Appendices

A.1 Proofs for Chapter 2

Proposition 2.4.1. Let $G$ be a simple CG such that $\text{norm}(G)$ exists. Then:
1. $G \leq_{i} \text{norm}(G)$.
2. If $G \leq_{i} G^{*}$ and $G^{*}$ is normal, then $\text{norm}(G) \leq_{i} G^{*}$.

Proof
1. According to CG normalization, if $S$ is a set of vertices in $G$ that are joined into a vertex $e$ in $\text{norm}(G)$, then the conjunctive concept or relation type in $e$ is the least upper bound of all the corresponding conjunctive concept or relation types of the vertices in $S$. So, there exists a CG projection $\pi$ from $G$ to $\text{norm}(G)$ that maps the vertices in such a set $S$ in $G$ to the corresponding vertex $e$ in $\text{norm}(G)$, and the other vertices in $G$ to themselves in $\text{norm}(G)$, whence $G \leq_{i} \text{norm}(G)$ by $\pi$.
2. Let $G^{*}$ be normal and $G \leq_{i} G^{*}$ by $\pi^{*}$. Since $G^{*}$ is normal, all the vertices in a set $S$ in $G$ that are joined into a vertex $e$ in $\text{norm}(G)$ have to be mapped to one vertex $e^{*}$ in $G^{*}$ by $\pi^{*}$, where the conjunctive concept or relation type in $e^{*}$ is an upper bound of all the corresponding conjunctive concept or relation types of the vertices in $S$ and, thus, a subtype of the conjunctive concept or relation type in $e$. So, there exists a CG projection $\pi_{0}$ from $\text{norm}(G)$ to $G^{*}$ that maps such a vertex $e$ in $\text{norm}(G)$ to the corresponding $e^{*}$ in $G^{*}$, and the other vertices in $\text{norm}(G)$ to the same vertices in $G^{*}$ that their identical ones in $G$ are mapped to by $\pi^{*}$, whence $\text{norm}(G) \leq_{i} G^{*}$ by $\pi_{0}$.

Proposition 2.5.1. For every fuzzy set $A$ and $\alpha \in [0, 1]$, $A+(1-\alpha)$ is the least specific solution for $A^{*}$ such that $N(A | A^{*}) \geq \alpha$.

Proof
Let $U$ be the domain of $A$ and $A^{*}$. Firstly, from Definition 2.5.4 and Equation 2.5.4, one has:
\[
N(A | A+(1-\alpha)) = 1 - \sup_{u \in U} \{ \max \{ 0, \mu_{A+(1-\alpha)}(u) - \mu_{A}(u) \} \} \\
= 1 - \sup_{u \in U} \{ \max \{ 0, \min \{ 1, \mu_{A}(u) + (1-\alpha) \} - \mu_{A}(u) \} \} \\
= 1 - \sup_{u \in U} \{ \min \{ 1, \mu_{A}(u) + (1-\alpha) \} - \mu_{A}(u) \} \\
= 1 - \sup_{u \in U} \{ \min \{ 1 - \mu_{A}(u), 1 - \alpha \} \} \\
= \inf_{u \in U} \{ \max \{ \mu_{A}(u), \alpha \} \} \\
= \max \{ \inf_{u \in U} \{ \mu_{A}(u) \}, \alpha \} \\
\geq \alpha.
\]
One has \( N(A \mid A+(1-\alpha)) = \alpha \) iff \( \inf_{u \in U} \{ \mu_A(u) \} \leq \alpha \). In particular, this occurs when \( \exists u \in U: \mu_A(u) < \alpha \), that is, \( A+(1-\alpha) \neq \bot \).

We now prove that, if \( N(A \mid A^*) \geq \alpha \) then \( A^* \subseteq A+(1-\alpha) \). From Definition 2.5.4, one has:
\[
N(A \mid A^*) = 1 - \sup_{u \in U} \{ \max \{ 0, \mu_A(u) - \mu_A(u) \} \}
= \inf_{u \in U} \{ \min \{ 1, 1 - \mu_A(u) + \mu_A(u) \} \}.
\]
Thus, if \( N(A \mid A^*) \geq \alpha \) then \( \forall u \in U: \min \{ 1, 1 - \mu_A(u) + \mu_A(u) \} \geq \alpha \), whence \( \forall u \in U: \mu_A(u) \leq \min \{ 1, \mu_A(u)+(1-\alpha) \} \), that is, \( A^* \subseteq A+(1-\alpha) \).

**Proposition 2.5.2.** Let \( A, A^*, A_1 \) and \( A_2 \) be fuzzy sets on the same domain. Then the following properties hold:
1. \( \Delta(A \mid A^*) = 0 \) iff \( A \subseteq A^* \), i.e., \( A^* \subseteq A \).
2. If \( A_1 \leq A_2 \) then \( \Delta(A \mid A_2) \leq \Delta(A \mid A_1) \).
3. \( A+\varepsilon \leq A^* \) iff \( \Delta(A \mid A^*) \leq \varepsilon \), for every \( \varepsilon \in [0, 1] \).

**Proof**
Properties 1 and 2 are straightforward from Definition 2.5.5.

For property 3, by Proposition 2.5.1, if \( \Delta(A \mid A^*) \leq \varepsilon \) then \( A+\varepsilon \leq A^* \). On the other hand, if \( A+\varepsilon \leq A^* \) then, by property 2, \( \Delta(A \mid A^*) \leq \Delta(A \mid A+\varepsilon) \). Since \( \Delta(A \mid A+\varepsilon) \leq \varepsilon \) by Proposition 2.5.1, one has \( \Delta(A \mid A^*) \leq \varepsilon \).

**Proposition 2.6.1.** The fuzzy subtype relation is a partial order.

**Proof**
1. Reflexivity: it is obvious that the relation is reflexive (case 1 of Definition 2.6.3).
2. Transitivity: from Definition 2.6.3, if \( (t_1, v_1) \leq (t_2, v_2) \) then \( v_1 \leq v_2 \). Thus, supposing that \( (t_1, v_1) \leq (t_2, v_2) \leq (t_3, v_3) \), one has \( v_1 \leq v_2 \leq v_3 \). Regarding the basic subtype relations between \( t_1 \) and \( t_2 \) and between \( t_2 \) and \( t_3 \), there are nine cases in total. We now prove that \( (t_1, v_1) \leq (t_3, v_3) \) by considering them case by case:

   (a) \( t_1 = t_2 \):
   - \( t_2 = t_3 \): one has \( (t_1, v_1) \leq (t_3, v_3) \) by case 1 of Definition 2.6.3.
   - \( t_2 \leq t_3 \) and \( v_2 \leq \text{lub} \{ v \in T \mid v \leq v_3 \} \): one has \( t_1 \leq t_3 \) and \( v_1 \leq v_2 \leq \text{lub} \{ v \in T \mid v \leq v_3 \} \), whence \( (t_1, v_1) \leq (t_3, v_3) \) by case 2 of Definition 2.6.3.
   - \( t_2 \geq t_3 \) and \( v_2 \leq \text{lub} \{ v \in F \mid v \leq v_3 \} \): one has \( t_3 \leq t_1 \) and \( v_1 \leq v_2 \leq \text{lub} \{ v \in F \mid v \leq v_3 \} \), whence \( (t_1, v_1) \leq (t_3, v_3) \) by case 3 of Definition 2.6.3.

   (b) \( t_1 \leq t_2 \) and \( v_1 \leq \text{lub} \{ v \in T \mid v \leq v_2 \} \):
   - \( t_2 = t_3 \): one has \( t_1 \leq t_3 \) and \( v_1 \leq \text{lub} \{ v \in T \mid v \leq v_2 \} \leq \text{lub} \{ v \in T \mid v \leq v_3 \} \), whence \( (t_1, v_1) \leq (t_3, v_3) \) by case 2 of Definition 2.6.3.
   - \( t_2 \leq t_3 \) and \( v_2 \leq \text{lub} \{ v \in T \mid v \leq v_3 \} \): one has \( t_1 \leq t_3 \) and \( v_1 \leq \text{lub} \{ v \in T \mid v \leq v_2 \} \leq \text{lub} \{ v \in T \mid v \leq v_3 \} \), whence \( (t_1, v_1) \leq (t_3, v_3) \) by case 2 of Definition 2.6.3.
• $t_2 >_t t_3$ and $v_2 \leq \text{lub}\{v \in F \mid v \leq v_3\}$: this case does not occur because it would lead to $\text{lub}\{v \in T \mid v \leq v_2\} \leq v_2 \leq \text{lub}\{v \in F \mid v \leq v_3\}$, which would violate Definition 2.6.1 requiring that a TRUE-characteristic fuzzy truth-value and a FALSE-characteristic one are not comparable.

(c) $t_2 <_t t_1$ and $v_1 \leq \text{lub}\{v \in F \mid v \leq v_2\}$:

• $t_2 = t_3$: one has $t_3 <_t t_1$ and $v_1 \leq \text{lub}\{v \in F \mid v \leq v_2\} \leq \text{lub}\{v \in F \mid v \leq v_3\}$, whence $(t_1, v_1) \leq (t_3, v_3)$ by case 3 of Definition 2.6.3.

• $t_2 <_t t_3$ and $v_2 \leq \text{lub}\{v \in T \mid v \leq v_3\}$: this case does not occur because it would lead to $\text{lub}\{v \in F \mid v \leq v_2\} \leq v_2 \leq \text{lub}\{v \in T \mid v \leq v_3\}$, which would violate Definition 2.6.1 of a fuzzy truth-value lattice.

• $t_2 >_t t_3$ and $v_2 \leq \text{lub}\{v \in F \mid v \leq v_3\}$: one has $t_3 <_t t_1$ and $v_1 \leq \text{lub}\{v \in F \mid v \leq v_3\}$, whence $(t_1, v_1) \leq (t_3, v_3)$ by case 3 of Definition 2.6.3.

3. Anti-symmetry: supposing that $(t_1, v_1) \leq (t_2, v_2)$ and $(t_2, v_2) \leq (t_1, v_1)$, one has $v_1 \leq v_2$ and $v_2 \leq v_1$, whence $v_1 = v_2$. Furthermore:

(a) if $t_1 <_t t_2$ then $v_1 = v_2 \leq \text{lub}\{v \in T \mid v \leq v_2\}$ due to $(t_1, v_1) \leq (t_2, v_2)$, and $v_1 = v_2 \leq \text{lub}\{v \in F \mid v \leq v_1\}$ due to $(t_2, v_2) \leq (t_1, v_1)$, whence $v_1 = v_2 = \text{lub}\{v \in T \mid v \leq v_2\} = \text{lub}\{v \in F \mid v \leq v_1\}$, which would violate Definition 2.6.1.

(b) if $t_1 >_t t_2$ then $v_1 = v_2 \leq \text{lub}\{v \in F \mid v \leq v_2\}$ due to $(t_1, v_1) \leq (t_2, v_2)$, and $v_1 = v_2 \leq \text{lub}\{v \in T \mid v \leq v_1\}$ due to $(t_2, v_2) \leq (t_1, v_1)$, whence $v_1 = v_2 = \text{lub}\{v \in F \mid v \leq v_2\} = \text{lub}\{v \in T \mid v \leq v_1\}$, which would also violate Definition 2.6.1.

Thus, $t_1 = t_2$, whence $(t_1, v_1) = (t_2, v_2)$.

**Proposition 2.6.2.** The conjunctive fuzzy subtype relation is a partial order.

**Proof**

1. Reflexivity: it is obvious that the relation is reflexive.

2. Transitivity: Supposing that $T_1 \leq T_2 \leq T_3$, one has $\forall \tau_1 \in T_1 \exists \tau_2 \in T_2: \tau_1 \leq \tau_2$ and $\forall \tau_2 \in T_2 \exists \tau_3 \in T_3: \tau_2 \leq \tau_3$, whence $\forall \tau_1 \in T_1 \exists \tau_3 \in T_3: \tau_1 \leq \tau_3$, which means $T_1 \subseteq T_3$.

3. Anti-symmetry: supposing that $T_1 \leq T_2$ and $T_2 \leq T_1$, one has $\forall \tau_1 \in T_1 \exists \tau_2 \in T_2: (\tau_1 \leq \tau_2 \text{ and } \exists \tau_1^* \in T_1: \tau_2 \leq \tau_1^*)$, where $\tau_1 = \tau_1^*$ and thus $\tau_1 = \tau_2$, because otherwise $\tau_1 = \tau_1^*$, which would violate Definition 2.6.4. Thus, $\forall \tau_1 \in T_1 \exists \tau_2 \in T_2: \tau_1 = \tau_2$, which means $T_1 \subseteq T_2$. Similarly, one has $T_2 \subseteq T_1$, whence $T_1 = T_2$.

**Proposition 2.6.3.** The set of all conjunctive fuzzy types, defined over a partially ordered set of basic types and a fuzzy truth-value lattice, forms an upper semi-lattice under the conjunctive fuzzy subtype relation where, for two conjunctive fuzzy types $T_1$ and $T_2$, $\text{lub}\{T_1, T_2\} = \text{con}(T_1 \cup T_2)$.

**Proof**

As it is described, $\text{con}(T_1 \cup T_2)$ is constructed from $T_1 \cup T_2$ by just removing the elements that are less specific than others in $T_1 \cup T_2$. Thus, for every element $\tau$ in
∀τ∈T con

1. Let (Proof

2. Non-negative, or

1. Constructed from the same basic type, or

Thus, if T is an upper bound of \{T_1, T_2\}, then for every element τ in con(T_1∪T_2) there is an element τ* in T such that τ ≤ τ*, whence con(T_1∪T_2) is the least upper bound of \{T_1, T_2\}.

Proposition 2.6.4. Let T_1 and T_2 be two conjunctive fuzzy types such that ∀τ_1∈T_1 \; ∀τ_2∈T_2: glb{τ_1, τ_2} exists if τ_1 and τ_2 have a common lower bound. Then, glb(T_1, T_2) = con{glb{τ_1, τ_2} l τ_1∈T_1, τ_2∈T_2 and glb{τ_1, τ_2} exists} if T_1 and T_2 have a common lower bound.

Proof

Let T_0 = con{glb{τ_1, τ_2} l τ_1∈T_1, τ_2∈T_2 and glb{τ_1, τ_2} exists}. It is obvious that T_0 \subseteq T_1 and T_0 \subseteq T_2. Also, if T is a lower bound of \{T_1, T_2\}, then ∀τ∈T \; ∃τ_1∈T_1 \; ∃τ_2∈T_2: τ ≤ τ_1 and τ ≤ τ_2. Thus, ∀τ∈T \; ∃τ_1∈T_1 \; ∃τ_2∈T_2: glb{τ_1, τ_2} exists and τ ≤ glb{τ_1, τ_2}, whence T ≤ T_0. Therefore, glb(T_1, T_2) = T_0.

Proposition 2.6.5. Let τ_1 and τ_2 be two fuzzy types, defined over a basic type lattice and a fuzzy truth-value lattice, such that both are either:

1. Constructed from the same basic type, or
2. Non-negative, or
3. Non-positive

Then glb{τ_1, τ_2} exists if τ_1 and τ_2 have a common lower bound.

Proof

1. Let (t, v_1) and (t, v_2) be two fuzzy types constructed from the same basic type t.

It is obvious that (t, glb{v_1, v_2}) is a lower bound of \{(t, v_1), (t, v_2)\}. We now prove that for any lower bound (t_0, v_0) of \{(t, v_1), (t, v_2)\}, one has (t_0, v_0) ≤ (t, glb{v_1, v_2}): (a) \( t_0 = t \): one has v_0 ≤ v_1 and v_0 ≤ v_2, whence v_0 ≤ glb{v_1, v_2} and thus (t_0, v_0) ≤ (t, glb{v_1, v_2}).

(b) \( t_0 < t \): one has v_0 ≤ lub{v∈T l v ≤ v_1} ≤ v_1 and v_0 ≤ lub{v∈T l v ≤ v_2} ≤ v_2, whence v_0 ≤ glb{v∈T l v ≤ v_1}, lub{v∈T l v ≤ v_2} ≤ glb{v_1, v_2}. Since glb{lub{v∈T l v ≤ v_1}, lub{v∈T l v ≤ v_2}} ∈ T, one has v_0 ≤ lub{v∈T l v ≤ glb{v_1, v_2}} and thus (t_0, v_0) ≤ (t, glb{v_1, v_2}).

(c) \( t_0 > t \): one has v_0 ≤ lub{v∈F l v ≤ v_1} ≤ v_1 and v_0 ≤ lub{v∈F l v ≤ v_2} ≤ v_2, whence v_0 ≤ glb{lub{v∈F l v ≤ v_1}, lub{v∈F l v ≤ v_2}} ≤ glb{v_1, v_2}. Since glb{lub{v∈F l v ≤ v_1}, lub{v∈F l v ≤ v_2}} ∈ F, one has v_0 ≤ lub{v∈F l v ≤ glb{v_1, v_2}} and thus (t_0, v_0) ≤ (t, glb{v_1, v_2}).

Therefore, glb{(t, v_1), (t, v_2)} = (t, glb{v_1, v_2}).

2. Let (t_0, v_0) be a lower bound of \{(t_1, v_1), (t_2, v_2)\}. Regarding the basic subtype relations between t_0 and t_1 and between t_0 and t_2, there are totally nine cases:

(a) \( t_0 = t_1 = t_2 \) and v_0 ≤ v_1 and v_0 ≤ v_2.

(b) \( t_0 = t_1 = t_2 \) and v_0 ≤ v_1 and v_0 ≤ lub{v∈T l v ≤ v_2}.
(c) \( t_0 = t_1 > t_2 \) and \( v_0 \leq v_1 \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_2\} \).

(d) \( t_0 = t_2 < t_1 \) and \( v_0 \leq \text{lub}\{v \in T \mid v \leq v_1\} \) and \( v_0 \leq v_2 \).

(e) \( t_0 = t_2 > t_1 \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_1\} \) and \( v_0 \leq v_2 \).

(f) \( t_0 < t_1 \) and \( t_0 < t_2 \) and \( v_0 \leq \text{lub}\{v \in T \mid v \leq v_1\} \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_2\} \).

(g) \( t_0 > t_1 \) and \( t_0 > t_2 \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_1\} \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_2\} \).

(h) \( t_1 < t_0 < t_2 \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_1\} \) and \( v_0 \leq \text{lub}\{v \in T \mid v \leq v_2\} \).

(i) \( t_1 > t_0 > t_2 \) and \( v_0 \leq \text{lub}\{v \in T \mid v \leq v_1\} \) and \( v_0 \leq \text{lub}\{v \in F \mid v \leq v_2\} \).

With \((t_1, v_1)\) and \((t_2, v_2)\) being non-negative fuzzy types, cases (c), (e), (g), (h) and (i) do not occur, so \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} \) is defined as follows:

- \( t_1 = t_2 \): only cases (a) and (f) are involved, whence \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (t_1, \text{glb}\{v_1, v_2\}) = (t_2, \text{glb}\{v_1, v_2\}) \), or else
- \( t_1 < t_2 \): only cases (b) and (f) are involved, whence \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (t_1, \text{glb}\{v_1, \text{lub}\{v \in T \mid v \leq v_2\}\}) \), or else
- \( t_1 > t_2 \): only cases (d) and (f) are involved, whence \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (t_2, \text{glb}\{\text{lub}\{v \in T \mid v \leq v_1\}, v_2\}) \), or else
- \( t_1 \) and \( t_2 \) are not comparable: only case (f) is involved, whence \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (\text{glb}\{t_1, t_2\}, \text{glb}\{\text{lub}\{v \in T \mid v \leq v_1\}, \text{lub}\{v \in T \mid v \leq v_2\}\}) \).

Similarly, with \((t_1, v_1)\) and \((t_2, v_2)\) being non-positive fuzzy types, \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} \) is defined as follows:

- \( t_1 = t_2 \): \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (t_1, \text{glb}\{v_1, v_2\}) = (t_2, \text{glb}\{v_1, v_2\}) \), or else
- \( t_1 < t_2 \): \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (t_2, \text{glb}\{\text{lub}\{v \in F \mid v \leq v_1\}, v_2\}) \), or else
- \( t_1 > t_2 \): \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (t_1, \text{glb}\{\text{lub}\{v \in F \mid v \leq v_2\}, v_2\}) \), or else
- \( t_1 \) and \( t_2 \) are not comparable: \( \text{glb}\{(t_1, v_1), (t_2, v_2)\} = (\text{lub}\{t_1, t_2\}, \text{glb}\{\text{lub}\{v \in F \mid v \leq v_1\}, \text{lub}\{v \in F \mid v \leq v_2\}\}) \).

**Proposition 2.6.6.** For every fuzzy type \( \tau_1 \) and \( \varepsilon \in [0, 1) \), \( \tau_1 + \varepsilon \) is the least specific solution for \( \tau_2 \) such that \( \Delta(\tau_1 \mid \tau_2) \leq \varepsilon \).

**Proof.**

Firstly, one has \( \Delta(\tau_1 \mid \tau_1 + \varepsilon) = \Delta(v_1 \mid v_1 + \varepsilon) \). By Proposition 2.5.2, \( \Delta(v_1 \mid v_1 + \varepsilon) \leq \varepsilon \), whence \( \Delta(\tau_1 \mid \tau_1 + \varepsilon) \leq \varepsilon \), for every \( \varepsilon \in [0, 1] \).

We now prove that, if \( \Delta(\tau_1 \mid \tau_2) \leq \varepsilon \) then \( \tau_1 + \varepsilon \leq \tau_2 \). Let \( \tau_1 = (t_1, v_1) \) and \( \tau_2 = (t_2, v_2) \). By Definition 2.6.7, there are three cases:

1. \( t_1 = t_2 \): one has \( \Delta(\tau_1 \mid \tau_2) = \Delta(v_1 \mid v_2) \leq \varepsilon \), whence \( v_1 + \varepsilon \leq v_2 \) (Proposition 2.5.2) and thus \( \tau_1 + \varepsilon \leq \tau_2 \) by case 1 of Definition 2.6.3.

2. \( t_1 < t_2 \) and \( \exists v \in T : v \leq v_2 \): one has \( \Delta(\tau_1 \mid \tau_2) = \Delta(v_1 \mid \text{lub}\{v \in T \mid v \leq v_2\}) \) \leq \varepsilon \), whence \( v_1 + \varepsilon \leq \text{lub}\{v \in T \mid v \leq v_2\} \) and thus \( \tau_1 + \varepsilon \leq \tau_2 \) by case 2 of Definition 2.6.3.

3. \( t_1 > t_2 \) and \( \exists v \in F : v \leq v_2 \): one has \( \Delta(\tau_1 \mid \tau_2) = \Delta(v_1 \mid \text{lub}\{v \in F \mid v \leq v_2\}) \) \leq \varepsilon \), whence \( v_1 + \varepsilon \leq \text{lub}\{v \in F \mid v \leq v_2\} \) and thus \( \tau_1 + \varepsilon \leq \tau_2 \) by case 3 of Definition 2.6.3.

**Proposition 2.6.7.** Let \( \tau_1, \tau_2 \) and \( \tau_3 \) be fuzzy types, defined over a partially ordered set of basic types and a fuzzy truth-value lattice, such that \( \tau_1 \) is matchable to \( \tau_2 \). Then the following properties hold:
1. $\Delta(t_1 \mid t_2) = 0$ iff $t_1 \leq t_2$.
2. If $t_2 \leq t_3$, then $t_1$ is matchable to $t_3$ and $\Delta(t_1 \mid t_3) \leq \Delta(t_1 \mid t_2)$.
3. $t_1 + \varepsilon \leq t_2$ iff $\Delta(t_1 \mid t_2) \leq \varepsilon$, for every $\varepsilon \in [0, 1]$.

**Proof**

1. This property is straightforward from Definition 2.6.3 and Definition 2.6.7.
2. Let $t_1 = (t_1, v_1)$, $t_2 = (t_2, v_2)$, $t_3 = (t_3, v_3)$ and $t_2 \leq t_3$. Regarding the basic subtype relations between $t_1$ and $t_2$ and between $t_2$ and $t_3$, there are nine cases in total.

   We now prove that $\Delta(t_1 \mid t_3) \leq \Delta(t_1 \mid t_2)$ by considering them case by case:
   
   (a) $t_1 = t_2$: one has $\Delta(t_1 \mid t_2) = \Delta(v_1 \mid v_2)$.
      - $t_2 = t_3$ and $v_2 \leq v_3$: one has $\Delta(t_1 \mid t_3) = \Delta(v_1 \mid v_3) \leq \Delta(v_1 \mid v_2) = \Delta(t_1 \mid t_2)$.
      - $t_2 < t_3$ and $v_2 \leq v_3$: one has $\Delta(t_1 \mid t_3) = \Delta(v_1 \mid v_3) \leq \Delta(v_1 \mid v_2) = \Delta(t_1 \mid t_2)$.

   (b) $t_1 \leq t_2$ and $\exists v \in T$: $v \leq v_2$.
      - $t_2 = t_3$ and $v_2 \leq v_3$: one has $\Delta(t_1 \mid t_3) = \Delta(v_1 \mid v_3) \leq \Delta(v_1 \mid v_2) = \Delta(t_1 \mid t_2)$.
      - $t_2 < t_3$ and $v_2 \leq v_3$: one has $\Delta(t_1 \mid t_3) = \Delta(v_1 \mid v_3) \leq \Delta(v_1 \mid v_2) = \Delta(t_1 \mid t_2)$.

   (c) $t_1 > t_2$ and $\exists v \in F$: $v \leq v_2$.
      - $t_2 = t_3$ and $v_2 \leq v_3$: one has $\Delta(t_1 \mid t_3) = \Delta(v_1 \mid v_3) \leq \Delta(v_1 \mid v_2) = \Delta(t_1 \mid t_2)$.
      - $t_2 < t_3$ and $v_2 \leq v_3$: this case does not occur because it would lead to $\exists v \in T$: $v \leq v_2$.

3. By Proposition 2.6.4, if $\Delta(t_1 \mid t_2) \leq \varepsilon$ then $t_1 + \varepsilon \leq t_2$. On the other hand, if $t_1 + \varepsilon \leq t_2$, then, by property 2 above, $\Delta(t_1 \mid t_2) \leq \Delta(t_1 \mid t_1 + \varepsilon)$. Since $\Delta(t_1 \mid t_1 + \varepsilon) \leq \varepsilon$ by Proposition 2.6.6, one has $\Delta(t_1 \mid t_2) \leq \varepsilon$.

**Proposition 2.6.8.** For every conjunctive fuzzy type $T_1$ and $\varepsilon \in [0, 1]$, $T_1 + \varepsilon$ is the least specific solution for $T_2$ such that $\Delta(T_1 \mid T_2) \leq \varepsilon$.

**Proof**

Let $S = \{t_1 + \varepsilon \mid t_1 \in T_1\}$. By Definition 2.6.9, one has $\Delta(T_1 \mid T_1 + \varepsilon) = \max_{\varepsilon \in T} \min_{\varepsilon T + \varepsilon} \Delta(t_1 \mid t_2)$. Since $T_1 + \varepsilon =$ $\text{con}(S)$ is obtained from $S$ by just removing the elements that are less specific than others in $S$, one has $\forall t_2 \in T_2: t_2 \ast \leq t_2$. By property 2 in Proposition 2.6.7, for any $t_1$, if $t_1$ is matchable to $t_2$, then $t_1$
is matchable to $\tau_2$ and $\Delta(\tau_1 \mid \tau_2) \leq \Delta(\tau_1 \mid \tau_2^*)$. Thus, for every $\tau_1 \in T_1$, $\min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\} = \min_{\tau_1 \in \tau \in T} \{\Delta(\tau_1 \mid \tau_2^*)\} \leq \Delta(\tau_1 \mid \tau_1 + \epsilon) \leq \epsilon$, whence $\Delta(T_1 \mid T_1 + \epsilon) = \max_{T \in T} \min_{\tau \in \tau \in T} \{\Delta(\tau_1 \mid \tau_2)\} \leq \epsilon$.

We now prove that, if $\Delta(T_1 \mid T_2) \leq \epsilon$ then $T_1 + \epsilon \leq T_2$. One has $\Delta(T_1 \mid T_2) = \max_{T \in T} \min_{\tau \in \tau \in T} \{\Delta(\tau_1 \mid \tau_2)\} \leq \epsilon$ whence $\forall \tau_1 \in T_1: \min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\} \leq \epsilon$. Thus, $\forall \tau! \in T_1 \exists \tau_2 \in T_2: \Delta(\tau_1 \mid \tau_2) \leq \epsilon$ whence $\tau_1 + \epsilon \leq \tau_2$, which means $T_1 + \epsilon \leq T_2$.

**Proposition 2.6.9.** Let $T_1$, $T_2$ and $T_3$ be conjunctive fuzzy types, defined over a partially ordered set of basic types and a fuzzy truth-value lattice, such that $T_1$ is matchable to $T_2$. Then the following properties hold:

1. $\Delta(T_1 \mid T_2) = 0$ iff $T_1 \leq T_2$.
2. If $T_2 \leq T_3$, then $T_1$ is matchable to $T_3$ and $\Delta(T_1 \mid T_3) \leq \Delta(T_1 \mid T_2)$.
3. $T_1 + \epsilon \leq T_2$ iff $\Delta(T_1 \mid T_2) \leq \epsilon$, for every $\epsilon \in [0, 1]$.

**Proof**

1. $\Delta(T_1 \mid T_2) = 0$ iff $\max_{T \in T} \min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\} = 0$ iff $\forall \tau_1 \in T_1: \min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\} = 0$ iff $\forall \tau_2 \in T_2: \Delta(\tau_1 \mid \tau_2) = 0$ iff $\forall \tau_1 \in T_1 \exists \tau_2 \in T_2: \Delta(\tau_1 \mid \tau_2) = 0$. Thus, $\forall \tau_1 \in T_1: \min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\} = 0$.

2. If $T_2 \leq T_3$, then $\forall \tau_2 \in T_2 \exists \tau_3 \in T_3: \tau_2 \leq \tau_3$ whence, by property 2 in Proposition 2.6.7, $\Delta(\tau_1 \mid \tau_3) \leq \Delta(\tau_1 \mid \tau_2)$ for any $\tau_1$ that is matchable to $\tau_2$. Thus, $\forall \tau_1 \in T_1: \min_{T \in T} \{\Delta(\tau_1 \mid \tau_3)\} \leq \min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\}$, whence $\Delta(T_1 \mid T_3) = \max_{T \in T} \min_{T \in T} \{\Delta(\tau_1 \mid \tau_3)\} = \max_{T \in T} \min_{T \in T} \{\Delta(\tau_1 \mid \tau_2)\} = \Delta(T_1 \mid T_2)$.

3. By Proposition 2.6.8, if $\Delta(T_1 \mid T_2) \leq \epsilon$ then $T_1 + \epsilon \leq T_2$. On the other hand, if $T_1 + \epsilon \leq T_2$ then, by property 2 above, $\Delta(T_1 \mid T_2) \leq \Delta(T_1 \mid T_1 + \epsilon)$. Since $\Delta(T_1 \mid T_1 + \epsilon) \leq \epsilon$ by Proposition 2.6.8, one has $\Delta(T_1 \mid T_2) \leq \epsilon$.

**Proposition 2.7.1.** There exists an infinite CG such that there is no irredundant CG that is equivalent to it.

**Proof**

We prove this property by proving that the infinite CG $G$ in Figure 2.7.6 is redundant and there is no irredundant CG that is equivalent to it.

It is obvious that $G$ is redundant, because there is a projection from $G$ to its strict subgraph $g$, for instance, that is obtained from $G$ by deleting the first concept and the first relation.

Let $H$ be a CG that is equivalent to $G$ and $\pi$ be a projection from $G$ to $H$. If any two concepts in $G$ were mapped to one concept in $H$, then $H$ would have a closed path. Since $H$ also has a projection to $G$, it would entail that $G$ also had a closed path, which is not the case. Thus, no two concepts in $G$ are mapped to one concept in $H$, i.e., $\pi$ is injective. Further, in $\pi G$ all the concept labels must be the same as $(T, *)$ and all the relation labels the same as $R$, because they are so in $G$, whence $G$ and $\pi G$ are identical. Therefore, $H$ is redundant, because either $\pi G$ is a strict subgraph of $H$ or otherwise $G$ and $H$ are identical.

**Proposition 2.7.2.** There exist two irredundant infinite CGs that are equivalent but are not identical.
Proof
Assuming an infinite type lattice that contains the infinite chain \( \{ T_q \mid z \in \mathbb{Z} \text{ and } T_p \leq T_q \text{ iff } q \leq p \} \), we prove the property by proving that the two infinite CGs \( G \) and \( H \) in Figure 2.7.7, which are obviously not identical, are equivalent and irredundant.

The two CGs are equivalent because there are projections from one to the other, as shown in the figure.

On the other hand, the least reduction of \( H \) is obtained by deleting a relation vertex in it, as shown in the figure, for instance, producing \( z \) as shown in the figure.

Similarly, if the concept \([ T_{1/2}: *]\) in \( H \) were projected to a concept in \( h_1 \), then there would be no path in \([ h_1 \ h_2]\) that could correspond to the right part of \( H \) from \([ T_{1/2}: *]\), due to the disconnection of \( h_1 \) and \( h_2 \).

Thus, there cannot be a projection from \( H \) to \([ h_1 \ h_2]\), that is, \( H \) is irredundant. Similarly, \( G \) is also irredundant.

A.2 Proofs for Chapter 3

Proposition 3.4.1. \( R_p \) and \( T_p \) are monotonic, that is:
1. If \( I_1 \leq I_2 \) then \( R_p(I_1) \leq R_p(I_2) \), where \( I_1 \), \( I_2 \) are \( r \)-interpretations.
2. If \( I_1 \leq I_2 \) then \( T_p(I_1) \leq T_p(I_2) \), where \( I_1 \), \( I_2 \) are \( g \)-interpretations.

Proof
1. See Definition 3.4.1. Let \( S_1 \) and \( S_2 \) be the sets \( \{ H+\varepsilon \mid \ldots \} \) corresponding to \( I_1 \) and \( I_2 \). Since \( I_1 \leq I_2 \), one has \( I_1(Obj) \leq I_2(Obj) \) whence, by property 2 in Proposition 2.5.3, one has \( \Delta(B_i \mid I_2(Obj)) \leq \Delta(B_i \mid I_1(Obj)) \). Thus, if \( H+\varepsilon_1 \in S_1 \) then there exists \( H+\varepsilon_2 \in S_2 \) with \( \varepsilon_2 \leq \varepsilon_1 \), i.e., \( H+\varepsilon_1 \leq H+\varepsilon_2 \), whence \( R_p(I_1) \leq R_p(I_2) \).

2. See Definition 3.4.2. Let \( S_1 \) and \( S_2 \) be the sets \( \{ H+\varepsilon \mid \ldots \} \) corresponding to \( I_1 \) and \( I_2 \). Since \( I_1 \leq I_2 \), if \( B_i \varepsilon \in I_1(Obj) \) then \( B_i \varepsilon \in I_2(Obj) \). Thus, if \( H+\varepsilon \in S_1 \) then \( H+\varepsilon \in S_2 \), whence \( T_p(I_1) \leq T_p(I_2) \).

Proposition 3.4.2. Let \( P \) be an AFLP and \( I_r \) and \( I_g \) be an \( r \)-interpretation and a \( g \)-interpretation, respectively. Then:
1. \( I_r \) is an \( r \)-model of \( P \) iff \( R_p(I_r) \leq I_r \).
2. \( I_g \) is a \( g \)-model of \( P \) iff \( T_p(I_g) \leq I_g \).

Proof
1. See Definition 3.4.1. Let \( S_r \) be the set \( \{ H+\varepsilon \mid \ldots \} \) corresponding to \( I_r \) and \( Obj \bl_a \). If \( I_r \) is an \( r \)-model of \( P \) then, by Definition 3.3.5, for every \( H+\varepsilon \in S_r \), one has \( H+\varepsilon \leq I_r(Obj) \) whence \( lub(S_r) \leq I_r(Obj) \) and thus \( R_p(I_r) \leq I_r \). On the other hand, if \( R_p(I_r) \leq I_r \) then \( lub(S_r) \leq I_r(Obj) \), whence \( I_r \) satisfies every ground instance of every clause in \( P \), i.e., \( I_r \) is an \( r \)-model of \( P \).
2. See Definition 3.4.2. Let \( S_g \) be the set \( \{ H+\varepsilon \mid \ldots \} \) corresponding to \( I_{g} \) and \( \text{Obj}_{B_{L}} \). If \( I_{g} \) is a g-model of \( P \) then, by Definition 3.3.6, for every \( H+\varepsilon \in S_g \), one has \( H+\varepsilon \in I_{g}(\text{Obj}) \), i.e., \( S_g \subseteq I_{g}(\text{Obj}) \), whence \( T_{R}(I_{g}) \subseteq I_{g} \). On the other hand, if \( T_{R}(I_{g}) \leq I_{g} \) then \( S_g \subseteq I_{g}(\text{Obj}) \), whence \( I_{g} \) satisfies every ground instance of every clause in \( P \), i.e., \( I_{g} \) is a g-model of \( P \).

**Proposition 3.4.4.** \( R_{P} \) and \( T_{P} \) are continuous, that is:
1. \( R_{P}(\text{lp}(I_{j})) = \text{lp}(R_{P}(I_{j})) \), for all sequences \( I_{1} \leq I_{2} \leq \ldots \) of \( r \)-interpretations.
2. \( T_{P}(\text{lp}(I_{j})) = \text{lp}(T_{P}(I_{j})) \), for all sequences \( I_{1} \leq I_{2} \leq \ldots \) of \( g \)-interpretations.

**Proof**
1. Let us use \( \alpha \) to enumerate the ground instances of clauses in \( P \) having \( \text{Obj} \in B_L \) in their heads. We denote the \( \alpha \)-th such ground instance by \( \text{Obj}: H_{\alpha} \leftarrow \text{Obj}_{\alpha 1}: B_{\alpha 1} \& \text{Obj}_{\alpha 2}: B_{\alpha 2} \& \ldots \& \text{Obj}_{\alpha n}: B_{\alpha n} \).

Then, the expression in Definition 3.4.1 can be rewritten as follows:

\[
\text{R}_{P}(I)(\text{Obj}) = \text{lp}_{\alpha}(H_{\alpha} + \text{max}_{i=1,n}\{\Delta(B_{\alpha i} \mid I(\text{Obj}_{\alpha i}))\})
\]

whence

\[
\text{R}_{P}(\text{lp}(I_{j}))(\text{Obj}) = \text{lp}_{\alpha}(H_{\alpha} + \text{max}_{i=1,n}\{\Delta(B_{\alpha i} \mid \text{lp}(I_{j})(\text{Obj}_{\alpha i}))\})
\]

Since \( I_{1} \leq I_{2} \leq \ldots \), for each \( \alpha \) and each \( i \) from 1 to \( n_{\alpha} \), one has \( \Delta(B_{\alpha i} \mid I_{1}(\text{Obj}_{\alpha i})) \geq \Delta(B_{\alpha i} \mid I_{2}(\text{Obj}_{\alpha i})) \geq \ldots \) (Property 2 in Proposition 2.5.3), and the following holds:

\[
\Delta(B_{\alpha i} \mid \text{lp}(I_{j})(\text{Obj}_{\alpha i})) = \inf_{j}\{\Delta(B_{\alpha i} \mid I_{j}(\text{Obj}_{\alpha i}))\}
\]

and then

\[
\text{max}_{i=1,n}\{\Delta(B_{\alpha i} \mid \text{lp}(I_{j})(\text{Obj}_{\alpha i}))\} = \text{max}_{i=1,n}\inf_{j}\{\Delta(B_{\alpha i} \mid I_{j}(\text{Obj}_{\alpha i}))\}
\]

whence

\[
\text{R}_{P}(\text{lp}(I_{j}))(\text{Obj}) = \text{lp}_{\alpha}(H_{\alpha} + \inf_{j}\text{max}_{i=1,n}\{\Delta(B_{\alpha i} \mid I_{j}(\text{Obj}_{\alpha i}))\})
\]

Similarly, one has:

\[
\text{lp}(\text{R}_{P}(I_{j}))(\text{Obj}) = \text{lp}(\text{R}_{P}(I_{j}))(\text{Obj})
\]

Thus \( \text{R}_{P}(\text{lp}(I_{j})) = \text{lp}(R_{P}(I_{j})) \).

2. Let \( S \) and \( S_j \) be the sets \( \{ H+\varepsilon \mid \ldots \} \) in Definition 3.4.2 corresponding to \( T_{P}(\text{lp}(I_{j}))(\text{Obj}) \) and \( T_{R}(I_{j})(\text{Obj}) \):

\[
S = \{ H+\varepsilon \mid \text{Obj}: H \leftarrow \text{Obj}_{1}: B_{1} \& \text{Obj}_{2}: B_{2} \& \ldots \& \text{Obj}_{n}: B_{n} \text{ is a ground instance of a clause in } P, \text{ and } \varepsilon = \text{max}_{i=1,n}\{\Delta(B_{i} \mid B_{i}^{*})\} \text{ where } B_{i}^{*} = \text{lp}(I_{j})(\text{Obj}_{i}) \text{ for every } i \text{ from } 1 \text{ to } n \},
\]

\[
S_j = \{ H+\varepsilon \mid \text{Obj}: H \leftarrow \text{Obj}_{1}: B_{1} \& \text{Obj}_{2}: B_{2} \& \ldots \& \text{Obj}_{n}: B_{n} \text{ is a ground instance of a clause in } P, \text{ and } \varepsilon = \text{max}_{i=1,n}\{\Delta(B_{i} \mid B_{i}^{*})\} \text{ where } B_{i}^{*} = I_{k}(\text{Obj}_{i}) \text{ for every } i \text{ from } 1 \text{ to } n \}
\]

It is sufficient to prove that \( S = \uplus S_j \). It is obvious that \( \uplus S_j \subseteq S \). We now prove that \( S \subseteq \uplus S_j \). Since \( I_{1} \leq I_{2} \leq \ldots \), for each \( i \) from 1 to \( n \), if \( B_{i}^{*} = \text{lp}(I_{j})(\text{Obj}_{i}) \), then there exists \( k_{i} \in \mathbb{N} \) such that \( B_{i}^{*} = I_{k_{i}}(\text{Obj}_{i}) \). With \( k = \text{max}_{i=1,n}\{k_{i}\} \), one has \( B_{i}^{*} = I_{k}(\text{Obj}_{i}) \) for every \( i \) from 1 to \( n \). Thus, if \( H+\varepsilon \in S \) then \( H+\varepsilon \in S_k \) for some \( k \in \mathbb{N} \), that is, \( S \subseteq \uplus S_j \).
Proposition 3.4.5. If $P$ is a restricted AFLP, then $P$ has the finite fixpoint reachability property with respect to the restricted semantics.

Proof
We prove that, if $A \leq \text{lfp}(R_P)(\text{Obj}) = \text{lub}\{R_P^\uparrow n(\text{Obj}) \mid n \in \mathbb{N}\}$ then $\exists k \in \mathbb{N}: A \leq R_P^\uparrow k(\text{Obj}).$

If $A = \bot$, then $A \leq R_P^\uparrow k(\text{Obj})$ is easily satisfied with $k = 0$. Thus, only cases with $\bot < A$ need to be considered.

If $A \leq \text{lub}\{R_P^\uparrow n(\text{Obj}) \mid n \in \mathbb{N}\}$, then $A \leq \text{lub}\{R_P^\uparrow n(\text{Obj}) \mid n \in \mathbb{N} \text{ and } A^* \leq \text{lub}\{R_P^\uparrow n(\text{Obj})\} \text{ for any } A^* \text{ such that } \bot < A^* < A$. It is sufficient to prove that the set $\{R_P^\uparrow n(\text{Obj}) \mid n \in \mathbb{N} \text{ and } A^* \leq \text{lub}\{R_P^\uparrow n(\text{Obj})\}, \text{where } \bot < A^* < A, \text{ is finite.}$

Firstly, we define a set $E$ of fuzzy set mismatching degrees as follows:

1. If $\text{Obj}: B$ is in the body of a clause in $P$, then $\Delta(B \mid \bot) \in E$.
2. If $\text{Obj}_1: B_1$ and $\text{Obj}_2: H$ is the head of a clause in $P$, and $\text{Obj}_1$ and $\text{Obj}_2$ are unifiable, then $\Delta(B \mid H) \in E$.
3. If $\text{Obj}_1: H_1 \text{ and } \text{Obj}_2: H_2$ are the heads of two clauses in $P$, and $\text{Obj}_1$ and $\text{Obj}_2$ are unifiable, where $H_1 = H_2 + \epsilon$ or $H_2 = H_1 + \epsilon$, then $\epsilon \in E$.

It is clear that $E$ is finite, because $P$ is finite.

Secondly, for every $\text{Obj} \in B_L$, let $H_{\text{Obj}}$ be the greatest annotation value such that $\text{Obj}: H_{\text{Obj}}$ is the head of a ground instance of a clause in $P$. That is, for any $H$ such that $\text{Obj}: H$ is the head of a ground instance of a clause in $P$, $H = H_{\text{Obj}} + \epsilon$ for some $\epsilon \in [0, 1]$.

As noted after Proposition 2.5.2, if $A + \epsilon \neq \bot$, then $\Delta(A \mid A + \epsilon) = \epsilon$. Thus, one has $R_P^\uparrow n(\text{Obj}) = H_{\text{Obj}} + \epsilon_1 + \epsilon_2 + \ldots + \epsilon_m$ with $\epsilon_i \in E$ ($1 \leq i \leq m$). Since $E$ is finite, the number of values of $H_{\text{Obj}} + \epsilon_1 + \epsilon_2 + \ldots + \epsilon_m$ such that $\bot < A^* \leq H_{\text{Obj}} + \epsilon_1 + \epsilon_2 + \ldots + \epsilon_m$ is also finite. That is, the set $\{R_P^\uparrow n(\text{Obj}) \mid n \in \mathbb{N} \text{ and } A^* \leq \text{lub}\{R_P^\uparrow n(\text{Obj})\}\}$ is finite.

Proposition 3.5.1. Let $P$ be an AFLP, $\text{Obj}: H \leftarrow \text{Obj}_1: B_1 \text{ and } \text{Obj}_2: B_2 \ldots \text{ and } \text{Obj}_n: B_n$ be an annotation variable-free instance of a reductant of $P$, and $\theta$ be a substitution for object variables in this reductant. If $\text{(Obj}_1: B_1 \text{ and } \text{Obj}_2: B_2 \ldots \text{ and } \text{Obj}_n: B_n)\theta$ is a logical consequence of $P$, then so is $(\text{Obj}: H)\theta$.

Proof
It is sufficient to prove that, for every ground (i.e., both annotation variable-free and object variable-free) instance $C$ of a reductant of $P$, if the body of $C$ is a logical consequence of $P$, then so is its head.

Let $C$ be the following ground instance of a reductant of $P$:

$$\text{Obj: } \text{lub}\{H_1 + \xi_1, H_2 + \xi_2, \ldots, H_m + \xi_m\}$$

$$\leftarrow \text{Obj}_{11}: B_{11} + \xi_1 \text{ and } \text{Obj}_{12}: B_{12} + \xi_1 \text{ and } \ldots \text{ and } \text{Obj}_{1n}: B_{1n} + \xi_1 \text{ and } \text{Obj}_{21}: B_{21} + \xi_2 \text{ and } \text{Obj}_{22}: B_{22} + \xi_2 \text{ and } \ldots \text{ and } \text{Obj}_{2n}: B_{2n} + \xi_2 \text{ and } \ldots \text{ and } \text{Obj}_{mn}: B_{mn} + \xi_m \text{ and } \text{Obj}_{mn}: B_{mn} + \xi_m \text{ and } \ldots \text{ and } \text{Obj}_{mn}: B_{mn} + \xi_m,$$

where $\xi_i \in [0, 1]$ and $\text{Obj}: H_k \leftarrow \text{Obj}_{k1}: B_{k1} \text{ and } \text{Obj}_{k2}: B_{k2} \ldots \text{ and } \text{Obj}_{kn}: B_{kn}$ is a ground instance of a clause in $P (1 \leq k \leq m)$.

Let $I$ be a $g$-model of $P$. If the body of $C$ is a logical consequence of $P$ then, by Definition 3.3.6, $B_{ki} + \xi_k \in I(\text{Obj}_{ki}) (1 \leq k \leq m, 1 \leq i \leq n_k)$. Since $\text{Obj}: H_k \leftarrow \text{Obj}_{k1}:$
Proposition 3.5.2. Any normal AFLP constraint is solvable.

Proof
Let $C = \sigma_1 \leq_1 \phi_1 \& \sigma_2 \leq_2 \phi_2 \& \ldots \& \sigma_m \leq_m \phi_m$ be a normal AFLP constraint. One can assume that, if $\sigma_i$ and $\sigma_j$ ($i < j$) have the same variable then, for every $k$ from $i$ to $j$, $\sigma_k$ also has this variable. This can be achieved by the re-grouping operation presented in Kifer and Subrahmanian (1992), which preserves the normality of constraints. The following algorithm for testing the satisfiability of $C$ is adapted from that work for constraints on fuzzy set values:

1. If $C$ is empty, then return **satisfiable**.
2. Let $j_0$ be the maximal integer such that all $\sigma_i$’s, for $k$ from 1 to $j_0$, are either constants or have the same variable. Substitute 0 for each real number variable and for each fuzzy set variable occurring $\phi_1$, $\phi_2$, ..., $\phi_j$. This substitution is applied throughout $C$. After this step, all $\phi_1$, $\phi_2$, ..., $\phi_j$ are ground.
3. If $\sigma_k$’s are constants ($k$ from 1 to $j_0$):
   (a) If $\sigma_1 \leq_1 \phi_1 \& \ldots \& \sigma_j \leq_j \phi_j$, then set $C$ to $\sigma_{j+1} \leq_{j+1} \phi_{j+1} \& \ldots \& \sigma_m \leq_m \phi_m$. Re-index $\sigma_i$’s and $\phi_i$’s so that their indexes will start with 1. Go back to step 1.
   (b) Otherwise, return **unsatisfiable**.
4. If $\sigma_k = \Delta(A_k | \phi_k)$ ($k$ from 1 to $j_0$):
   (a) Set $\xi$ to $\max_{k=1,j} \{ \Delta(A_k | \phi_k) \}$ (this setting is applied throughout $C$), then set $C$ to $\sigma_{j+1} \leq_{j+1} \phi_{j+1} \& \ldots \& \sigma_m \leq_m \phi_m$. Re-index $\sigma_i$’s and $\phi_i$’s so that their indexes will start with 1. Go back to step 1.
5. If $\sigma_k = X (k$ from 1 to $j_0$):
   (a) Set $X$ to $\text{glb}_{k=1,j} \{ \phi_k \}$ (this setting is applied throughout $C$), then set $C$ to $\sigma_{j+1} \leq_{j+1} \phi_{j+1} \& \ldots \& \sigma_m \leq_m \phi_m$. Re-index $\sigma_i$’s and $\phi_i$’s so that their indexes will start with 1. Go back to step 1.

The correctness of the substitution in step 2 and the termination of the algorithm are respectively due to the monotonicity and the computability of functions, if any, in $\phi_i$’s. The correctness of the calculation of $\xi$ in step 4(a) is due to property 3 in Proposition 2.5.3.

Theorem 3.6.1. (AFLP Resolution Soundness) Let $P$ be an AFLP and $G$ be an AFLP goal. If $G < C_1 \theta_1 G_1 < C_2 \theta_2 \ldots G_{n-1} < C_n \theta_n G_n$ is a refutation of $G$ and $P$, and $\emptyset$ is a solution for $C_G$, then $<\theta_1 \theta_2 \ldots \theta_n \emptyset>$ is a correct answer for $G$ wrt $P$. 
Lemma 3.6.1. Let $G$ and $C$ be respectively a goal and a reductant of an AFLP. If $G$ is a normal AFLP goal, then any (unrestricted) resolvent of $G$ and $C$ is also a normal AFLP goal.

Proof
The proof is similar to the corresponding proof in Kifer and Subrahmanian (1992).

See Definition 3.6.3. Since $G$ and $C$ have no variable in common, if $\sigma_k$ contains a variable then this variable does not occur in $\phi$. By the same reason, any variable on the left-hand side of inequalities in $C_G$ does not occur in $\phi$. Thus, if $C_G$ is normal (i.e., $G$ is normal) then $\sigma_k \leq \phi$ and $C_G$ is also normal. Here, the order of $\sigma_k \leq \phi$ and $C_G$ is significant, as noted after Definition 3.6.3.

Lemma 3.6.1. (AFLP Mgu Lemma) Let $P$ be an AFLP and $G$ be an AFLP goal. If there exists an unrestricted refutation of $G$ and $P$, then there exists a refutation of $G$ and $P$. 

Proof
We prove the theorem by induction on refutation lengths.

1. $n = 1$: $Q_G$ must have only one annotated object, namely $O_1: \sigma_1$, and $C_1$ is a reductant $Obj: \phi$, where $\phi$ is ground, such that $\theta_1$ is a most general unifier of $O_1$ and $Obj$, and $\phi$ is a solution for $C_G = \sigma_1 \leq \phi$.

By Proposition 3.5.1, $Obj: \phi$ is a logical consequence of $P$, whence $Obj \theta_1: \phi$ is a logical consequence of $P$. Since $\theta_1$ is a most general unifier of $O_1$ and $Obj$, and $\phi$ is a solution for $\sigma_1 \leq \phi$, every annotation variable-free instance of $Obj \theta_1: \phi = (O_1: \sigma_1) \theta_1 \phi = Q_G \theta_1 \phi$ is a logical consequence of $P$. Thus, $\theta_1, \phi$ is a correct answer for $G$ wrt $P$.

2. Induction hypothesis: Suppose that it holds in the case of refutation length $n - 1$.

3. Let $G = O_1: \sigma_1 \& O_2: \sigma_2 \& ... \& O_m: \sigma_m \parallel C_G$ and $C_1 = Obj: \phi \leftarrow Obj: \phi_1 \& O bj: \phi_2 \& ... \& Obj: \phi_r$, and suppose that $O_k: \sigma_k$ is resolved by $C_1$. Then $G_1 = \{O_1: \sigma_1 \& ... \& O_{k-1}: \sigma_{k-1} \& (Obj_1: \phi_1 \& Obj_2: \phi_2 \& ... \& Obj_2: \phi_r) \& O_{k+1}: \sigma_{k+1} \& ... \& O_m: \sigma_m \parallel \sigma_1 \leq \phi \parallel C_G\}$. 

On the basis of the induction hypothesis, $\theta_2 \& ... \& \theta_m, \phi$ is a correct answer for $G_1$ wrt $P$. That is, every annotation variable-free instance of $[O_1: \sigma_1 \& ... \& O_{k-1}: \sigma_{k-1} \& (Obj_1: \phi_1 \& Obj_2: \phi_2 \& ... \& Obj_2: \phi_r) \& O_{k+1}: \sigma_{k+1} \& ... \& O_m: \sigma_m] \theta_1 \theta_2 \& ... \& \theta_m \phi$ is a logical consequence of $P$, and $\phi$ is a solution for $\sigma_1 \leq \phi \parallel C_G$.

Then, by Proposition 3.5.1, every annotation variable-free instance of $(Obj: \phi) \theta_1 \theta_2 \& ... \& \theta_m \phi$ is a logical consequence of $P$. Since $\theta_1$ is a most general unifier of $Obj$ and $O_k$, and $\phi$ is a solution for $\sigma_k \leq \phi$, one has that every annotation variable-free instance of $(O_k: \sigma_k) \theta_1 \theta_2 \& ... \& \theta_m \phi$ and thus every one of $Q_G \theta_1 \theta_2 \& ... \& \theta_m \phi$ are logical consequences of $P$. Therefore, $\theta_1 \theta_2 \& ... \& \theta_m \phi$ is a correct answer for $G$ wrt $P$. 

Proposition 3.6.1. Let $G$ and $C$ be respectively a goal and a reductant of an AFLP. If $G$ is a normal AFLP goal, then any (unrestricted) resolvent of $G$ and $C$ is also a normal AFLP goal.
Proof
Let $G < C_1, \theta_1 > G_1 < C_2, \theta_2 > \ldots G_{n-1} < C_n, \theta_n > G_n$ be an unrestricted refutation of $G$ and $P$. From the proof of the Mgu lemma in Lloyd (1987), there also exists the derivation $G < C_1, \theta_1 * > G_1 * < C_2, \theta_2 * > \ldots G_{n-1} * < C_n, \theta_n * > G_n *$ where, for each $i$ from 1 to $n$, $\theta_i *$ is a most general unifier, $G_i * = R_\theta(G_{i-1} *, C_i)$ $(G_0 * = G)$, and $G_i = G_i * \rho_i$, with $\rho_i$ being a substitution for object variables. Here, $Q_{G *} = Q_0 *$ which is empty and $C_{G *} = C_G$ which is solvable and has a solution, whence $G < C_1, \theta_1 * > G_1 * < C_2, \theta_2 * > \ldots G_{n-1} * < C_n, \theta_n * > G_n *$ is a refutation of $G$ and $P$.

Lemma 3.6.2. (AFLP Lifting Lemma) Let $P$ be an AFLP, $G$ be a normal AFLP goal, and $< \theta, \phi >$ be an answer for $G$ wrt $P$. If there exists a refutation of $G \theta \phi$ and $P$, then there exists a refutation of $G$ and $P$.

Proof
Let $G \theta \phi < C_1, \theta_1 > G_1 < C_2, \theta_2 > \ldots G_{n-1} < C_n, \theta_n > G_n$ be a refutation of $G \theta \phi$ and $P$. Then there exists the derivation $G \theta < C_1, \theta_1 > G_1 * < C_2, \theta_2 > \ldots G_{n-1} * < C_n, \theta_n > G_n *$ where, for each $i$ from 1 to $n$, $G_i * = R_\theta(G_{i-1} *, C_i)$ $(G_0 * = G \theta)$ and $G_i = G_i * \phi$. Here, $Q_{G *} = Q_G$ which is empty and $C_{G *} = C_G \phi$, whence $C_{G *}$ has a solution, because $C_G$ does. It is also solvable, because $G_0 *$ is a normal AFLP goal due to the normality of $G$ and Proposition 3.6.1. Thus, $G \theta < C_1, \theta_1 > G_1 * < C_2, \theta_2 > \ldots G_{n-1} * < C_n, \theta_n > G_n *$ is a refutation of $G \theta$ and $P$.

Further, from the proof of the Lifting lemma in Lloyd (1987), one obtains $G < C_1, \theta \theta_1 > G_1 * < C_2, \theta_2 > \ldots G_{n-1} * < C_n, \theta_n > G_n *$ as an unrestricted refutation of $G$ and $P$, whence there exists a refutation of $G$ and $P$, on the basis of the AFLP Mgu lemma.

Theorem 3.6.2. (AFLP Resolution Completeness) Let $P$ be an AFLP and $G$ be a normal AFLP goal. If there exists a correct answer for $G$ wrt $P$, then there exists a refutation of $G$ and $P$.

Proof
In the proof, we apply the Mgu and Lifting lemmas for annotated fuzzy logic programs presented above (Lemmas 3.6.1 and 3.6.2), as in proving the completeness of SLD-resolution style proof procedures for classical logic programs and annotated logic programs.

Since there exists a correct answer for $G$ wrt $P$, there also exists a ground correct answer, i.e., ground substitutions $\theta$ and $\phi$ such that $G \theta \phi$ is variable-free and $< \theta, \phi >$ is a correct answer for $G$ wrt $P$. Firstly, we prove that there exists an unrestricted refutation of $G \theta \phi$ and $P$. Let $Q_G = O_1: \sigma_1 \land O_2: \sigma_2 \land \ldots \land O_m: \sigma_m$. Since $Q_G \theta \phi$ is a logical consequence of $P$, there exists $n \geq 1$ such that $\sigma_i \phi \in T_P^\uparrow m(O_\theta)$ for every $i$ from 1 to $m$, on the basis of Theorem 3.4.1 and the finite fixpoint reachability of AFLPs with respect to the general semantics. We prove that there exists an unrestricted refutation of $G \theta \phi$ and $P$ by induction on such a number of upward iterations of $T_P$:

1. $n = 1$: one has $\sigma_i \phi \in T_P^\uparrow 1(O_\theta)$, for every $i$ from 1 to $m$. Hence, there must exist reductants $Obj_j$: $\phi_i$'s of $P$, where $\phi_i$'s are ground, and unifiers $\theta_i$'s such
that, for every \( i \) from 1 to \( m \), \( \text{Obj}_i \theta_i = O \theta \) and \( \sigma_i \varphi \leq \phi_i \). Then the following unrestricted refutation of \( G \varphi \) and \( P \) can be constructed:

\[
G \varphi, C_1, \theta_1, G_1 < C_2, \theta_2, \ldots, G_{m-1} < C_m, \theta_m > G_m
\]

where \( C_i = \text{Obj}; \phi_i (1 \leq i \leq m), G_i = O_{\theta_i}; \sigma_{i+1} \varphi \) and \( \& O_{\theta}; \sigma_m \varphi \) \( \& \sigma_i \varphi \leq \phi_i \) and \( \& \sigma_1 \varphi \leq \phi_1 \) and \( C_P \) is empty, and \( C_G = \sigma_m \varphi \leq \phi_m \) and \( \& \sigma_1 \varphi \leq \phi_1 \) and \( C_P \varphi \) is ground and holds.

2. Induction hypothesis: Suppose that it holds in the case of \( n - 1 \) upward iterations of \( T_P \).

3. One has \( \sigma_i \varphi \in T_P \uparrow n(O \theta) \), for every \( i \) from 1 to \( m \). Thus, there must exist reductants \( \text{Obj}; \phi_i \leftarrow \text{Obj}_{i+1}; \phi_{i+1} \& \text{Obj}_{i+2}; \phi_{i+2} \& \ldots \& \text{Obj}_{i+1}; \phi_{i+1} \) of \( P \), unifiers \( \theta_i \)’s and substitutions \( \varphi_i \)’s such that, for every \( i \) from 1 to \( m \): (1) \( \text{Obj} \theta_i = O \theta, \varphi_i \) is ground and \( \sigma_i \varphi \leq \phi_i \varphi \); and (2) for every \( j \) from 1 to \( r_i, \text{Obj} \theta_j; \phi_j \varphi \) is ground and \( \phi_j \varphi \in T_P \uparrow (n - 1)(\text{Obj} \theta_i) \). After \( m \) sequential resolution steps on \( O \theta; \sigma_i \varphi \), \( \ldots, O \theta; \sigma_m \varphi \), one obtains the following unrestricted resolvent:

\[
G^* = [(\text{Obj}_{i+1}; \phi_{i+1} \& \text{Obj}_{i+2}; \phi_{i+2} \& \ldots \& \text{Obj}_{i+1}; \phi_{i+1}) \& \ldots \& (\text{Obj}_{m+1}; \phi_{m+1} \& \text{Obj}_{m+2}; \phi_{m+2} \& \ldots \& \text{Obj}_{m+1}; \phi_{m+1})] \theta_1 \ldots \theta_m \| \sigma_m \varphi \leq \phi_m \) and \( \& \sigma_1 \varphi \leq \phi_1 \) and \( C_P \varphi \).

On the basis of the induction hypothesis, there exists an unrestricted refutation of \( G^* \varphi \) and \( P \). Applying the AFLP Mgu lemma and then the AFLP Lifting lemma, one has the result that there exists a refutation of \( G^* \) and \( P \). Thus, there exists an unrestricted refutation of \( G \varphi \) and \( P \).

Finally, applying the AFLP Mgu lemma and then the AFLP Lifting lemma again, one has the result that there exists a refutation of \( G \) and \( P \).

**Proposition 3.8.1.** Let \( P \) be a generalized AFLP, \( \text{Obj}; H \leftarrow \text{Obj}_1; B_1 \& \text{Obj}_2; B_2 \& \ldots \& \text{Obj}_n; B_n \) be an annotation variable-free instance of a reductant of \( P \), and \( \theta \) be a substitution for object variables in this reductant. If \( (\text{Obj}; B_1 \& \text{Obj}_2; B_2 \& \ldots \& \text{Obj}_n; B_n) \theta \) is a logical consequence of \( P \), then so is \( (\text{Obj}; H) \theta \).

**Proof**

It is sufficient to prove that, for every ground (i.e., both annotation variable-free and object variable-free) instance \( C \) of a reductant of \( P \), if the body of \( C \) is a logical consequence of \( P \), then so is its head.

Let \( C \) be the following ground instance of a reductant of \( P \):

\[
\text{Obj}; \text{lb}\{H_x; + \theta(1 - \alpha_x, \xi_x) \}, H_z; + \theta(1 - \alpha_z, \xi_z) \ldots, H_m; + \theta(1 - \alpha_m, \xi_m)\}
\]

\[
\leftarrow \text{Obj}_{11}; B_{11} + \xi_{11} \& \text{Obj}_{12}; B_{12} + \xi_{12} \& \ldots \& \text{Obj}_{1i}; B_{1i} + \xi_{1i} \& \text{Obj}_{21}; B_{21} + \xi_{21} \& \text{Obj}_{22}; B_{22} + \xi_{22} \& \ldots \& \text{Obj}_{2j}; B_{2j} + \xi_{2j} \& \ldots \& \text{Obj}_{mn}; B_{mn} + \xi_{mn} \& \text{Obj}_{mn}; B_{mn} + \xi_{mn}
\]

where \( \xi_x \in [0, 1] \) and \( \text{Obj}; H_k \leftarrow \text{Obj}_{k1}; B_{k1} \& \text{Obj}_{k2}; B_{k2} \& \ldots \& \text{Obj}_{kn}; B_{kn}; \alpha_k \) is a ground instance of a clause in \( P \) (1 \( \leq k \leq m \)).

Let \( I \) be a model of \( P \). If the body of \( C \) is a logical consequence of \( P \) then, by Definition 3.8.2, \( B_k + \xi_k \in I(\text{Obj}_{ki}) \) (1 \( \leq k \leq m, 1 \leq i \leq n_k \)). Since \( \text{Obj}; H_k \leftarrow \text{Obj}_{ki}; B_{ki} \& \text{Obj}_{ki2}; B_{k2} \& \ldots \& \text{Obj}_{kin}; B_{kin}; \alpha_k \) is a ground instance of a clause in \( P \), also by Definition 3.8.2, one has \( H_k + \theta(1 - \alpha_k, \max_{i=1,n}(\Delta(B_{ki} \& B_{ki} + \xi_k))) \in I(\text{Obj}) \) (1 \( \leq k \leq m \)).
By Propositions 2.5.2, 2.6.6 and 2.6.8, \( \Delta(B_{ki} | B_{ki} + \xi_k) \leq \xi_k \), whence \( \max_{i=1,n}(\Delta(B_{ki} | B_{ki} + \xi_k)) \leq \xi_k \). Then, due to the monotonicity of the \( t \)-conorm \( \oplus \), \( \ominus(1 - \alpha_k, \max_{i=1,n}(\Delta(B_{ki} | B_{ki} + \xi_k))) \leq \Theta(1 - \alpha_k, \xi_k) \), whence \( H_k + \Theta(1 - \alpha_k, \xi_k) \leq H_k + \Theta(1 - \alpha_k, \max_{i=1,n}(\Delta(B_{ki} | B_{ki} + \xi_k))) \). Since \( I \) as an ideal is downward closed, one has \( H_k + \Theta(1 - \alpha_k, \xi_k) \in I(\text{Obj}) \) (1 \( \leq k \leq m \)). Also as an ideal, \( I(\text{Obj}) \) is closed under finite least upper bounds, whence \( \text{lb}(H_1 + \Theta(1 - \alpha_1, \xi_1), H_2 + \Theta(1 - \alpha_2, \xi_2), ..., H_m + \Theta(1 - \alpha_m, \xi_m)) \in I(\text{Obj}) \), that is, \( \text{Obj}: \) \( \text{lb}(H_1 + \Theta(1 - \alpha_1, \xi_1), H_2 + \Theta(1 - \alpha_2, \xi_2), ..., H_m + \Theta(1 - \alpha_m, \xi_m)) \) is satisfied by \( I \).

Since \( I \) is an arbitrary model of \( P \), this means \( \text{Obj}: \text{lb}(H_1 + \Theta(1 - \alpha_1, \xi_1), H_2 + \Theta(1 - \alpha_2, \xi_2), ..., H_m + \Theta(1 - \alpha_m, \xi_m)) \) is a logical consequence of \( P \).

### A.3 Proofs for Chapter 4

#### Proposition 4.4.1

For any FCGP \( P, T_P \) is monotonic, that is, if \( I_1 \leq I_2 \) then \( T_P(I_1) \leq T_P(I_2) \).

**Proof**

See Definition 4.4.1. Let if \( u \) then \( v \) be a clause in \( P \) and \( \pi_{12} \) be an ideal FCG projection from \( I_1 \) to \( I_2 \). Suppose that \( v + \varepsilon_\pi \in S_P(I_1) \) where \( \pi_1 \) is an FCG projection from \( u \) to a principal instance \( g_1 \) of \( I_1 \). Since \( I_1 \leq I_2 \), there must exist a principal instance \( g_2 \) of \( I_2 \) such that a fuzzy value in each vertex in \( g_1 \) is the same as that in the corresponding vertex in \( g_2 \) and \( g_1 \leq g_2 \) by \( \pi_{12} \). Thus, there exists \( \pi_2 = \pi_{12} \cdot \pi_1 \) as an FCG projection from \( u \) to \( g_2 \) where \( \varepsilon_\pi = \varepsilon_\pi \), whence \( v + \varepsilon_\pi = v + \varepsilon_\pi \in S_P(I_2) \).

Furthermore, if \( c \in \text{VC}, c^* \in \text{VC} \) and \( \text{coref}(c, c^*) \), then \( \text{coref}(c, \pi_0(c)) \) in \( S_P(I_1) \cup I_1 \), and \( \text{coref}(c, \pi_0(c)) \) in \( S_P(I_2) \cup I_2 \). Thus, \( S_P(I_1) \cup I_1 \leq S_P(I_2) \cup I_2 \) by \( \pi_{12} \cup \pi_{0} \), where \( \pi_0 \) is an ideal FCG projection that maps each \( v + \varepsilon_\pi \in S_P(I_1) \) to the corresponding \( v + \varepsilon_\pi \in S_P(I_2) \), whence \( T_P(I_1) \leq T_P(I_2) \).

#### Proposition 4.4.2

Let \( P \) be an FCGP and \( I \) be an interpretation. Then \( I \) is a model of \( P \) iff \( T_P(I) \leq I \) by an ideal FCG projection \( \pi_I \) that maps the occurrence of \( I \) in \( T_P(I) \) to \( I \) itself.

**Proof**

Let if \( u \) then \( v \) be a clause in \( P \) and \( \pi \) be an FCG projection from \( u \) to a principal instance of \( I \). From Definition 4.3.6 and Definition 4.4.1, if \( I \) is a model of \( P \), then there is an ideal FCG projection \( \pi^* \) from \( v + \varepsilon_\pi \in S_P(I) \) to \( I \). Moreover, if \( c \in \text{VC}, c^* \in \text{VC} \) and \( \text{coref}(c, c^*) \), then \( \text{coref}(c, \pi_0(c)) \) in \( S_P(I) \cup I \) by Definition 4.4.1, and \( \pi^* \cdot c^* = \pi_0(c) \) by Definition 4.3.6. Thus, there is an ideal FCG projection \( \pi_I \) from \( T_P(I) = \text{norm}(S_P(I) \cup I) \) to \( I \) that comprises those \( \pi^* \)'s and maps the occurrence of \( I \) in \( T_P(I) \) to \( I \) itself.

On the other hand, let \( \pi_I \) be an ideal FCG projection from \( T_P(I) \) to \( I \) that maps the occurrence of \( I \) in \( T_P(I) \) to \( I \) itself. If \( c \in \text{VC}, c^* \in \text{VC} \) and \( \text{coref}(c, c^*) \), then \( \text{coref}(c, \pi_0(c)) \) in \( S_P(I) \cup I \) by Definition 4.4.1, whence \( \pi^* \cdot \pi_0(c) = \pi_0(c) \) by the property of \( \pi_0 \). Thus, by Definition 4.3.6, \( I \) satisfies if \( u \) then \( v \) as an arbitrary clause in \( P \), whence \( I \) is a model of \( P \).
**Theorem 4.4.1.** For any FCGP $P$, $T_P \uparrow \omega$ is the minimal model (modulo ideal FCG equivalence) of $P$.

**Proof**
We recall that $T_P \uparrow 0 \leq_1 T_P \uparrow 1 \leq_1 \ldots$ and, in $T_P \uparrow \omega = \text{norm}(T_P \uparrow n \mid n \in \mathbb{N})$, there are coreference links between concept vertices in $T_P \uparrow n$ and corresponding ones in the occurrence of $T_P \uparrow n$ in $T_P \uparrow (n + 1)$.

Firstly, we prove that $T_P \uparrow \omega$ is a model of $P$. Let if $u$ then $v$ be a clause in $P$ and $\pi$ be an FCG projection from $u$ to a principal instance of $T_P \uparrow \omega = \text{norm}(T_P \uparrow n \mid n \in \mathbb{N})$. Since $u$ is finite and all fuzzy values in $u$ are principal ideals, there must exist an FCG projection $\pi_k$ from $u$ to a principal instance of some $T_P \uparrow k$ with $\varepsilon_{\pi} = \varepsilon_{\pi_k}$. Thus, $S_P(\text{norm}(T_P \uparrow n)) \cup \text{norm}(T_P \uparrow n) \leq_1 \text{norm}((\cup S_P(T_P \uparrow n)) \cup \text{norm}(T_P \uparrow n))$ by an ideal FCG projection that maps $\text{norm}(T_P \uparrow n)$ to itself and each $v + \varepsilon_{\pi_k} \in S_P(\text{norm}(T_P \uparrow n))$ to the corresponding $v + \varepsilon_{\pi} \in S_P(T_P \uparrow k)$ for some $k \in \mathbb{N}$. Consequently, one has:

$$
T_P(\text{norm}(T_P \uparrow n))
= \text{norm}(S_P(\text{norm}(T_P \uparrow n))) \cup \text{norm}(T_P \uparrow n)
\leq_1 \text{norm}((\cup S_P(T_P \uparrow n)) \cup \text{norm}(T_P \uparrow n))
= \text{norm}(S_P(T_P \uparrow n) \cup T_P \uparrow n)
= \text{norm}(T_P \uparrow n)
= \text{norm}(T_P \uparrow (n + 1))
= \text{norm}(T_P \uparrow n)
$$

That is, $T_P(T_P \uparrow \omega) \leq_1 T_P \uparrow \omega$ by an ideal FCG projection that maps the occurrence of $T_P \uparrow \omega$ in $T_P(T_P \uparrow \omega)$ to itself. Thus, by Proposition 4.4.2, $T_P \uparrow \omega$ is a model of $P$.

We now prove that $T_P \uparrow \omega$ is the minimal model (modulo ideal FCG equivalence) of $P$. Let $I$ be a model of $P$. It is obvious that $T_P \uparrow 0 = [] \leq_1 I$. Supposing that $T_P \uparrow n \leq_1 I$, one has $T_P \uparrow (n + 1) = T_P(T_P \uparrow n) \leq_1 T_P(I)$, because $T_P$ is monotonic. Since $I$ is a model of $P$, by Proposition 4.4.2, one has $T_P(I) \leq_1 I$, whence $T_P \uparrow (n + 1) \leq_1 I$. Thus, by induction, $T_P \uparrow n \leq_1 I$ for every $n \in \mathbb{N}$. Therefore, $T_P \uparrow \omega = \text{norm}(T_P \uparrow n) \leq_1 I$, whence $T_P \uparrow \omega$ is the minimal model (modulo ideal FCG equivalence) of $P$.

**Proposition 4.6.1.** Let $P$ be an FCGP, if if $u$ then $v$ be an annotation variable-free instance of a reductant of $P$, and $\rho$ and $\omega$ be respectively a referent specialization and a coreference partition on concepts with VAR generic markers in this reductant. If $\rho \bowtie u$ is a logical consequence of $P$, then so is $\rho \bowtie v$.

**Proof**
The main concern is fuzzy values and real number variables in an FCGP reductant. They have the same declarative semantics as those in an AFLP reductant or order-sorted AFLP reductant, on the basis of the same fuzzy rule model. Thus, the proof, as well as that of Proposition 3.7.1 for order-sorted AFLP reductants, is similar to the proof of Proposition 3.5.1 for AFLP reductants.
Suppose the following annotation variable-free instance of a reductant of $P$:

$$\text{if } \rho_0[1 + \xi_1 \ 2 + \xi_2 \ ... \ m + \xi_m] \text{ then } \sigma_0[v_1 + \xi_1 \ v_2 + \xi_2 \ ... \ v_m + \xi_m]$$

where $\xi_k \in [0, 1]$ and if $u_k$ then $v_k$ is a clause in $P (1 \leq k \leq m)$.

Let $I$ be a model of $P$. If $\rho_0[1 + \xi_1 \ 2 + \xi_2 \ ... \ m + \xi_m]$ is a logical consequence of $P$ then, by Definition 4.3.6, $\rho_0[u_k + \xi_k] \leq I$ by some ideal FCG projection $\pi_k (1 \leq k \leq m)$. Since $I$ is an ideal FCG, there exists a principal instance $g_k$ of $I$ such that each fuzzy value in $g_k$ is the same as the corresponding one (wrt $\pi_k$) in $\rho_0(u_k + \xi_k)$. Thus, $\pi_k$ also defines an FCG projection from $\rho_0g_k$ to $g_k$ with mismatching degree $\varepsilon_\pi$ where, on the basis of Propositions 2.5.2, 2.6.6 and 2.6.8, $\varepsilon_\pi \leq \xi_k$.

Since if $u_k$ then $v_k$ is a clause in $P$, also by Definition 4.3.6, one has $\rho_0[v_k + \varepsilon_\pi_k] \leq I$, whence $\rho_0[v_k + \xi_k] \leq I (1 \leq k \leq m)$ and thus $\rho_0[v_1 + \xi_1 \ v_2 + \xi_2 \ ... \ v_m + \xi_m]$ is satisfied by $I$. This means $\rho_0[v_1 + \xi_1 \ v_2 + \xi_2 \ ... \ v_m + \xi_m]$ is a logical consequence of $P$, because $I$ is an arbitrary model of $P$.

**Proposition 4.7.1.** The condition on $\theta$ in Definition 4.7.3 is necessary and sufficient so that NON-VAR generic markers do not occur in FCGP resolvents.

**Proof**

Necessary: if the condition does not hold, then NON-VAR generic markers will occur in $\rho_0\sigma_0\delta_0G$.

Sufficient: The condition guarantees that NON-VAR generic markers do not occur in $\rho_0\sigma_0\delta_0G$, if they do not occur in $G$ before (e.g. when $G$ is a query, i.e., an original goal). By Definition 4.6.3, for a reductant if $u$ then $v$, no VAR generic marker in $v$ is bound to a NON-VAR one. This guarantees that NON-VAR generic markers do not occur in $\rho_0\sigma_0\delta_0u$.

**Theorem 4.7.1.** (FCGP Resolution Soundness) Let $P$ be an FCGP and $G$ be an FCGP goal. If $G < C_1, \theta_1 > C_1 < C_2, \theta_2 > ... < C_n, \theta_n > G_n$ is a refutation of $G$ and $P$, and $\rho$ is a solution for $C_G$, then $< \rho_0\rho_0 ... \rho_0, \sigma_0\sigma_0 ... \sigma_0, \phi >$ is a correct answer for $G$ wrt $P$.

**Proof**

We prove by induction on refutation lengths that $\phi$ is a solution for $C_G$ and every annotation variable-free instance of $\rho_0\rho_0 ... \rho_0\sigma_0\sigma_0 ... \sigma_0\phiQ_G$ is a logical consequence of $P$:

1. $n = 1$: $Q_G$ is empty, $C_G \models C_\theta$ & $C_G$ is satisfied by $\phi$, whence $\phi$ is a solution for $C_G$, and $C_1$ must have no body, whence all generic markers in $C_1$ are NON-VAR ones (Definition 4.6.1). Thus, $\theta_1$ also defines an FCG projection with mismatching degree 0 from each annotation variable-free instance of $\rho_0\sigma_0\phiQ_G$ to $\text{norm}(C_1)$. By Proposition 4.6.1, $C_1$ is a logical consequence of $P$, whence every annotation variable-free instance of $\rho_0\sigma_0\phiQ_G$ is also a logical consequence of $P$. Thus, $< \rho_0, \sigma_0, \phi >$ is a correct answer for $G$ wrt $P$.

2. Induction hypothesis: Suppose that it holds in the case of refutation length $n - 1$. 
3. Let $G = Q_G \parallel C_G$, $C_1 = \text{if } u \text{ then } v, \rho = \rho_0 \rho_1 \ldots \rho_n$ and $\tau = \tau_0 \tau_1 \ldots \tau_n$. On the basis of the induction hypothesis, $\phi$ is a solution for $C_0 \& C_G$, whence for $C_G$, and every annotation variable-free instance of $\rho \odot \phi[\delta_0 Q_G \ u]$ is a logical consequence of $P$, whence every one of $\rho \odot \phi v$ is also a logical consequence of $P$. By Proposition 4.6.1, every annotation variable-free instance of $\rho \odot \phi v$ is a logical consequence of $P$, whence every one of $\rho \odot \phi[\delta_0 Q_G \ v]$ is also a logical consequence of $P$.

What remains is to prove that every annotation variable-free instance of $\rho \odot \phi Q_G$ has an FCG projection with mismatching degree 0 to the corresponding annotation variable-free instance of $\text{norm}(\rho \odot \phi[\delta_0 Q_G \ v])$. In fact, the vertices in $\rho \odot \phi Q_G$ that remain after the first step of the refutation can be projected to themselves in $\rho \odot \phi \delta_0 Q_G$, with mismatching degree 0. Meanwhile, those that are deleted by $\delta_0$ can be projected to the vertices in $\text{norm}(\rho \odot \phi v)$ that they are unified with by $\theta_1$. The mismatching degrees of these projections are also 0, because $\phi$ is a solution for $C_0$. Thus, every annotation variable-free instance of $\rho \odot \phi Q_G$ is a logical consequence of $P$, whence $\langle \rho, \tau, \phi \rangle$ is a correct answer for $G \text{ wrt } P$.

**Lemma 4.7.1.** (FCGP Lifting Lemma) Let $P$ be a normal FCGP, $G$ be a normal FCGP goal, and $\langle \rho, \tau, \phi \rangle$ be an answer for $G \text{ wrt } P$. If there exists a refutation of $\rho \odot \phi G$ and $P$, then there exists a refutation of $G$ and $P$.

**Proof**

Let $\rho \odot \phi G \prec C_1, \theta_1, G_1 \prec C_2, \theta_2 \ldots \prec C_n, \theta_n, G_n$ be a refutation of $\rho \odot \phi G$ and $P$. Then there exists the derivation $G \prec C_1, \theta_1, G_1 \ast \prec C_2, \theta_2 \ldots \prec C_n, \theta_n, G_n \ast$ where, for each $i$ from 1 to $n$, $G_i \ast = R_u(G_{i-1} \ast, C_i) (G_0 \ast = G)$. $Q_G = \rho \odot \phi Q_G \ast$ and $C_G = \phi C_G \ast$. Here, $Q_G \ast = Q_G$ which is empty, and $C_G = \phi C_G \ast$ whence $C_G \ast$ has a solution, because $C_G$ does, and is solvable, because $C_G$ is a normal FCGP constraint due to the normality of $G$ and $P$ and Proposition 4.7.2. Thus, $G \prec C_1, \theta_1, G_1 \ast \prec C_2, \theta_2 \ldots \prec C_n, \theta_n, G_n \ast$ is a refutation of $G$ and $P$.

**Theorem 4.7.2.** (FCGP Resolution Completeness) Let $P$ be a normal FCGP and $G$ be a normal FCGP goal. If there exists a correct answer for $G \text{ wrt } P$, then there exists a refutation of $G$ and $P$.

**Proof**

In the proof, we apply the Lifting lemma for FCG programs presented above (Lemma 4.7.1).

Since there exists a correct answer for $G \text{ wrt } P$, there also exists an annotation variable-free correct answer, i.e., an answer $\langle \rho, \tau, \phi \rangle$ such that $\rho \odot \phi G$ is annotation variable-free, $\phi C_G$ holds and $\rho \odot \phi Q_G$ is a logical consequence of $P$. First, we prove that there exists a refutation of $\rho \odot \phi G$ and $P$.

Since $\rho \odot \phi Q_G$ is a logical consequence of $P$, there exists $n \geq 1$ such that $\rho \odot \phi Q_G \leq T_P \downarrow n$, on the basis of Theorem 4.4.1. We now prove that there exists a refutation of $\rho \odot \phi G$ and $P$ by induction on such a number of upward iterations of $T_P$:
1. \( n = 1 \): \( \rho \sigma \varphi Q_G \leq T_P \uparrow 1 \). Thus, there exist reductants \( C_1, C_2, \ldots, C_m \) of \( P \) which have no bodies and are annotation variable-free such that \( \rho \sigma \varphi Q_G \leq norm(\{C_1 C_2 \ldots C_m\}) \) by an FCG projection \( \pi \) (with mismatching degree 0).

   For every \( i \) from 1 to \( m \), let \( \theta_i \) be the unification with \( norm(C_i) \) of the normalized subgraph of \( \rho \sigma \varphi Q_G \) that is projected to \( norm(C_i) \) by \( \pi \), where \( C_0 \) holds. Then, the following refutation of \( \rho \sigma \varphi G \) and \( P \) can be constructed:

\[
\rho \sigma \varphi <C_1, \theta_1> G_1 <C_2, \theta_2> \ldots G_{m-1} <C_m, \theta_m> G_m
\]

where \( Q_G \) is empty, and \( C_G = C_0 \& C_0 \& \ldots \& C_0 \& \varphi C_G \) is annotation variable-free and holds.

2. Induction hypothesis: Suppose that it holds in the case of \( n - 1 \) upward iterations of \( T_P \).

3. Assume \( \rho \sigma \varphi Q_G \leq T_P \uparrow n \). Then, there must exist reductants \( C_i = \text{if } u_i \text{ then } v_i \) of \( P \) (\( 1 \leq i \leq m \)), a referent specialization \( \rho^* \) and a coreference partition \( \sigma^* \) on concepts with \( \text{VAR} \) generic markers in \( C_i \)'s, and a substitution \( \varphi^* \) for annotation variables in \( C_i \)'s such that: (1) \( \varphi^* C_i \)'s are annotation variable-free and \( \rho \sigma \varphi Q_G \leq norm(\rho^* \sigma^* \varphi^*[v_1 v_2 \ldots v_m]) \) by an FCG projection \( \pi \) (with mismatching degree 0); and (2) \( norm(\rho^* \sigma^* \varphi^*[u_1 u_2 \ldots u_m]) \leq T_P \uparrow (n - 1) \).

For every \( i \) from 1 to \( m \), let \( \theta_i \) be the unification with \( norm(v_i) \) of the normalized subgraph of \( \rho \sigma \varphi Q_G \) that is projected to \( norm(\rho^* \sigma^* \varphi^* v_i) \) by \( \pi \), where \( \varphi C_0 \) holds.

Let \( \rho' \) and \( \sigma' \) respectively denote \( \rho_0 \rho_0 \ldots \rho_0 \) and \( \sigma_0 \sigma_0 \ldots \sigma_0 \). If \( \rho' \) binds an individual marker in \( \rho \sigma \varphi Q_G \) to a concept with a \( \text{VAR} \) generic marker in \( v_i \)'s, then \( \rho^* \) must also bind this individual marker to the concept, because \( \rho \sigma \varphi Q_G \leq norm(\rho^* \sigma^* \varphi^*[v_1 v_2 \ldots v_m]) \). By the same reason, if \( \sigma' \) groups two concepts in \( v_i \)'s in one coreference partition, then \( \sigma^* \) must also group the two concepts in one coreference partition. Therefore, one has \( \rho' \sigma' \varphi^*[v_1 v_2 \ldots v_m] \leq norm(\rho^* \sigma^* \varphi^*[v_1 v_2 \ldots v_m]) \), whence \( \rho' \sigma' \varphi^*[u_1 u_2 \ldots u_m] \leq norm(\rho^* \sigma^* \varphi^*[u_1 u_2 \ldots u_m]) \).

After \( m \) sequential resolution steps on \( C_1, C_2, \ldots, C_m \), one obtains the resolvent:

\[
G^* = \rho' \sigma' [u_1 u_2 \ldots u_m] \parallel C_0 \& C_0 \& \ldots \& C_0 \& \varphi C_G.
\]

Here, \( \varphi G^* \) is annotation variable-free. Also, \( \varphi Q_{G^*} = \rho' \sigma' \varphi^*[u_1 u_2 \ldots u_m] \leq norm(\rho^* \sigma^* \varphi^*[u_1 u_2 \ldots u_m]) \leq T_P \uparrow (n - 1) \) and \( \varphi C_{G^*} = \varphi^*(C_0 \& C_0 \& \ldots \& C_0 \& \varphi C_G) \) holds. Thus, on the basis of the induction hypothesis, there exists a refutation of \( \varphi G^* \) and \( P \). Then, by the FCGP Lifting lemma, there exists a refutation of \( G^* \) and \( P \), whence there exists a refutation of \( \rho \sigma \varphi G \) and \( P \).

Finally, applying the FCGP Lifting lemma again, one has the result that there exists a refutation of \( G \) and \( P \).

### A.4 Proofs for Chapter 5

**Proposition 5.2.1.** For any fuzzy number \( A \):

1. \( A \leq \text{at least } A \) and \( B \subseteq \text{at least } A \) for every \( B \) such that \( A \leq B \), and
2. \( \text{at most } A \leq A \) and \( B \subseteq \text{at most } A \) for every \( B \) such that \( B \leq A \).
1. (a) Suppose the extension principle is applied for fuzzy arithmetic. According to Definition 5.2.1, one has:
\[ \forall x \in \mathbb{R}: \mu_{at\,\,least\,\,A}(z) = \sup \{ \mu_A(y) \mid y \leq z \} \geq \mu_A(z) \]

Meanwhile, for every \( x \in \mathbb{R} \):
\[ \mu_{\min\{A, \, at\,\,least\,\,A\}}(z) = \sup \{ \min \{ \mu_A(x), \mu_{at\,\,least\,\,A}(y) \} \mid z = \min \{ x, y \} \} \]
\[ = \max \{ \sup \{ \min \{ \mu_A(z), \mu_{at\,\,least\,\,A}(y) \} \mid z \leq y \}, \sup \{ \min \{ \mu_A(x), \mu_{at\,\,least\,\,A}(z) \} \mid z \leq x \} \} \]
\[ = \max \{ \mu_A(z), \sup \{ \min \{ \mu_A(x), \mu_{at\,\,least\,\,A}(z) \} \mid z \leq x \} \} \]

For every point \( (z, \mu_A(z)) \) in the non-decreasing part of \( \mu_A \), \( \sup \{ \min \{ \mu_A(x), \mu_{at\,\,least\,\,A}(z) \} \mid z \leq x \} = \mu_A(\zeta) \). For every other point \( (z, \mu_A(z)) \) in the decreasing part of \( \mu_A \), \( \sup \{ \min \{ \mu_A(x), \mu_{at\,\,least\,\,A}(z) \} \mid z \leq x \} = \mu_A(z) \). Therefore \( \forall x \in \mathbb{R}: \mu_{\min\{A, \, at\,\,least\,\,A\}}(z) = \mu_A(z) \), i.e., \( \min\{A, \, at\,\,least\,\,A\} = A \) or \( A \leq at\,\,least\,\,A \).

We now prove that, if \( B \subseteq at\,\,least\,\,A \) does not hold, then neither does \( A \leq B \). If \( \exists x \in \mathbb{R}: B(z) > at\,\,least\,\,A(z) \), then the point \( (z, \mu_A(z)) \) must be in the non-decreasing part of \( \mu_A \), because \( at\,\,least\,\,A(z) = 1 \) otherwise. One has:
\[ \mu_{\min\{A, \, B\}}(z) = \sup \{ \min \{ \mu_A(x), \mu_B(y) \} \mid z = \min \{ x, y \} \} \]
\[ = \max \{ \sup \{ \min \{ \mu_A(z), \mu_B(y) \} \mid z \leq y \}, \sup \{ \min \{ \mu_A(x), \mu_B(z) \} \mid z \leq x \} \} \]
\[ \geq \sup \{ \min \{ \mu_A(x), \mu_B(z) \} \mid z \leq x \} \]
\[ > \mu_A(z) \]
and thus \( \min\{A, \, B\} \neq A \), i.e., \( A \leq B \) does not hold.

(b) If the \( \alpha \)-cut interval arithmetic method is applied and \( A^\alpha = [a, b] \), then:
\[ \min\{A, \, at\,\,least\,\,A\}^\alpha = \min\{A^\alpha, \, at\,\,least\,\,A^\alpha\} \]
\[ = \min\{[a, b], \{[a, +\infty]\} = [a, b] = A^\alpha, \]
whence \( \min\{A, \, at\,\,least\,\,A\} = A \), i.e., \( A \leq at\,\,least\,\,A \).

Meanwhile, if \( A \leq B, A^\alpha = [a, b], \) and \( B^\alpha = [c, d] \), then \( A^\alpha \subseteq B^\alpha \). Therefore \( a \leq c \) and thus \( B^\alpha \subseteq at\,\,least\,\,A^\alpha \), whence \( B \subseteq at\,\,least\,\,A \).

2. The proof is similar.

**Proposition 5.2.2.** For any fuzzy number \( A \):

1. at least \( A = A \) if \( A \) is a non-decreasing fuzzy number, and
2. at most \( A = A \) if \( A \) is a non-increasing fuzzy number.

**Proof**

1. (a) Suppose the extension principle is applied for fuzzy arithmetic. According to Definition 5.2.1, one has:
\[ \forall x \in \mathbb{R}: \mu_{at\,\,least\,\,A}(x) = \sup \{ \mu_A(y) \mid y \leq x \} = \mu_A(x) \]
because \( A \) is a non-decreasing fuzzy number. Thus \( at\,\,least\,\,A = A \).
2. The proof is similar.

**Proposition 5.2.3.** For any fuzzy number $A$ and positive fuzzy number $B$:

1. at least $(A/B) = (at least A) / B$, and
2. at most $(A/B) = (at most A) / B$.

**Proof**

1. (a) Suppose the extension principle is applied for fuzzy arithmetic.

   Since $B$ is positive, only $\mu_B(y)$ where $y > 0$ need to be counted in the following:

   \[
   \forall x \in \mathbb{R}: \mu_{at \ least \ (A/B)}(t) = \sup \{\mu_{A/B}(z) \mid z \leq t\} = \sup \{\sup \{\min \{\mu_A(x), \mu_B(y)\} \mid x/y \leq t\} =
   \sup \{\sup \{\min \{\mu_A(z), \mu_B(y)\} \mid z \leq x\} \mid t = x/y\}
   = \sup \{\min \{\sup \{\mu_A(z) \mid z \leq x\}, \mu_B(y)\} \mid t = x/y\} = \mu_{at \ least \ A / B}(t), \text{ whence at least } (A/B) = (at least A) / B.
   \]

   (b) If the $\alpha$-cut interval arithmetic method is applied, then let $A^\alpha = [a, b]$ and $B^\alpha = [c, d]$. Then one has:

   \[
   (at least (A/B))^\alpha = at least (A^\alpha/B^\alpha) = at least ([a, b]/[c, d]) = at least ([a/d, b/c]) = [a/d, +\infty) = [a, +\infty]/[c, d] = (at least A^\alpha) / B^\alpha = (at least A) / B^\alpha.
   \]

2. The proof is similar.

**Theorem 5.4.1.** Let $G$ and $H$ be two plain CGs with no coreference links. If $G \leq H$, then $G$ is a logical consequence of $H$.

**Proof**

It is proved by showing that the expansion of $G$ can be derived from the expansion of $H$ using the direct inference rules on CGs given in Sowa (1984).

1. If both $G$ and $H$ are existential CGs, then the theorem holds on the basis of Theorem 4.3.7 in Sowa (1984). Note that, when CGs are existentially quantified, the universal CG projection performs as the existential one.

2. Otherwise, in general, let $G_e$ and $H_e$ be respectively the defining expansions of $G$ and $H$, where the negative context could be empty (i.e., in the case of an existential CG):

   \[
   G_e = \neg[ e_{11} e_{32} \ldots e_{3m} \neg[G_3]] \quad H_e = \neg[ e'_{11} e'_{32} \ldots e'_{3m} \ldots e'_{3n} \neg[H_3]]
   \]

   According to Definition 5.4.2, one has $n \geq m$, each universal concept $e_i$ in $G$ can be projected to one or more universal concepts $e'_j$ in $H$, and some existential concepts in $G$ can be projected to some universal concepts $e'_j$ in $H$. For each partition $S$ of universal concepts in $H$ that are coreferent after projection:
(i) If \( \Pi^{-1}S \) contains a universal concept in \( G \), then the corresponding subset of \( S \) in \( \{ c'_1, c'_2, \ldots, c'_{\exists m}, \ldots, c'_{\exists n} \} \) is unified and type-restricted into one existential concept whose type is equal to that of the universal concept in \( \Pi^{-1}S \), or

(ii) If \( \Pi^{-1}S \) contains only existential concepts in \( G \), then the corresponding subset of \( S \) in \( \{ c'_1, c'_2, \ldots, c'_{\exists m}, \ldots, c'_{\exists n} \} \) is unified and type-restricted into one existential concept whose type is equal to the intersection of the types of the concepts \( S \) and the types of the existential concepts in \( G \). Since that type intersection is not absurd, the resulting concept can be erased from the negative context of \( H_e \).

Then the coreferent concepts in \( H_e \) are joined and type-restricted accordingly, deriving \( \neg[ c_{\exists 1} c_{\exists 2} \ldots c_{\exists m} \neg[H'_{\exists}]] \) where \( H'_{\exists} \leq G_{\exists} \), which entails \( \neg[ c_{\exists 1} c_{\exists 2} \ldots c_{\exists m} \neg[G_{\exists}]] = G_e \) with coreference links attached to concepts in \( H'_{\exists} \) are transferred to the corresponding concepts in \( G_{\exists} \).

**Lemma 5.4.1.** If \( \neg[u \neg[v]] \) is true, where \( u \) and \( v \) are existential CGs and there may be coreference links between them, then the following inference rules are sound:

1. In a positive (i.e., evenly enclosed context), \( u \) is replaced by \( v \) and coreference links attached to concepts in \( u \) are transferred to the corresponding concepts in \( v \).
2. In a negative (i.e., oddly enclosed context), \( v \) is replaced by \( u \) and coreference links attached to concepts in \( v \) are transferred to the corresponding concepts in \( u \).

**Proof**

1. The proof is similar to that of a case in Theorem 4.3.7 in Sowa (1984).
2. This is dual to case 1.

**Theorem 5.4.2.** Let \( G \) be a universal CG and \( H \) be any plain CG, with no coreference links in \( G \) and \( H \). Then, any \( J_{\forall}(G, H) \) is a logical consequence of \( \{G, H\} \).

**Proof**

Assume that \( H \) is also a universal CG. The proof is similar otherwise. Let \( G_e \) and \( H_e \) be respectively the defining expansions of \( G \) and \( H \):

\[
G_e = \neg[ c_{\exists 1} \ldots c_{\exists i} c_{\exists i+1} \ldots c_{\exists j} \ldots c_{\exists n} \neg[G_{\exists}]]
\]

\[
H_e = \neg[ c'_{\exists 1} \ldots c'_{\exists i} \ldots c'_{\exists m} \neg[H_{\exists}]]
\]

supposing that \( c_1, \ldots, c_i \) are the universal concepts in \( G \) that are joined with the universal concepts \( c'_1, \ldots, c'_i \) in \( H \), and \( c_{i+1}, \ldots, c_j \) are the universal concepts in \( G \) that are joined with existential concepts in \( H \). So, the defining expansion of \( J_{\forall}(G, H) \) is:

\[
J_{\forall}(G, H)_e = \neg[ c'_{\exists 1} \ldots c'_{\exists i} \ldots c'_{\exists m} c_{\exists j+1} \ldots c_{\exists n} \neg[J_{\exists}(G_{\exists}, H_{\exists})]]
\]

where \( J_{\exists}(G_{\exists}, H_{\exists}) \) is the conventional join of \( G_{\exists} \) and \( H_{\exists} \) that corresponds to \( J_{\forall}(G, H) \).
Proposition 5.6.2. Then $\gamma(G, H)_e$ is provable from $G_e$ and $H_e$ by the CG inference rules as follows:

\[
\{ G_e, H_e \} \Rightarrow \neg[H_3 c_{j1} \ldots c_{j3} \ldots c_{j3j+1} \ldots c_{j3j+n} \neg[Q_3]] \quad (\text{Insertion})
\]

$H_3$ is inserted into the negative context of $G_e$

\[
\Rightarrow \neg[H_3 c_{j1} \ldots c_{j3} \ldots c_{j3j+1} \ldots c_{j3j+n} \neg[Q_3 H_3]] \quad (\text{Iteration})
\]

$H_3$ is copied into $\neg[Q_3]$

\[
\Rightarrow \neg[H_3 c_{j3j+1} \ldots c_{j3n} \neg[Q_3 H_3]] \quad (\text{Coreferent Join})
\]

$c_{j1}, \ldots, c_{j3}$ are type-restricted to and joined with $c'_{j1} \ldots c'_{j3}$ in $H_3$

$c_{j3j+1}, \ldots, c_{j3j+n}$ are type-restricted to and joined with the corresponding existential concepts in $H_3$, then $G_3$ is type-restricted accordingly to become $G'_{3}$

\[
\Rightarrow \neg[H_3 c_{j3j+1} \ldots c_{j3n} \neg[J_3(G_3, H_3)]] \quad (\text{Coreferent Join})
\]

$G'_{3}$ and $H_3$ are coreferent-joined accordingly

\[
\Rightarrow \neg[c'_{j1} \ldots c'_{j3} \ldots c'_{j3m} c_{j3j+1} \ldots c_{j3j+n} \neg[J_3(G_3, H_3)]] \quad (\text{Lemma 5.4.1})
\]

$H_3$ is replaced by $c'_{j1} \ldots c'_{j3} \ldots c'_{j3m}$ and coreference links attached to concepts in $v_3$ are transferred accordingly to $c'_{j1}, \ldots, c'_{j3}, \ldots, c'_{j3m}$.

**Proposition 5.6.1.** A generally quantified FCG $G$ with a relative quantifier $Q$ entails a generally quantified FCG $G^*$ obtained from $G$ by generalizing it except for its generally quantified concept, and replacing $Q$ with at least $Q$.

**Proof**

Let the expansion of $G$ be if $F$ then $H [Q]$. Then the expansion of $G^*$ is if $F$ then $H^* [at least Q]$, where $H^*$ is a generalization of $H$. One has $Pr(H^* \mid F) \geq Pr(H \mid F)$, whence $Pr(H \mid F) = Q$ entails $Pr(H^* \mid F) = at least Q$. Thus if $F$ then $H [Q]$ entails if $F$ then $H^* [at least Q]$, i.e., $G$ entails $G^*$.

**Proposition 5.6.2.** A generally quantified FCG $G$ with a relative quantifier $Q$ entails a generally quantified FCG $G^*$ obtained from $G$ by specializing it except for its generally quantified concept, and replacing $Q$ with at most $Q$.

**Proof**

Let the expansion of $G$ be if $F$ then $H [Q]$. Then the expansion of $G^*$ is if $F$ then $H^* [at most Q]$, where $H^*$ is a specialization of $H$. One has $Pr(H^* \mid F) \leq Pr(H \mid F)$, whence $Pr(H \mid F) = Q$ entails $Pr(H^* \mid F) = at most Q$. Thus if $F$ then $H [Q]$ entails if $F$ then $H^* [at most Q]$, i.e., $G$ entails $G^*$.

**Proposition 5.6.3.** A universally quantified FCG $G$ entails a universally quantified FCG $G^*$ obtained from $G$ by specializing its lambda FCG.

**Proof**

Let the expansion of $G$ be if $F$ then $H [1]$. Then the expansion of $G^*$ is if $F^*$ then $H^* [1]$, where $F^*$ and $H^*$ are specializations of $F$ and $H$, respectively, with their
two co-referent concepts corresponding to the lambda concept in $G$. More precisely, if the type of the lambda concept in $G$ is specialized in $G^*$, then $H^*$ is only different from $H$ in that type specialization; otherwise, $H^*$ and $H$ are identical. As such, $H^* \wedge F^*$ is semantically equivalent to $H \wedge F^*$, whence $Pr(H^* | F^*) = Pr(H^* \wedge F^* | F^*) / Pr(F^*) = Pr(H \wedge F^* | F^*) / Pr(F^*) = (Pr(F^*) - Pr(\neg H \wedge F^*)) / Pr(F^*)$. Furthermore, one has $Pr(\neg H \wedge F^*) \leq Pr(\neg H \wedge F) = Pr(\neg H | F) \cdot Pr(F) = (1 - Pr(H | F)) \cdot Pr(F) = 0$, as $Pr(H | F) = 1$, whence $Pr(H^* | F^*) = 1$. That is, $Pr(H | F) = 1$ entails $Pr(H^* | F^*) = 1$ or, in other words, if $F$ then $H [1]$ entails if $F^*$ then $H^*$ [1], i.e., $G$ entails $G^*$.

**Proposition 5.6.4.** A generally quantified FCG $G$ with an absolute quantifier $Q$ entails a generally quantified FCG $G^*$ obtained from $G$ by generalizing it, including its lambda FCG, and replacing $Q$ with at least $Q$.

**Proof**

Let the expansion of $G$ be $F$ then $H [Q_F]$, where $F$ corresponds to $G$’s lambda FCG, and $Q_F = Q/|F|$ with $|F|$ being the cardinality of the denotation set in a universe of discourse of the type that $F$ represents, as noted in Section 5.5. Then the expansion of $G^*$ is if $F^*$ then $H^*$ [at least $Q / |F^*|$], where $F^*$ and $H^*$ are respectively generalizations of $F$ and $H$. One has $Pr(H^* | F^*) = Pr(H^* \wedge F^* | F^*) / Pr(F^*) \geq Pr(H \wedge F) / Pr(F^*) = Pr(H | F) \cdot Pr(F) / Pr(F^*) = (Pr(F | F^*) / Pr(F^*)) \cdot Pr(F^*)$, and $Pr(F) = Pr(F \wedge F^*) = Pr(F | F^*) \cdot Pr(F^*) = (|F| / |F^*|) \cdot Pr(F^*)$, whence $Pr(H^* | F^*) \geq Q / |F^*|$. Thus $Pr(H | F) = Q / |F|$ entails $Pr(H^* | F^*) = at least (Q / |F^*|)$. According to Proposition 5.2.3, at least $(Q / |F^*|) = at least Q / |F^*|$, whence if $F$ then $H [Q / |F|]$ entails if $F^*$ then $H^* [at least Q / |F^*|]$, i.e., $G$ entails $G^*$.

**Proposition 5.6.5.** A generally quantified FCG $G$ with an absolute quantifier $Q$ entails a generally quantified FCG $G^*$ obtained from $G$ by specializing it, including its lambda FCG, and replacing $Q$ with at most $Q$.

**Proof**

Let the expansion of $G$ be if $F$ then $H [Q/|F|]$, where $F$ corresponds to $G$’s lambda FCG, and $|F|$ is the cardinality of the denotation set in a universe of discourse of the type that $F$ represents. Then the expansion of $G^*$ is if $F^*$ then $H^*$ [at most $Q / |F^*|$], where $F^*$ and $H^*$ are respectively specializations of $F$ and $H$. One has $Pr(H^* | F^*) = Pr(H^* \wedge F^* | F^*) / Pr(F^*) \leq Pr(H \wedge F) / Pr(F^*) = Pr(H | F) \cdot Pr(F) / Pr(F^*) = ((Q / |F|) \cdot Pr(F) / Pr(F^*))$, and $Pr(F^*) = Pr(F^* \wedge F) = Pr(F^* | F) \cdot Pr(F) = (|F^*| / |F|) \cdot Pr(F)$, whence $Pr(H^* | F^*) \leq Q / |F^*|$. Thus $Pr(H | F) = Q / |F|$ entails $Pr(H^* | F^*) = at most (Q / |F^*|)$. According to Proposition 5.2.3, at at most $(Q / |F^*|) = at most Q / |F^*|$, whence if $F$ then $H [Q / |F|]$ entails if $F^*$ then $H^* [at most Q / |F^*|]$, i.e., $G$ entails $G^*$.

**Proposition 5.6.6.** Let $G$ be a generally quantified FCG with a relative quantifier $Q$, and $G^*$ be a simple FCG such that there is a probabilistic FCG projection $\pi$ from the lambda FCG in $G$ to $G^*$. Then Jeffrey’s rule derives the simple FCG $H^*$ with the probability (at least $(Q \cdot \varepsilon_{\pi}) \cap (at most (Q \cdot \varepsilon_{\pi} + (1 - \varepsilon_{\pi})))$ where $H^*$ is
obtained from $G$ by replacing its generally quantified concept with its lambda concept $c$ whose referent $\lambda$ is replaced with $\text{referent}(\pi c)$.

**Proof**

Let the expansion of $G$ be $\textbf{if } F \textbf{ then } H [Q]$, where $F$ corresponds to $G$’s lambda FCG. Let $F^*$ be the simple FCG obtained from $G$’s lambda FCG by replacing the referent of its lambda concept $c$ with $\text{referent}(\pi c)$. As such, $F$ and $F^*$ are different only in that referent, whence $\pi F^* = \pi F$ and $Pr(F^* | \pi F^*) = Pr(F | \pi F) = \varepsilon_{\pi}$. Jeffrey’s rule gives $Pr(H^*) = Pr(H | F).Pr(F^*) + Pr(H | \neg F).Pr(\neg F^*)$, from which one has $Pr(H | F).Pr(F^*) \leq Pr(H^*) \leq Pr(H | F).Pr(F^*) + (1 - Pr(F^*))$, assuming that $Pr(H | \neg F)$ is totally unknown, i.e., only $0 \leq Pr(H | \neg F) \leq 1$ is known. Here $Pr(H | F) = Q$, and $Pr(F^*) = Pr(F^* | \pi F^*) = \varepsilon_{\pi}$ as $Pr(\pi F^*) = Pr(G^*) = 1$. Thus $Pr(H^*)$ is at least $(Q.\varepsilon_{\pi})$ and at most $(Q.\varepsilon_{\pi} + (1 - \varepsilon_{\pi}))$, whence $Pr(H^*) = (\text{at least } (Q.\varepsilon_{\pi})) \cap (\text{at most } (Q.\varepsilon_{\pi} + (1 - \varepsilon_{\pi})))$ on the basis of the principle of minimum specificity.
References

References