with lifetime \(\zeta\) constructed from \(\xi\) and the strictly positive multiplicative functional \(t \mapsto \exp\{-K_t(b)\}\). Then \(\hat{\xi}\) is also a right process; see Sharpe (1988, p.287). Let \(\hat{\xi} = (\Omega, \mathcal{G}, \mathcal{G}_t, \hat{\xi}_t, \mathbb{P}_x)\) be the concatenation defined from an infinite sequence of copies of \(\hat{\xi}\) and the transfer kernel \(\eta(\omega, dy) := \hat{\gamma}(\xi(\omega), \omega, dy)\) as in Sharpe (1988, p.82). The intuitive idea of this concatenation is described as follows. The process \(\hat{\xi}\) evolves as \(\xi\) until time \(\zeta\), it is then revived by means of the kernel \(\eta\), and evolves again as \(\xi\) and so on. It is known that \(\hat{\xi}\) is also a right process; see Sharpe (1988, p.79 and p.82). Suppose that every \(f \in b\mathcal{B}(E)\) is extended trivially to \(\bar{E}\ \setminus E\). Then we have the renewal equation

where the expectations of \(\xi_t\) or \(\bar{\omega}\) are taken with respect to \(\mathbb{P}_x\) or \(\mathbb{P}_{\eta(\omega, \cdot)}\) and those of \(\omega\) are taken with respect to \(\mathbb{P}_x\). By Sharpe (1988, p.210), we have \(\mathbb{P}_x\{\text{the path } t \mapsto \xi_t \text{ has at most countably many jumps}\} = 1\). Let \((\bar{P}_t)_{t \geq 0}\) denote the transition semigroup of \(\hat{\xi}\). The above equation can be rewritten as

Then \((t, x) \mapsto \bar{P}_t f(x)\) is a solution of (A.32). Since the processes \(\xi\) and \(\hat{\xi}\) coincide during the time interval \([0, \zeta]\), they induce identical fine topologies on \(E\).

**Theorem A.43** If \(b(x) \geq \gamma(x, 1)\) for every \(x \in E\), then the semigroup \((\pi_t)_{t \geq 0}\) defined by (A.32) is a right semigroup which induces the same fine topology on \(E\) as \((P_t)_{t \geq 0}\). Moreover, if \((P_t)_{t \geq 0}\) has a Hunt realization, so does the semigroup \((\pi_t)_{t \geq 0}\).

**Proof.** The first assertion follows from the arguments given above and the uniqueness of the solution of (A.32). Since \(t \mapsto \exp\{-K_t(b)\}\) is continuous and strictly
positive, the lifetime \( \zeta \) of the subprocess \( \tilde{\zeta} \) is totally inaccessible. Then \( \tilde{\zeta} \) is a Hunt process if so is \( \xi \). That proves the second assertion.

Let \( b\mathcal{C}_\xi(E) \) be the set of functions \( f \in b\mathcal{B}(E) \) that are finely continuous relative to \( \xi \). Theorem A.20 implies that \( t \mapsto P_t f(x) \) is right continuous pointwise for every \( f \in b\mathcal{C}_\xi(E) \). Let \( (U^\alpha)_{\alpha > 0} \) denote the resolvent of \( \xi \).

**Lemma A.44** The set of functions \( U^\beta b\mathcal{C}_\xi(E) \) is independent of \( \beta > 0 \). Moreover, if \( g_1, g_2 \in b\mathcal{C}_\xi(E) \) and \( U^\beta g_1 = U^\beta g_2 \) for some \( \beta > 0 \), then \( g_1 = g_2 \).

**Proof.** Let us consider two constants \( \alpha, \beta > 0 \). If \( f \in U^\beta b\mathcal{C}_\xi(E) \), we have \( f = U^\beta g \) for some \( g \in b\mathcal{C}_\xi(E) \). Then the resolvent equation implies that

\[
f = U^\alpha g - (\beta - \alpha)U^\alpha U^\beta g = U^\alpha h,
\]

where \( h = g - (\beta - \alpha)U^\beta g \in b\mathcal{C}_\xi(E) \). It follows that \( U^\beta b\mathcal{C}_\xi(E) \subset U^\alpha b\mathcal{C}_\xi(E) \).

By symmetry we have \( U^\alpha b\mathcal{C}_\xi(E) \subset U^\beta b\mathcal{C}_\xi(E) \). That proves the first assertion. Suppose that \( g_1, g_2 \in b\mathcal{C}_\xi(E) \) and \( U^\beta g_1 = U^\beta g_2 \) for some \( \beta > 0 \). By the resolvent equation we have \( U^\alpha g_1 = U^\alpha g_2 \) for every \( \alpha > 0 \). Since \( t \mapsto P_t g_1(x) \) and \( t \mapsto P_t g_2(x) \) are right continuous, we have \( g_1 = g_2 \) by the uniqueness of Laplace transforms. \( \square \)

Fix \( \beta > 0 \) and let \( \mathcal{D}(A) = U^\beta b\mathcal{C}_\xi(E) \). For \( f = U^\beta g \in \mathcal{D}(A) \) with \( g \in b\mathcal{C}_\xi(E) \) set \( Af = \beta f - g \), which is well-defined by Lemma A.44. We call \( (A, \mathcal{D}(A)) \) the weak generator of \( (P_t)_{t \geq 0} \). By the resolvent equation of \( (U^\alpha)_{\alpha > 0} \) it is easy to show that \( (A, \mathcal{D}(A)) \) is independent of the choice of \( \beta > 0 \). We can also define a multi-valued version of the weak generator following Ethier and Kurtz (1986). Let \( \tilde{\mathcal{D}}(\tilde{A}) = U^\beta b\mathcal{B}(E) \) and for any \( f \in \tilde{\mathcal{D}}(\tilde{A}) \) let \( \tilde{A} f = \{ \beta f - g : g \in b\mathcal{B}(E) \} \). It is easy to show that \( (\tilde{A}, \tilde{\mathcal{D}}(\tilde{A})) \) is also independent of the choice of \( \beta > 0 \). In particular, for any \( f \in \tilde{\mathcal{D}}(\tilde{A}) \) we have \( Af \in \tilde{A} f \).

**Proposition A.45** Let \( \alpha > 0 \). Then \( U^\alpha(\alpha - A)f = f \) for every \( f \in \mathcal{D}(A) \) and \( (\alpha - A)U^\alpha f = f \) for every \( f \in b\mathcal{C}_\xi(E) \).

**Proof.** For any \( f \in \mathcal{D}(A) \) there is \( g \in b\mathcal{C}_\xi(E) \) so that \( f = U^\beta g \). Then the definition of \( Af \) and the resolvent equation yields

\[
U^\alpha(\alpha - A)f = U^\alpha(\alpha f - \beta f + g) = (\alpha - \beta)U^\alpha U^\beta g + U^\alpha g = U^\beta g = f,
\]

giving the first assertion. For any \( f \in b\mathcal{C}_\xi(E) \) we first use the resolvent equation to see

\[
U^\alpha f = U^\beta f + (\beta - \alpha)U^\beta U^\alpha f = U^\beta h,
\]

where \( h = f + (\beta - \alpha)U^\alpha f \). Therefore

\[
(\alpha - A)U^\alpha f = \alpha U^\alpha f - AU^\beta h = \alpha U^\alpha f - \beta U^\beta h + h = f.
\]

That gives the second assertion. \( \square \)
Theorem A.46 Let \((A, \mathcal{D}(A))\) be the weak generator of \((P_t)_{t \geq 0}\). Then for \(f \in \mathcal{D}(A)\) we have

\[
P_tf(x) = f(x) + \int_0^t P_sAf(x)ds, \quad t \geq 0, x \in E. \tag{A.34}
\]

Proof. Suppose that \(f = U^\beta g\) for \(g \in bC_\xi(E)\). Then \(U^\alpha Af = \alpha U^\alpha f - f\) for \(\alpha > 0\) by Proposition A.45. Using this relation it is easy to show

\[
\int_0^{\infty} e^{-\alpha t} dt \int_0^t P_sAf(x)ds = \int_0^\infty e^{-\alpha t}(P_tf - f)(x)dt.
\]

Since \(f\) is finely continuous relative to \(\xi\), the function \(t \mapsto P_tf(x)\) is right continuous for every \(x \in E\). Then (A.34) follows by the uniqueness of Laplace transforms. \(\square\)

Corollary A.47 Let \((A, \mathcal{D}(A))\) be the weak generator of \((P_t)_{t \geq 0}\). Then for \(f \in \mathcal{D}(A)\) we have

\[
Af(x) = \lim_{t \to 0} \frac{1}{t} [P_tf(x) - f(x)], \quad x \in E. \tag{A.35}
\]

Proof. Since \(Af \in bC_\xi(E)\), we have (A.35) from (A.34). \(\square\)

In the remainder of this section we consider the semigroup \((\pi_t)_{t \geq 0}\) defined by (A.32) in the special case with \(K(ds) = ds\) being the Lebesgue measure. Given a function \(b \in bB(E)\), we define a locally bounded semigroup of Borel kernels \((P^b_t)_{t \geq 0}\) on \(E\) by the following Feynman–Kac formula:

\[
P^b_tf(x) = P_x \left[ e^{-\int_0^t b(\xi_s)ds} f(\xi_t) \right], \quad x \in E, f \in bB(E). \tag{A.36}
\]

Then we can rewrite (A.32) into

\[
\pi_tf(x) = P^b_tf(x) + \int_0^t P^b_{t-s} \gamma \pi_s f(x)ds, \quad t \geq 0, x \in E. \tag{A.37}
\]

Lemma A.48 (Gronwall’s inequality) Suppose that \(t \mapsto g(t) \geq 0\) and \(t \mapsto h(t)\) are integrable functions on the interval \([0, T]\). If there is a constant \(C > 0\) such that

\[
g(t) \leq h(t) + C \int_0^t g(s)ds, \quad 0 \leq t \leq T, \tag{A.38}
\]

then

\[
g(t) \leq h(t) + C \int_0^t e^{C(t-s)}h(s)ds, \quad 0 \leq t \leq T. \tag{A.39}
\]

Proof. Let \(f(t)\) denote the right-hand side of (A.39). By integration by parts,
\[
\int_0^t f(s) ds = \int_0^t h(s) ds + C \int_0^t \left[ e^{Cu} \int_0^u e^{-Cs} h(s) ds \right] du
\]
\[
= \int_0^t h(s) ds + e^{Ct} \int_0^t e^{-Cs} h(s) ds - \int_0^t h(s) ds
\]
\[
= \int_0^t e^{C(t-s)} h(s) ds.
\]

It follows that
\[
f(t) = h(t) + C \int_0^t f(s) ds, \quad 0 \leq t \leq T. \tag{A.40}
\]

Let \( \Delta(t) = f(t) - g(t) \). From (A.38) and (A.40) we have
\[
\Delta(t) \geq C \int_0^t \Delta(s) ds \geq C^2 \int_0^t ds \int_0^s \Delta(r) dr = C^2 \int_0^t (t-r) \Delta(r) dr
\]
\[
\geq C^3 \int_0^t (t-r) dr \int_0^r \Delta(s) ds = \frac{C^3}{2} \int_0^t (t-s)^2 \Delta(s) ds
\]
\[
\geq \cdots
\]
\[
\geq \frac{C^n}{(n-1)!} \int_0^t (t-s)^{n-1} \Delta(s) ds.
\]
The right-hand side goes to zero as \( n \to \infty \). Then \( \Delta(t) \geq 0 \) and (A.39) follows. \( \square \)

**Proposition A.49** For any \( f \in b\mathcal{B}(E) \) the solution to (A.37) is also the unique locally bounded solution to
\[
\pi_t f(x) = P_t f(x) + \int_0^t P_{t-s}(\gamma - b) \pi_s f(x) ds, \quad t \geq 0, x \in E. \tag{A.41}
\]

Moreover, we have \( \| \pi_t f \| \leq \| f \| e^{c_0 t} \) for all \( t \geq 0 \), where \( c_0 = \| b^- \| + \| \gamma(\cdot,1) \| \) and \( b^- = 0 \lor (-b) \).

**Proof.** Let \((t,x) \mapsto \pi_t f(x)\) be the unique locally bounded solution of (A.37). We can use the Markov property of \( \xi \) and Fubini’s theorem to write
\[
\int_0^t ds \int_E b(y) P_s^b f(y) P_{t-s}(x,dy)
\]
\[
= \int_0^t P_x \left\{ b(\xi_{t-s}) P_{\xi_{t-s}} \left[ e^{-\int_0^{t-s} b(\xi_u) du} f(\xi_s) \right] \right\} ds
\]
\[
= \int_0^t P_x \left[ b(\xi_{t-s}) e^{-\int_0^{t-s} b(\xi_u) du} f(\xi_t) \right] ds
\]
\[
= P_x \left[ f(\xi_t) \int_0^t b(\xi_{t-s}) e^{-\int_0^{t-s} b(\xi_u) du} ds \right]
\]
\[
= P_x \left\{ 1 - e^{-\int_0^t b(\xi_u) du} \right\} f(\xi_t)
\]
\[
= P_t f(x) - P_t^b f(x).
\]

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By similar calculations,
\[
\int_0^t ds \int_E \left[ b(y) \int_0^s P_{s-r}^b \gamma_{r} f(y) \, dr \right] P_{t-s}(x, dy) \\
= \int_0^t ds \int_0^s E \{ b(\xi_{t-s}) P_\xi \left[ e^{-\int_0^{s-r} b(\xi_u) \, du} \gamma_r f(\xi_{s-r}) \right] \} \, dr \\
= \int_0^t ds \int_0^s P_x \left[ \int_0^t \gamma_r f(\xi_t) \, dr \int_{t-r}^t b(\xi_t-r) e^{-\int_0^{t-r} b(\xi_u) \, du} \, ds \right] \\
= \int_0^t P_{t-r} \gamma_r f(x) \, dr - \int_0^t P_{t-r}^b \gamma_r f(x) \, dr.
\]

Then we can add up the two equations and use (A.37) to get (A.41). For any solution \((t, x) \mapsto \pi_t f(x)\) of (A.41) we have
\[
\|\pi_t f\| \leq \|f\| + c_0 \int_0^t \|\pi_{t-s} f\| \, ds = \|f\| + c_0 \int_0^t \|\pi_s f\| \, ds.
\]

Then Gronwall’s inequality implies \(\|\pi_t f\| \leq \|f\| e^{c_0 t}\). That gives the uniqueness of the solution since (A.41) is a linear equation.

Now we prove some analytic properties of the semigroup \((\pi_t)_{t \geq 0}\) defined by (A.37) or (A.41). By Proposition A.49 we have \(\|\pi_t f\| \leq \|f\| e^{c_0 t}\) for \(t \geq 0\) and \(f \in \mathfrak{b}(E)\). Then we can define the operators \((R^\alpha)_{\alpha > c_0}\) on \(\mathfrak{b}(E)\) by
\[
R^\alpha f(x) = \int_0^\infty e^{-\alpha t} \pi_t f(x) \, dt, \quad x \in E, \ f \in \mathfrak{b}(E).
\]

**Proposition A.50** *For every \(\alpha > c_0\) and \(f \in \mathfrak{b}(E)\) we have*
\[
R^\alpha f(x) = U^\alpha f(x) + U^\alpha (\gamma - b) R^\alpha f(x), \quad x \in E.
\]

**Proof.** By taking the Laplace transforms of both sides of (A.41) we have
\[
R^\alpha f(x) = U^\alpha f(x) + \int_0^\infty e^{-\alpha t} \int_0^t P_{t-s} (\gamma - b) \pi_s f(x) \, ds \\
= U^\alpha f(x) + \int_0^\infty ds \int_0^\infty e^{-\alpha t} P_{t-s} (\gamma - b) \pi_s f(x) \, dt \\
= U^\alpha f(x) + \int_0^\infty e^{-\alpha s} U^\alpha (\gamma - b) \pi_s f(x) \, ds \\
= U^\alpha f(x) + U^\alpha (\gamma - b) R^\alpha f(x).
\]
That proves the desired equation.
Proposition A.51 Let \( f \in b\mathcal{B}(E) \). Then we have \( f \in (\alpha - \tilde{A})U^\alpha f \) for \( \alpha > 0 \) and \( f \in (\alpha - \tilde{A} - \gamma + b)R^\alpha f \) for \( \alpha > c_0 \).

Proof. Let \( h = f + (\beta - \alpha)U^\alpha f \). By the resolvent equation we have \( U^\alpha f = U^\beta h \). Then the definition of \( \tilde{A} \) implies \( f = (\alpha - \beta)U^\alpha f + h \in \{ \alpha U^\alpha f - \beta U^\alpha f + g : g \in b\mathcal{B}(E) \} \) and \( U^\beta g = U^\alpha f \), which gives the first assertion. For \( \alpha > c_0 \) we get from Proposition A.50 that

\[
(\alpha - \tilde{A} - \gamma + b)R^\alpha f = (\alpha - \tilde{A})R^\alpha f - (\gamma - b)R^\alpha f
= (\alpha - \tilde{A})U^\alpha [f + (\gamma - b)R^\alpha f] - (\gamma - b)R^\alpha f.
\]

By the first assertion, the set represented by the first term on the right-hand side includes \( f + (\gamma - b)R^\alpha f \). Then \((\alpha - \tilde{A} - \gamma + b)R^\alpha f\) includes \( f \), proving the second assertion. \( \square \)

Lemma A.52 Let \( \alpha > c_0 \) and \( f \in \mathcal{D}(\tilde{A}) \). Then for any \( h \in (\alpha - \tilde{A} - \gamma + b)f \) we have \( \|h\| \geq (\alpha - c_0)\|f\| \).

Proof. For \( h \in (\alpha - \tilde{A} - \gamma + b)f \) we have \( \alpha f - \gamma f + bf - h \in \tilde{A} f \). By the definition of \( \tilde{A} \), there exist \( g \in b\mathcal{B}(E) \) so that \( U^\alpha g = f \) and \( \alpha f - \gamma f + bf - h = \alpha f - g \). Therefore \( \|f\| \leq \alpha^{-1}\|g\| \) and \( h = g - \gamma f + bf \). It follows that \( \|h\| \geq \|g\| - \|(\gamma + b^-)f\| \geq (\alpha - c_0)\|f\| \). \( \square \)

Lemma A.53 If \( f_1 \) and \( f_2 \) are distinct functions from \( \mathcal{D}(\tilde{A}) \), then for any \( \alpha > c_0 \) the intersection \((\alpha - \tilde{A} - \gamma + b)f_1 \cap (\alpha - \tilde{A} - \gamma + b)f_2 \) is empty.

Proof. Suppose that \( h \in (\alpha - \tilde{A} - \gamma + b)f_1 \cap (\alpha - \tilde{A} - \gamma + b)f_2 \). Then there exist \( h_1 \in \tilde{A} f_1 \) and \( h_2 \in \tilde{A} f_2 \) so that \( h = (\alpha - \gamma + b)f_1 - h_1 = (\alpha - \gamma + b)f_2 - h_2 \). By the definition of \( \tilde{A} \), there exist \( g_1, g_2 \in b\mathcal{B}(E) \) so that \( f_i = U^\beta g_i \) and \( h_i = \beta f_i - g_i \) for \( i = 1 \) and \( 2 \). It follows that \( (f_2 - f_1) = U^\beta (g_2 - g_1) \) and \( h_2 - h_1 = \beta (f_2 - f_1) - (g_2 - g_1) \). Those imply \( h_2 - h_1 \in \tilde{A} (f_2 - f_1) \), and so

\[
0 = (\alpha - \gamma + b)(f_2 - f_1) - (h_2 - h_1) \in (\alpha - \gamma + b - \tilde{A})(f_2 - f_1),
\]

which is in contradiction to Lemma A.52. \( \square \)

Lemma A.54 For any \( \alpha > c_0 \) and \( f \in \mathcal{D}(A) \) we have \( f = R^\alpha (\alpha - A - \gamma + b)f \).

Proof. Clearly, for \( f \in \mathcal{D}(A) \) the set \((\alpha - \tilde{A} - \gamma + b)f\) contains the function \( h := (\alpha - A - \gamma + b)f \). By Proposition A.51 we have \( h \in (\alpha - \tilde{A} - \gamma + b)R^\alpha h \). Then the sets \((\alpha - \tilde{A} - \gamma + b)f\) and \((\alpha - \tilde{A} - \gamma + b)R^\alpha h\) have a non-empty intersection. Thus Lemma A.53 implies that \( f = R^\alpha h \). \( \square \)

Theorem A.55 Let \( f \in \mathcal{D}(A) \) and let \( (t, x) \mapsto \pi_t f(x) \) be defined by (A.37) or (A.41). Then we have

\[
\pi_t f(x) = f(x) + \int_0^t \pi_s(A + \gamma - b)f(x)ds, \quad t \geq 0, x \in E.
\]
**Proof.** By Theorems A.16 and A.20, any $f \in \mathcal{D}(A)$ is finely continuous relative to $\xi$, so $t \mapsto P_t f(x)$ is right continuous pointwise. Then Proposition A.42 implies $t \mapsto \pi_t f(x)$ is right continuous pointwise. By integration by parts it is easy to show
\[
\int_0^\infty e^{-\alpha t} dt \int_0^t \pi_s(A + \gamma - b) f ds = \frac{1}{\alpha} R^{\alpha}(A + \gamma - b)f.
\]
Using Lemma A.54 one can see the above value is equal to
\[
R^{\alpha} f - \frac{1}{\alpha} f = \int_0^\infty e^{-\alpha t} (\pi_t f - f) dt.
\]
Then the desired equation follows by the uniqueness of the Laplace transform. \qed

### A.7 Time–Space Processes

In this section we discuss briefly time–space processes associated with inhomogeneous Markov processes. For simplicity we only consider those processes with Borel transition semigroups. Suppose that $I \subset \mathbb{R}$ is an interval and $F$ is a Lusin topological space. Let $\tilde{E}$ be a Borel subset of $I \times F$. For $t \in I$ let $E_t = \{x \in F : (t, x) \in \tilde{E}\}$. Then each $E_t$ is a Lusin topological space. We fix an abstract point $\partial \notin I \times F$ and assume all functions on $\tilde{E} \cup \{\partial\}$ have been extended trivially to $\tilde{E}$. Suppose that for each pair $r \leq t \in I$ there is a Markov kernel $P_{r,t}$ from $(E_r, \mathcal{B}(E_r))$ to $(E_t, \mathcal{B}(E_t))$. The family $(P_{r,t} : r \leq t \in I)$ is called an inhomogeneous transition semigroup with global state space $\tilde{E}$ if it satisfies the Chapman–Kolmogorov equation
\[
P_{r,t}(x, B) = \int_{E_s} P_{r,s}(x, dy) P_{s,t}(y, B)
\]
for all $r \leq s \leq t \in I$, $x \in E_r$ and $B \in \mathcal{B}(E_t)$. In this work, we assume for every $f \in b\mathcal{B}(\tilde{E})$ the function
\[
(r, x, t) \mapsto 1_{\{r \leq t\}} \int_{E_t} f(t, y) P_{r,t}(x, dy)
\]
is measurable with respect to the $\sigma$-algebra $\mathcal{B}(\tilde{E} \times I)$.

**Definition A.56** The collection $\xi = (\Omega, \mathcal{G}, \mathcal{G}_{r,t}, \xi_t, P_{r,x})$ is called an inhomogeneous Markov process with global state space $\tilde{E}$ and transition semigroup $(P_{r,t} : r \leq t \in I)$ if the following conditions are satisfied:

1. For every $r \in I$, $(\Omega, \mathcal{G}, \mathcal{G}_{r,t} : t \in I \cap [r, \infty))$ is a filtered measurable space so that $\mathcal{G}_{s,t} \subset \mathcal{G}_{r,u}$ for $r \leq s \leq t \leq u \in I$. 


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Let $(\tilde{\eta}, \mathcal{B}(E))$ be an inhomogeneous time–space process in $E$

Proof.

Let $r, x \in \tilde{E}$, $P_{r,x}$ is a probability measure on $(\Omega, \mathcal{G})$ such that for every $H \in \mathcal{B}$ the function $(r, x) \mapsto P_{r,x}(H)$ is $\mathcal{B}(\tilde{E})$-measurable.

For every $(r, x) \in \tilde{E}$ we have $P_{r,x}\{\xi_t = x\} = 1$ and the following simple Markov property holds:

\[ P_{r,x}[f(\xi_t)|\mathcal{G}_s] = P_{s,t}f(\xi_s), \quad r \leq s \leq t \in I, f \in b\mathcal{B}(E_t). \]

We say $\xi$ is right continuous if $t \mapsto \xi_t(\omega)$ is right continuous for every $\omega \in \Omega$.

Given an inhomogeneous transition semigroup $(P_{r,t} : r \leq t \in I)$ with global state space $\tilde{E}$, we can define a homogeneous Borel transition semigroup $(\tilde{P}_t)_{t \geq 0}$ on $\tilde{E}$ by

\[ \tilde{P}_tf(r, x) = 1_t(r + t) \int_{E_{r+t}} f(r + t, y)P_{r,r+t}(x, dy), \quad \text{(A.44)} \]

where $t \geq 0$, $(r, x) \in \tilde{E}$ and $f \in b\mathcal{B}(\tilde{E})$. We call $(\tilde{P}_t)_{t \geq 0}$ the time–space semigroup associated with $(P_{r,t} : r \leq t \in I)$. Suppose that $\xi = (\Omega, \mathcal{G}, \mathcal{G}_{r,t}, \xi_t, P_{r,x})$ is a right continuous inhomogeneous Markov process with transition semigroup $(P_{r,t} : r \leq t \in I)$. Let $\tilde{\Omega} = I \times \Omega$. For $(v, \omega) \in \tilde{\Omega}$ define

\[ \tilde{\xi}_t(v, \omega) = \begin{cases} (v + t, \xi_{v+t}(\omega)) & \text{if } t \geq 0 \text{ and } v + t \in I, \\ \partial & \text{if } t \geq 0 \text{ and } v + t \notin I. \end{cases} \quad \text{(A.45)} \]

Let $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t)$ be the $\mathcal{B}(\tilde{E})$-natural $\sigma$-algebras generated by \{\tilde{\xi}_t : t \geq 0\}. For $(r, x) \in \tilde{E}$ let $\tilde{P}_{r,x}$ be the probability measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ induced by $P_{r,x}$ via the mapping $\omega \mapsto (r, \omega)$.

**Theorem A.57** The system $\tilde{\xi} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\xi}_t, \tilde{P}_{r,x})$ is a right continuous Markov process in $\tilde{E}$ with transition semigroup $(\tilde{P}_t)_{t \geq 0}$.

**Proof.** Let $(r, x) \in \tilde{E}$ and $t \geq s \geq 0$. Let $f = f_v(x) = f(v, x)$ be a bounded Borel function on $\tilde{E}$. Since $\tilde{P}_{r,x}$ is concentrated on $\{r\} \times \Omega$, if $r + t \in I$, we have

\[ \tilde{P}_{r,x}[f(\tilde{\xi}_t)|\tilde{\mathcal{F}}_s] = \tilde{P}_{r,x}[f(r + t, \xi_{r+t})|\sigma(\{\xi_{r+u} : 0 \leq u \leq s\})] \\
= \tilde{P}_{r,x}[f(r + t, \xi_{r+t})|\mathcal{G}_{r,r+s}] = P_{r+s,r+t}(\xi_{r+s}) \\
= \tilde{P}_{t-s}f(r + s, \xi_{r+s}) = \tilde{P}_{t-s}f(\xi_s). \]

If $r + t \notin I$, both sides of the above equality are equal to zero. Then $\tilde{\xi}$ is a Markov process with transition semigroup $(\tilde{P}_t)_{t \geq 0}$.

We call $\tilde{\xi} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\xi}_t, \tilde{P}_{r,x})$ the time–space process of $\xi$. By Theorem A.57, the study of the inhomogeneous process $\xi$ can be reduced to that of the homogeneous time–space process $\tilde{\xi}$. If $\tilde{\xi}$ has a right realization, we call $(P_{r,t} : r \leq t \in I)$
an inhomogeneous right transition semigroup. The following theorem shows that the terminology is consistent with that in the homogeneous case.

**Theorem A.58** Suppose that $\tilde{E} = [0, \infty) \times E$ for a Lusin topological space $E$ and there is a homogeneous Borel transition semigroup $(P_t)_{t \geq 0}$ on $E$ so that $P_{r,t} = P_{t-r}$ for $t \geq r \geq 0$. Then $(\tilde{P}_t)_{t \geq 0}$ is a right semigroup if and only if $(P_t)_{t \geq 0}$ is a right semigroup.

**Proof.** Suppose that $(\tilde{P}_t)_{t \geq 0}$ is a right semigroup. Let $\xi = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, (\alpha_t, \xi_t), \tilde{P}_r,x)$ be a right realization of $(\tilde{P}_t)_{t \geq 0}$. One can use Theorem A.16 to see that $\xi = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \eta_t, \tilde{P}_{0,x})$ is a right process with transition semigroup $(\tilde{P}_t)_{t \geq 0}$. Then $(\tilde{P}_t)_{t \geq 0}$ is a Borel right semigroup. The converse was obtained in Sharpe (1988, p.86).

Starting from a realization of the corresponding time–space process we can also reconstruct the inhomogeneous process with transition semigroup $(P_{r,t} : t \geq r \in I)$. For this purpose, let us consider a right continuous realization $\tilde{\xi} = (\tilde{\Omega}, \tilde{\mathcal{F}}_r \times \tilde{\mathcal{F}}_t, (\alpha_t, y_t), \tilde{P}_{r,x})$ of $(\tilde{P}_t)_{t \geq 0}$. In view of (A.44) we have $\alpha_t = \alpha_0 + t$ for $t \geq 0$. For $\omega \in \Omega$ define

$$\xi_t(\omega) = \begin{cases} y_{t-\alpha_0(\omega)}(\omega) & \text{if } t \in I \cap [\alpha_0(\omega), \infty), \\ \infty & \text{if } t \in I \cap (-\infty, \alpha_0(\omega)). \end{cases}$$

(A.46)

Let $\mathcal{F} = \sigma(\{\xi_t : t \in I\})$ and let $\mathcal{F}_{r,t} = \sigma(\{\xi_s : r \leq s \leq t\})$ for $t \geq r \in I$.

**Theorem A.59** The system $\xi = (\Omega, \mathcal{F}_{r,t}, \xi_t, \mathcal{P}_{r,x})$ is a right continuous inhomogeneous Markov process with transition semigroup $(P_{r,t} : t \geq r \in I)$.

**Proof.** Let $(r, x) \in \tilde{E}$ and $r \leq s \leq t \in I$. Since $P_{r,x}\{\alpha_0 = r\} = 1$, for any $f \in \mathcal{B}(E_t)$ we have

$$P_{r,x}[f(\xi_t)|\mathcal{F}_{r,s}] = P_{r,x}[f(\xi_t)|\sigma(\{\xi_u : r \leq u \leq s\})]$$

$$= P_{r,x}[f(y_{t-r})|\sigma(\{y_{u-r} : r \leq u \leq s\})]$$

$$= P_{r,x}[f(y_{t-r})|G_{s-r}] = \tilde{P}_{t-s}f(\alpha_{s-r}, y_{s-r})$$

$$= \tilde{P}_{t-s}f(s, \xi_s) = P_{s,t}f(\xi_s).$$

That gives the desired Markov property of $\xi$. 

**Example A.2** Suppose that $E$ is a complete separable metric space. Let $D_E := D([0, \infty), E)$ be the space of càdlàg paths from $[0, \infty)$ to $E$ equipped with the usual Skorokhod metric. Then $D_E$ is also a complete separable metric space. Suppose that $(P_t)_{t \geq 0}$ is a Borel right semigroup on $E$ with a càdlàg realization. For simplicity we consider the canonical realization $\xi = (D_E, \mathcal{F}_0^E, \mathcal{F}_t^E, \xi_t, \mathcal{P}_x)$, where $(\mathcal{F}_0^E, \mathcal{F}_t^E)$ are the natural $\sigma$-algebras of $D_E$ and $\xi_t(\omega) = \omega(t)$ for $t \geq 0$ and $\omega \in D_E$. Let $y^* \in (t, y) \in [0, \infty) \times D_E : y = y^*$ for $s, t \geq 0$ and $y \in D_E$. Let $S = \{(t, y) \in [0, \infty) \times D_E : y = y^*\}$. For $t \geq 0$ let

$$D^t_E = \{y \in D_E : y = y^t\} = \{y \in D_E : (t, y) \in S\}.$$
Then we have \( D_E^s \subset D_E^t \) for \( t \geq s \geq 0 \). Given \( r \geq 0 \) and \( y_1, y_2 \in D_E \) we define \( y_1/r/y_2 \in D_E \) by

\[
(y_1/r/y_2)(t) = \begin{cases} 
    y_1(t) & \text{if } 0 \leq t < r, \\
    y_2(t - r) & \text{if } r \leq t < \infty.
\end{cases}
\]

The operators \( y \mapsto y^s \) and \( (r, y_1, y_2) \mapsto y_1/r/y_2 \) are Borel measurable; see Dellacherie and Meyer (1978, p.146). We can define an inhomogeneous Borel transition semigroup \( (\bar{P}_{r,t} : t \geq r \geq 0) \) with global state space \( S \) by

\[
\bar{P}_{r,t}f(y) = P_y(r)[f(y/r/\xi^{t-r})], \quad y \in D_E^r, f \in \mathcal{B}(D_E^t). \tag{A.47}
\]

From Proposition 2.1.2 of Dawson and Perkins (1991, p.14) it follows that \( (\bar{P}_{r,t} : t \geq r \geq 0) \) is a right transition semigroup. For \( \omega \in D_E \) and \( t \geq 0 \) let \( \xi_t(\omega) = \omega^t \in D_E^t \). It is easy to see \( \mathcal{F}_{r,t}^0 := \sigma(\{\xi_s : r \leq s \leq t\}) = \mathcal{F}_t^0 \) for \( t \geq r \geq 0 \). For \( r \geq 0 \) and \( y \in D_E^r \) define the probability measure \( \bar{P}_{r,y} \) on \( (D_E, \mathcal{F}_r^0) \) by

\[
\bar{P}_{r,y}(A) = P_{y(r)}(\{\omega \in D_E : y/r/\omega \in A\}), \quad A \in \mathcal{F}_r^0. \tag{A.48}
\]

Then \( \bar{\xi} = (D_E, \mathcal{F}_r^0, \mathcal{F}_{r,t}^0, \xi_t, \bar{P}_{r,y}) \) is a c\`adl\`ag realization of \( (P_{r,t} : t \geq r \geq 0) \). This process is called the path process of \( \xi \); see Dawson and Perkins (1991). Clearly, the path process records all the information of the history of the sample path of \( \xi \).
References


References


References

Mitoma, I. (1983): Tightness of probabilities on \( C([0, 1], \mathcal{F}) \) and \( D([0, 1], \mathcal{F}) \). Ann. Probab. 11, 989–999.
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