Hilbert’s dream in particular, and the formalistic axiomatic program in general, was to ground mathematics by a finite formal system – a set of axioms and deterministic rules of derivation, the latter (by the Curry-Howard correspondence) operating like an algorithm on the former like an input, which would proof all true theorems of mathematics. Gödel and Turing, among others, put an end to this formalistic dream [160, 273], as vividly expressed by Gödel in a postscript, dated from June 3, 1964 [243, pp. 369–370]: “due to A. M. Turing’s work, a precise and unquestionably adequate definition of the general concept of formal system can now be given, the existence of undecidable arithmetical propositions and the non-demonstrability of the consistency of a system in the same system can now be proved rigorously for every consistent formal system containing a certain amount of finitary number theory.

Turing’s work gives an analysis of the concept of “mechanical procedure” (alias “algorithm” or “computation procedure” or “finite combinatorial procedure”). This concept is shown to be equivalent with that of a “Turing machine.” A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas.”

We shall present a very brief survey of the consequences of these findings, and first hint on the fact that almost all elements of the continuum, and, in particular, almost all reals, are incomputable. That is, they are inaccessibly to any computation.

Then we head on to modern, algorithmic, definitions of randomness, and, in particular, of random reals. That is, random reals are algorithmically incompressible, and cannot be produced by any program whose code has much smaller length (as the original phenotype or number).

Thereby we shall mention quantitative incompleteness theorems and introduce the busy beaver function. In a certain sense, those constructions give a glimpse on how fast a computation may diverge, and how difficult it is to compute or represent certain objects. We shall also speculate how primordial chaos may give rise to unbounded complexities.

Finally we consider Chaitin’s halting probability Omega, which serves as a sort of Rosetta stone for comprehending at least mild forms of random reals in perplexing ways.
A.1 Abundance of Incomputable Reals

For the sake of an orientation for the reader a very brief expose of formal definitions related to indeterminism and randomness is offered. Mostly the concepts will be presented without proofs.

Let us start with the set $\mathbb{R}$ of real numbers [see for instance, Refs. [179, 194, 488], or Ref. [281] (in German)]. Reals will be coded by, or written as, infinite decimals.

By Cantor’s theorem using diagonalization [488, Sect. 3.5.1, pp. 70–71] the set of real numbers $\mathbb{R}$ (or, say, the real unit interval $[0, 1]$) is nondenumerable. More generally – because one needs not proceed along the diagonal and process the entries therein to produce a real which does not occur in any type of enumeration of reals – the same result can be obtained by applying self-reference and the existence of some map without any fixed points [579, p. 368], As a result $\mathbb{R}$ cannot be brought into a one-to-one correspondence with the natural numbers $\mathbb{N}$ (such as, for instance, the integers $\mathbb{Z}$ or the rationals $\mathbb{Q}$). Sets of this $\mathbb{R}$ type will be called continua. Sets of type $\mathbb{N}$ will be called denumerable.

But we can go quantitatively further than that: we can show that, from the point of view of measure theory, denumerable sets are “meagre;” that is, almost all reals are not in any such set of denumerable numbers [488, Sect. 3.5.2, pp. 71–72]. For a sketch of a proof suppose that we are “covering” the $i$’th element of the denumerable set with an interval $\delta^{-i}\varepsilon$, with $1 < \delta < \infty$ and $\varepsilon$ a tiny number. Summing over all such cover intervals can be readily performed, as the respective set is denumerable. By the geometric series summation formula, the entire length covered is at most (for nonoverlapping intervals) $\varepsilon \delta^{-1} / (1 - \delta^{-1})$. We can make this covering length arbitrary small by making either $\delta$ larger or $\varepsilon$ smaller. That is, in this measure theoretic, quantitative, sense, “almost all” reals are not in any particular denumerable set.

Next we define a computable real by the property that it is produced – that is, it is the output of – some algorithm “running” on a (supposedly universal) computer. The set of computable reals is denumerable because we can find ways to enumerate all of them: for a sketch of the idea how to perform this task, imagine the numbers produced by successively generated (by their code lengths in lexicographic order) algorithms at successive times.

As a consequence we find that “almost all” reals are incomputable. That is, if one considers the real unit interval as a “continuum urn” – one needs the axiom of choice in order to “draw” a general element of this urn, as no computable “handle” exists to fetch it – then with probability 1 it will be incomputable.

It may appear amazing that all denumerable sets are so “meagre,” and its members so “thinly distributed and embedded” in the real continuum. In particular, such sets – such as the rational numbers and also the computable ones which include “many more” irrational numbers – are dense in the sense that in-between two arbitrary numbers $a$ and $b$ with $a < b$ of a dense set there always lies another number $c$ in that set, such that $c$ is larger than $a$ but smaller than $b$; that is, $a < c < b$. 
A.2 Random Reals

A real can be defined to be random if its (decimal) expansion cannot be algorithmically compressed \([103, 122, 127, 178, 315, 316, 325, 326, 337]\). In particular \([122]\), “it may perhaps not appear entirely arbitrary to define a patternless or random finite binary sequence as a sequence which, in order to be calculated, requires, roughly speaking, at least as long a program as any other binary sequence of the same length.”

Randomness via algorithmic incompressibility implies that the respective random sequences or random reals \([353]\) “possess all conceivable computable statistical properties of randomness”; that is, they pass all conceivable computable statistical tests of randomness. What is such a “conceivable computable statistical test?” It is based on all conceivable computable laws – that is, all algorithms. More precisely, a single conceivable computable statistical test is based on a single algorithm: it is the hypothesis that the random sequences or random real cannot be generated by this algorithm. Because if it were, it would be algorithmically compressible by that algorithm. Herein lies the connection between statistical test and algorithm: that any algorithm constitutes an algorithmic test against nonrandomness (algorithmic compressibility); and vice versa, every computable statistical test is representable by an algorithm.

A.3 Algorithmic Information

A.3.1 Definition

Let us be more precise and, for the sake of avoiding difficulties related to subadditivity \([127]\), restrict ourselves to prefix or instantaneous program codes \([126, 335]\) which have the “prefix (free) property.” This property requires that there is no whole code word in the system that is a prefix (initial segment) of any other code word in that same system.

Define the algorithmic information (content), or, used synonymously, the (Kolmogorov) program-size complexity, or the information-theoretic complexity of an individual object is a measure or criterion how difficult it is to algorithmically specify (but not in terms of time it takes to produce) that object \([127]\). In particular, the algorithmic information content \(I(x)\) of binary string \(x\) as the size/length (encoded in bits, that is, binary digits) of the shortest/smallest program running on some (Turing-type) universal computer \(U\) to calculate \(x\), plus the information content \(I(|s|)\) of the length \(|s|\) of this sequence (since this also contributes); that is, if \(|s|\) stands for the length of the binary sequence \(s_n\) in bits, and the order \(O(f)\) of \(f\) stands for a function whose absolute value is bounded by a constant times \(f\) [and thus \(O(1)\) just stands for a constant], then
\[ I(x) = |s| + I(|s|) + O(1) = |s| + O(\log_2 |s|). \] (A.1)

A.3.2 Algorithmic Information of a Single Random Sequence

Instead of delving into joint and mutual information content we head straight to a formal definition of randomness. A finite random binary sequence \( s_n \) of length \( n \) is defined to be (nearly) algorithmically incompressible; that is, its algorithmic information content \( I(s_n) \) is not (much) less than \( n \). An infinite binary sequence \( s \) is random if its initial segments \( s_n \) are random finite binary sequences. That is, \( s \) is random if and only if there exists some constant \( c \), such that, for all \( n \in \mathbb{N} \), the algorithmic information content of its initial segments \( s_n \) is bounded from below by \( n - c \); that is,

\[ s \text{ is random } \iff \exists c \forall n \left[ I(s_n) > n - c \right]. \] (A.2)

A random real (in arbitrary base notation) is one whose base 2 expansion of its fractional part (forgetting the integer part as long as it is finite) is a random infinite binary sequence.

A.3.3 Bounds from Above

Let us go a little further and mention a bound from above on the algorithmic information content of a string of length \( n \): it must be less than \( n + O(1) \). Because, to paraphrase Chaitin [124, p. 11], “the algorithmic information content of a string of length \( n \) must be less than \( n + O(1) \), because any string of length \( n \) can be calculated by putting it directly into a program as a table. This requires \( n \) bits, to which must be added \( O(1) \) bits of instructions for printing the table. In other words, if nothing betters occurs to us, the string itself can be used as its definition, and this requires only a few more bits than its length.”

A.3.4 Abundance of Random Reals

Almost all reals of the continuum are not only incomputable, as we have argued previously by a measure theoretic argument, but they are also random. Rather than rephrasing the argument, Chaitin’s argument [124, p. 11] can be paraphrased as follows: “the algorithmic information content of the great majority of strings of length \( n \) is approximately \( n \), and very few strings of length \( n \) are of algorithmic information content much less than \( n \). The reason is simply that there are much fewer programs of length appreciably less than \( n \) than strings of length \( n \). More exactly, there are \( 2^n \) binary strings of length \( n \), and less than \( 2^{n-k} \) binary encoded programs...
of length less than $n - k$. Thus the number of strings of length $n$ and algorithmic information content less than $n - k$ decreases exponentially as $k$ increases, and increases exponentially as $n$ increases."

As a consequence, most of the sequences of length $n$ are of algorithmic information content close to $n$, and, according to the definition earlier, appear random. Therefore, if one chooses some nonalgorithmic method generating such sequences – if they are operational, that is, they can be produced by some physically process; say, by tossing a fair coin and hoping for the best that this process is not deterministic or biased as alleged in Ref. [169] – then chances are high that the algorithmic information content of such a string will be as long as the length of that sequence. Pointedly stated: “grabbing and picking” a random real from the continuum with nonalgorithmic means, facilitated by the axiom of choice, will almost always yield a random real.

### A.4 Information-Theoretic Limitations of Formal Systems

By reduction to the halting problem it can be argued that the algorithmic information content $I$ in general is incomputable. Because computability of the algorithmic information content $I(s_n)$ would require that it would be possible to compute whether or not particular programs of length up to $n + O(n)$ halt (after output of $s_n$). But this is clearly impossible for large enough (and even for small) $n$; see also the busy beaver function discussed later.

It is therefore impossible that in general it is possible to prove (non)randomness or (in)computability of a particular individual infinite sequence. (Any particular finite sequence is provable computable, as by the earlier mentioned tabulation technique, an algorithm outputting it can be constructed by putting it directly into a program as a table. This is no contradiction to the earlier definition of randomness of a finite sequence because this is means relative and not absolute.) That is, all statements (e.g., Ref. [589]) such as “this string is irreducibly random,” at least as far as they relate to ontology, are provably unprovable hypotheses. Epistemically they are inclinations at best, as expressed by Born’s statement [68, p. 866] (English translation in [570, p. 54]) “I myself am inclined to give up determinism in the world of atoms.” At worst they are ideologies which remain unfalsifiable.

In a quantitative sense one could go beyond the Gödel–Turing reduction. Let us follow Chaitin [124] and employing Berry’s paradox, as reported by Russell [273, p. 153, contradiction (4), footnote 3]: “But ‘the least integer not nameable in fewer than nineteen syllables’ is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction.”

Chaitin’s paradoxical construction which is based upon the Berry paradox can be expressed by the following sentence [124, 127] which cannot be valid: “the program which yields the shortest proof that its algorithmic information content is much greater than its length, say, 1 billion bits.”
Because such a program, if it existed, purports to prove that its algorithmic information content is much greater than its length, say, 1 billion bits. And yet, this statement is quite short, and certainly less than a billion bits. This is contradictory; and as a consequence no such program can exist. Stated differently: For every formal system deriving statements of the form $I(s) > n$, there is a number $k$ such that no such statement is provable using the given rules of that formal system for any $n > k$ [161, pp. 265–266]. $k$ can be called the “strength” of such a formal system. This strength is a limiting measure for the capacity of the formal system to prove statements about the algorithmic information content of sequences.

One way of interpreting this result is in terms of independence: because more axioms specify more theorems, different axioms specify different theorems. That is, it is the choice of the formalist which deductive mathematical universe is created by the assumptions [270, p. 38].

### A.5 Abundance of True Yet Unprovable Statements

In view of the aforementioned incompleteness and independence result one may ask [132, p. 148]: “How common is incompleteness and unprovability? Is it a very bizarre pathological case, or is it pervasive and quite common?”

Indeed, just as the set of computable reals is “meagre” in the set of reals, so is the set of provable (by constructive, algorithmic methods) theorems “meagre” with respect to all true theorems of mathematics. This has been proven in a topological sense of “meagre” in the context of Gödel–Turing type incompleteness [105] (and not in the sense of independence of, say, the continuum hypothesis).

Rice’s theorem [432] asserts that all non-trivial, (semantic) functional properties of programs are undecidable. A functional property is one (i) describing how some functions performs in terms of its input/output behavior, and (ii) which is non-trivial in the sense that some (of all perceivable) programs which have this input/output behavior, and other programs which don’t. Rice’s theorem can be algorithmically proven by reduction to the halting problem: Suppose there is an algorithm $A$ deciding whether or not any given function or algorithm $B$ has any functional property. Then we can define another program $C$ which first solves the halting problem for some other arbitrary function $D$, clears the memory, and consecutively executes a program $E$ with has the respective functional property decided by $A$. As long as $D$ halts, all may go well. But if $D$ does not halt, the program $C$ never clears the memory, and can never execute a program $E$ with the respective functional property. Now, if one inserts $C$ into $A$, in order to be able to decide (positively) about the functional property of $E$ – and thus of $C − A$ would have to be able to solve the halting problem for $D$ first; a task which is provable impossible by algorithmic means.

As Yanofsky observes this bears some similarity to the downward Löwenheim-Skolem [580, footnote 20, p. 375], “stating that if there is a consistent way of using a language [[statements in mathematics ‥ written with a finite set of symbols]] to talk about such a system [[with an uncountably infinite number of elements]], then
that language might very well be talking about a system with only a countably infinite number of elements. That is, the axioms might be intended for discussing something uncountably infinite, but we really cannot show that it is has more than a countably infinite number of elements.”

A.6 Halting Probability $\Omega$

The program lengths $|p_i|$ of $q$ algorithms $p_i$, $i = 1, \ldots, q$ encoded by binary prefix free codes on a universal computer satisfies the Kraft inequality

$$0 \leq \sum_{i=1}^{q} 2^{-|p_i|} \leq 1.$$ (A.3)

These bounds are motivation to define Chaitin’s halting probability $[126, 135, 136] \Omega$ by the sum of the weighted length of all binary prefix free encoded programs $p$ on a given universal computer which halt:

$$\Omega = \sum_{p \text{ halt}} 2^{-|p|}.$$ (A.4)

$\Omega$ is Borel normal in any base $[123]$, and “highly incomputable” as it requires the solution to all halting problems. Conversely, knowledge of $\Omega$, at least up to some degree, entails the solution of decision problems associated with halting problems: for instance, the Goldbach conjecture (“every even number greater than 2 can be represented as the sum of two primes”) can be rephrased as a halting problem by parsing through all cases and halting if one of them fails.

Despite this obvious computational hardness, the initial bits of $\Omega$ can be computed, or at least estimated up to some small degree $[111, 129, 134]$. This is related to the fact that very small-size programs still “converge fast” – that is, they soon halt – if ever; and, because of the exponentially decreasing weight with length, those programs contribute more to $\Omega$ as longer ones. But because of the recursive unsolvability of the halting problem there exists no computable rate of convergence. In particular, as we shall see next, halting times grow faster than every computable function of program length.

A.7 Busy Beaver Function and Maximal Execution and Recurrence Time

Suppose one considers all programs (on a particular computer) up to length $n$. The busy beaver function $\Sigma(n)$ of $n$ is the largest number producible by such a programs
of length \( n \) before halting [71, 168, 426]. (Note that non-halting programs, possibly producing infinite numbers, for example by a non-terminating loop, do not apply.)

Alternatively, in terms of algorithmic information content, the busy beaver function \( \Sigma(n) \) can be defined as the largest number (of bits) whose algorithmic information content is less than or equal to \( n \) [125, 128]; that is, \( \Sigma(n) = \max_{I(k) \leq n} k \) [125, Definition 5.3, p. 414].

\( \Sigma(n) \) grows faster than any computable function of \( n \) and therefore is incomputable. Let us follow Chaitin [128] and suppose that \( n \) is greater than \( I(f) + O(1) \), the algorithmic information content of \( f \) (in terms of its binary code) plus a positive constant.

For the computation of \( f(n) + 1 \) it suffices to know a minimal-size program to compute \( f \), as well as the value of \( n \), or, even more economically, the value of \( n - I(f) \). Thus, \( I(f(n) + 1) \leq I(f) + I(\log_2 |n - I(f)|) \leq I(f) + I(\log_2 (I(f) + O(1) - I(f))) = I(f) + I(\log_2 O(1)) < I(f) + O(1) < n \). Therefore, by the definition of \( \Sigma(n) \), \( f(n) + 1 \) is included in \( \Sigma(n) \); that is, \( \Sigma(n) \geq f(n) + 1 \) if \( n > I(f) + O(1) \).

A related question is about the maximal execution or run-time of a halting algorithm of length smaller than or equal to \( n \); what is minimum time \( S(n) \) – or, alternatively, recurrence time – such that all programs of length at most \( n \) bits which halt have done so; that is, have terminated or, alternatively, are recurring?

An answer to this question will explain just how long it may take for the most time-consuming program of length \( n \) bits to halt. That, of course, is a worst-case scenario. Many programs of length \( n \) bits will have halted long before this maximal halting time [115]. \( S(n) \) can be estimated in terms of \( \Sigma(n) \) by two bounds; one from below and one from above. The bound from below is rather straightforward: Since the printout of any symbol requires at least a unit time step, \( \Sigma(n) \) can be interpreted as a sort of counter variable. Thus a first estimate is \( S(n) \geq \Sigma(n) \); that is, \( S(n) \) grows faster than any computable function of \( n \).

A bound from above can be conceptualized in terms of an “inner dialogue” of an algorithm which can be published – that is, printed – with little algorithmic overhead. Stated differently, every “algorithmic contemplation” or “symbolic computation” could, with a little overhead [symbolized by “of the order of” \( O(\cdot) \)], be transformed into a “printout” of this “monologue” directed towards the outside world, whose size in turn cannot exceed \( \Sigma(n + O(1)) \) [125]. This yields a bound from above \( S(n) \leq O(\Sigma(n + O(1))) \) [101]. Thus the busy beaver function can serve as some sort of measure of what some algorithm can(not) do before it halts. In this sense “expressing something to the world” can be equated to “contemplating internally.”

A simulation of the original computation yields bounds \( \Sigma(3n + O(1)) \) [51, 52] for any program of size \( n \) bits to either halt, or else never to halt.

Knowledge of the maximal halting time – in particular, some computable upper bound on \( \Sigma \) – would solve the halting problem quantitatively because if the maximal halting time were known and bounded by any computable function of the program size of \( n \) bits, one would have to wait just a little longer than the maximal halting time to make sure that every program of length \( n \) – also this particular program, if it is destined for termination – has terminated. Otherwise, the program would run
forever. Hence, because of the recursive unsolvability of the halting problem the maximal halting time cannot be a computable function.

As a consequence, the upper bounds for the recurrence of any kind of physical behaviour can be expressed in terms of the busy beaver function [499]. In particular, for deterministic systems representable by \( n \) bits the maximal recurrence time grows faster than any computable number of \( n \). This maximal estimate related to possible behaviours may be interpreted quite generally as a measure of the impossibility to predict and forecast such behaviours by algorithmic means.

Just as for \( \Omega \), knowledge of busy beaver function and thus the maximal halting time at least up to some degree, entails the solution of decision problems associated with halting problems. But these capacities, at least with computable means, are forever blocked by recursion theoretic incomputability.

### A.8 Some Speculations on Primordial Chaos and Unlimited Information Content

Chaitin’s independence theorem discussed earlier imposes quantitative bounds on formal expressability: essentially [that is, up to \( O(1) \)] it is impossible to “squeeze out” (in terms of proofs of theorems, and with a caveat [325]) of a formal system much more than one has put in. If one wants more provable theorems one has to assume more. There appears to be no “pay once eat all scenario” envisaged by Hilbert. This, flamboyantly speaking, “garbage in, garbage out” situation fits well with means relativity and intrinsic perception of embedded observers: as there is no external point of view from which to execute omniscience the only possibility is to perform relative to the (intrinsic) means available.

There is, however, another option not excluded by Chaitin’s limiting theorems: the possibility to obtain a system of arbitrary algorithmic information content by considering subsequences of infinite sequences interpreted as formal systems, or as axioms thereof. There are two extreme scenarios which could be imagined: in the first scenario, a “primordial chaos” is taken as a resource. In the second scenario, the continuum of all infinite binary sequences \( 2^\omega \) is approximated by a nonterminating process of generating it.

In both cases partial sequences could in principle be taken as a basis representation for axiomatic systems. And in both cases the encoded axiomatic systems are potentially infinite, with an unlimited algorithmic information content.
Appendix B
Two Particle Correlations and Expectations

B.1 Two Two-State Particle Correlations and Expectations

As has already been pointed out earlier, due to the Einstein–Podolsky–Rosen explosion type setup [196] in certain (singlet) states allowing for uniqueness [448, 508, 514] through counterfactual reasoning, second order correlations appear feasible (subject to counterfactual existence).

B.1.1 Classical Correlations with Dichotomic Observables in a “Singlet” State

For dichotomic observables with two outcomes \{0, 1\} the classical expectations in the plane perpendicular to the direction connecting the two particles is a linear function of the measurement angle [388]. Assume the uniform distribution of (opposite but otherwise) identical “angular momenta” shared by the two particles and lying on the circumference of the unit circle, as depicted in Fig. B.1; and consider only the sign of these angular momenta.

The arc lengths on the unit circle \(A_+(\theta_1, \theta_2)\) and \(A_- (\theta_1, \theta_2)\), normalized by the circumference of the unit circle, correspond to the frequency of occurrence of even (“++” and “−−”) and odd (“+−” and “−+) parity pairs of events, respectively. Thus, \(A_+(\theta_1, \theta_2)\) and \(A_- (\theta_1, \theta_2)\) are proportional to the positive and negative contributions to the correlation coefficient. One obtains for \(0 \leq \theta = |\theta_1 - \theta_2| \leq \pi\); i.e.,

\[
E_{c,2,2}(\theta) = E_{c,2,2}(\theta_1, \theta_2) = \frac{1}{2\pi} \left[ A_+(\theta_1, \theta_2) - A_- (\theta_1, \theta_2) \right] = \frac{1}{2\pi} \left[ 2A_+(\theta_1, \theta_2) - 2\pi \right] = \frac{2}{\pi} |\theta_1 - \theta_2| - 1 = \frac{2}{\pi} \theta - 1,
\] (B.1)
where the subscripts stand for the number of mutually exclusive measurement outcomes per particle, and for the number of particles, respectively. Note that $A_+(\theta_1, \theta_2) + A_- (\theta_1, \theta_2) = 2\pi$.

**B.1.2 Quantum Dichotomic Case**

The two spin one-half particle case is one of the standard quantum mechanical exercises, although it is seldom computed explicitly. For the sake of completeness and with the prospect to generalize the results to more particles of higher spin, this case will be enumerated explicitly. In what follows, we shall use the following notation: Let $|+\rangle$ denote the pure state corresponding to $(1, 0)^\top$, and $|−\rangle$ denote the orthogonal pure state corresponding to $(0, 1)^\top$.

**B.1.2.1 Single Particle Observables and Projection Operators**

Let us start with the spin one-half angular momentum observables of a single particle along an arbitrary direction in spherical co-ordinates $\theta$ and $\varphi$ in units of $\hbar$ [447]; that is,

\[
M_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad M_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (B.2)

The angular momentum operator in some direction specified by $\theta, \varphi$ is given by the spectral decomposition
Appendix B: Two Particle Correlations and Expectations

\[ S^2_{\frac{1}{2}}(\theta, \varphi) = xM_x + yM_y + zM_z \]

\[ = M_x \sin \theta \cos \varphi + M_y \sin \theta \sin \varphi + M_z \cos \theta \]

\[ = \frac{1}{2} \sigma(\theta, \varphi) = \frac{1}{2} \left( e^{i\varphi} \sin \theta \cos \varphi \right) \]

\[ = \frac{1}{2} \left( \sin^2 \frac{\theta}{2} - \frac{1}{2} e^{-i\varphi} \sin \theta \right) \]  

\[ + \frac{1}{2} \left( \cos^2 \frac{\theta}{2} + \frac{1}{2} e^{-i\varphi} \sin \theta \right) \]

\[ = \frac{1}{2} \left\{ \frac{1}{2} \left[ I_2 + \sigma(\theta, \varphi) \right] - \frac{1}{2} \left[ I_2 - \sigma(\theta, \varphi) \right] \right\} \]

\[ = \frac{1}{2} \left[ F_+(\theta, \varphi) - F_-(\theta, \varphi) \right]. \]  

The orthonormal eigenstates (eigenvectors) associated with the eigenvalues \(-\frac{1}{2}\) and \(+\frac{1}{2}\) of \(S^2_{\frac{1}{2}}(\theta, \varphi)\) in Eq. (B.3) are

\[ |+\rangle_{\theta, \varphi} = e^{i\delta_+} \left( e^{-i\frac{\varphi}{2}} \cos \frac{\theta}{2}, e^{i\frac{\varphi}{2}} \sin \frac{\theta}{2} \right)^T, \]

\[ |-\rangle_{\theta, \varphi} = e^{i\delta_-} \left( -e^{-i\frac{\varphi}{2}} \sin \frac{\theta}{2}, e^{i\frac{\varphi}{2}} \cos \frac{\theta}{2} \right)^T, \]  

respectively. \(\delta_+\) and \(\delta_-\) are arbitrary phases. These orthogonal unit vectors correspond to the two orthogonal projectors

\[ F_{\pm}(\theta, \varphi) = |\pm\rangle_{\theta, \varphi} \langle \pm|_{\theta, \varphi} = \frac{1}{2} \left[ I_2 \pm \sigma(\theta, \varphi) \right] \]  

for the “+” and “−” states along \(\theta\) and \(\varphi\), respectively. By setting all the phases and angles to zero, one obtains the original orthonormalized basis \(|-\rangle, |+\rangle\).

B.1.2.2 Substitution Rules for Probabilities and Correlations

In order to evaluate Boole’s classical conditions of possible experience, and check for quantum violations of them, the classical probabilities and correlations entering those classical conditions of possible experience have to be compared to, and substituted by, quantum probabilities and correlations derived earlier. For example, for \(n\) spin-\(\frac{1}{2}\) particles in states (subscript \(i\) refers to the \(i\)th particle) “+\(i\)” or “−\(i\)” along the directions \(\theta_1, \varphi_1, \theta_2, \varphi_2, \ldots, \theta_n, \varphi_n\), the classical-to-quantum substitutions are [212, 448, 513]:
\[ p_{1,\pm_1} \rightarrow q_{1,\pm_1} = \frac{1}{2} \left[ \mathbb{I}_2 \pm \sigma(\theta_1, \varphi_1) \right] \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2, \quad \text{n-1 factors} \]

\[ p_{2,\pm_2} \rightarrow q_{2,\pm_2} = \mathbb{I}_2 \otimes \left[ \mathbb{I}_2 \pm \sigma(\theta_2, \varphi_2) \right] \otimes \cdots \otimes \mathbb{I}_2, \quad \text{n-2 factors} \]

\[ \vdots \]

\[ p_{1,\pm_1,\pm_2,\pm_2} \rightarrow q_{1,\pm_1,\pm_2,\pm_2} = \frac{1}{2} \left[ \mathbb{I}_2 \pm \sigma(\theta_1, \varphi_1) \right] \otimes \frac{1}{2} \left[ \mathbb{I}_2 \pm \sigma(\theta_2, \varphi_2) \right] \otimes \cdots \otimes \mathbb{I}_2, \quad \text{n-2 factors} \]

\[ \vdots \]

\[ p_{1,\pm_1,\pm_2,\pm_2,\ldots,(n-1),\pm_{n-1},n,\pm_n} \rightarrow q_{1,\pm_1,\pm_2,\pm_2,\ldots,(n-1),\pm_{n-1},n,\pm_n} = \]

\[ = \frac{1}{2} \left[ \mathbb{I}_2 \pm \sigma(\theta_1, \varphi_1) \right] \otimes \frac{1}{2} \left[ \mathbb{I}_2 \pm \sigma(\theta_2, \varphi_2) \right] \otimes \cdots \]

\[ \cdots \otimes \frac{1}{2} \left[ \mathbb{I}_2 \pm \sigma(\theta_n, \varphi_n) \right] ; \quad (B.6) \]

with \( \sigma(\theta, \varphi) \) defined in Eq. (B.3).

### B.1.2.3 Quantum Correlations for the Singlet State

The two-partite quantum expectations corresponding to the classical expectation value \( E_{c,2,2} \) in Eq. (B.1) can be defined to be the difference between the probabilities to find the two particles in identical spin states (along arbitrary directions) minus the probabilities to find the two particles in different spin states (along those directions); that is, \( E_{q,2,2} = q_{++} + q_{--} - (q_{+-} + q_{-+}) \), or \( q_{q} = q_{++} + q_{--} = \frac{1}{2} \left( 1 + E_{q,2,2} \right) \) and \( q_{\neq} = q_{+-} + q_{-+} = \frac{1}{2} \left( 1 - E_{q,2,2} \right) \).

In what follows, singlet states \( |\Psi_{d,n,i}\rangle \) will be labelled by three numbers \( d, n \) and \( i \), denoting the number \( d \) of outcomes associated with the dimension of Hilbert space per particle, the number \( n \) of participating particles, and the state count \( i \) in an enumeration of all possible singlet states of \( n \) particles of spin \( j = (d - 1)/2 \), respectively. For \( n = 2 \), there is only one singlet state (see Ref. [448] for more general cases).

Consider the singlet “Bell” state of two spin-\( \frac{1}{2} \) particles
\[
\Psi_{2,2,1} = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle)
\]

\begin{align*}
\Psi_{2,2,1} &= \frac{1}{\sqrt{2}} \left( (1, 0)^T \otimes (0, 1)^T - (0, 1)^T \otimes (1, 0)^T \right) \\
&= \left( 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right)^T.
\end{align*}

(B.7)

The density operator \( \rho_{\Psi_{2,2,1}} = |\Psi_{2,2,1}\rangle \langle \Psi_{2,2,1}| \) is just the projector of the dyadic product of this vector.

Singlet states are form invariant with respect to arbitrary unitary transformations in the single-particle Hilbert spaces and thus also rotationally invariant in configuration space, in particular under the rotations [29, Eqs. (7)–(49)]

\[
|+\rangle = e^{i\frac{\theta}{2}} (\cos \frac{\theta}{2} |+\rangle - \sin \frac{\theta}{2} |\rangle) \quad \text{and} \quad |-\rangle = e^{-i\frac{\theta}{2}} (\sin \frac{\theta}{2} |+\rangle + \cos \frac{\theta}{2} |\rangle).
\]

The Bell singlet state satisfies the uniqueness property [508] in the sense that the outcome of a spin state measurement along a particular direction on one particle “fixes” also the opposite outcome of a spin state measurement along the same direction on its “partner” particle: (assuming lossless devices) whenever a “plus” or a “minus” is recorded on one side, a “minus” or a “plus” is recorded on the other side, and vice versa.

**B.1.2.4 Quantum Predictions**

We now turn to the calculation of quantum predictions. The joint probability to register the spins of the two particles in state \( \rho_{\Psi_{2,2,1}} \) aligned or anti-aligned along the directions defined by \((\theta_1, \varphi_1)\) and \((\theta_2, \varphi_2)\) can be evaluated by a straightforward calculation of

\[q_{\Psi_{2,2,1}} = \frac{1}{4} \left\{ 1 - (\pm 1) (\pm 1) \left[ \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2) \right] \right\}.
\]

(B.8)

Since \( q_+ + q_- = 1 \) and \( E_{\Psi_{2,2,1}} = q_+ - q_- \), the joint probabilities to find the two particles in an even or in an odd number of spin-“\(-\frac{1}{2}\)”-states when measured along \((\theta_1, \varphi_1)\) and \((\theta_2, \varphi_2)\) are in terms of the correlation coefficient given by
\[ q_+ = q_{++} + q_{--} = \frac{1}{2} (1 + E_{q,2,2}) \]

\[ = \frac{1}{2} \{ 1 - [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)] \}, \quad (B.9) \]

\[ q_\neq = q_{+-} + q_{-+} = \frac{1}{2} (1 - E_{q,2,2}) \]

\[ = \frac{1}{2} \{ 1 + [\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2)] \}. \]

Finally, the quantum mechanical correlation is obtained by the difference \( q_+ - q_\neq \); i.e.,

\[ E_{q,2,2} (\theta_1, \varphi_1, \theta_2, \varphi_2) = \]

\[ = - [\cos \theta_1 \cos \theta_2 + \cos(\varphi_1 - \varphi_2) \sin \theta_1 \sin \theta_2]. \quad (B.10) \]

By setting either the azimuthal angle differences equal to zero, or by assuming measurements in the plane perpendicular to the direction of particle propagation, i.e., with \( \theta_1 = \theta_2 = \frac{\pi}{2} \), one obtains

\[ E_{q,2,2} (\theta_1, \theta_2) = -\cos(\theta_1 - \theta_2), \]

\[ E_{q,2,2} \left( \frac{\pi}{2}, \frac{\pi}{2}, \varphi_1, \varphi_2 \right) = -\cos(\varphi_1 - \varphi_2). \quad (B.11) \]

### B.2 Two Three-State Particles

#### B.2.1 Observables

The single particle spin one angular momentum observables in units of \( \hbar \) are given by \([447]\)

\[ M_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \]

\[ M_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (B.12) \]

Again, the angular momentum operator in arbitrary direction \( \theta, \varphi \) is given by its spectral decomposition
\[
\mathbf{S}_1(\theta, \varphi) = x\mathbf{M}_x + y\mathbf{M}_y + z\mathbf{M}_z \\
= \mathbf{M}_x \sin \theta \cos \varphi + \mathbf{M}_y \sin \theta \sin \varphi + \mathbf{M}_z \cos \theta
\]

\[
= \begin{pmatrix}
\cos \theta & e^{-i\varphi} \sin \theta & 0 \\
e^{i\varphi} \sin \theta & \frac{\sqrt{2}}{\sqrt{2}} & 0 \\
0 & \frac{e^{-i\varphi} \sin \theta}{\sqrt{2}} & -\cos \theta
\end{pmatrix}
\tag{B.13}
\]

\[
= -F_- (\theta, \varphi) + 0 \cdot F_0 (\theta, \varphi) + F_+ (\theta, \varphi),
\]

where the orthogonal projectors associated with the eigenstates of \(\mathbf{S}_1(\theta, \varphi)\) are

\[
F_- = \begin{pmatrix}
\frac{\sin^2 \theta}{2} & -\frac{e^{-i\varphi} \cos \theta \sin \theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} e^{-i\varphi} \sin^2 \theta \\
-\frac{e^{i\varphi} \cos \theta \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} & e^{-i\varphi} \frac{\cos \theta \sin \theta}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} e^{2i\varphi} \sin^2 \theta & e^{i\varphi} \frac{\cos \theta \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \frac{\cos \theta \sin \theta}{\sqrt{2}}
\end{pmatrix}
\]

\[
F_0 = \begin{pmatrix}
\frac{\cos^2 \theta}{2} & \frac{e^{-i\varphi} \cos^2 \frac{\varphi}{2} \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{-i\varphi} \sin^2 \theta \\
\frac{e^{i\varphi} \cos \theta \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} & e^{-i\varphi} \frac{\cos \theta \sin \theta}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} e^{2i\varphi} \sin^2 \theta & e^{i\varphi} \frac{\cos \theta \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \frac{\cos \theta \sin \theta}{\sqrt{2}}
\end{pmatrix}
\tag{B.14}
\]

\[
F_+ = \begin{pmatrix}
\frac{\sin^2 \theta}{2} & -\frac{e^{-i\varphi} \cos^2 \frac{\varphi}{2} \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} e^{-i\varphi} \sin^2 \theta \\
\frac{e^{i\varphi} \cos \theta \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} & e^{-i\varphi} \frac{\cos \theta \sin \theta}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} e^{2i\varphi} \sin^2 \theta & e^{i\varphi} \frac{\cos \theta \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \frac{\cos \theta \sin \theta}{\sqrt{2}}
\end{pmatrix}
\]

The orthonormal eigenstates associated with the eigenvalues +1, 0, −1 of \(\mathbf{S}_1(\theta, \varphi)\) in Eq. (B.13) are

\[
|\!-\rangle_{\theta, \varphi} = e^{i\delta_0} \left(-\frac{1}{\sqrt{2}} e^{-i\varphi} \sin \theta, \cos \theta, \frac{1}{\sqrt{2}} e^{i\varphi} \sin \theta\right)^T,
\]

\[
|0\rangle_{\theta, \varphi} = e^{i\delta_1} \left(e^{-i\varphi} \cos^2 \frac{\varphi}{2}, \frac{1}{\sqrt{2}} \sin \theta, e^{i\varphi} \sin^2 \frac{\varphi}{2}\right)^T,
\tag{B.15}
\]

\[
|+\rangle_{\theta, \varphi} = e^{i\delta_2} \left(e^{-i\varphi} \sin^2 \frac{\varphi}{2}, -\frac{1}{\sqrt{2}} \sin \theta, e^{i\varphi} \cos^2 \frac{\varphi}{2}\right)^T,
\]

respectively. For vanishing angles \(\theta = \varphi = 0\), \(|-\rangle = (0, 1, 0)^T, |0\rangle = (1, 0, 0)^T, \text{ and } |+\rangle = (0, 0, 1)^T\), respectively.

### B.2.2 Singlet State

Consider the two spin-one particle singlet state

\[
|\Psi_{3,2,1}\rangle = \frac{1}{\sqrt{3}} (-|00\rangle + |0-\rangle + |1+\rangle).
\tag{B.16}
\]
Its vector space representation can be explicitly enumerated by taking the direction \( \theta = \phi = 0 \) and recalling that \( |+\rangle = (1, 0, 0)^T \), \( |0\rangle = (0, 1, 0)^T \), and \(|-\rangle = (0, 0, 1)^T \); i.e.,

\[
|\Psi_{3,2,1}\rangle = \frac{1}{\sqrt{3}} (0, 0, 1, 0, -1, 0, 1, 0, 0)^T.
\]  

(B.17)

### B.3 Two Four-State Particles

#### B.3.1 Observables

The spin three-half angular momentum observables in units of \( \hbar \) are given by [447]

\[
M_x = \frac{1}{2} \begin{pmatrix}
0 & -\sqrt{3}i & 0 & 0 \\
\sqrt{3}i & 0 & 2i & 0 \\
0 & 2i & 0 & \sqrt{3}i \\
0 & 0 & \sqrt{3}i & 0
\end{pmatrix},
\]

\[
M_y = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & \sqrt{3}i \\
0 & \sqrt{3}i & 0 & 0 \\
\sqrt{3}i & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{3}i
\end{pmatrix},
\]

\[
M_z = \frac{1}{2} \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix}.
\]  

(B.18)

Again, the angular momentum operator in arbitrary direction \( \theta, \phi \) can be written in its spectral form

\[
S_{\frac{3}{2}}(\theta, \phi) = xM_x + yM_y + zM_z
\]

\[= M_x \sin \theta \cos \phi + M_y \sin \theta \sin \phi + M_z \cos \theta
\]

\[
= \begin{pmatrix}
\frac{3\cos \theta}{2} & \sqrt{\frac{3}{2}} e^{-i\phi} \sin \theta & 0 & 0 \\
\frac{3}{2} e^{i\phi} \sin \theta & \cos \theta \sin \phi & e^{-i\phi} \sin \theta & 0 \\
0 & e^{i\phi} \sin \theta & \cos \theta \sin \phi & \sqrt{3} e^{-i\phi} \sin \theta \\
0 & 0 & \sqrt{3} e^{i\phi} \sin \theta & \frac{3\cos \theta}{2}
\end{pmatrix}
\]

\[= -\frac{3}{2} F_{-\frac{1}{2}} - \frac{1}{2} F_{-\frac{1}{2}} + \frac{1}{2} F_{\frac{1}{2}} + \frac{3}{2} F_{\frac{3}{2}}.
\]  

(B.19)
B.3.2 Singlet State

The singlet state of two spin-3/2 observables can be found by the general methods developed in Ref. [448]. In this case, this amounts to summing all possible two-partite states yielding zero angular momentum, multiplied with the corresponding Clebsch-Gordan coefficients

\[
\langle j_1 m_1, j_2 m_2 | 00 \rangle = \delta_{j_1, j_2} \delta_{m_1, -m_2} \frac{(-1)^{j_1 - m_1}}{\sqrt{2j_1 + 1}} \tag{B.20}
\]

of mutually negative single particle states resulting in total angular momentum zero. More explicitly, for \( j_1 = j_2 = \frac{3}{2} \), \(|\psi_{4,2,1}\rangle\) can be written as

\[
\frac{1}{2} \left( \left| \frac{3}{2}, -\frac{3}{2} \right\rangle - \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle + \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right). \tag{B.21}
\]

Again, this two-partite singlet state satisfies the uniqueness property. The four different spin states can be identified with the Cartesian basis of 4-dimensional Hilbert space \(|\frac{3}{2}\rangle = (1, 0, 0, 0)^T\), \(|\frac{1}{2}\rangle = (0, 1, 0, 0)^T\), \(|-\frac{1}{2}\rangle = (0, 0, 1, 0)^T\), and \(|-\frac{3}{2}\rangle = (0, 0, 1, 1)^T\), respectively, so that

\[|\psi_{4,2,1}\rangle = (0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)^T. \tag{B.22}\]

B.4 General Case of Two Spin \( j \) Particles in a Singlet State

The general case of spin correlation values of two particles with arbitrary spin \( j \) (see the Appendix of Ref. [321] for a group theoretic derivation) can be directly calculated in an analogous way, yielding

\[
E_{\psi_{2,2j+1,1}}(\theta_1, \varphi_1; \theta_2, \varphi_2) \propto \text{Tr} \left\{ \rho_{\psi_{2,2j+1,1}} \left[ S_j(\theta_1, \varphi_1) \otimes S_j(\theta_2, \varphi_2) \right] \right\} \tag{B.23}
\]

\[=- \frac{j(1 + j)}{3} \left[ \cos \theta_1 \cos \theta_2 + \cos(\varphi_1 - \varphi_2) \sin \theta_1 \sin \theta_2 \right]. \]

Thus, the functional form of the two-particle correlation coefficients based on spin state observables is independent of the absolute spin value.
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