Appendix A
Floquet Theory

The stability of solutions to linear periodic differential systems can be studied using Floquet theory. Consider a linear first-order system

\[
\dot{x}(t) = A(t) x(t),
\]

(A.1)

where \( A(t) \) is a \( n \times n \) periodic matrix,

\[
A(t + T) = A(t),
\]

(A.2)

with period \( T \).

The principle matrix solution of Eq. (A.1), \( \Phi(t, t_0) \), is a matrix whose columns are linearly independent solutions of Eq. (A.1) and at some time \( t_0 \) \( \Phi(t_0, t_0) = I \), where \( I \) is the identity matrix. This matrix satisfies

\[
\dot{\Phi}(t, t_0) = A(t) \Phi(t, t_0)
\]

(A.3)

and

\[
\Phi(t + T, t_0 + T) = \Phi(t, t_0)
\]

(A.4)

and can be written as

\[
\Phi(t, t_0) = P(t, t_0) e^{(t-t_0)Q(t_0)},
\]

(A.5)

where \( Q(t_0) \) is an \( n \times n \) matrix and \( P(t + T, t_0) = P(t, t_0) \) is \( T \)-periodic. The principle matrix describes the evolution of solutions to Eq. (A.1), i.e. if \( x(t + t_0) \) is a solution to Eq. (A.1) then the evolution from time \( t_0 \) is given by

\[
x(t + t_0) = \Phi(t, t_0) x(t_0).
\]

(A.6)

The stability of the solution \( x(t + t_0) \) is then given by the eigenvalues of the so called monodromy matrix which is defined as
\[ M(t_0) = \Phi(t_0 + T, t_0) \]  
(A.7)

and describes the evolution over one period \( T \). The monodromy matrix can be written in the form

\[ M(t_0) = e^{TQ(t_0)} . \]  
(A.8)

The eigenvalues \( \mu_j \) of \( M(t_0) \) are referred to as the Floquet multipliers (or characteristic multipliers) of the solution to Eq. (A.1) and the eigenvalues \( \lambda_j \) of \( Q(t_0) \) are the Floquet exponents (or characteristic exponents). A solution is stable if \( |\mu_j| \leq 1 \) \( (\text{Re}[\lambda_j] \leq 0) \) for all \( j \). For a more detailed introduction to Floquet theory see, for example, [1].

For linear delay differential equations, a complete Floquet theory does not exist [2], however the characteristic multipliers and exponents can be defined in similar way. Consider a linear delay differential equation

\[ \dot{x}(t) = A(t)x(t) - B(t - \tau)x(t - \tau) , \]  
(A.9)

with periodic matrices \( A(t + T) = A(t) \) and \( B(t + T) = B(t) \). Let \( U(t, t_0) \) be an operator that describes the time evolution of a solution \( x(t + t_0) \) to Eq. (A.9), i.e.

\[ x(t + t_0) = U(t, t_0) x(t_0) . \]  
(A.10)

Then the monodromy operator in defined as

\[ M(t_0) = U(t_0 + T, t_0) \]  
(A.11)

and the characteristic multipliers \( \mu_j \) are the eigenvalues of \( M(t_0) \). Details on the stability analysis of periodic linear delay differential can be found in [2].

Note that in both the linear ordinary and delay differential equations case the characteristic multipliers are independent of the choice of \( t_0 \) and both systems (Eqs. (A.1) and (A.9)) have solutions of the form

\[ x(t) = p(t) e^{\lambda_j t} , \]  
(A.12)

where \( p(t + T) = p(t) \) is \( T \)-periodic.

References
Appendix B
Linearised Mode-Locked Laser System

B.1 Linearised DDE Model

The DDE system for the mode-locked laser (Eqs. \((4.10)-(4.12)\)) linearised about a periodic solution \(\psi_0(t)\), is given by

\[
\frac{d}{dt} \delta \psi(t) = A(t) \delta \psi(t) + \sum_{n=0}^{N} B_n (t - \tau_n') \delta \psi(t - \tau_n') + Dw(t), \tag{B.1}
\]

where \(\delta \psi(t) = (\text{Re} \delta \mathcal{E}(t), \text{Im} \delta \mathcal{E}(t), \delta \mathcal{G}(t), \delta \mathcal{Q}(t))^T, w(t) = (\xi_1(t), \xi_2(t), 0, 0)^T\),

\(A\) and \(B_n\) are \(T_0\)-periodic Jacobi matrices of the linearisation and the delay times are defined as \(\tau'_n = T\) and \(\tau'_n = T + \tau_n\) for \(n \geq 1\). The Jacobi matrices are given by

\[
A(t) = \begin{pmatrix}
-\gamma & -\omega & 0 & 0 \\
\omega & -\gamma & 0 & 0 \\
\frac{e^{-q_0(t)} \mathcal{G}(t) 2\mathcal{E}^R_0(t) - e^{-q_0(t)} \mathcal{G}(t) 2\mathcal{E}^I_0(t) - \gamma \mathcal{E}^R_0(t) - \gamma \mathcal{E}^R_0(t) |\mathcal{E}_0(t)|^2 - e^{-q_0(t)} \mathcal{G}(t) |\mathcal{E}_0(t)|^2)}{\mathcal{Q}(t) 2\mathcal{E}^I_0(t)} & 0 & -\gamma - r_s e^{-q_0(t)} \mathcal{E}_0(t) \\
\frac{e^{-q_0(t)} \mathcal{Q}(t) 2\mathcal{E}^I_0(t) - e^{-q_0(t)} \mathcal{Q}(t) 2\mathcal{E}^I_0(t) - \gamma \mathcal{E}^R_0(t) - \gamma \mathcal{E}^R_0(t) |\mathcal{E}_0(t)|^2 - e^{-q_0(t)} \mathcal{G}(t) |\mathcal{E}_0(t)|^2)}{\mathcal{Q}(t) 2\mathcal{E}^I_0(t)} & 0 & -\gamma - r_s e^{-q_0(t)} \mathcal{E}_0(t) \\
\end{pmatrix}, \tag{B.2}
\]

with \(\mathcal{G}(t) = 1 - e^{G_0(t)}\) and \(\mathcal{Q}(t) = 1 - e^{Q_0(t)}\), and

\[
B_n (s) = K_0 \gamma \begin{pmatrix}
R_0^R (s) - R_0^I (s) & R_0^R (s) \mathcal{E}_{RI}^R (s) - R_0^R (s) \mathcal{E}_{RI}^I (s) - R_0^R (s) \mathcal{E}_{RI}^R (s) + R_0^R (s) \mathcal{E}_{RI}^I (s) \\
0 & R_0^R (s) \mathcal{E}_{RI}^R (s) + R_0^R (s) \mathcal{E}_{RI}^I (s) - R_0^R (s) \mathcal{E}_{RI}^R (s) - R_0^R (s) \mathcal{E}_{RI}^I (s) \\
\end{pmatrix}, \tag{B.3}
\]

for \(s = t - \tau'_m\) and \(K_0 = 1\), with \(\mathcal{E}_{RI}^R = \text{Re} \mathcal{E}_0, \mathcal{E}_{RI}^I = \text{Im} \mathcal{E}_0, R_0 (s) = \sqrt{\text{Re}^2 \left( 1 - i \alpha_s \right) G_0(s) - \frac{1}{2} (1 - 2i \alpha_s) Q_0(s) + i C_m} - i (\Delta \Omega + \omega)(s - t), R_0^R = \text{Re} R_0, R_0^I = \text{Im} R_0, C_0 = 0, \mathcal{E}_{RI}^R (s) = \frac{1}{2} \left( \mathcal{E}^R_0 (s) + \alpha_s \mathcal{E}^I_0 (s) \right), \mathcal{E}_{RI}^I (s) = \frac{1}{2} \left( \mathcal{E}^I_0 (s) - \alpha_s \mathcal{E}^R_0 (s) \right), \mathcal{E}_{RI}^q (s) = \frac{1}{2} \left( \mathcal{E}^R_0 (s) + \alpha_q \mathcal{E}^I_0 (s) \right) \text{ and } \mathcal{E}_{RI}^q (s) = \frac{1}{2} \left( \mathcal{E}^I_0 (s) - \alpha_q \mathcal{E}^R_0 (s) \right).
These matrices are obtained by calculating the partial derivatives of the right-hand side of Eqs. (4.10), (2.43) and (2.44) with respect to the dynamical variables, whereby delayed variables are treated separately.

**B.2 Adjoint System**

The linearised system Eqs. (4.13) ((B.1)) can be expressed as

\[
\dot{\delta \psi}(t) = L \delta \psi(t),
\]

for the appropriate operation \( L \). The adjoint problem to the homogeneous \((D = 0)\) version of Eq. (4.13) is defined as the function

\[
\dot{\delta \psi}^\dagger(t) = L^\dagger \delta \psi^\dagger(t),
\]

that fulfils

\[
[L \delta \psi(t), \delta \psi^\dagger(t)] = [\dot{\delta \psi}(t), L^\dagger \delta \psi^\dagger(t)],
\]

where the square brackets represent the bilinear form (Eq. 4.16). Substituting Eq. (4.13) into the left-hand side this Eq. (B.6) and writing out the bilinear form for this side, and comparing this with the bilinear form written out for the right-hand side of Eq. (B.6), one can see that the linear system adjoint to Eq. 4.13, for \( D = 0 \), is given by

\[
\frac{d}{dt} \delta \psi^\dagger(t) + \delta \psi^\dagger(t) A(t) + \sum_{n=0}^{N} \delta \psi^\dagger(t + \tau'_n) B_n(t) = 0,
\]

where \( \delta \psi^\dagger(t) = (\delta \psi_1^\dagger, \delta \psi_2^\dagger, \delta \psi_3^\dagger, \delta \psi_4^\dagger) \) is a row vector.
Appendix C
Suppression of Noise-Induced Modulations

C.1 FitzHugh–Nagumo Oscillator

In this section we show that the characteristic equation Eq. (5.19) accurately describes the dominant Floquet exponents of a FitzHugh–Nagumo oscillator subject to resonant feedback from two non-invasive feedback terms.\(^1\)

The FitzHugh–Nagumo oscillator subject to two non-invasive feedback terms is given by

\[
\dot{u} = \frac{1}{\varepsilon} \left( u - \frac{u^3}{3} - v + K_1 (u (t - \tau_1) - u) + K_2 (u (t - \tau_2) - u) \right) \quad (C.1)
\]

and

\[
\dot{v} = u + a, \quad (C.2)
\]

where \(u\) is the fast activator variable, \(v\) is the slow inhibitor variable, \(\varepsilon\) is the time separation parameter and \(a\) is the excitability parameter. We choose \(a = 0.8\) (oscillatory regime) and \(\varepsilon = 0.01\).

In Fig. C.1 the white symbols indicate the three dominant Floquet exponents obtained from the DDE-BIFTOOL calculations, plotted behind these are the results of the fitted characteristic equation Eq. (5.19) with the red circles indicating the most dominant Floquet exponent. The only fit parameter is the effective total feedback strength, which is \(K_{\text{TOT}}^{\text{eff}} = 0.101\). Figure C.1 shows excellent agreement between the Floquet exponents given by Eq. (5.19) and those calculated using DDE-BIFTOOL.

\(^1\)For examples of the dynamics and parameter dependence of the FitzHugh–Nagumo oscillator see [1].
Fig. C.1 Real (a) and imaginary (b) parts of the four dominant Floquet exponents of a FitzHugh–Nagumo oscillator (FHN) with two feedback terms. The white markers indicate the numerically calculated values and the coloured markers indicate the results of the fitted characteristic equation. Parameters: $K_1 = K_2 = 0.005$, $K_{\text{eff}}^\text{TOT} = 0.101$, $\tau_1 = 25T_0$, $a = 0.8$ and $\varepsilon = 0.01$

Reference

1. P. Hövel, Control of complex nonlinear systems with delay, Ph.D. thesis (Technische Universität Berlin, 2009)