Appendix A

A.1 Dedekind Reals in a Topos

In Sect. 9.2 we gave the definition of the internal natural number object $\mathbb{Z}$, and the internal rational number object $\mathbb{Q}$. These will now be utilised to define the internal Dedekind reals $\mathbb{R}$. To this end, let us consider the set $\mathbb{R}$ of ordered reals, each real number $r \in \mathbb{R}$ defines two disjoint subsets in $\mathbb{Q}$, namely

$$L = \{q \in \mathbb{Q} | q < r\}$$
$$U = \{q \in \mathbb{Q} | q > r\}.$$

These subsets have the following properties:

1. Each subset is non-empty.
2. $L$ is a downwards closed set but has no largest element.
3. $U$ is an upwards closed set but has no smallest element.
4. If $x$ is a rational number then $L \cup U \subseteq \mathbb{Q}$, otherwise $L \cup U = \mathbb{Q}$.

Given the above we can now define the notion of a Dedekind cut.

**Definition A.1.1** A Dedekind cut is a pair of disjoint subsets $(L, U)$ of $\mathbb{Q}$ such that the following conditions hold:

1. Non-degenerate:
   $$\exists p \in \mathbb{Q} \ (p \in L), \quad \exists q \in \mathbb{Q} \ (q \in U).$$

2. Inward-closed:
   $$\forall p, q \in \mathbb{Q} \ (p < q \land q \in L \Rightarrow p \in L);$$
   $$\forall p, q \in \mathbb{Q} \ (q < p \land q \in U \Rightarrow p \in U).$$
3. Outward-open:
\[ \forall q \in Q \ (q \in L \Rightarrow \exists p \in Q (p \in L \land q < p)) ; \]
\[ \forall q \in Q \ (q \in U \Rightarrow \exists p \in Q (p \in U \land p < q)) . \]

4. Located:
\[ \forall p, q \in Q \ (q < p \Rightarrow (q \in L \lor p \in U)). \]

5. Mutually exclusive:
\[ L \cap U = \emptyset . \]

It is possible to internalise the above definition of Dedekind cut in any topos with a natural number object. This is done by first defining the ordering relation \(<\) in \(\mathbb{Q}_\text{FS} / \text{STX}\) as a sub-object of \(\mathbb{Q}_\text{FS} / \text{STX}\). In particular, given two elements \(m/n, p/q \in \mathbb{Q}_\text{FS} / \text{STX}\), then \(\leq \) \(\{ (m/n, p/q) \in \mathbb{Q}_\text{FS} / \text{STX} \mid m \cdot q < p \cdot n \} \) where \(m \cdot q < p \cdot n\) is the order relation on the integers. We can then re-write all of the above conditions in Definition A.1.1 in terms of the internal language of \(\tau\), by simply replacing \(\mathbb{Q}\) by \(\mathbb{Q}_\text{FS}\), and interpreting all the logical symbols in term of the internal language of \(\tau\).

The internal Dedekind reals \(\overline{\mathbb{R}}\) is then defined as follows:
\[ \overline{\mathbb{R}} = \{ (L, U) \in P(\mathbb{Q}_\text{FS}) \times P(\mathbb{Q}_\text{FS}) \mid (L, U) \text{ is a Dedekind cut} \} . \]

It is also possible to define the internal Dedekind reals in terms of a geometric theory [50, D4.7.4]. In particular, we consider the geometrical theory \(\mathbb{T}_\mathbb{R}\) generated by the symbols \((p, q) / \mathbb{Q}_\text{FS} / \text{STX}\). These formal symbols undergo an ordering defined as follows: \((p, q) \leq (p', q')\) iff \(p \leq p'\) and \(q \leq q'\). The axioms of the theory \(\mathbb{T}_\mathbb{R}\) are

1. \((p_1, q_1) \land (p_2, q_2) = \begin{cases} \{\max\{p_1, p_2\}, \min\{q_1, q_2\}\} & \text{if } \max\{p_1, p_2\} \leq \min\{q_1, q_2\} \\ \bot & \text{otherwise} \end{cases} .\]
2. \((p, q) = \bigvee\{(p', q') \mid p < p' \land q' < q\} .\]
3. \(\top = \bigvee\{(p, q) \mid p < q\} .\]
4. \((p, q) = (p_1, q_1) \lor (p_1, q)\) if \(p \leq p_1 \leq q_1 \leq q\).

Given a topos \(\tau\), an interpretation of the theory \(\mathbb{T}_\mathbb{R}\) in \(\tau\) gives rise to a locale \(\mathbb{R}_\tau\) with associated frame \(\mathcal{O}(\mathbb{R}_\tau)\). The points of the locale \(\mathbb{R}_\tau\), i.e. the maps \(p^{-1} : \mathcal{O}(\mathbb{R}_\tau) \rightarrow \Omega_\tau\), are in bijective correspondence with Dedekind cuts defined in Definition A.1.1.
A.2 Scott’s Interval Domain

We will start by giving a brief description of Scott’s interval domain $\mathbb{I}R$ in $\textbf{Sets}$. In particular, as a set, $\mathbb{I}R$ consists of all compact subsets of the form $[a, b]$ with $a, b \in \mathbb{R}$ and $a \leq b$. In $\mathbb{I}R$ are included also the singletons $[a, a] = \{a\}$ for each $a \in \mathbb{R}$. $\mathbb{I}R$ is a poset under reverse inclusion. In this sense, elements of $\mathbb{I}R$ can be thought of as approximations of real numbers. The smaller the subsets the better it approximates the corresponding real number. It is possible to equip $\mathbb{I}R$ with the so called Scott topology which is defined as follows:

**Definition A.2.1** Given a subset $U \subseteq \mathbb{I}R$, we say that $U$ is Scott open if the following conditions hold:

1. If $[a, b] \in U$ and $[a, b] \subseteq [a', b']$ then $[a', b'] \in U$. This means that $U$ is upwards closed.
2. If all directed sets $S$ with supremum in $U$ have non-empty intersection with $U$, i.e. for any directed subset $S \subseteq \mathbb{I}R$ with supremum $\bigvee S$, if $\bigvee S \in U$, then there exists a $W \in S$ such that $W \in U$. This property means that $U$ is inaccessible by directed joins.

Clearly the complement of a Scott open is Scott closed. These can be defined as follows:

**Definition A.2.2** Given a subset $U \subseteq \mathbb{I}R$, we say that $U$ is Scott closed if the following two condition hold:

1. $U$ is a down set.
2. If $S$ is a directed set contained in $U$ and the supremum ($\text{Sup}(S)$) of $S$ exists, then $\text{Sup}(S) \in U$.

Given the above definition, a basis for the Scott topology is given by the collection of the following subsets

$$(p, q)_S := \{[r, s] | p < r \leq s < q\}, \ p, q \in \mathbb{Q} \text{ and } p < q.$$ 

We will denote the set $\mathbb{I}R$, equipped with the Scott topology, by $\mathcal{O}(\mathbb{I}R)$.

Next we would like to internalise the object $\mathbb{I}R$ in the topos $[\mathcal{C}(A), \textbf{Sets}]$. This can be done utilising the technique elucidated in Sect. 9.3. In particular, we recall that the category $\mathcal{C}(A)$ is equipped with the upwards Alexandroff topology, such that the product $\mathcal{C}(A) \times \mathbb{I}R$ is given the product topology. We then consider the continuous projection map $\pi : \mathcal{C}(A) \times \mathbb{I}R \rightarrow \mathcal{C}(A)$, $(C, [a, b]) \mapsto C$. The associated frame map is then $\pi^{-1} : \mathcal{O}(\mathcal{C}(a)) \rightarrow \mathcal{O}(\mathcal{C}(A) \times \mathbb{I}R)$.

Such a map describes the internal locale $\mathbb{T}_{\mathbb{I}R}$.

Similarly, as for the internal Dedekind reals, also the internal interval domain can be defined in terms of a geometric theory $\mathbb{T}_{\mathbb{I}R}$. In particular, the generating symbols for $\mathbb{T}_{\mathbb{I}R}$ are $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ with $p < q$. These formal symbols undergo an ordering
defined as follows: \((p, q) \leq (p', q')\) iff \(p \leq p'\) and \(q \leq q'\). The axioms of the theory \(T_{IR}\) are

1. \((p_1, q_1) \land (p_2, q_2) = \begin{cases} \max\{p_1, p_2\}, \min\{q_1, q_2\} & \text{if } \max\{p_1, p_2\} \leq \min\{q_1, q_2\} \\ \bot & \text{otherwise} \end{cases} .

2. \((p, q) = \bigvee\{(p', q') \mid p < p' < q' < q\} .

3. \(\top = \bigvee\{(p, q) \mid p < q\} .

Given a topos \(\tau\), an interpretation of the theory \(T_{IR}\) in \(\tau\) gives rise to a locale \(IR_\tau\) with associated frame \(O(\IR_\tau)\). The points of the locale \(IR_\tau\), i.e. the maps \(p^{-1} : O(\IR_\tau) \to \Omega_\tau\) are in bijective correspondence with elements of \(IR\) as defined above.

As one can see from the above definitions, the Scott interval domain is closely related to Dedekind cuts, in fact, only axiom (4.) in the definition of \(T_R\) fails to hold for \(T_{IR}\).

### A.3 Properties of Daseinised Projections

In [26, Sec. 10.2] various properties of the daseinisation map where introduced. In this context it was also shown that sub-objects \(\delta(\hat{P})\) of the spectral presheaf \(\Sigma\) are ‘special’ in the sense that they are the only elements in \(\text{Sub}_{\text{cl}}(\Sigma)\) for which the presheaf maps are surjective. In particular, in the definition of a sub-object of \(\hat{P}\) we have the condition that, for each \(V' \subseteq V\), the respective presheaf map

\[
\Sigma(i_{V'}: \Sigma_V \to \Sigma_{V'}) \tag{A.3.1}
\]

is such that, for a given subset \(S_V \subseteq \Sigma_V\), then

\[
\Sigma(i_{V'}: \Sigma_{V'}) \subseteq S_{V'} \subseteq \Sigma_{V'} . \tag{A.3.2}
\]

However, for subobjects of the form \(\delta(\hat{P})\) we obtain an equality in (A.3.2). This property is encoded in the following theorem:

**Theorem A.3.1** Given any projection operator \(\hat{P}\) and any two contexts \(V' \subseteq V\), then the following relation holds:

\[
S_{Q(i_{V'}V)}\delta(\hat{P})_V = \Sigma(i_{V'}V)(S_{\delta(\hat{P}V)}V) . \tag{A.3.3}
\]

**Proof** As a first step we will show that the map

\[
\Sigma(i_{V'}V) : P(\Sigma_V) \to P(\Sigma_{V'}) \tag{A.3.4}
\]

\(S \mapsto r_{V'}VS := \{\lambda_{V'}V | \lambda \in S\} ,
\]
is continuous, closed and open. For notational simplicity we will write $r = \sum(i_{V'V})$ and Eq. (A.3.3) becomes

$$r(S_{g^{\delta}(\hat{P})}) = S_{\delta^0(\hat{P})} = S_{g^{\delta}(\hat{P})}.$$ \hspace{1cm} (A.3.5)

Let us first show that such a map is continuous. Consider an open basis set $R \in \sum_V$, we know that $\sum_V := \{ \lambda : V \to \mathbb{C} | \lambda(\hat{1}) = 1 \}$ and, similarly, $\sum_{V'} := \{ \lambda : V' \to \mathbb{C} | \lambda(\hat{1}) = 1 \}$. Moreover if $\lambda \in \sum_V$ then from the definition of the presheaf maps it follows that $\lambda|_{V'} \in \sum_{V'}$ when $V' \subseteq V$. We can then define, for any $R \in P(\sum_V)$ the following:

$$r^{-1}(R) := R \cap \sum_V.$$ \hspace{1cm} (A.3.6)

Since the intersection of open sets is open, $r^{-1}(R)$ is open.

Next we need to show that $r$ is closed. Consider a closed subset $S \subseteq \sum_V$. Since $\sum_V$ is compact so is $S$ and, since $r$ is continuous, then $r(S)$ is compact in $\sum_{V'}$. But $\sum_{V'}$ is Hausdorff thus $r(S)$ is closed.

To show that $r$ is open we note that since every $\lambda_{V'} \in \sum_{V'}$ is of the form $\lambda_V|_{V'}$ for some $\lambda_V \in S \subseteq \sum_V$, then

$$r(S) = \{ \lambda_{V'} | \lambda \in S \} = S \cap \sum_{V'}.$$ \hspace{1cm} (A.3.7)

If $S$ is open, then $S \cap \sum_{V'}$ is the intersection of two opens thus it is itself open.

Given the above properties of $r$, a clopen subset $S_{g^{\delta}(\hat{P})} \subseteq \sum_V$ gets mapped to the clopen subset $r(S_{g^{\delta}(\hat{P})}) \subseteq \sum_{V'}$. Such a subset is

$$r(S_{g^{\delta}(\hat{P})}) = \text{int} \big\{ S_{\hat{Q}} \in \text{Sub}(\sum_{V'}) | r(S_{g^{\delta}(\hat{P})}) \subseteq S_{\hat{Q}} \big\} \hspace{1cm} (A.3.8)$$

thus $r(S_{g^{\delta}(\hat{P})}) \subseteq S_{\hat{Q}}$. We now need to show that $\hat{Q} \geq \delta^0(\hat{P})$. We prove this by contradiction. Assume that $\delta^0(\hat{P}) \geq \hat{Q}$ and define $\hat{R} := \delta(\hat{P}) - \hat{Q} \in P(V)$, such that $\lambda \in S_{\hat{R}}$. It follows that $\lambda \in S_{g^{\delta}(\hat{P})}$ but $\lambda \notin S_{\hat{Q}} \subseteq \sum_{V'}$.

However if $\delta^0(\hat{P}) = \delta(\hat{P})$ then $r(S_{g^{\delta}(\hat{P})}) = S_{g^{\delta}(\hat{P})}$. In fact, given an element $\lambda \in \delta(\hat{P})$ by definition $\lambda(\delta(\hat{P})) = 1$. Since $\delta^0(\hat{P}) \geq \delta^0(\hat{P}) (V' \subseteq$

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1 Note that for each $V \in \mathcal{V}(\mathcal{H})$, $\sum_V$ has the spectral topology (being the spectrum of $V$) which is compact and Hausdorff. The details of such a topology are not necessary to prove continuity. It is worth saying, though, that it can be shown that a basis for this topology is the collection of clopen subsets. This renders the prof of continuity easier, however we will not use it here. On the other hand, when proving closeness of $r$ we will use the fact that $\sum_{V'}$ is a Hausdorff compact space.

2 Note that the $\text{int}$ operation is needed for the subset to be clopen, otherwise it would only be closed.
are such that the presheaf $\lambda_{|V'} \in S_{\delta^o(\hat{P})_{|V'}}$. On the other hand, if $\lambda \notin S_{\delta^o(\hat{P})_{|V'}}$ then $\lambda(\delta^o(\hat{P})_{|V'}) = 0$. Since $\delta^o(\hat{P})_{|V'} = \delta^o(\hat{P})_{|V'}$ then $\lambda_{|V'} \notin S_{\delta^o(\hat{P})_{|V'}}$.

Given the fact that every $\lambda_{|V'} \in \sum_{V'}$ is of the form $\lambda_{|V'} = r(\lambda_{|V'})$ for some $\lambda \in S_{\delta^o(\hat{P})_{|V'}}$, then $r(S_{\delta^o(\hat{P})_{|V'}}) = S_{\delta^o(\hat{P})_{|V'}}$ and $r\left((S_{\delta^o(\hat{P})_{|V'}})^c\right) = S_{\delta^o(\hat{P})_{|V'}}^c$.

It follows that in our case we have $r(S_{\delta(\hat{P})}) = S_{\delta^o(\hat{P})_{|V'}}$ and $r\left((S_{\delta(\hat{P})})^c\right) = (S_{\delta(\hat{P})})^c$. We have shown that, $\lambda_{|V'} \notin S_{\delta^o(\hat{P})_{|V'}}$ and $\lambda \in r(S_{\delta^o(\hat{P})_{|V'}})$, what this means is that

$$\text{if } \hat{Q} \leq \delta^o(\hat{P})_{|V'} \text{ then } r(S_{\delta^o(\hat{P})_{|V'}}) \not\subseteq S_{\delta(\hat{P})}.$$  

(A.3.9)

However, this is a contradiction, therefore it must be the case that $\hat{Q} \geq \delta^o(\hat{P})_{|V'}$. We can now write $r(S_{\delta^o(\hat{P})_{|V'}})$ as

$$r(S_{\delta^o(\hat{P})_{|V'}}) = \text{int} \bigcap \left\{ S_{\hat{Q}} \in \text{Sub}_{\text{cl}}(\sum_{V'}) | \hat{Q} \geq \delta^o(\hat{P})_{|V'} \right\}$$  

(A.3.10)

$$= S_{\hat{Q} \in P'(V')} | \hat{Q} \geq \delta^o(\hat{P})_{|V'} = S_{\delta^o(\hat{P})_{|V'}}.$$

(A.3.11)

therefore

$$\sum_{V'}(i_{V'} : S_{\delta^o(\hat{P})_{|V'}} \to S_{\delta^o(\hat{P})_{|V'}}).$$

(A.3.12)

It follows that the clopen sub-objects of the form $\delta(\hat{P})$ are such that the presheaf maps are also surjective.

A.4 Connection Between Sheaves and Etalé Bundles

In this Section we will investigate the connection between sheaves and an étalé bundles is. To this end we need to introduce the notion of a germ of a function. Once we have introduced such a notion, it can be shown that each sheaf is a sheaf of cross sections of a suitable bundle. All this will become clear as we proceed. First of all: what is a germ? Germs represent constructions which define local properties of functions. In particular they indicate how similar two functions are locally. Because of this locality requirement, germs are generally defined on functions acting on topological spaces, such that the word local acquire meaning. For example, one can consider measure of ‘locality’ to be a power series expansion of a function around some fixed point. Thus, one can say that two holomorphic functions $f, g : U \to \mathbb{C}$ have the same germ at a point $a \in U$ iff the power series expansions around that point are the same. Thus, $f, g$ agree on some neighbourhood of $a$, i.e., with respect to that neighbourhood they “look” the same.

This definition obviously holds only if a power series expansion exists, however it is possible to generalise such a definition in a way that it only requires topological.
properties of the spaces involved. For example two functions \( f, g : X \to E \) have the same germ at \( x \in X \) if there exist some neighbourhood of \( x \) on which they agree. In this case we write \(^3\) \( \text{germ}_x f = \text{germ}_x g \) which implies that \( f(x) = g(x) \). However the converse is not true.

How do we generalise such a definition of germs in the case of presheaves? Let us consider a presheaf \( P : \mathcal{O}(X) \to \text{Sets} \in \text{Sets}^{\mathcal{O}(X)^{op}} \), where \( X \) is a topological space and \( \mathcal{O}(X)^{op} \) is the category of open sets with reverse ordering to the inclusion ordering. Given a point \( x \in X \) and two neighbourhoods \( U \) and \( V \) of \( x \), the presheaf \( P \) assigns two sets \( P(U) \) and \( P(V) \). Now consider two points \( t \in P(V) \) and \( s \in P(U) \), we then say that \( t \) and \( s \) have the same germ at \( x \) iff there exists some open \( W \subseteq U \cap V \), such that \( x \in W \) and \( s|_W = t|_W \in P(W) \).

The condition of having the same germ at \( x \) defines an equivalence class which is denoted as \( \text{germ}_x ,s \). Thus \( t \in \text{germ}_x ,s \) iff, given two opens \( U, V \ni x \) then there exists some \( W \subseteq U \cap V \) such that \( x \in W \) and \( t|_W = s|_W \in P(W) \), where \( s \in P(U) \) and \( t \in P(V) \). It follows that the set of all elements obtained through the \( P \) presheaf get ‘quotient’ through the equivalence relation of “belonging to the same germ”. Therefore, for each point \( x \in X \) there will exist a collection of germs at \( x \), i.e., a collection of equivalence classes:

\[
P_x := \{ \text{germ}_x ,s \mid s \in P(U), x \in U \text{ open in } X \} . \tag{A.4.1}\]

We can now collect all these set of germs for all points \( x \in X \), defining

\[
\Lambda_P = \bigsqcup_{x \in X} P_x = \{ \text{all germ}_x ,s \mid s \in X, s \in P(U) \} . \tag{A.4.2}\]

What we have done so far is, basically, to divide the presheaf space in equivalence classes. We can now define the map

\[
p : \Lambda_P \to X \tag{A.4.3}\]

\[
\text{germ}_x ,s \mapsto x
\]

\[
\text{germ}_y ,s \mapsto y
\]

which sends each germ to the point in which it is taken. It follows that each \( s \in P(U) \) defines a function

\[
\hat{s} : U \to \Lambda_P \tag{A.4.4}
\]

\[
x \mapsto \text{germ}_x ,s .
\]

\(^3\)This should be read as: the germ of \( f \) at \( x \) is the same as the germ of \( g \) at \( x \).
It is straightforward to see that $\hat{s}$ is a section of $p : \Lambda_P \rightarrow X$. Since the assignments $s \mapsto \hat{s}$ is unique, it is possible to replace each element $s$ in the original presheaf with a section $\hat{s}$ on the set of germs $\Lambda_P$.

We now define a topology on $\Lambda_P$ by considering as basis of open sets all the image sets $\hat{s}(U) \subseteq \Lambda_P$ for $U$ open in $X$, i.e. open sets are unions of images of sections. Such a topology obviously makes $p$ continuous. In fact, given an open set $U \subseteq X$ then $p^{-1}(U)$ is open by definition of the topology on $\Lambda_P$, since $p^{-1}(U) = \bigcup_{s \in P(U)} \hat{s}(U)$.

On the other hand it is also possible to show that the sections $\hat{s}$, as defined above, are continuous with respect to the topology on $\Lambda_P$. To understand this consider two elements $t \in P(V)$ and $s \in P(U)$ such that $i(x) = \hat{s}(x)$, i.e. $\text{germ}_s(t) = \text{germ}_t(s)$ where $x \in V \cap U$. It then follows that there exists an open set $W \ni x$ such that $W \subseteq V \cap U$. If we consider all those elements $y \in V \cap U \subseteq X$ for which $\hat{s}(y) = \hat{t}(y)$, then all such elements will comprise the open set $W \subseteq V \cap U$. Given this reasoning we want to show that for any open $\mathcal{O} \in \Lambda_P$, then $\hat{s}^{-1}(\mathcal{O})$ is open in $X$. Without loss of generality we can choose $\mathcal{O}$ to be a basis set, i.e.

$$\hat{s}(W) = \{\text{germ}_s(x) | \forall x \in W\}. \tag{A.4.5}$$

Thus $\hat{s}^{-1}\hat{s}(W) = W$ consists of all those points $x$ such that $\hat{s}(x) = i(x)$ for $t, s \in \text{germs}_s(s)$. It follows that $W$ is open.

One can also show that $\hat{s}$ is open and an injection. The property of being open follows directly from the definition of topology on $\Lambda_P$ since the basis of open sets are all the image sets $\hat{s}(U) \subseteq \Lambda_P$ for $U$ open in $X$. To show that it is injective we need to show that if $\text{germ}_s, s = \text{germ}_t, s$ then $x = y$. This follows from the definition of germs at a point. Putting all these results together we show that $\hat{s} : U \rightarrow \hat{s}(U)$ is a homeomorphism. So we have managed to construct a bundle $p : \Lambda_P \rightarrow X$ which is a local homeomorphism, since each point $\text{germ}_s(s) \in \Lambda_P$ has an open neighbourhood $\hat{s}(U)$ so that $p$, restricted to $\hat{s}(U)$, $p : \hat{s}(U) \rightarrow X$ has a two sided inverse $\hat{s} : U \rightarrow \hat{s}(U)$:

$$p \circ \hat{s} = \text{id}_X; \quad \hat{s} \circ p = \text{id}_{\Lambda_P} \tag{A.4.6}$$

hence $p$ is a local homeomorphism.

The above reasoning shows how, given a presheaf $P$ it is possible to construct an etalé bundle $p : \Lambda_P \rightarrow X$ out of it. Given such a bundle, it is then possible to construct a sheaf in terms of it. In particular we have the following theorem:

**Theorem A.4.1** The presheaf

$$\Gamma(\Lambda_P) : \mathcal{O}^{op} \rightarrow \text{Sets} \tag{A.4.7}$$

$$U \mapsto \{\hat{s} | s \in P(U)\}$$

is a sheaf.
**Proof** In the presheaf

\[
\Gamma(\Lambda_p) : \mathcal{O}(X)^{op} \rightarrow \text{Sets} \\
\Gamma(\Lambda_p) : \mathcal{O}(X)^{op} \rightarrow \text{Sets}
\]

\[
U \mapsto \{s \mid s \in P(U)\} \\
U_i \mapsto \{s_i \mid s_i \in P(U_i)\}
\]

the maps are defined by restriction, i.e. given \(U_i \subseteq U\), then

\[
\Gamma(\Lambda_p) : \mathcal{O}(X)^{op} \rightarrow \text{Sets} \\
U_i \mapsto \{s_i \mid s_i \in P(U_i)\}
\]

where \(s \mapsto s_i\) is defined via \(s_i = P(i_{U_i/U})s\). Now since

\[
\dot{s} : U \rightarrow \Lambda_p(U) \\
x \mapsto \text{germ}_x s
\]

while

\[
\dot{s}_i : U_i \rightarrow \Lambda_p(U_i) \\
y \mapsto \text{germ}_y s_i.
\]

Since \(U_i \subseteq U\), then

\[
\dot{s} : U_i \rightarrow \Lambda_p(U_i) \\
y \mapsto \text{germ}_y s_i.
\]

it follows that \(\dot{s}_i = \dot{s}_{\mid U_i}\).

In order to show that the above is indeed a sheaf we need to show that the diagram

\[
\begin{array}{ccc}
\Gamma(\Lambda_p(U)) & \xrightarrow{e} & \prod_i \Gamma(\Lambda_p(U_i)) \\
\Gamma(\Lambda_p(U_i)) & \xrightarrow{p} & \prod_{i,j} \Gamma(\Lambda_p(U_i \cap U_j))
\end{array}
\]

is an equaliser. By applying the definition of the sheaf maps we obtain

\[
e : \Gamma(\Lambda_p(U)) \rightarrow \prod_i \Gamma(\Lambda_p(U_i)) \quad \text{(A.4.13)}
\]

\[
e(\dot{s}) = \{s_i \mid i \in I\} = \{s_i \mid i \in I\}
\]

On the other hand

\[
p(\dot{s}_i) = \{s_i \mid U_i \cap U_j\} = \{s_i \mid U_i \cap U_j\} \quad \text{(A.4.14)}
\]
while

\[ q(\hat{s}_j) = \{s_j|_{U_i \cap U_j}\} = \{s|_{U_i \cap U_j}\}. \]  \hspace{1cm} (A.4.15)

\[ \square \]

\( \Gamma(\Lambda_P) \) is called the sheaf of cross sections of the bundle \( p : \Lambda_P \rightarrow X \). We can now define a map

\[ \eta : P \rightarrow \Gamma \circ \Lambda_P \]  \hspace{1cm} (A.4.16)

such that for each context \( U \in O(X)^{op} \) we obtain

\[ \eta_U : P_U \rightarrow \Gamma(\Lambda_P)(U) \]  \hspace{1cm} (A.4.17)

\[ s \mapsto \hat{s}. \]

**Theorem A.4.2** If \( P \) is a sheaf then \( \eta \) is an isomorphism.

**Proof** We need to show that \( \eta \) is 1:1 and onto.

1. One to one:

We want to show that if \( \hat{s} = \hat{t} \) then \( t = s \). Given \( t,s \in P(U) \), \( \hat{s} = \hat{t} \) means that \( \text{germ}_x(s) = \text{germ}_x(t) \) for all \( x \in U \). Therefore there exists opens \( V_x \subseteq U \) such that \( x \in V_x \) and \( t|_{V_x} = s|_{V_x} \). The collection of these opens \( V_x \) for all \( x \in U \) form a cover of \( U \) such that \( s|_{V_x} = t|_{V_x} \). This implies that \( s,t \) agree on the map \( P(U) \rightarrow \bigsqcup_{x \in U} P(V_x) \). From the sheaf requirements it follows that \( t = s \).

2. Onto:

We want to show that any section \( h : U \rightarrow \Lambda_P \) is of the form \( \eta_U(s) = \hat{s} \) for some \( s \in P(U) \). Let us consider a section \( h : U \rightarrow \Lambda_P \), this will pick for each \( x \in U \) an element, say \( h(x) = \text{germ}_x(s_x) \). Therefore for each \( x \in U \) there will exist an open \( U_x \ni x \) such that \( s_x \in P(U_x) \). By definition \( \text{germ}_x(s_x) = \hat{s}_x(x) \) where \( \hat{s}_x \) is a continuous section, therefore for each open \( U_x \) we get \( \hat{s}_x(U_x) = \{\text{germ}_x(s_x)\} \forall x \in U_x \), which is open by definition. It follows that for each \( x \in U_x \) there will exist some \( t,s \in \text{germ}_x(s) \), such that \( \hat{s}(x) = \hat{t}(x) \). This implies that there exists some open set \( W_x \) for which \( x \in W_x \subseteq U_x \subseteq U \) and such that \( t|_{W_x} = s|_{W_x} \). These open sets \( W_x \) form a covering of \( U \), i.e. \( U = \bigsqcup_{x \in U_x} W_x \) with \( s|_{W_x} \in P(W_x) \) for each \( P(W_x) \). Moreover, since \( h(x) = \text{germ}_x(s_x) \) for \( x \in U_x \) it follows that \( h = \hat{s}_x \) for each \( W_x \). Now consider two sections \( \hat{s}_x \) and \( \hat{s}_y \) for \( x \in P(W_x) \) and \( y \in P(W_y) \), then on the intersection \( W_x \cap W_y \), \( h \) agrees with both \( \hat{s}_x \) and \( \hat{s}_y \), therefore the latter agrees in the intersection. This means that \( \text{germ}_z(s_x) = \text{germ}_z(s_y) \) for \( z \in W_x \cap W_y \), therefore \( s_x|_{W_x \cap W_y} = s_y|_{W_y \cap W_x} \).

We thus obtain a family of elements \( s_x \) for each \( x \in U_x \) such that they agree on both maps \( P(U_x) \Rightarrow \bigsqcup_{x \in U_x} P(W_x) \cap P(W_y) \). From the condition of being a sheaf it follows that there exists an \( s \in P(U) \), such that \( s|_{W_x} = s_x \). Then at each \( x \in U \) we have \( h(x) = \text{germ}_x(s_x) = \text{germ}_x(s) = \hat{s}(x) \), therefore \( h = \hat{s} \).
It follows that all sheaves are sheaves of cross sections of some bundle. Moreover it is possible to generalise the above process and define the following pair of functors

\[
\text{Sets}^{\mathcal{O}(X)^{op}} \xrightarrow{\Lambda} \text{Bund}(X) \xrightarrow{\Gamma} \text{Sh}(X),
\]

which if we combine together we get the so called sheafification functor:

\[
\Gamma \Lambda : \text{Sets}^{\mathcal{O}(X)^{op}} \to \text{Sh}(X).
\]

Such a functor sends each presheaf \( P \) on \( X \) to the “best approximation” \( \Gamma \Lambda P \) of \( P \) by a sheaf.

In the case of etalé bundles we then obtain the following equivalence of categories:

\[
\begin{array}{c}
\text{Etalé}(X) \\
\xleftarrow{\Lambda} \\
\xrightarrow{\Gamma} \\
\text{Sh}(X)
\end{array}
\]

The pair of functors \( \Gamma \) and \( \Lambda \) are an adjoint pair (see Sect. A.5). Here we have restricted the functors to act on \( \text{Sh}(X) \subseteq \text{Sets}^{\mathcal{O}(X)^{op}} \).

### A.5 The Adjoint Pair

As discussed in Chapter 14 of [26], given a map \( f : X \to Y \) between topological spaces \( X \) and \( Y \) we obtain a geometric morphism, whose inverse and direct image are, respectively,

\[
f^* : \text{Sh}(Y) \to \text{Sh}(X)
\]

\[
f_* : \text{Sh}(X) \to \text{Sh}(Y).\]

We also know that \( f^* \dashv f_* \), i.e., \( f^* \) is the left adjoint of \( f_* \). If \( f \) is an etalé map, however, there also exists the left adjoint \( f! \) to \( f^* \), namely

\[
f! : \text{Sh}(X) \to \text{Sh}(Y)
\]

with \( f! \dashv f^* \dashv f_* \).

In Theorem A.5.1, below, we will show that

\[
f!(p_A : A \to X) = f \circ p_A : A \to Y
\]
so that we combine the étalé bundle \( p_A : A \to X \) with the étalé map \( f : X \to Y \),
to give the étalé bundle \( f \circ p_A : A \to Y \). Here we have used the fact that sheaves
can be defined in terms of étalé bundles. In fact in Chapter 14 of [26] it was shown
that there exists an equivalence of categories \( Sh(X) \simeq Etale(X) \) for any topological
space \( X \).

**Theorem A.5.1** Given the étalé map \( f : X \to Y \), the left adjoint functor \( f! : Sh(X) \to Sh(Y) \) is defined as follows:

\[
f!(p_A : A \to X) = f \circ p_A : A \to Y \tag{A.5.4}
\]

for \( p_A : A \to Y \) being an étalé bundle.

**Proof** In the proof we will first define the functor \( f! \) for general presheaf situation,
then we will restrict our attention to the case of sheaves \( (Sh(X) \subseteq Sets^{X^{op}}) \) and \( f 
étalé.

Consider the map \( f : X \to Y \), this gives rise to the functor \( f! : Sets^{X^{op}} \to Sets^{Y^{op}} \).
The standard definition of \( f! \) is as follows:

\[
f! := - \otimes_X (f^{\ast}) \tag{A.5.5}
\]
such that, for any object \( A \in Sets^{X^{op}} \) we have

\[
A \otimes_X (f^{\ast}). \tag{A.5.6}
\]

This is a presheaf in \( Sets^{Y^{op}} \), thus for each element \( y \in Y \) we obtain the set

\[
(A \otimes_X (f^{\ast}))y := A \otimes_X (f^{\ast})(-, y) \tag{A.5.7}
\]

where \( (f^{\ast}) \) is the presheaf

\[
(f^{\ast}) : X \times Y^{op} \to Sets. \tag{A.5.8}
\]

This presheaf derives from the composition of \( f \times id_{Y^{op}} : X \times Y^{op} \to Y \times Y^{op} \)
\((f \times id_{Y^{op}})^{\ast} : Sets^{X \times Y^{op}} \to Sets^{X \times Y^{op}} \) with the bi-functor \( \ast Y^{\ast} : Y \times Y^{op} \to Sets; \)
\((y, y') \mapsto Hom_{Y}(y', y) \), i.e.,

\[
(f^{\ast}) := (f \times id_{Y^{op}})^{\ast} (\ast Y^{\ast}) = \ast Y^{\ast} \circ (f \times id_{Y^{op}}). \tag{A.5.9}
\]

Now coming back to our situation we then have the restricted functor

\[
(f^{\ast})(-, y) : (X, y) \to Sets \tag{A.5.10}
\]

\[(x, y) \mapsto (f^{\ast})(x, y) \]
which, from the definition given above is

\[(fY^*)(x, y) = Y^* \circ (f \times id_{yp})(x, y) = Y^*(f(x), y) = Hom_Y(y, f(x)).\]

(A.5.11)

Therefore, putting all the results together we have that for each \(y \in Y\) we obtain

\[A \otimes_X (f^Y)(-, y),\]

defined for each \(x \in X\) as

\[A(-) \otimes_X (f^Y)(x, y) := A(x) \otimes_X Hom_Y(y, f(x)).\]

(A.5.12)

This represents the presheaf \(A\) defined over the element \(x\), plus a collection of maps in \(Y\) mapping the original \(y\) to the image of \(x\) via \(f\).

In particular \(A(x) \otimes_X (f^X) = A(x) \otimes_X Hom_Y(y, f(-))\) represents the following equaliser:

\[
\begin{array}{ccc}
\prod_{x,x'} A(x) \times Hom_X(x', x) \times Hom_Y(y, f(x')) & \xrightarrow{\tau} & \prod_x A(x) \times Hom_Y(y, f(x)) \\
\theta & \simeq & \\
\sigma & & \\
& & A(-) \otimes_X Hom_Y(y, f(-))
\end{array}
\]

such that, given a triplet \((a, g, h) \in A(x) \times Hom_X(x', x) \times Hom_Y(y, f(x'))\), we then obtain that

\[\tau(a, g, h) = (ag, h) = \theta(a, g, h) = (a, gh).\]

(A.5.13)

Therefore, from the above equivalence conditions, \(A(-) \otimes_X Hom_Y(y, f(-))\) is the quotient space of \(\prod_x A(x) \times Hom_Y(y, f(x))\).

We now consider the situation in which \(A\) is a sheaf on \(X\), in particular it is an étale bundle \(p_A : A \rightarrow X \) and \(f\) is an étale map which means that it is a local homeomorphism, i.e. for each \(x \in X\) there is an open set \(V\), such that \(x \in V\) and \(f|_V : V \rightarrow f(V)\) is a homeomorphism. It follows that for each \(x_i \in V\) there is a unique element \(y_i\) such that \(f|_V(x_i) = y_i\). In particular for each \(V \subset X\) then \(f|_V(V) = U\) for some \(U \subset Y\).

Note that, since the condition of being a homeomorphism is only local, it can be the case that \(f|_V(V_i) = f|_V(V_j)\) even if \(V_i \neq V_j\). However in these cases the restricted étale maps have to agree on the intersections, i.e. \(f|_{V_i}(V_i \cap V_j) = f|_{V_j}(V_j \cap V_j)\).

Let us now consider an open set \(V\) with local homeomorphism \(f|_V\). In this setting each element \(y_i \in f|_V(V)\) will be of the form \(f(x_i)\) for a unique \(x_i\). Moreover, if we consider two open sets \(V_1, V_2 \subset V\), then to each map \(V_1 \rightarrow V_2\) in \(X\), with associated
bundle map \( A(V_2) \to A(V_1) \), there corresponds a map \( f_{V}(V_1) \to f_{V}(V_2) \) in \( Y \). Therefore, evaluating \( A(\cdot) \otimes_X Hom_Y(\cdot, f(\cdot)) \) at the open set \( f_{V}(V) \subset Y \) we get, for each \( V_i \subseteq V \), the equivalence classes

\[
[A(V_i) \times_X Hom_Y(f_{V}(V), f_{V}(V_i))].
\]

The equivalence relation is such that

\[
A(V_j) \times_X Hom_Y(f_{V}(V), f_{V}(V_j)) \simeq A(V_k) \times_X Hom_Y(f_{V}(V), f_{V}(V_k))
\]

iff: (1) there exists a map \( f_{Vj}(V_j) \to f_{Vk}(V_k) \) which combines with \( F_{Vj} \to f_{Vj}(V_j) \), giving \( f_{V}(V) \to f_{V}(V_k) \); and (2) the corresponding bundle map \( A(V_k) \to A(V_j) \to A(V) \) is given by the map \( V \to V_j \to V_k \) in \( X \). A moment of thought reveals that such an equivalence class is nothing but \( p_X^{-1}(V) \) (the fibre of \( p_A \) at \( V \)) with associated fibre maps induced by the base maps.

We will now denote such an equivalence class by \([A(V) \times_X Hom_Y(f_{V}(V), f_{V}(V_i))]\) since, obviously, in each equivalence class there will be the element \( A(V) \times_X Hom_Y(f_{V}(V), f_{V}(V_i)) \).

Applying the same procedure for each open set \( V_i \subset X \) we can obtain two cases:

(i) \( f_{V}(V_i) = U \neq f_{V}(V) \). In this case we simply get an independent equivalence class for \( U \).

(ii) If \( f_{V}(V_i) = U = f_{V}(V) \) and there is no map \( i : V \to V_i \) in \( X \) then, in this case, for \( U \), we obtain two distinct equivalence classes \([A(V_i) \times_X Hom_Y(f_{V}(V_i), f_{V}(V_i))]\) and \([A(V) \times_X Hom_Y(f_{V}(V), f_{V}(V))]\).

Thus the sheaf \( A(\cdot) \otimes_X (Y^*) \) is defined for each open set \( f_{V}(V) \subset Y \) as the set

\[
[A(V) \times_X Hom_Y(f_{V}(V), f_{V}(V))] \simeq A(V)),
\]

while, for each map \( f_{V'}(V') \to f_{V}(V) \) in \( Y \) (with associated map \( V' \to V \) in \( X \), there is associated the map

\[
[A(V) \times_X Hom_Y(f_{V}(V), f_{V}(V))] \simeq A(V) \to [A(V') \times_X Hom_Y(f_{V'}(V'), f_{V'}(V'))] \simeq A(V').
\]

This is precisely what the étale bundle \( f \circ p_A : A \to Y \) is.

Now that we understand the action of \( f! \) on sheaves we will try to understand its action on functions. To this end, let us go back to étale bundles. Given a map \( \alpha : A \to B \) of étale bundles over \( X \), we obtain the map \( f!(\alpha) : f!(A) \to f!(B) \) which is defined as follows: we start with the collection of fibre maps \( \alpha_x : A_x \to B_x \), \( x \in X \), where \( A_x := p^{-1}(\{x\}) \). Then, for each \( y \in Y \) we want to define the maps \( f!(\alpha)_y : f!(A)_y \to f!(B)_y \), i.e., \( f!(\alpha)_y : p^{-1}(A(f^{-1}(\{y\}))) \to p^{-1}(B(f^{-1}(\{y\}))) \). This
are defined as
\[ f!(\alpha)_y(a) := \alpha_{p^{a}(a)}(a) \quad (A.5.14) \]
for all \( a \in f! (A)_y = p^{-1}(A(f^{-1}\{a\})) \).

### A.6 Lawvere-Tierney Topology and Closure Operator

In this section we will give a very brief review of a Lawvere-Tierney Topology and of the closure operator. Essentially a Lawvere-Tierney topology is a topology on a topos. We will choose the topos to be \( \text{Sets}^{\mathcal{V}(\mathcal{H})^{op}} \) since it is the topos we are interested in.

**Definition A.6.1** Given the topos \( \text{Sets}^{\mathcal{V}(\mathcal{H})^{op}} \) with sub-object classifier \( \Omega \), a Lawvere-Tierney Topology on \( \text{Sets}^{\mathcal{V}(\mathcal{H})^{op}} \) is a map
\[ j : \Omega \rightarrow \Omega \]
in \( \text{Sets}^{\mathcal{V}(\mathcal{H})^{op}} \) which satisfies the following properties:

1. \( j \circ \text{true} = \text{true} \), i.e. the diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{\text{true}} & \Omega \\
\downarrow{\text{true}} & & \downarrow{j} \\
\Omega & & \Omega \\
\end{array}
\]
commutes.

2. \( j \circ j = j \), i.e. the diagram

\[
\begin{array}{ccc}
\Omega & \xrightarrow{j} & \Omega \\
\downarrow{j} & & \downarrow{j} \\
\Omega & & \Omega \\
\end{array}
\]
commutes.
3. \( j \circ \wedge = \wedge \circ (j \times j) \), i.e. the diagram

\[
\begin{array}{ccc}
\Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\
\downarrow{j \times j} & & \downarrow{j} \\
\Omega \times \Omega & \xrightarrow{\wedge} & \Omega
\end{array}
\]

commutes. Here \( \wedge : \Omega \times \Omega \to \Omega \) is defined so that for each \( V \in \mathcal{V}(\mathcal{H}) \) we have

\[
\wedge_V : \Omega_V \times \Omega_V \to \Omega_V \\
(w_1, w_2) \mapsto \wedge_V(w_1, w_2) := w_1 \cap w_2.
\]

**Definition A.6.2** A closure operator \( \overline{\cdot} \) is a map such that, for every \( P \in \text{Sets}^{\mathcal{V}(\mathcal{H})^{op}} \), it maps a sub-object \( S \subseteq P \) (\( S \in \text{Sub}(P) \)) to another sub-object \( \overline{S} \subseteq P \). This assignment is such that, given any two sub-objects \( S, T \in \text{Sub}(Q) \) then the following conditions hold:

\[
\begin{align*}
S & \subseteq \overline{S} \\
\overline{\overline{S}} & = S \\
\overline{S} \cap \overline{T} & = \overline{S \cap T}.
\end{align*}
\]

In [55] it was shown that the Lawvere-Tierney topology, the closure operator and the Grothendieck topology are equivalent to each other in the sense that each of them implies the other.

**Proof**

1. Lawvere-Tierney topology \( \Longrightarrow \) the closure operator: let us assume we have a Lawvere-Tierney topology \( j \). We then construct the following pullback

\[
\begin{array}{ccc}
S & \xrightarrow{!} & 1 \\
\downarrow{i} & & \downarrow{\text{true}} \\
Q & \xrightarrow{\chi} & \Omega
\end{array}
\]
This is utilised to construct the closure $\overline{S}$ of $S$ as the sub-object of $Q$, whose characteristic morphism is $j \circ \chi$, i.e. such that the outer square is a pullback:

2. Closure operator $\Rightarrow$ Grothendieck topology: let us assume we have a closure operator $(\cdot)$, then the Grothendieck topology is given in terms of the closure of the terminal object $1$, i.e.

$$J \leftarrow \Omega := \overline{1} \cong \Omega.$$  

3. Grothendieck topology $\Rightarrow$ Lawvere-Tierney topology: let us assume we have a Grothendieck topology $J$, then it is possible to construct a Lawvere-Tierney Topology $j$ in terms of the characteristic morphism of $J$. In particular we defined $j$ to be the morphism which would make the following diagram a pullback:

It is straightforward to see that if we reiterate the procedures $1 \to 2 \to 3 \to 1$ we would end up with the Lawvere-Tierney topology we started with. □

A.7 Yoneda Lemma

In this section we will give a very brief review of the Yoneda Lemma since it is used through out the book.
Lemma A.7.1 Preliminary: if $C$ is a locally small category,\textsuperscript{4} then each object $A$ of $C$ induces a natural contravariant functor from $C$ to \textbf{Sets} called a hom-functor $y(A) := \text{Hom}_C(-, A)$.\textsuperscript{5} Such a functor is defined on objects $C \in C$ as

$$y(A) : C \to \text{Sets}$$

$$C \mapsto \text{Hom}_C(C, A)$$

on $C$-morphisms $f : C \to B$ as

$$y(A)(f) : \text{Hom}_C(B, A) \to \text{Hom}_C(C, A)$$

$$g \mapsto y(A)(f)(g) := g \circ f .$$

A very simple graphical example of the above is the following:

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (4,0) {$B$};
  \node (C) at (2,2) {$E$};
  \node (D) at (2,-2) {$D$};
  \draw[->] (A) -- (B) node[midway,above] {$\theta$};
  \draw[->] (A) -- (D) node[midway, below] {$\alpha$};
  \draw[->] (B) -- (C) node[midway,above] {$\varphi$};
  \draw[->] (B) -- (D) node[midway, below] {$\gamma$};
  \draw[->] (A) -- (C) node[midway, above] {$h$};
  \draw[->] (D) -- (C) node[midway, below] {$g$};
\end{tikzpicture}
\end{center}

\[ \Downarrow y(A) \]

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$\text{Hom}_C(A, A)$};
  \node (B) at (2,2) {$\text{Hom}_C(E, A)$};
  \node (C) at (4,0) {$\text{Hom}_C(B, A)$};
  \node (D) at (0,-2) {$\text{Hom}_C(D, A)$};
  \draw[->] (A) -- (B) node[midway, above] {$y(A)(\vartheta)$};
  \draw[->] (A) -- (D) node[midway, below] {$y(A)(\vartheta)$};
  \draw[->] (B) -- (C) node[midway, above] {$y(A)(\varphi)$};
  \draw[->] (B) -- (D) node[midway, below] {$y(A)(\varphi)$};
\end{tikzpicture}
\end{center}

Lemma A.7.2 Yoneda lemma: Given an arbitrary presheaf $P$ on $C$ there exists a bijective correspondence between natural transformations $y(A) \to P$ and elements

\textsuperscript{4}A category $C$ is said to be locally small iff its collection of morphisms form a proper set.

\textsuperscript{5}We have already encountered this in Example 5.10 of [51].
of the set \( P(A) \, (A \in C) \) defined as an arrow

\[
\theta : \text{Nat}_C(y(A), P) \xrightarrow{\sim} P(A) \\
(\alpha : y(A) \to P) \mapsto \theta(\alpha) = \alpha_A(id_A).
\]

where \( \alpha_A : y(A)(A) \to P(A); \text{Hom}_C(A, A) \to P(A). \)
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