Appendix A
Abstract Graphs

The purpose of this Appendix is twofold. First we want to develop a terminology
common to both abstract and profinite graphs that is appropriate for this book. Every
profinite graph has the underlying structure of a graph in the abstract sense, if we
dispense with the topology. The second purpose is to establish some basic results
that are needed in parts of this work, using this common terminology. The choice of
these results is dictated by our needs in the book.

A.1 The Fundamental Group of an Abstract Graph

An abstract (oriented) graph consists of a set $\Delta$ together with a nonempty subset
$V = V(\Delta)$ (the vertices of the graph) and two incidence maps

$$d_0, d_1 : \Delta \longrightarrow V$$

whose restriction to $V$ are the identity map on $V$. The set $E = E(\Delta) = \Delta - V$ is
the set of edges of the graph. If $e \in E$, $d_0(e)$ is the initial vertex (or origin) of $e$,
and $d_1(e)$ is the terminal vertex (or terminus or end vertex) of $e$. A graph with
only one edge $e$ and one vertex $v$ with $d_0(e) = d_1(e) = v$ is called a loop based
at $v$. A nonempty subset $\Delta'$ of $\Delta$ is called a subgraph if whenever $m \in \Delta'$, then
d_0(m), d_1(m) \in \Delta'$; observe that then $\Delta'$ is a graph in a natural way. A morphism
(or a ‘map of abstract graphs’) $\alpha : \Delta \longrightarrow \Lambda$ of abstract graphs is a map such that
d_i(\alpha(e)) = \alpha(d_i(e)) (i = 0, 1) and such that $\alpha(e) \in E(\Lambda)$, for every $e \in E(\Delta)$; in
other words, $\alpha$ sends vertices to vertices and edges to edges and $\alpha$ preserves the
graph structure.

A group $G$ acts on a graph $\Delta$ if it acts on the set $\Delta$ and $d_i(gm) = gd_i(m)$, for all
g $\in G$, m $\in \Delta$ and i = 0, 1 (note that then $G$ acts on $V(\Delta)$ and on $E(\Delta)$). If a group
$G$ acts on a graph $\Delta$, the quotient $G \backslash \Delta$ is a graph in a natural way: $V(G \backslash \Delta) = G \backslash V$
and $d_i(Gm) = Gd_i(m)$ (m $\in \Delta$, i = 0, 1).
**The Star of a Vertex**

Let $v \in V = V(\Delta)$. Define

$$\text{Star}_0(v) = \{ e \in E(\Delta) \mid d_0(e) = v \} \quad \text{and} \quad \text{Star}_1(v) = \{ e \in E(\Delta) \mid d_1(e) = v \}. $$

Observe that $\text{Star}_0(v)$ and $\text{Star}_1(v)$ are subsets of $\Delta$, and they are not disjoint if and only if $\Delta$ has a loop based at $v$. Define the ‘star’ of the vertex $v$ to be

$$\text{Star}_\Delta(v) = \text{Star}(v) = \text{Star}_0(v) \cup \text{Star}_1(v)$$

The **valency** of $v$ in $\Delta$ is the cardinality of $\text{Star}_\Delta(v)$ (so, the valency of $v$ is the number of edges $e \in E(\Delta)$ incident with $v$, i.e., having $v$ as a vertex, where a loop is counted twice). Note that if

$$\alpha : \Delta \longrightarrow \Lambda$$

is a morphism of abstract, then $\alpha(\text{Star}_i(v)) \subseteq \text{Star}_i(\alpha(v)) \ (i = 0, 1)$. Therefore $\alpha$ induces a map of sets

$$\alpha_v : \text{Star}_\Delta(v) \longrightarrow \text{Star}_\Lambda(\alpha(v)),$$

for each $v \in V(\Delta)$.

We say that $\alpha$ is an **immersion** if for each $v \in V(\Delta)$, the induced map

$$\alpha_v : \text{Star}_\Delta(v) \longrightarrow \text{Star}_\Lambda(\alpha(v))$$

is injective.

Observe that if $\alpha$ fails to be an immersion, it is because there exists a pair of different edges $\{e_1, e_2\}$ such that either $d_0(e_1) = d_0(e_2)$ or $d_1(e_1) = d_1(e_2)$ and $\alpha(e_1) = \alpha(e_2)$.

We say that $\alpha$ is a **covering** of abstract graphs if $\alpha_v$ is a bijection, for all $v \in V(\Delta)$.

**Paths**

To make the notion of ‘path’ easier to describe, one introduces new formal symbols $e^{\pm 1}$ for each $e \in E$. For convenience, we make the identification $e^1 = e$, for $e \in E(\Delta)$. One extends the incidence maps $d_i$ as follows: $d_0(e^{-1}) = d_1(e)$ and $d_1(e^{-1}) = d_0(e)$; we also refer to the $e^{-1}$ as the ‘inverse edge’ of $e$. Put $E^{-1} = \{ e^{-1} \mid e \in E \}$; we assume that $E^{-1} \cap E = \emptyset$. If $\alpha : \Delta \longrightarrow \Lambda$ is a morphism of graphs, we extend $\alpha$ to the formal edges $e^{-1}$ by setting $\alpha(e^{-1}) = \alpha(e)^{-1}$.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1 \ (i = 1, \ldots, n)$ and $n \geq 1$ is a natural number. Define $\text{Circ}_n(\varepsilon)$ to be a graph with $n$ vertices (that we take to be the elements of
\( \mathbb{Z}/n\mathbb{Z} \) and \( n \) edges \( e_1, \ldots, e_n \)

\[
\text{Circ}_n(\varepsilon) :
\]

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
n-1 \\
e_n \\
e_1 \\
e_2 \\
e_3 \\
\end{array}
\]

such that \( d_0(e_i) = i - 1 \) and \( d_1(e_i) = i \), if \( \varepsilon_i = 1 \), and \( d_0(e_i) = i \) and \( d_1(e_i) = i - 1 \), if \( \varepsilon_i = -1 \). We refer to a graph of the form \( \text{Circ}_n(\varepsilon) \) as a circuit of length \( n \) (or \( n \)-circuit). Note that a circuit of length 1 is a loop.

A standard arc \( A_n(\varepsilon) \) of length \( n \) is a graph with \( n + 1 \) vertices \( 0, 1, \ldots, n \), and \( n \) edges \( e_1, \ldots, e_n \)

\[
\text{A}_n(\varepsilon) :
\]

\[
\begin{array}{c}
0 \\
1 \\
2 \\
\ldots \\
n-1 \\
n \\
e_1 \\
e_2 \\
e_3 \\
e_n \\
\end{array}
\]

such that \( d_0(e_i) = i - 1 \) and \( d_1(e_i) = i \), if \( \varepsilon_i = 1 \), and \( d_0(e_i) = i \) and \( d_1(e_i) = i - 1 \), if \( \varepsilon_i = -1 \). We call 0 and \( n \) the initial and terminal vertices of the standard arc \( A_n(\varepsilon) \). Note that one obtains the circuit \( \text{Circ}_n(\varepsilon) \) by identifying the vertices 0 and \( n \) in \( A_n(\varepsilon) \).

Define a path \( p = p_{v,w} \) from a vertex \( v \) to a vertex \( w \) of \( \Delta \) to be a finite sequence \( e_{\varepsilon_1}^1, \ldots, e_{\varepsilon_r}^r \) of edges such that \( d_0(e_{\varepsilon_1}^1) = v \), \( d_1(e_{\varepsilon_r}^r) = w \) and \( d_1(e_{\varepsilon_i}^i) = d_0(e_{\varepsilon_{i+1}}^{i+1}) \) for all \( i < r \). The length \( |p_{v,w}| \) of such a path is \( r \). We refer to \( v \) and \( w \) as the initial and terminal vertices, respectively, of the path \( p = p_{v,w} \). Observe that an equivalent way of specifying a path in \( \Delta \) from \( v \) to \( w \) of length \( r \) is by giving a morphism of graphs

\[
f : A_r(\varepsilon) \longrightarrow \Delta
\]

such that \( f(0) = v \) and \( f(r) = w \). This way of defining a path will be useful later in Sect. A.4 dealing with certain algorithms in free abstract groups.

The underlying graph of the path \( p_{v,w} \) is the subgraph of \( \Delta \) with edges \( e_1, \ldots, e_r \) (note that these may not be distinct) and their vertices. The path \( p_{v,w} \) is called reduced if it contains no subsequence \( e_{\varepsilon_i}^i, e_{\varepsilon_{i+1}}^{i+1} \) of the form \( e^\varepsilon, e^{-\varepsilon} \) (\( e \in E(\Delta), \varepsilon = \pm 1 \)). Note that if the path is defined as a morphism from a standard arc, as in (A.1), then saying that it is reduced is the same as saying that \( f \) is an immersion.

If \( v = w \), then a path \( p_{v,v} \) is called a cycle based at \( v \) or a closed path based at \( v \) (a cycle of length 0 is just a vertex; this is also called the ‘empty cycle’ based at \( v \)).
The underlying graph of a reduced cycle of length greater than 0 has a subgraph which is a circuit.

If \( p = e_1^{\varepsilon_1}, \ldots, e_r^{\varepsilon_r} \) and \( p' = e_1^{\varepsilon_1}, \ldots, e_r^{\varepsilon_r} \) are paths in \( \Delta \), we say that \( p \) and \( p' \) are elementary homotopic if one can pass from \( p \) to \( p' \) by deleting or inserting adjacent pairs of the form \( e^\varepsilon, e^{-\varepsilon} \) \((e \in E(\Delta))\); observe that if \( p \) and \( p' \) are elementary homotopic, then they have the same initial and the same terminal vertices. This generates an equivalence relation on the set of paths, which is called homotopy; explicitly: two paths \( p \) and \( p' \) in \( \Delta \) are homotopic if there exists a finite sequence of paths \( p = p_1, \ldots, p_t = q \) such that any two adjacent paths \( p_i, p_{i+1} \) are elementary homotopic. Note that every path is homotopic to a reduced path.

Let \( v \) be a fixed vertex of an abstract graph \( \Delta \). Given two cycles \( p_{vv} \) and \( p'_{vv} \) based at a vertex \( v \), define their product \( p_{vv}p'_{vv} \) by concatenation. This product is compatible with the homotopy relation defined above. Let \( \pi_{1}^{\text{abs}}(\Delta, v) \) denote the set of homotopy classes of cycles based at \( v \). Then it easily follows that \( \pi_{1}^{\text{abs}}(\Delta, v) \) is a group whose identity element is the class represented by the empty cycle based at \( v \): this is called the fundamental group of \( \Delta \) based at \( v \).

A more algebraic approach is the following. Put \( Y = \{ ye \mid e \in E(\Delta) \} \), and let \( \Phi = \Phi(Y) \) denote the free abstract group with basis \( Y \). Then \( \pi_{1}^{\text{abs}}(\Delta, v) \) is the subgroup of \( \Phi \) consisting of those elements represented by words of the form \( ye_1^{\varepsilon_1} \cdots ye_r^{\varepsilon_r} \) such that \( e_1^{\varepsilon_1}, \ldots, e_r^{\varepsilon_r} \) is a cycle in \( \Delta \) based at \( v \); this interpretation of \( \pi_{1}^{\text{abs}}(\Delta, v) \) is justified because homotopy in paths corresponds to the usual equivalence of words. In particular, one deduces that \( \pi_{1}^{\text{abs}}(\Delta, v) \) is a free abstract group. One also obtains the following result from this interpretation and from the classical treatment of free groups (see Sect. B.2 in Appendix B).

**Lemma A.1.1** An element of \( \pi_{1}^{\text{abs}}(\Delta, v) \) is represented by a unique reduced cycle based at \( v \), or equivalently, if \( p_1 \) and \( p_2 \) are reduced homotopic cycles based at \( v \), then \( p_1 = p_2 \).

An abstract graph \( \Delta \) is connected if for any given pair of vertices \( v, w \), there exists a path in \( \Delta \) from \( v \) to \( w \). For an abstract graph \( \Delta \) define a relation \( R \) on \( \Delta \) as follows: for \( m, m' \in \Delta \), we say that \( mRm' \) if there exists a path from \( d_0(m) \) to \( d_0(m') \); this is an equivalence relation. The equivalence classes of \( R \) are connected subgraphs of \( \Delta \), which we call the ‘connected components’ of \( \Delta \).

One easily checks that the image of a connected abstract graph under a morphism is connected. A connected abstract graph \( \Delta \) is a tree if it contains no circuits, or equivalently, if the only reduced cycles in \( \Delta \) have length 0.

**Exercise A.1.2** Let \( \Delta \) be a abstract graph and let \( R \) be a ring (with an identity element 1). For a set \( X \) denote by \( [RX] \) the free \( R \)-module with basis \( X \). Consider the sequence of free \( R \)-modules and homomorphisms

\[
0 \longrightarrow [RE(\Delta)] \overset{d}{\longrightarrow} [RV(\Delta)] \overset{e}{\longrightarrow} R \longrightarrow 0.
\]  

(A.2)
where \( d(e) = d_1(e) - d_0(e) \) and \( \varepsilon(v) = 1 \ (e \in E(\Delta), v \in V(\Delta)) \).

(a) Show that \( \Delta \) is connected if and only if the sequence (A.2) is exact at \( [RV(\Delta)] \).

(See Proposition 2.3.2.)

(b) Show that \( \Delta \) is a tree if and only if the short exact sequence (A.2) is exact.

If \( v \) and \( w \) are vertices in a tree \( T \), it is not hard to see that there is a unique reduced path in the tree from \( v \) to \( w \); we denote by \([v, w]\) the underlying graph of such a reduced path: it is a finite subtree of \( T \), and we refer to \([v, w]\) as the chain determined by \( v \) and \( w \).

If \( \Delta \) is connected, then the fundamental group \( \pi_1^{\text{abs}}(\Delta, v) \) is independent of the chosen vertex \( v \). This is a consequence of the following result, which also provides a basis for the free group \( \pi_1^{\text{abs}}(\Delta, v) \). Using Zorn’s Lemma, one sees that every abstract connected graph \( \Delta \) contains a subtree \( T \) with \( V(\Delta) = V(T) \) (a ‘maximal tree’).

**Proposition A.1.3** Let \( \Delta \) be a connected abstract graph and let \( T \) be a maximal tree of \( \Delta \). Choose \( v \in V(\Delta) \); let \( Y \) be as above and put

\[
X = \{ y_e \mid e \in E(\Delta) - E(T) \}.
\]

Then the restriction to \( \pi_1^{\text{abs}}(\Delta, v) \) of the natural homomorphism of free abstract groups

\[
\varphi : \Phi(Y) \longrightarrow \Phi(X)
\]

which sends \( X \) to \( X \) identically and sends each \( y \in Y - X \) to \( 1 \), is an isomorphism from \( \pi_1^{\text{abs}}(\Delta, v) \) onto \( \Phi(X) \).

**Proof** Given \( e \in E(\Delta) \), define

\[
y'_e = \rho_{d_0(e)} y_e \rho_{d_1(e)}^{-1},
\]

where, for a given vertex \( w \in V(T) \), we set \( \rho_w = y_{e_1} \cdots y_{e_r} \), if \( e_1, \ldots, e_r \) is the unique reduced path in \( T \) from \( v \) to \( w \). Note that \( y'_e \) represents an element of \( \pi_1^{\text{abs}}(\Delta, v) \) (by abuse of notation we write \( y'_e \in \pi_1^{\text{abs}}(\Delta, v) \)), and if \( e \in E(T) \), then \( y'_e = 1 \). Define

\[
\psi : \Phi(X) \longrightarrow \pi_1^{\text{abs}}(\Delta, v) \leq \Phi(Y)
\]

to be the homomorphism that sends \( x = y_e \in X \) to \( y'_e (e \in E(\Delta) - E(T)) \). Then clearly \( \varphi|_{\pi_1^{\text{abs}}(\Delta, v)} \psi = \text{id}_\Phi(X) \).

Next we need to show that \( \psi \varphi|_{\pi_1^{\text{abs}}(\Delta, v)} = \text{id}_{\pi_1^{\text{abs}}(\Delta, v)} \). To prove this it is first convenient to extend our notation as follows: if \( e \in E(\Delta) \), we define \( y_{e^{-1}} = y_e^{-1} \). Then we have

\[
(y'_e)^{-1} = (\rho_{d_0(e)} y_e \rho_{d_1(e)}^{-1})^{-1} = \rho_{d_1(e)} y_e^{-1} \rho_{d_0(e)}^{-1} = \rho_{d_0(e^{-1})} y_{e^{-1}} \rho_{d_1(e^{-1})}.
\]
Hence, if we define $y'_e^{-1} = (y'_e)^{-1}$, we have
\[ y'_e = \rho d_0(e) y_e \rho d_1^{-1}(e), \quad \varepsilon = \pm 1, \ e \in E(\Delta). \]

Consider an element $\pi = y_{e_1}^{e_1} \cdots y_{e_n}^{e_n} = y_{e_1}^{e_1} \cdots y_{e_n}^{e_n} \in \pi_1^{\text{abs}}(\Delta, v)$ corresponding to a cycle $e_1^{e_1}, \ldots, e_n^{e_n}$ in $\Delta$ based at $v$. Then we have
\[ \pi = y_{e_1}^{e_1} \cdots y_{e_n}^{e_n} = \rho d_0^{-1}(e_1) y_{e_1}^{e_1} \cdots y_{e_n}^{e_n} \rho d_1(e_n) = y_{e_1}^{e_1} \cdots y_{e_n}^{e_n}, \]

since $\rho d_0(e_1) = \rho d_1(e_n) = 1$, because $d_0(e_1) = d_1(e_n)$. Thus
\[ \psi \varphi|_{\pi_1^{\text{abs}}(\Delta, v)}(\pi) = \pi, \]

(since $y_{e_i}^{e_i} = 1$, when $e_i \in E(T)$), as desired. \qed

The following corollary is now clear.

**Corollary A.1.4** Let $\Delta$ be a connected abstract graph. The group $\pi_1^{\text{abs}}(\Delta, v)$ is a free group which is independent, up to isomorphism, of the choice of the vertex $v$; furthermore, its rank is the cardinality of the set $E(\Delta) - E(T)$, where $T$ is a maximal tree of $\Delta$, and this is independent of the chosen maximal tree.

In view of this corollary, one sometimes uses the notation $\pi_1^{\text{abs}}(\Delta)$, rather than $\pi_1^{\text{abs}}(\Delta, v)$, when $\Delta$ is a connected graph, and we refer to $\pi_1^{\text{abs}}(\Delta)$ as the fundamental group of the abstract graph $\Delta$.

**Corollary A.1.5** Let $\Lambda$ be a connected subgraph of a connected graph $\Delta$. Choose $v \in V(\Lambda) \subseteq V(\Delta)$. Then $\pi_1^{\text{abs}}(\Lambda, v)$ is naturally embedded into $\pi_1^{\text{abs}}(\Delta, v)$ as a free factor (i.e., a basis of the subgroup $\pi_1^{\text{abs}}(\Lambda, v)$ of $\pi_1^{\text{abs}}(\Delta, v)$ can be extended to a basis of $\pi_1^{\text{abs}}(\Delta, v)$).

**Proof** Choose a maximal tree $T'$ of $\Lambda$. Extend $T'$ to a maximal tree $T$ of $\Delta$. Then $\Lambda - T' \subseteq \Delta - T$. Continuing with the notation in the proof of Proposition A.1.3, we deduce that the basis $\{y'_e \mid e \in E(\Lambda) - E(T')\}$ of $\pi_1^{\text{abs}}(\Lambda, v)$ is a subset of the basis $\{y'_e \mid e \in E(\Delta) - E(T)\}$ of $\pi_1^{\text{abs}}(\Delta, v)$. \qed

The following algorithm is clear from the proof of Proposition A.1.3.

**Algorithm A.1.6** Construction of a basis for $\pi_1^{\text{abs}}(\Delta, v)$ when $\Delta$ is a connected finite graph.

Explicitly: We assume that $\Delta$ is specified by its vertices, edges and explicitly given incident functions $d_0, d_1$. Since a maximal tree of $\Delta$ is a subtree that covers all its vertices, one can construct algorithmically one such maximal tree $T$. For each
edge $e \in \Delta - T$, construct a cycle $\tilde{e}$ based at $v$ as the path defined as follows: take the unique reduced path in $T$ from $v$ to $d_0(e)$, followed by $e$, followed by the unique path in $T$ from $d_1(e)$ to $v$. Then $\tilde{e}$ represents an element of $\pi_1^{\text{abs}}(\Delta, v)$, and the set of all those elements is a basis for $\pi_1^{\text{abs}}(\Delta, v)$.

**Example A.1.7** Let $G$ be an abstract group and let $X$ be a subset of $G$ with $1 \notin X$. The Cayley graph $\Gamma = \Gamma(G, X)$ of $G$ with respect to $X$ is defined as follows: $\Gamma = V(\Gamma) \cup E(\Gamma)$, where $V(\Gamma) = G$, $E(\Gamma) = G \times X$, $d_0(g, x) = g$ and $d_1(g, x) = gx$ ($g \in G, x \in X$). Clearly $\Gamma$ is connected if and only if $G = \langle X \rangle$.

## A.2 Coverings of Abstract Graphs

In this section we construct a ‘universal covering’ $\tilde{\Delta}$ of a connected graph $\Delta$ and show how any other covering of $\Delta$ appears as an image of $\tilde{\Delta}$. We begin with some general properties of immersions and coverings.

**Proposition A.2.1** Let $\zeta : \Delta' \longrightarrow \Delta$ be an immersion of abstract connected graphs and let $v \in V(\Delta')$.

(a) Let $p$ and $p'$ be paths in $\Delta'$ with the same initial vertex $v$. If $\zeta(p) = \zeta(p')$, then $p = p'$.

(b) If $p$ is a reduced path in $\Delta'$, then $\zeta(p)$ is a reduced path in $\Delta$.

**Proof** (a) Say $p = \{e_1^{\epsilon_1}, \ldots, e_r^{\epsilon_r}\}$ and $p' = \{(e'_1)^{\epsilon'_1}, \ldots, (e'_s)^{\epsilon'_s}\}$. Since $\zeta(p) = \zeta(p')$, we have $\{\zeta(e_1)^{\epsilon_1}, \ldots, \zeta(e_r)^{\epsilon_r}\} = \{(\zeta(e'_1)^{\epsilon'_1}, \ldots, \zeta(e'_s)^{\epsilon'_s}\}$. Hence $r = s$, $\zeta(e_i) = \zeta(e'_i)$ and $\epsilon_i = \epsilon'_i$, for all $i = 1, \ldots, r$. Assume first that $r = 1$, i.e., $p = \{e_1^{\epsilon_1}\}$ and $p' = \{(e'_1)^{\epsilon'_1}\}$. Since also $v = d_0(e_1^{\epsilon_1}) = d_0((e'_1)^{\epsilon'_1})$, we deduce that either $d_0(e_1) = d_0(e'_1)$ or $d_1(e_1) = d_1(e'_1)$. Hence, since $\zeta$ induces an injection on Star($v$), we also have that $e_1 = e'_1$; therefore $p = \{e_1^{\epsilon_1}\} = \{(e'_1)^{\epsilon'_1}\} = p'$, proving the result for paths of length 1. The general result now follows by an easy induction.

(b) We think of the path $p$ as a morphism from a standard arc, $p : A(\epsilon) \rightarrow \Delta'$. Since $p$ is reduced, the morphism $p$ is an immersion. So the result reduces to the obvious statement that the composition of immersions is an immersion.

**Proposition A.2.2** Let $\zeta : \Delta' \longrightarrow \Delta$ be a covering of connected abstract graphs and let $v' \in V(\Delta')$.

(a) If $p$ is a path in $\Delta$ with initial vertex $\zeta(v')$, then there exists a unique path $\tilde{p}$ (the ‘lifting’ of $p$) in $\Delta'$ with initial vertex $v'$ such that $\zeta(\tilde{p}) = p$.

(b) If $p_1$ and $p_2$ are homotopic paths in $\Delta$ with initial vertex $\zeta(v')$, then $\tilde{p}_1$ and $\tilde{p}_2$ are homotopic paths in $\Delta'$.

(c) If $\zeta$ has a section (i.e., a graph morphism $\sigma : \Delta \longrightarrow \Delta'$ such that $\zeta \sigma = id_\Delta$), then $\zeta$ is an isomorphism.
Proof Parts (a) and (b) are easy to prove. Part (c) follows from (a) and the connectedness of $\Delta'$.

Proposition A.2.3 Let $\xi : \Delta' \longrightarrow \Delta$ be an immersion of connected abstract graphs and let $v' \in V(\Delta')$. Then the natural homomorphism

$$
\xi : \pi_1^{\text{abs}}(\Delta', v') \longrightarrow \pi_1^{\text{abs}}(\Delta, \xi(v'))
$$

is injective.

Proof Let $1 \neq \alpha \in \pi_1^{\text{abs}}(\Delta', v')$; by Lemma A.1.1, $\alpha$ is represented by a unique cycle $p$ based at $v'$ in reduced form. Since $\alpha \neq 1$, $|p| \geq 1$. Hence $|\xi(p)| \geq 1$. By Proposition A.2.1(b), $\xi(p)$ is in reduced form; therefore, $\xi(\alpha) \neq 1$.

Associated with a connected abstract graph $\Delta$ one can define an abstract ‘universal covering graph’ $v : \tilde{\Delta} \longrightarrow \Delta$ of $\Delta$ as follows. The morphism $v$ is surjective and a covering of the abstract graph $\Delta$, and the following universal property is satisfied: given a surjective covering graph $\zeta : \Gamma' \longrightarrow \Gamma$ of an abstract graph $\Gamma$, vertices $v \in V(\tilde{\Delta})$, $w' \in V(\Gamma')$ and a morphism of graphs $\varphi : \Delta \longrightarrow \Gamma$ such that $\varphi v(v) = \zeta(w')$, then there exists a unique morphism of graphs $\tilde{\varphi} : \tilde{\Delta} \longrightarrow \Gamma'$ such that $\tilde{\varphi}(v) = w'$.

From this definition it follows easily that an abstract universal covering graph $v : \tilde{\Delta} \longrightarrow \Delta$ is unique up to isomorphism, if it exists. We proceed to its construction.

To define $\tilde{\Delta}$ first we choose a maximal tree $T$ of $\Delta$, and identify $\pi_1^{\text{abs}}(\Delta)$ with the free group $\Phi(X)$ as in Proposition A.1.3, where

$$
X = \{y_e \mid e \in E(\Delta) - E(T)\}.
$$

Furthermore, we require the existence of a morphism of graphs $v : \tilde{\Delta} \longrightarrow \Delta$ and we want the fundamental group $\pi_1^{\text{abs}}(\Delta)$ to act freely on $\tilde{\Delta}$ in such a way that $v$ induces an isomorphism of the quotient graph $\pi_1^{\text{abs}}(\Delta) \backslash \tilde{\Delta}$ and $\Delta$. This forces

$$
\tilde{\Delta} = \pi_1^{\text{abs}}(\Delta) \times \Delta \quad \text{and} \quad V(\tilde{\Delta}) = \pi_1^{\text{abs}}(\Delta) \times V(\Delta).
$$

To complete the requirements we need to specify the incidence maps $d_i : \tilde{\Delta} \longrightarrow V(\tilde{\Delta})$ $(i = 0, 1)$ on the edges of $\tilde{\Delta}$. Put

$$
d_0(g, e) = (g, d_0(e)),
$$

$$
d_1(g, e) = (g\tilde{y}_e, d_1(e)), \quad (g \in \pi_1^{\text{abs}}(\Delta), e \in E(\Delta)),
$$

where $\tilde{y}_e$ is the image of $y_e$ in $\pi_1^{\text{abs}}(\Delta) = \Phi(X)$, i.e., $\tilde{y}_e$ is 1 if $e \in E(T)$, and it is $y_e$ otherwise. To facilitate the calculations later it is convenient to observe that the
above definitions for $d_0$ and $d_1$ extend to the following formulas valid also for the inverse edges:

\[
d_0(g, e)^\varepsilon = (g_{\bar{y}^\varepsilon(-\eta(\varepsilon))}, d_0(e^\varepsilon)),
\]

\[
d_1(g, e)^\varepsilon = (g_{\bar{y}^\varepsilon(1-\eta(\varepsilon))}, d_1(e^\varepsilon)) ,
\]

where $g \in \pi_1^{\text{abs}}(\Delta)$, $e \in E(\Delta)$, $\varepsilon = \pm 1$, $\eta(1) = 0$ and $\eta(-1) = 1$.

Define

\[\nu : \tilde{\Delta} \longrightarrow \Delta\]

to be the natural projection. The group $\pi_1^{\text{abs}}(\Delta) = \Phi(X)$ acts freely on $\tilde{\Delta}$ by multiplication on the first component. Note that $\Lambda = \{(1, m) \mid m \in \Delta\}$ is a connected subgraph of $\tilde{\Delta}$: it contains an isomorphic copy $T' = \{ (1, m) \mid m \in T \}$ of $T$, and $d_0(1, m) \in T'$ for all $(1, m) \in \Lambda$. We shall show in Proposition A.2.5 that $\nu : \tilde{\Delta} \longrightarrow \Delta$, as constructed above, is indeed a universal covering graph of $\Delta$.

**Proposition A.2.4**

(a) For every subgroup $H$ of $\pi_1^{\text{abs}}(\Delta)$, the induced morphism

\[\nu_H : H \backslash \tilde{\Delta} \longrightarrow \Delta\]

is a covering of abstract graphs.

(b) $\tilde{\Delta}$ is a tree.

**Proof** (a) is clear. Let $\Lambda$ be as defined above. To prove (b) first observe that if $e \in X = E(\Delta) - E(T)$, then $\Lambda \cup y_e \Lambda$ is a connected subgraph of $\tilde{\Delta}$ because $d_0(1, e) \in T'$ and $d_1(1, e) \in y_e T'$; consequently, $\Lambda \cup y_e^{-1} \Lambda$ is also connected. It follows inductively that

\[\Lambda \cup y_{e_1} \Lambda \cup y_{e_1} y_{e_2} \Lambda \cup \cdots \cup y_{e_1} \cdots y_{e_r} \Lambda\]

is connected, for $y_{e_1}, \ldots, y_{e_r} \in X$, $e_i = \pm 1$. Since $\tilde{\Delta} = \pi_1^{\text{abs}}(\Delta) \Lambda$ and since $X$ generates $\pi_1^{\text{abs}}(\Delta)$, one deduces that $\tilde{\Delta}$ is connected (see Lemma 2.2.4(a)).

To show that $\tilde{\Delta}$ does not contain a circuit, we proceed by contradiction. Assume that there exists a nontrivial reduced cycle

\[ (g_1, e_1)^{\varepsilon_1}, \ldots, (g_r, e_r)^{\varepsilon_r} \quad \text{(A.3)} \]

in $\tilde{\Delta}$ of length $r > 0$ ($e_i \in E(\Delta)$, $g_i \in \pi_1^{\text{abs}}(\Delta)$, $e_i = \pm 1$). Since $d_1(g_i, e_i)^{\varepsilon_i} = d_0(g_{i+1}, e_{i+1})^{\varepsilon_{i+1}}$ ($i = 1, \ldots, r - 1$) and $d_1(g_r, e_r)^{\varepsilon_r} = d_0(g_1, e_1)^{\varepsilon_1}$, we deduce that

\[ (g_i y_{e_i}^{\varepsilon_i(1-\eta(e_i))}, d_1(e_i^{\varepsilon_i})) = (g_{i+1} y_{e_{i+1}}^{\varepsilon_{i+1}(1-\eta(e_{i+1}))}, d_0(e_{i+1}^{\varepsilon_{i+1}})) \quad (i = 1, \ldots, r - 1),\]

\[ (g_r y_{e_r}^{\varepsilon_r(1-\eta(e_r))}, d_1(e_r^{\varepsilon_r})) = (g_1 y_{e_1}^{\varepsilon_1(1-\eta(e_1))}, d_0(e_1^{\varepsilon_1})).\]
Therefore, putting \( a_i = g_i y_{e_i}^{\varepsilon_i}(-\eta(\varepsilon_i)) \), we obtain
\[
\begin{align*}
& a_i y_{e_i}^{\varepsilon_i} = a_{i+1} \quad (i = 1, \ldots, r - 1), \\
& a_r y_{e_r}^{\varepsilon_r} = a_1.
\end{align*}
\]

Hence, in \( \pi_1^{\text{abs}}(\Delta) \) we have
\[
\bar{y}_{e_1} \cdots \bar{y}_{e_r} = 1.
\]

Consider now the element \( y_{e_1}^{\varepsilon_1} \cdots y_{e_r}^{\varepsilon_r} \) of \( \Phi(Y) \). Since \( e_1^{\varepsilon_1}, \ldots, e_r^{\varepsilon_r} \) is a closed path in \( \Delta \), we deduce from Proposition A.1.3 that \( y_{e_1}^{\varepsilon_1} \cdots y_{e_r}^{\varepsilon_r} = 1 \) in \( \Phi(Y) \). To obtain the desired contradiction we shall show that \( y_{e_1}^{\varepsilon_1} \cdots y_{e_r}^{\varepsilon_r} \) is in reduced form. Indeed, if we had \( y_{e_i}^{\varepsilon_i} = y_{e_i}^{-\varepsilon_i + 1} \), then \( e_i = e_{i+1} \) and \( \varepsilon_i = -\varepsilon_{i+1} \). Then \( d_1(q_i, e_i)^{\varepsilon_i} = d_0(q_{i+1}, e_{i+1})^{\varepsilon_{i+1}} \) implies that \( q_i y_{e_i}^{(1-\eta(\varepsilon_i))} = q_{i+1} y_{e_i}^{(-\varepsilon_i)(-\eta(\varepsilon_i))} \). Hence \( q_i = q_{i+1} \), and therefore the edges \((q_i, e_i)^\varepsilon\) and \((q_{i+1}, e_{i+1})^{\varepsilon_{i+1}}\) would be inverse to each other, contrary to our assumption that (A.3) is reduced.

\[\Box\]

**Proposition A.2.5**  Let \( \Delta \) be a connected abstract graph, and let \( \tilde{\Delta} \) be the graph constructed above.

(a) \( \nu : \tilde{\Delta} \rightarrow \Delta \) is an abstract universal covering of \( \Delta \).

(b) Let \( \zeta : \Gamma' \rightarrow \Gamma \) be a covering of abstract graphs, and let \( \varphi : \Delta \rightarrow \Gamma \) be a morphism of graphs. Let \( \nu' \in V(\Gamma') \) and \( v \in V(\Delta) \) be vertices such that \( \varphi(v) = \zeta(\nu') \). Assume that \( \varphi(\pi_1^{\text{abs}}(\Delta, v)) \leq \pi_1^{\text{abs}}(\Gamma', \nu') \); then there exists a unique morphism \( \varphi' : \Delta \rightarrow \Gamma' \) such that \( \zeta \circ \varphi' = \varphi \) and \( \varphi'(v) = \nu' \).

**Proof** Since \( \tilde{\Delta} \) is a tree, \( \pi_1^{\text{abs}}(\tilde{\Delta}) = 1 \). So (a) is a special instance of (b). Let \( u \in V(\Delta) \) and let \( p_{v,u} \) be a path in \( \Delta \) from \( v \) to \( u \). Since \( \zeta \) is a covering, there exists a unique lifting of the path \( \varphi(p_{v,u}) \) in \( \Gamma \) to a path \( p' \) in \( \Gamma' \) with initial vertex \( \nu' \) (see Proposition A.2.2(a)). Define \( \varphi'(u) \) to be the terminal vertex of \( p' \). Observe that \( \varphi'(u) \) is well-defined because of the condition \( \varphi(\pi_1^{\text{abs}}(\Delta, v)) \leq \pi_1^{\text{abs}}(\Gamma', \nu') \). Finally, if \( e \) is an edge in \( \Delta \) incident with \( u \), define \( \varphi'(e) \) to be the unique edge \( e' \) in \( \Gamma' \) incident with \( \varphi'(u) \) such that \( \zeta(e') = \varphi(e) \).

\[\Box\]

**Proposition A.2.6**  Let \( \Delta \) be a connected graph and let \( H \) be a subgroup of \( \pi_1^{\text{abs}}(\Delta, v) \), where \( v \) is a fixed, but arbitrary, vertex of \( \Delta \). Then there exists a covering
\[\zeta : \Delta' \rightarrow \Delta\]

such that \( \zeta(\pi_1^{\text{abs}}(\Delta', v')) = H \), where \( \zeta(v') = v \), and the index of \( H \) in \( \pi_1^{\text{abs}}(\Delta, v) \) is \( |\zeta^{-1}(v)| \). In particular, \( \pi_1^{\text{abs}}(\Delta', v') \cong H \) (see Proposition A.2.3). Furthermore, this covering is unique up to an isomorphism commuting with \( \zeta \). Explicitly,
\[\Delta' = H \setminus \tilde{\Delta} = (H \setminus \pi_1^{\text{abs}}(\Delta, v)) \times \Delta.\]
Proof The uniqueness follows from Proposition A.2.5(b). Denote by 

\[ \nu_H : \tilde{\Delta} \longrightarrow H \backslash \Delta = (H \backslash \pi_1^{\text{abs}}(\Delta, (H_1, v)) \times \Delta \]

the natural epimorphism of graphs; observe that \( \nu_H \) is a covering of abstract graphs, and so it respects path homotopy. For \( h \in H \), define \( \rho_H(h) \in \pi_1^{\text{abs}}(H \backslash \Delta, (H_1, v)) \) to be the homotopy class of cycles based at \( (H_1, v) \) represented by the cycle \( \nu_H(p(1, v), (h, v)) \), where \( p(1, v), (h, v) \) is a path from \( (1, v) \) to \( (h, v) \) in \( \tilde{\Delta} \). Then \( \rho_H \) is well-defined since \( \nu_H \) respects path homotopy. Furthermore, \( \rho_H \) is a homomorphism. Note that, since \( \tilde{\Delta} \) is a tree, we can choose \( p(1, v), (h, v) \) in the above definition to be the unique reduced path from \( (1, v) \) to \( (h, v) \).

Since \( \nu_H \) is a covering, a reduced cycle in \( H \backslash \Delta \) based at \( (H_1, v) \) can be lifted uniquely to a reduced path in \( \tilde{\Delta} \) with initial vertex at \( (1, v) \) (which necessarily ends at a vertex of the form \( (h, v) \), for some \( h \in H \)). Therefore, \( \rho_H \) is a bijection. It is now clear that the projection \( (H \backslash \pi_1^{\text{abs}}(\Delta, v)) \times \Delta \longrightarrow \Delta \) is the desired covering. □

Example A.2.7 Let \( X \) be a set and consider the graph \( \Delta \) consisting of a single vertex \( v \) and a set of edges \( \{e_x \mid x \in X\} \) in a one-to-one correspondence with \( X \), so that \( d_i(e_x) = v \), where \( x \in X \) and \( i = 0, 1 \) (i.e., \( \Delta \) is a bouquet of \( |X| \) loops). Then the fundamental group \( \pi_1^{\text{abs}}(\Delta) \) of \( \Delta \) is the free group \( \Phi = \Phi(X) \) with basis \( X \). Moreover, the abstract universal covering graph \( \tilde{\Delta} \) coincides with the Cayley graph \( \Gamma(\Phi, X) \) of the free group \( \Phi \) with respect to \( X \): its vertices are the elements of \( \Phi \) (which we identify with \( \Phi \times \{v\} \)) and its edges are the pairs \( (f, x) \in \Phi \times X \) with \( d_0(f, x) = f \) and \( d_1(f, x) = fx \). As we have shown above, \( \Gamma(\Phi, X) \) is a tree.

Using the notation in this example, Proposition A.2.6 translates into the following corollary.

Corollary A.2.8 Let \( \Phi = \Phi(X) \) be a free abstract group with basis the set \( X \), and let \( \Gamma = \Gamma(\Phi, X) \) be the Cayley graph of \( \Phi \) with respect to \( X \).

(a) Given a subgroup \( H \) of \( \Phi \), consider the map

\[ \rho_H : H \longrightarrow \pi_1^{\text{abs}}(H \backslash \Gamma) = \pi_1^{\text{abs}}(H \backslash \Gamma, H1) \]

that sends \( h \in H \) to the element \( \rho_H(h) \in \pi_1^{\text{abs}}(H \backslash \Gamma) \) determined by the image in \( H \backslash \Gamma \) of the unique reduced path \( p_{1,h} \) from 1 to \( h \) in \( \Gamma(\Phi, X) \). Then \( \rho_H \) is an isomorphism.

(b) If \( H \leq K \) are subgroups of \( \Phi \), the following diagram commutes

\[
\begin{array}{ccc}
\pi_1^{\text{abs}}(H \backslash \Gamma) & \longrightarrow & \pi_1^{\text{abs}}(K \backslash \Gamma) \\
\rho_H \uparrow & & \rho_K \uparrow \\
H & \hookrightarrow & K
\end{array}
\]
where the upper horizontal map is the homomorphism induced by the natural projection $H\backslash\Gamma \longrightarrow K\backslash\Gamma$.

It is worth noticing that the Nielsen–Schreier subgroup theorem, which asserts that a subgroup of an abstract free group is also free, is an immediate consequence of part (a) in the above corollary and Corollary A.1.4.

We end this section with a result that shows how a given immersion can be ‘enlarged’ to a covering, in certain cases.

**Lemma A.2.9** Let

$$\zeta : \Delta' \longrightarrow \Delta$$

be an immersion of connected graphs. Assume that $\Delta$ has a unique vertex $w$, and that the set of vertices $V(\Delta')$ of $\Delta'$ is finite. Then there exists a covering

$$\bar{\zeta} : \bar{\Delta} \longrightarrow \Delta$$

of abstract graphs such that

(a) $\Delta'$ is a subgraph of $\bar{\Delta}$ with $V(\Delta') = V(\bar{\Delta})$, and

(b) $\bar{\zeta}$ extends $\zeta$.

**Proof** Put $V' = V(\Delta')$, $E' = E(\Delta')$. Given an edge $e \in E(\Delta)$, define

$$R_e = \{ (u', v') \in V' \times V' \mid \exists e' \in E' \text{ with } d_0(e') = u', d_1(e') = v', \zeta(e') = e \}.$$

Note that $R_e$ may be the empty set. Since $\zeta$ is an immersion, $R_e$ defines an injective map from a subset of $V'$ to a subset of $V'$. Since $V'$ is finite, we may choose a subset $S_e$ of $V' \times V'$ containing $R_e$ such that $S_e$ defines a bijection $V' \longrightarrow V'$. Define $\bar{\Delta}$ as follows: $\bar{V} = V(\bar{\Delta}) = V(\Delta') = V'$. The set of edges $\bar{E} = E(\bar{\Delta})$ of $\bar{\Delta}$ is

$$\bar{E} = \{ (u', v', e) \mid e \in E(\Delta), (u', v') \in S_e \}.$$

The incidence maps of $\bar{\Delta}$ are

$$d_0(u', v', e) = u', \quad d_1(u', v', e) = v' \quad ((u', v', e) \in \bar{E}).$$

Next define $\bar{\zeta} : \bar{\Delta} \longrightarrow \Delta$ by

$$\bar{\zeta}(w') = w, \quad \bar{\zeta}(u', v', e) = e \quad (w' \in \bar{V} = V', \ (u', v', e) \in \bar{E}),$$

where $w$ is the unique vertex of $\Delta$. Then $\bar{\zeta}$ is a morphism of graphs, and the construction shows that it is a covering.

Finally, define $\iota : \Delta' \longrightarrow \Delta$ by $\iota(v') = v'$, $\iota(e') = (d_0(e'), d_1(e'), \zeta(e'))$ ($v' \in V'$, $e' \in E'$). Clearly $\iota$ is an embedding of graphs, and $\zeta = \bar{\zeta} \iota$. \qed
A.3 Foldings

Let \( \alpha : \Delta \rightarrow \Lambda \) be a morphism of graphs. As was pointed out in Sect. A.1, if \( \alpha \) is not an immersion it is because there exists a pair of different edges \( \{ e_1, e_2 \} \) in \( \Delta \) such that either \( d_0(e_1) = d_0(e_2) \) or \( d_1(e_1) = d_1(e_2) \) and \( \alpha(e_1) = \alpha(e_2) \). Motivated by this observation, we make the following construction: given two edges \( e_1, e_2 \in E(\Delta) \) such that either \( d_0(e_1) = d_0(e_2) \) or \( d_1(e_1) = d_1(e_2) \), we define a new graph

\[
\Delta/[e_1 = e_2]
\]

obtained from \( \Delta \) by identifying \( e_1 \) with \( e_2 \), \( d_0(e_1) \) with \( d_0(e_2) \) and \( d_1(e_1) \) with \( d_1(e_2) \). Following a terminology due to Stallings, we call this construction a folding. The natural quotient map

\[
f = f_{e_1,e_2} : \Delta \rightarrow \Delta/[e_1 = e_2]
\]

is an epimorphism of graphs. The following proposition is clear.

**Proposition A.3.1** Let \( \alpha : \Delta \rightarrow \Lambda \) be a morphism of graphs.

(a) Let \( e_1, e_2 \in E(\Delta) \) be edges such that either \( d_0(e_1) = d_0(e_2) \) or \( d_1(e_1) = d_1(e_2) \) and \( \alpha(e_1) = \alpha(e_2) \). Then \( \alpha \) factors through the folding \( f_{e_1,e_2} \), i.e.,

\[
\alpha = \alpha' f_{e_1,e_2},
\]

for some unique morphism \( \alpha' : \Delta/[e_1 = e_2] \rightarrow \Lambda \).

(b) Assume that \( \Delta \) is finite. Then \( \alpha \) factors through a finite sequence of foldings and an immersion, i.e., there is a graph \( \Delta' \), a composition of foldings

\[
\Delta = \Delta_1 \xrightarrow{f_1} \Delta_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{r-1}} \Delta_r \xrightarrow{f_r} \Delta'
\]

and an immersion

\[
i : \Delta' \rightarrow \Lambda
\]

such that

\[
\alpha = i f_r \cdots f_1.
\]

The sequence of foldings in part (b) of the proposition above is in general not unique, but one shows easily by induction on the number of edges of \( \Delta \) that \( \Delta' \) and \( i \) are uniquely determined by \( \alpha \). Moreover, \( \Delta' \) and \( i \) are obtained algorithmically: at each stage \( i \) one simply has to check in the finite graph \( \Delta_i \) whether there is a pair \( e_1, e_2 \in E(\Delta) \) with either \( d_0(e_1) = d_0(e_2) \) or \( d_1(e_1) = d_1(e_2) \) and \( \alpha(e_1) = \alpha(e_2) \).

**Proposition A.3.2** Let \( \Delta \) be a connected graph and let \( e_1 \) and \( e_2 \) be edges of \( \Delta \) with a common vertex \( v \) such that either \( d_0(e_1) = d_0(e_2) \) or \( d_1(e_1) = d_1(e_2) \). Let

\[
f : \Delta \rightarrow \tilde{\Delta} = \Delta/[e_1 = e_2]
\]
be the corresponding folding map. Then \( f \) induces an epimorphism on the corresponding fundamental groups, i.e., \( f(\pi_1^{\text{abs}}(\Delta, v)) = \pi_1^{\text{abs}}(\tilde{\Delta}, f(v)) \).

**Proof** Define \( H = f(\pi_1^{\text{abs}}(\Delta, v)) \); then \( H \leq \pi_1^{\text{abs}}(\tilde{\Delta}, f(v)) \). By Proposition A.2.6 there exists a covering \( \zeta : \Delta' \to \tilde{\Delta} \) with \( \zeta(\pi_1^{\text{abs}}(\Delta', v')) = H \), where \( v' \in V(\Delta') \).

By Proposition A.2.5(b) there exists a unique morphism \( f' : \Delta \to \Delta' \) such that \( f'(v) = v' \) and \( \zeta f' = f \). Therefore, since \( \zeta \) is a covering, we have \( f'(e_1) = f'(e_2) \); hence by Proposition A.3.1(a), there exists a morphism \( \zeta' : \tilde{\Delta} \to \Delta' \) such that \( f' = \zeta' f \). Hence \( \zeta' \zeta f = \zeta f' = f \). So \( \zeta' = \text{id}_{\tilde{\Delta}} \). It follows from Proposition A.2.2(c) that \( \zeta \) is an isomorphism of graphs, and thus \( H = \pi_1^{\text{abs}}(\tilde{\Delta}, f(v)) \), as needed. \( \Box \)

**A.4 Algorithms**

In this section we describe several algorithms related to finitely generated subgroups of a free group \( \Phi \) of finite rank \( n \) with basis \( X \). We think of \( \Phi \) as the fundamental group \( \pi_1^{\text{abs}}(\Delta, u) \), where \( \Delta \) is a finite graph with a single vertex \( u \) and edges labeled by \( X \): a bouquet of \( n \) loops (see Example A.2.7). An element of \( \Phi \) is represented by a cycle in \( \Delta \) based at \( u \), and a subgroup of \( \Delta \) is represented by a finite set of generators of that subgroup.

**Algorithm A.4.1** Represents a finitely generated subgroup \( H \) of \( \Phi \) as an immersion \( \iota : \Gamma \to \Delta \) with \( \Gamma \) a finite connected graph.

We assume that \( H \) is generated by a finite subset \( \{h_1, \ldots, h_n\} \) of \( \pi_1^{\text{abs}}(\Delta, u) = \Phi \) and for each \( i \) we are given a cycle \( p_i \) in \( \Delta \) based at \( u \) representing \( h_i \). We think of \( p_i \) as a morphism of graphs

\[
p_i : B_i \to \Delta,
\]

where \( B_i \) is a standard arc (see (A.1)).

Construct a graph \( B \) to be the disjoint union of the graphs \( B_1, \ldots, B_n \), and define a new graph \( \Gamma_1 \) obtained from \( B \) by identifying all the initial and terminal vertices of all the \( B_i \) to a vertex that we will denote by \( v \). Denote the image of \( B_i \) in \( \Gamma_1 \) by \( C_i \); this is a circuit. Hence \( \Gamma_1 \) is a bouquet of the \( n \) circuits \( C_1, \ldots, C_n \) joined at the vertex \( v \). Therefore \( \pi_1^{\text{abs}}(B, v) \) is a free group of rank \( n \) with a basis whose elements are represented by the closed paths defined by the image of each of those circuits.

Since, for each \( i \), \( p_i \) maps the initial and terminal vertices of \( B_i \) to \( u \), the morphisms \( p_1, \ldots, p_n \) induce a morphism of graphs

\[
\alpha : \Gamma_1 \to \Delta
\]

that sends \( v \) to \( u \). It follows that

\[
\alpha(\pi_1^{\text{abs}}(B, v)) = H.
\]
Next apply Proposition A.3.1 to obtain a sequence of foldings $f_1, \ldots, f_r$ and an immersion $\iota$

$$
\Gamma_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{r-1}} \Gamma_r \xrightarrow{f_r} \Gamma \xrightarrow{\iota} \Delta
$$
such that $\alpha = \iota f_r \cdots f_1$. By Proposition A.3.2, a folding induces a surjection of fundamental groups. So, if we put $f_r \cdots f_1(v) = w$, we have $\iota \pi_1^{\text{abs}}(\Gamma, w) = H$. So $\iota$ is the desired immersion.

**Algorithm A.4.2** Obtains a basis of a subgroup $H$ of $\Phi$ when $H$ is given by a finite set of generators.

We continue with the above set-up. Since $\iota$ is an immersion, we deduce from Proposition A.2.3 that the induced map $\bar{\iota} : \pi_1^{\text{abs}}(\Gamma, w) \longrightarrow H$ is an isomorphism of groups. As pointed out above, the graph $\Gamma$ and the immersion $\iota$ can be obtained algorithmically. So it suffices to describe how to obtain a basis for $\pi_1^{\text{abs}}(\Gamma, w)$ algorithmically, and this is done in Algorithm A.1.6.

**Algorithm A.4.3** Decides whether an element of a free group is in a finitely generated subgroup.

We are given $n$ cycles $p_1, \ldots, p_n$ in $\Delta$ based at $u$. Each $p_i$ represents an element $h_i$ of $\pi_1^{\text{abs}}(\Delta, u) = \Phi$ $(i = 1, \ldots, n)$. Let $H$ be the subgroup of $\Phi$ generated by $h_1, \ldots, h_{n-1}$. We describe next an algorithm to decide whether $h_n \in H$ or not.

We continue with the notation in Algorithm A.4.1. Let $\Gamma_1'' = C_1 \cup \cdots \cup C_{n-1}$. Denote by $\Gamma'$ and $\Gamma''$ the images in $\Gamma$ of $\Gamma_1'$ and $C_n$, respectively. Let $\iota'$ be the restriction of $\iota$ to $\Gamma'$. Since $\iota$ is an immersion, so is $\iota'$, and hence the immersion $\iota' : \Gamma' \rightarrow \Delta$ represents $H$. It follows that $h_n \in H$ if and only if $\pi_1^{\text{abs}}(\Gamma', w) = \pi_1^{\text{abs}}(\Gamma, w)$, because $\iota$ is injective on $\pi_1^{\text{abs}}(\Gamma, w)$. And this can be decided by checking whether $\Gamma''$ determines a cycle in $\Gamma'$.

### Intersection of Finitely Generated Subgroups

Next we want to prove that the intersection of two finitely generated subgroups of a free group is finitely generated, and that a basis for it can be obtained algorithmically. Obviously one may assume that the free group has finite rank.
**Theorem A.4.4** Let \( H_1 \) and \( H_2 \) be finitely generated subgroups of a free abstract group \( \Phi \) of finite rank. Then

(a) the subgroup \( H_1 \cap H_2 \) of \( \Phi \) has finite rank, and
(b) there exists an algorithm to construct a basis for \( H_1 \cap H_2 \), assuming that \( H_1 \) and \( H_2 \) are given by explicit finite sets of generators.

**Proof** We assume as above that \( \Phi = \pi_1^{\text{abs}}(\Delta, u) \), where \( \Delta \) is a finite graph with a single vertex \( u \) and \( n \) loops. Using Algorithm A.4.1, we assume that \( H_i \) is represented by an immersion \( \iota_i : \Gamma_i \to \Delta \), where \( \Gamma_i \) is a finite connected graph \((i = 1, 2)\).

The first step is to show how to represent \( H_1 \cap H_2 \) by an immersion. To do this we begin by recalling what is the pullback diagram of \( \iota_1 \) and \( \iota_2 \):

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi_1} & \Gamma_1 \\
\varphi_2 \downarrow & & \downarrow \iota_1 \\
\Gamma_2 & \xrightarrow{\iota_2} & \Delta
\end{array}
\]

Here \( \Gamma \) is a graph defined as follows:

\[
\Gamma = \left\{ (m_1, m_2) \in \Gamma_1 \times \Gamma_2 \mid \iota_1(m_1) = \iota_2(m_2) \right\};
\]

its set of vertices is

\[
V(\Gamma) = \left\{ (v_1, v_2) \in V(\Gamma_1) \times V(\Gamma_2) \mid \iota_1(v_1) = \iota_2(v_2) \right\},
\]

and the incidence maps \( d_i : \Gamma \to V(\Gamma) \) are defined as

\[
d_0(m_1, m_2) = (d_0m_1, d_0m_2), \quad d_1(m_1, m_2) = (d_1m_1, d_1m_2).
\]

Finally, the morphisms \( \varphi_1 \) and \( \varphi_2 \) are just the projection maps. It is clear that this diagram satisfies the universal property of a pullback: if \( \psi_i : \Sigma \to \Gamma_i \) \((i = 1, 2)\) are maps of graphs with \( \iota_1 \psi_1 = \iota_2 \psi_2 \), then there is a unique map of graphs \( \psi : \Sigma \to \Gamma \) with \( \varphi_1 \psi = \psi_1 \) and \( \varphi_2 \psi = \psi_2 \).

This definition provides an algorithm to construct \( \Gamma \) from \( \Gamma_1 \) and \( \Gamma_2 \). In general \( \Gamma \) is not a connected graph.

Put \( \iota = \iota_1 \varphi_1 = \iota_2 \varphi_2 \). We claim that

\[
\iota : \Gamma \to \Delta
\]

is an immersion. Indeed, let \((m_1, m_2)\) and \((m'_1, m'_2)\) be edges in \( \Gamma \) with the same initial vertex (respectively, the same terminal point) \((v_1, v_2)\). If \((m_1, m_2) \neq (m'_1, m'_2)\), then either \(m_1 \neq m'_1\) or \(m_2 \neq m'_2\); since \(\iota_1\) and \(\iota_2\) are immersions, it follows that \(\iota(m_1, m_2) \neq \iota(m'_1, m'_2)\). On the other hand, if \((m_1, m_2)\) is a loop based at \((v_1, v_2)\), then both \(m_1\) and \(m_2\) are loops. It follows that \(\iota\) induces an injection on \(\text{Star}(v_1, v_2)\), proving the claim.
By assumption $\iota_1(\pi_1^{\text{abs}}(\Gamma_1, v_1)) = H_1$, $\iota_2(\pi_1^{\text{abs}}(\Gamma_2, v_2)) = H_2$, where $v_i \in \Gamma_i$ ($i = 1, 2$), $\iota_1(v_1) = \iota_1(v_2) = u$, and $u$ is the unique vertex of $\Delta$. Let $v = (v_1, v_2) \in \Gamma$. Then we clearly have $\iota(\pi_1^{\text{abs}}(\Gamma, v)) \subseteq H_1 \cap H_2$.

In fact, we assert that $\iota(\pi_1^{\text{abs}}(\Gamma, v)) = H_1 \cap H_2$. To see this assume that $h \in H_1 \cap H_2 \leq \Phi = \pi_1^{\text{abs}}(\Delta, u)$. Let $h_1 \in \pi_1^{\text{abs}}(\Gamma_1, v_1)$ be such that $\iota_1(h_1) = h$ ($i = 1, 2$). Represent $h_i$ by a reduced cycle $p_i$ based at $v_i$; since $\iota_i$ is an immersion, $\iota_i p_i$ is a reduced cycle of $\Delta$ based at $u$ ($i = 1, 2$) (see Proposition A.2.1(b)) representing $h$, i.e., the cycles $\iota_1 p_1$ and $\iota_2 p_2$ are homotopic and reduced. We deduce from Lemma A.1.1 that $\iota_1 p_1 = \iota_2 p_2$. Interpreting paths as morphisms from standard arcs (see (A.1)), this is equivalent to saying that there exist a standard arc $A_r(\epsilon)$ and morphisms

$$p_1 : A_r(\epsilon) \rightarrow \Gamma_1 \quad \text{and} \quad p_2 : A_r(\epsilon) \rightarrow \Gamma_2$$

such that the compositions $\iota_1 p_1$ and $\iota_2 p_2$ are equal. Using the pullback property, we deduce that there exists a morphism $\varphi : A_r(\epsilon) \rightarrow \Gamma$ with $\varphi_1 p = p_1$ and $\varphi_2 p = p_2$. Hence $p$ is a cycle in $\Gamma$ based at $v = (v_1, v_2)$, i.e., it represents an element $k$ of $\pi_1^{\text{abs}}(\Gamma, v)$; moreover, $\iota k = h$. This proves the assertion.

Next consider the connected component $\Gamma_3$ of $\Gamma$ containing $v = (v_1, v_2)$, and denote by $\iota_3 : \Gamma_3 \rightarrow \Delta$ the restriction of $\iota$ to $\Gamma_3$. Then $\iota_3$ is an immersion of finite connected graphs and so $\iota_3$ represents $H_1 \cap H_2$, i.e., $\iota_3$ induces an explicit isomorphism $\pi_1^{\text{abs}}(\Gamma_3, v) \rightarrow H_1 \cap H_2$. Therefore, using Algorithm A.1.6, one obtains a basis for $H_1 \cap H_2$. This proves both parts (a) and (b). \hfill \Box

### A.5 Notes, Comments and Further Reading

There is a very extensive mathematical literature on abstract graphs, often related to combinatorics. The point of view that we adopt here regarding abstract graphs is very much determined by the topics in this book. For more detailed treatments of abstract graphs and actions of groups on them one can consult Serre (1980), Dicks and Dunwoody (1989) or Stallings (1983). The ideas for the algorithms in Sect. A.4 are due to Stallings. Part (a) of Theorem A.4.4 is due to Howson (1954).

The concept of abstract graph that we use in this Appendix is substantially the same as that used in the Bass–Serre theory of groups acting on trees developed in Serre (1980). In the latter book a graph $\Delta$ includes by definition both the set of edges $E(\Delta)$ and the set of inverse edges $E(\Delta)^{-1}$; then what we denote by $E(\Delta)$ in our setting corresponds to a specific ‘orientation’ in the set-up of Serre. Our notion of action of a group on a graph corresponds to what Serre calls ‘action without inversion’, and our notion of morphism or map of graphs would correspond, in Serre’s set-up, to morphisms that preserve given orientations. The standard constructions of abstract graphs arising from ‘free constructions’ of abstract groups, such as the Cayley graph of a group, the tree canonically associated to a free product of abstract groups or an amalgamated product of abstract groups, or, more generally, the universal covering graph of a graph of abstract groups, all come equipped with a natural orientation (i.e., a graph in the sense used in this book).
Appendix B
Rational Sets in Free Groups and Automata

In this Appendix it is shown that a rational subset of an abstract free group can be described as the recognizable language of an automaton over an appropriately chosen alphabet.

Notation If $Y$ is a set, $Y^*$ denotes the free monoid on $Y$. The elements $w = y_1 \cdots y_m$ of $Y^*$ are called words on the alphabet $Y$ ($y_1, \ldots, y_m \in Y$). The length of $w = y_1 \cdots y_m$ is $m$. A language on the alphabet $A$ is simply a subset of $A^*$.

B.1 Finite State Automata: Review and Notation

A finite state automaton is a 4-tuple,
\[ \mathcal{A} = (A, Q, i, T) \]
where $Q$ and $A$ are finite sets, $i \in Q$, $T \subseteq Q$, together with a function (the next state function)
\[ Q \times A \rightarrow \mathcal{P}(Q) \tag{B.1} \]
that we denote by $(q, a) \mapsto qa$ ($q \in Q, a \in A$) [$\mathcal{P}(Q)$ denotes the set of subsets of $Q$].

The set $A$ is called the alphabet of the automaton $\mathcal{A}$; $Q$ is the set of states of $\mathcal{A}$; $i$ is the initial state of $\mathcal{A}$; and $T$ is the subset of terminal states of $\mathcal{A}$.

If $Q' \subseteq Q$ and $a \in A$, then $Q'a$ is defined to be the union
\[ Q'a = \bigcup_{q' \in Q'} q'a. \]
Then one extends the function (B.1) to a function
\[ Q \times A^* \rightarrow \mathcal{P}(Q) \]
by the formulas

\[ q_1 = q \quad \text{and} \quad q(wa) = (qw)a \quad (w \in A^*, a \in A). \]

The language recognized by the automaton \( A \) is the subset of \( A^* \)

\[ L(\mathcal{A}) = \{ w \in A^* \mid iw \cap T \neq \emptyset \}. \]

A subset \( L \) of \( A^* \) is termed recognizable if there exists some finite state automaton \( A \) with alphabet \( A \) such that \( L = L(\mathcal{A}) \). A fundamental result, due to Kleene, says that the recognizable subsets of \( A^* \) are precisely the rational subsets \( \text{Rat}(A^*) \) of \( A^* \) (see Theorem 12.3.2).

**Remark** The definition that we have given for a finite state automaton corresponds to what is usually called a ‘nondeterministic’ and ‘complete’ finite state automaton: nondeterministic because the values \( qa \) of the next state function (B.1) are subsets of \( Q \), rather than elements of \( Q \), i.e., singleton subsets of \( Q \); and complete because that function is defined on the whole set \( Q \times A \), rather than on just a subset.

Actually, this is not an essential distinction in the sense that a subset \( L \) of \( A^* \) is recognized by a nondeterministic and complete finite state automaton if and only it is recognized by some deterministic and not necessarily complete finite state automaton (see, for example, Eilenberg 1974).

### B.2 The Classical Function \( \rho \)

Let \( X = \{x_1, \ldots, x_n\} \) be a finite set of size \( n \). Define \( Z \) to be the finite set with \( 2n \) letters

\[ Z = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\} \quad (B.2) \]

where \( x \mapsto \bar{x} \) is an involution on \( Z \), i.e., \( \bar{x} = x \), for \( x \in Z \).

Two words in \( Z^* \) are equivalent if one can pass from one to the other by a finite sequence of insertions or deletions of subwords of the form \( x\bar{x} \) (\( x \in Z \)). A word \( w \in Z^* \) is reduced if it does not contain subwords of the form \( x\bar{x} \) (\( x \in Z \)). Define

\[ \rho : Z^* \longrightarrow Z^* \]

as follows: if \( w \in Z^* \), \( \rho(w) \) is the reduced word obtained by deleting, from left to right, all pairs \( x\bar{x} \) (\( x \in Z \)).

**Properties of \( \rho \)**

\( (\rho 1) \quad \rho(w_1w_2) = \rho(\rho(w_1)w_2) \) (\( w_1, w_2 \in Z^* \)).
\( (\rho 2) \quad \rho(\rho(w)) = \rho(w) \) (\( w \in Z^* \)).
\( (\rho 3) \quad \rho(w) = w, \) if \( w \in Z^* \) is reduced.
\( (\rho 4) \quad \rho(w_1x\bar{x}w_2) = \rho(w_1w_2) \) (\( w_1, w_2 \in Z^*, x \in Z \)).
\( (\rho 5) \quad \rho(w_1w_2) = \rho(\rho(w_1)\rho(w_2)) \) (\( w_1, w_2 \in Z^* \)).
Properties \((\rho_1), (\rho_2)\) and \((\rho_3)\) follow from the definition. Property \((\rho_4)\) follows from \((\rho_1)\). Property \((\rho_5)\) is proved by induction on the length of \(w_2\). If two words \(w_1\) and \(w_2\) are reduced and equivalent, one proves, using \((\rho_4)\), that \(\rho(w_1) = \rho(w_2)\); and by \((\rho_3)\), \(w_1 = w_2\). Therefore an equivalence class of words contains exactly one word which is reduced.

**Proposition B.2.1** The subset \(\rho(Z^*)\) of \(Z^*\) is recognizable.

**Proof** Note that \(\rho(Z^*) = Z^* - Z^*\{x\bar{x}, \bar{x}x \mid x \in X\}Z^*\). Since both \(Z^*\) and \(\{x\bar{x}, \bar{x}x \mid x \in X\}\) are recognizable, so is \(\rho(Z^*)\) (see Theorem 12.3.2(c)). \(\square\)

### B.3 Rational Subsets in Free Groups

Let \(B\) be a finite set and let \(Y\) be a subset of \(B^*\). Define a function

\[
\lambda_Y : B^* \rightarrow \mathcal{P}(B^*)
\]

from \(B^*\) into the set of subsets of \(B^*\) as follows: if \(w, w' \in B^*\), then \(w' = b_1 \cdots b_r \in \lambda_Y(w) (b_i \in B)\) if \(w\) has the form

\[
w = y_0b_1y_1 \cdots y_{r-1}b_my_r,
\]

where \(y_0, \ldots, y_r \in Y\) and \(r \geq 0\).

For example, if \(Y = \{1\}\), then \(\lambda_Y(w) = \{w\}\), for all \(w \in B^*\); while if \(B = Z = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}\) (as in Sect. B.2) and \(Y = \rho^{-1}(1)\), then the words in \(\lambda_Y(w)\) are those obtained from \(w\) by deleting subwords of the form \(x\bar{x}\) or \(\bar{x}x\) (\(x \in \{x_1, \ldots, x_n\}\)).

**Lemma B.3.1** Let \(B\) be a finite set and let \(Y \subseteq B^*\). Then the function \(\lambda_Y\) sends recognizable subsets of \(B^*\) to recognizable subsets of \(B^*\).

**Proof** Let \(K\) be a recognizable subset of \(B^*\). Say \(K = L(B)\), where \(B\) is a deterministic finite state automaton:

\[
B = (B, Q, i, T).
\]

For \(p, q \in Q\), define a subset of \(B^*\)

\[
K_{p,q} = \{w \in B^* \mid pw = q\}.
\]

To prove the result we shall construct a finite state automaton

\[
B' = (B, Q', i', T')
\]
on the same alphabet $B$ that recognizes $\lambda_Y(K)$. Let $Q' = Q \cup \{i'\}$, where $i'$ is a new element not in $Q$; we take $i'$ as the initial state of $B'$. The subset $T'$ of terminal states of $B'$ is

$$T' = \begin{cases} T, & \text{if } Y \cap K = \emptyset; \\ T \cup \{i'\}, & \text{if } Y \cap K \neq \emptyset. \end{cases}$$

To complete the description of the automaton $B'$ we need a next state function

$$Q' \times B \rightarrow \mathcal{P}(Q'),$$

which is defined by

$$q \in pb \quad \text{if and only if} \quad bY \cap K_{p,q} \neq \emptyset \quad (b \in B, p, q \in Q);$$

$$q \in i' b \quad \text{if and only if} \quad Y b Y \cap K_{p,q} \neq \emptyset \quad (b \in B, q \in Q).$$

Now one easily checks that $\lambda_Y(K) = L(B'). \square$

We continue with the notation in Sect. B.2:

$$X = \{x_1, \ldots, x_n\} \subseteq Z = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}.$$  

**Proposition B.3.2** The function $\rho : Z^* \rightarrow Z^*$ sends recognizable subsets to recognizable subsets: if $K$ is a recognizable subset of $Z^*$, then so is $\rho(K)$.

**Proof** Define $Y = \rho^{-1}(1)$. Let $w \in Z^*$; then, as pointed out above,

$$\lambda_Y(w)$$

is the subset of $Z^*$ consisting of those words obtained from $w$ by deleting subwords of the form $x \bar{x}$ or $\bar{x} x$ ($x \in \{x_1, \ldots, x_n\}$). Hence, $\rho(w)$ is the unique element in $\lambda_Y(w) \cap \rho(Z^*)$. Therefore, if $K$ is a recognizable subset of $Z^*$, then

$$\rho(K) = \lambda_Y(K) \cap \rho(Z^*)$$

is recognizable, because it is the intersection of recognizable subsets of $Z^*$ (see Theorem 12.3.2(c), Lemma B.3.1 and Proposition B.2.1). \qed

Let $\Phi = \Phi(X)$ be the free abstract group on $X$. We think of an element of $\Phi$ as an equivalence class $\tilde{w}$ of a word $w \in Z^*$ under the equivalence relation defined in Sect. B.2. With abuse of notation, one often denotes an element $\tilde{w}$ of $\Phi$ by $w$.

Define a map

$$\iota : \Phi \rightarrow Z^*$$

by $\iota(\tilde{w}) = \rho(w)$; this is well-defined because the equivalence class $\tilde{w}$ contains a unique reduced word. Then $\iota$ is an injection.
Consider the commutative diagram

\[
\begin{array}{ccc}
\Phi & \xrightarrow{\iota} & \mathbb{Z}^* \\
\downarrow{\psi} & & \downarrow{\rho} \\
\mathbb{Z}^* & & 
\end{array}
\]

where \(\psi\) is the unique morphism of monoids such that \(\psi(x) = x\) and \(\psi(\bar{x}) = x^{-1}\), for \(x \in X\).

**Theorem B.3.3** Let \(R\) be a rational subset of the free group \(\Phi = \Phi(X)\). Then \(\iota(R)\) is a recognizable subset of the free monoid \(\mathbb{Z}^*\).

**Proof** By Lemma 12.3.1, there exists some \(R' \in \text{Rat}(\mathbb{Z}^*)\) with \(\psi(R') = R\). By Kleene’s theorem (Theorem 12.3.2), \(R'\) is a recognizable subset of \(\mathbb{Z}^*\). Thus \(\iota(R) = \psi(R') = \rho(R')\) is recognizable, according to Proposition B.3.2. \(\square\)

**B.4 Notes, Comments and Further Reading**

There are many good general treatments of automata theory, such as Eilenberg (1974). Theorem B.3.3 is due to Benois (1969); the proof that we present here follows the treatment of Berstel (1979), Part III, Sect. 2, who in turn uses ideas from Fliess (1971). I am grateful to Benjamin Steinberg for bringing to my attention Benois’ paper and to Jean-Eric Pin for very precise information about several proofs of Benois’ theorem. A different approach to Benois’ theorem can be found in Gilman (1987) and Steinberg (2001a), Theorem 26.

Theorem B.3.3 provides an alternative to Algorithm A.4.3 to decide whether or not an element \(g\) of a free abstract group \(\Phi = \Phi(X)\), with finite basis \(X\), is in a given finitely generated subgroup of \(\Phi\). More generally, and this is what is in fact used in the proof of Theorem 12.3.10, if \(H_1, \ldots, H_n\) are finitely generated subgroups of \(\Phi\) (each of them given by a set of generators, i.e., words in \(X \cup X^{-1}\)), there is an explicitly constructed finite state automaton \(\mathcal{A}\) on the alphabet \(\mathbb{Z}\) (see (B.2)) such that \(\iota(H_1 \cdots H_n) = L(\mathcal{A})\). Hence, one can decide whether or not a given element \(g \in \Phi\) is in the subset \(H_1 \cdots H_n\) by checking whether or not \(\iota(g)\) is recognized by \(\mathcal{A}\).

The construction of \(\mathcal{A}\) follows from the proof of Theorem B.3.3 and standard facts in automata theory: (1) given a finite subset \(Y\) of \(\mathbb{Z}^*\) one can explicitly describe a finite state automaton over the alphabet \(\mathbb{Z}\) that recognizes \(Y^*\), and (2) given finitely many finite state automata \(\mathcal{A}_1, \ldots, \mathcal{A}_n\) over the alphabet \(\mathbb{Z}^*\), one can construct explicitly a finite state automaton \(\mathcal{A}\) over the alphabet \(\mathbb{Z}\) such that \(L(\mathcal{A}) = L(\mathcal{A}_1) \cdots L(\mathcal{A}_n)\).
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