Appendix A
Periodical Ateb Functions

In his classical paper Rosenberg 1963, introduced the so-called periodic Ateb functions concerning the problem of inversion of the half of the incomplete Beta function

\[ z \mapsto \frac{1}{2} B_z(a, b) = \frac{1}{2} \int_0^{0 \leq z \leq 1} t^{a-1}(1 - t)^{b-1} dt . \]  

(A.1)

Obviously, we are interested in the case where \( a = \frac{1}{\alpha+1} \) and \( b = \frac{1}{2} \), i.e.,

\[ z \mapsto \frac{1}{2} B_z \left( \frac{1}{\alpha+1}, \frac{1}{2} \right) = \frac{1}{2} \int_0^{0 \leq z \leq 1} \frac{dt}{(1 - t)^{1/2} t^{1/2} (\alpha+1)} . \]  

(A.2)

Senik in his article in 1969 shows that the Ateb functions are the solutions of the ordinary differential equations

\[ \dot{v} - u^\alpha = 0, \]
\[ \dot{u} + \frac{2}{\alpha + 1} v = 0 , \]  

(A.3)

Namely,

\[ v(z) = sa(1, \alpha, z), \quad u(z) = ca(\alpha, 1, z) . \]  

(A.4)

It can be easily verified that the inverse of \( \frac{1}{2} B_z \left( \frac{1}{\alpha+1}, \frac{1}{2} \right) \) and \( v(z) \) coincide on \([-\frac{1}{2} \Pi_\alpha, \frac{1}{2} \Pi_\alpha]\), where

\[ \Pi_\alpha := B \left( \frac{1}{\alpha+1}, \frac{1}{2} \right) . \]  

(A.5)

Having in mind the following set of properties:
\[ sa(1, \alpha, z) = \begin{cases} -sa(1, \alpha, -z) \\ \pm ca(\alpha, 1, \frac{1}{2} \Pi_\alpha \pm x) \\ \pm sa(1, \alpha, \Pi_\alpha \pm z) \\ \mp sa(1, \alpha, 2 \Pi_\alpha \mp z) \end{cases} \]  

(A.6)

we see that \( sa(\alpha, 1, z) \) is an odd function of \( z \in \mathbb{R} \); it is the so-called \( 2 \Pi_\alpha \)-periodic sine Ateb, i.e., \( sa \) function. Also there holds

\[ sa^2(1, \alpha, z) + ca^{\alpha+1}(\alpha, 1, z) = 1, \]  

(A.7)

and cosine Ateb, that is \( ca(1, \alpha, z) \) function, is even and \( 2 \Pi_\alpha \) periodic having properties:

\[ ca(\alpha, 1, z) = \begin{cases} ca(\alpha, 1, -z) \\ sa(1, \alpha, \frac{1}{2} \Pi_\alpha \pm z) \\ -ca(\alpha, 1, \Pi_\alpha \pm z) \\ ca(\alpha, 1, 2 \Pi_\alpha \pm z) \end{cases}. \]  

(A.8)

By these two sets of relations we see that functions \( sa, ca \) are defined on the whole range of \( \mathbb{R} \).

The first derivatives of the \( ca \) and \( sa \) Ateb functions are

\[
\frac{d}{dz} ca(\alpha, 1, z) = -\frac{2}{\alpha + 1} sa(1, \alpha, z) \\
\frac{d}{dz} sa(1, \alpha, z) = ca^\alpha(\alpha, 1, z).
\]  

(A.9)

Being cosine Ateb \( ca(n, m, z) \) even \( 2 \Pi_\alpha \) periodic function, it is a perfect candidate for a cosine Fourier series expansion. Let us mention that finite Fourier series approximation has been discussed in Droniuk and Nazarkevich (2010)\textsuperscript{1}, while in Droniuk and Nazarkevich (2010)\textsuperscript{2} the sine Ateb \( sa(n, m, z) \) has been approximated by its sine Fourier series. Applying the there exposed method to \( ca(1, \alpha, z) \), we conclude that

\[ ca(\alpha, 1, z) = \sum_{n=1}^{\infty} a_n \cos \frac{\pi n z}{\Pi_\alpha}, \]  

(A.10)

since obviously \( a_0 = 0 \) and

\[
a_n = \frac{2}{\Pi_\alpha} \int_{0}^{\Pi_\alpha} ca(\alpha, 1, z) \cdot \cos \frac{\pi n z}{\Pi_\alpha} \, dz = \frac{2}{\Pi_\alpha} \int_{0}^{\Pi_\alpha} \cos \frac{\pi n z}{\Pi_\alpha} \left\{ \int_{z}^{1} \frac{d\bar{u}}{(1 - \bar{u}^2)^{\alpha+1}} \right\} \, dz.
\]  

(A.11)

The \( a_n \) values we compute numerically according to the prescribed accuracy. Of course, it is enough to compute \( ca(\alpha, 1, z) \) for \( z \in [0, \Pi_\alpha/2] \), another values we calculate by means of formula (A.8) (see also Gricik et al. 2009). Another model in approximating Ateb functions is the Taylor series expansion, such that corresponds to the investigations by Gricik and Nazarkevich in 2007.
Now, inverting the half of the incomplete Beta function in (3.26):

\[
\frac{1}{2} B\left(\frac{\alpha + 1}{2}, \frac{\alpha}{2} + \frac{\sqrt{\alpha + 1} \, |c_\alpha|}{\sqrt{2}} A^{(\alpha-1)/2} t, \right),
\]

we clearly deduce

\[
x(t) = A \cdot sa \left(1, \alpha, \frac{\Pi_\alpha}{2} + \frac{\sqrt{\alpha + 1} \, |c_\alpha|}{\sqrt{2}} A^{(\alpha-1)/2} t\right).
\]

(A.12)

Having in mind the quarter period expansion formula, we arrive at

\[
x(t) = A \cdot ca \left(\alpha, 1, \frac{\sqrt{\alpha + 1} \, |c_\alpha|}{\sqrt{2}} A^{(\alpha-1)/2} t\right), \quad t \in \mathbb{R}.
\]

(A.13)

By \(ca(\alpha, 1, 0) = 1\), we see that the initial condition \(x(0) = A\) is satisfied as well.

Moreover, we have to point out a restricting characteristics of Rosenberg’s and Senik’s inversion (1969). Rosenberg (1963) pointed our the rule:

“Exponents \(n = \alpha + \frac{1}{2}\) and \(1/n\) behave like odd integers”, while Senik’s restriction to some rational values of \(\alpha\) constitutes the set of permitted \(\alpha\)-values:

Approximate methods by numerically obtained evaluations of Ateb functions have been performed by Droniuk and Nazarkevich (2010)\(_1\) and (2010)\(_2\), and the references therein. Further study on Ateb function integral was realized by Senik in 1969 (see Table A.1) and Drogomirecka in 1997.

**Table A.1** The values of \(\alpha = \frac{2\mu + 1}{2\nu + 1}\), \(\mu, \nu \in \mathbb{N}_0\), by Senik’s traces

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<td>1</td>
<td>7/5</td>
<td>9/5</td>
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<td>13/5</td>
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<td></td>
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<td>3/7</td>
<td>5/7</td>
<td>1</td>
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References

Appendix B
Fourier Series of the ca Ateb Function

Since the ca function is odd, its Fourier series comprises odd harmonics only, and it can be expressed as

$$\text{ca} (\alpha, 1, t) = \sum_{N=1}^{\infty} C_{2N-1} (\alpha) \cos \left[ (2N - 1) \frac{2\pi}{T} t \right],$$  \hspace{1cm} (B.1)

where the Fourier coefficients $C_{2N-1}$ depend on the parameter $\alpha$, and are defined by

$$C_{2N-1} (\alpha) = \frac{4}{T} \int_{0}^{T/2} \text{ca} (\alpha, 1, t) \cos \left[ (2N - 1) \frac{2\pi}{T} t \right] dt,$$  \hspace{1cm} (B.2)

where $T$ is the period.

To write this expression in a suitable form for further calculation, the procedure recently proposed in Belendez et al. (2015) is utilised. As a first step, the displacement is rescaled by the initial amplitude, $X = x/A$, yielding

$$C_{2N-1} (\alpha) = \frac{8}{T} \int_{0}^{T/4} X (\alpha, t) \cos \left[ (2N - 1) \frac{2\pi}{T} t \right] dt.$$  \hspace{1cm} (B.3)

Now, to find the expression for $dt$, the first integral is composed, and the following is derived

$$dt = \sqrt{\frac{\alpha + 1}{2c_{\alpha}^{2}}} |A|^{(1-\alpha)/2} \frac{dX}{\sqrt{1 - |X|^\alpha}}.$$  \hspace{1cm} (B.4)

This expression gives the possibility to determine how $t$ depends on $X$ (noting that this holds for $X \geq 0$):

$$t (X) = \sqrt{\frac{\alpha + 1}{2c_{\alpha}^{2}}} |A|^{(1-\alpha)/2} \int_{X}^{1} \frac{dy}{\sqrt{1 - y^{\alpha+1}}}.$$  \hspace{1cm} (B.5)
Performing some transformations, one can derive (see Belendez et al. 2015)

\[
t(X) = \sqrt{\frac{\pi}{2c_\alpha^2 (\alpha + 1)}} \frac{\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)} |A|^{(1-\alpha)/2} I\left(1 - X^{\alpha+1}, \frac{1}{2}, \frac{1}{\alpha+1}\right), \tag{B.6}
\]

where \(I\) stands for the regularized incomplete beta function.

Finally, substituting (B.4) into (B.3) as well as (B.6) into the argument of the cosine function in (B.3), one derives

\[
C_{2N-1}(\alpha) = \frac{2(\alpha + 1) \Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{\alpha+1}\right)} \int_0^1 \frac{X}{\sqrt{1 - X^{\alpha+1}}} \cos\left(\frac{(2n - 1) \pi}{2} I\left(1 - X^{\alpha+1}, \frac{1}{2}, \frac{1}{\alpha+1}\right)\right) dX. \tag{B.7}
\]

By using the substitution \(z = 1 - X^{\alpha+1}\), the following expression for the Fourier coefficients is obtained:

\[
C_{2N-1}(\alpha) = \frac{2\Gamma\left(\frac{\alpha+3}{2(\alpha+1)}\right)}{\sqrt{\pi} \Gamma\left(\frac{1}{\alpha+1}\right)} \int_0^1 (1 - z)^{(1-\alpha)/(1+\alpha)} \sqrt{z} \cos\left(\frac{(2N - 1) \pi}{2} I\left(z, \frac{1}{2}, \frac{1}{\alpha+1}\right)\right) dz. \tag{B.8}
\]

These values can be calculated by carrying out numerical integration. First four Fourier coefficients are calculated in this way by using (B.8) and plotted in Fig. B.1 as a function of the power \(\alpha\). It is seen that: \(C_1\) decreases from unity as \(\alpha\) increases; \(C_3\) and \(C_7\) are positive; \(C_5\) is negative for \(1 < \alpha < 2.34\), and positive otherwise.

---

**Fig. B.1** Fourier coefficients for \(ca(\alpha, 1, t)\) versus order of nonlinearity \(\alpha\): (a) first \(C_1\), (b) second \(C_2\), (c) third \(C_3\), (d) fourth \(C_4\)
Reference

Appendix C
Averaging of Ateb Functions

1. For $c^2_\alpha = 1$ and $A = 1$, the first integral (3.12) transforms into

$$\frac{\dot{x}^2}{2} = \frac{1}{\alpha + 1} (1 - x^{\alpha+1}). \quad (C.1)$$

Assuming

$$x = ca(\alpha, 1, \psi) \equiv ca, \quad (C.2)$$

and substituting it into the first integral (C.1), we have

$$\frac{\dot{x}^2}{2} = \frac{1}{\alpha + 1} (1 - ca^{\alpha+1}). \quad (C.3)$$

Using the relation for the sine and cosine Ateb functions

$$sa^2 + ca^{\alpha+1} = 1, \quad (C.4)$$

where $sa \equiv sa(1, \alpha, \psi)$, the expression (C.3) transforms into

$$sa^2 = \frac{\alpha + 1}{2} \dot{x}^2. \quad (C.5)$$

Averaging of the function $sa^2$ is done in the time interval $[0, \bar{T}/4]$, i.e., for the positive displacement $x$ in the interval $[0, 1]$

$$\langle sa^2 \rangle = \frac{1}{(\bar{T}/4)} \int_0^{\bar{T}/4} \frac{\alpha + 1}{2} \dot{x}^2 dt = \frac{1}{(\bar{T}/4)} \int_0^{\bar{T}/4} \frac{\alpha + 1}{2} \dot{x} dx. \quad (C.6)$$

where integrating the relation (C.3)
we obtain the quarter period of vibration (see Cveticanin and Pogany 2012)

\[
\frac{\bar{T}}{4} = \sqrt{\frac{1}{2(1+\alpha)}} B \left( \frac{1}{\alpha+1}, \frac{1}{2} \right).
\]  
(C.8)

Substituting (C.8) and (C.2) into (C.7) and after some transformation we obtain, finally,

\[
a_1 = \langle sa^2 \rangle = \frac{1 + \alpha}{3 + \alpha}.
\]  
(C.9)

For \( \alpha \in (0, \infty) \), the parameter \( a_1 \) increases in the interval \( a_1 \in (1/3, 1) \).

2. According to (C.3) and (C.5) we have

\[
sa^2 ca^2 = \frac{\alpha + 1}{2} \dot{x}^2 x^2 = (1 - x^\alpha) x^2.
\]  
(C.10)

The averaged product \( sa^2 ca^2 \) over the time period of 0 to \( \bar{T}/4 \), i.e., in the displacement interval \([0, 1]\) is

\[
\langle sa^2 ca^2 \rangle = \frac{1}{\bar{T}/4} \int_0^{\bar{T}/4} \frac{\alpha + 1}{2} \ddot{x}^2 x^2 dt = \frac{1}{\bar{T}/4} \int_0^{\bar{T}/4} \frac{\alpha + 1}{2} \ddot{x} dx.
\]  
(C.11)

Using (C.8), integrating the relation (C.11) and after some calculation we obtain

\[
a_2 = \langle sa^2 ca^2 \rangle = \frac{B \left( \frac{3}{2}, \frac{3}{1+\alpha} \right)}{B \left( \frac{1}{2}, \frac{1}{1+\alpha} \right)}.
\]  
(C.12)

Fig. C.1 \( a_1 - \alpha \) and \( a_2 - \alpha \) curves
In Fig. C.1, the $a_1 - \alpha$ and $a_2 - \alpha$ curves are plotted. It is shown that for $\alpha \in (0, \infty)$, the both parameters, $a_1$ and $a_2$ increase, but $a_2$ slower than $a_1$.

In Fig. C.2, the $\sqrt{a_1/a_2} - \alpha$ relation is plotted.

The curve decreases with increase of the nonlinearity order $\alpha$ from zero to infinity in a bounded region.

Reference

Appendix D
Jacobi Elliptic Functions

Jacobian elliptic functions are doubtless periodic functions defined over the complex plane. They represent the special case of periodical Ateb function, as is shown in Sect. 3.2. The fundamental three elliptic functions are the Jacobi elliptic sine \((sn(\psi, k^2) \equiv sn)\), cosine \((cn(\psi, k^2) \equiv cn)\) and delta \((dn(\psi, k^2) \equiv dn)\) functions with argument \(\psi\) and modulus \(k^2\). The elliptic functions \(sn\) and \(cn\) may be thought of as generalizations of sine and cosine trigonometric functions where their period depends on the modulus \(k^2\). For \(k^2 = 0\), the Jacobi elliptic functions transform into trigonometric ones

\[
sn(\psi, 0) = \sin \psi \quad cn(\psi, 0) = \cos \psi \quad dn(\psi, 0) = 1. \tag{D.1}
\]

The period of the \(sn\) and \(cn\) Jacobi elliptic functions is \(4K(k)\), while of the function \(dn\) it is \(2K(k)\), where \(K(k)\) is the complete elliptic integral of the first kind. The \(cn\) and \(dn\) Jacobi elliptic functions are even functions, while \(sn\) is an odd function.

In Fig. D.1 the \(sn(t, 1/2)\), \(cn(t, 1/2)\) and \(dn(t, 1/2)\) Jacobi elliptic functions are plotted.

The elliptic functions satisfy the following identities

\[
ca^2 + sa^2 = 1, \quad dn^2 + k^2 sn^2 = 1, \quad 1 - k^2 + k^2 cn^2 = dn^2. \tag{D.2}
\]

Only two of these three relations are independent.

The first time derivatives of the functions for the argument \(\psi\) are

\[
\frac{\partial}{\partial \psi}(cn) \equiv cn_\psi = -sn dn, \quad \frac{\partial}{\partial \psi}(sn) \equiv sn_\psi = cn dn, \\
\frac{\partial}{\partial \psi}(dn) \equiv dn_\psi = -k^2 sn cn. \tag{D.3}
\]

More about the Jacobi elliptic functions, the reader can find in the literature of the special functions (see Byrd and Friedman 1954, or Abramowitz and Stegun 1971).
Fig. D.1  Jacobi elliptic functions: $sn(t, 1/2)$ (dotted line), $cn(t, 1/2)$ (full line) and $dn(t, 1/2)$ (dashed line)

References


Appendix E
Euler’s Integrals of the First and Second Kind

Euler’s integral of the first kind also named Beta function, $B(p, q)$, is defined as (see Gradstein and Rjizhik 1971)

$$B(p, q) = \int_0^1 u^{p-1}(1-u)^{q-1} du, \quad (E.1)$$

which exists for

$$\text{Re}(p) > 0, \quad \text{Re}(q) > 0. \quad (E.2)$$

Introducing the new variable $x = 1 - u$ into (E.1), the Beta function is expressed as

$$B(p, q) = -\int_0^1 x^{q-1}(1-x)^{p-1} dx = \int_0^1 x^{q-1}(1-x)^{p-1} dx = B(q, p). \quad (E.3)$$

The Beta function is symmetric in $(p, q)$.

Euler’s integral of the second kind also called Gamma function is (see Mickens, 2004)

$$\Gamma(p) = \int_0^\infty u^{p-1}e^{-u} du, \quad (E.4)$$

where $p$ satisfies the relation (E.2). The connection between the Euler’s integrals of the first and second kind is

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}. \quad (E.5)$$

For $(p - 1) = n$, where $n$ is a whole positive number, the relation (E.4) modifies into
Appendix E: Euler’s Integrals of the First and Second Kind

\[ \Gamma(n + 1) = \int_0^{\infty} u^n e^{-u} du = n! \]  \hspace{1cm} (E.6)

Thus,

\[ \Gamma(n) = (n - 1)! \]  \hspace{1cm} (E.7)

and the relation between (E.6) and (E.7) is

\[ \Gamma(n + 1) = n(n - 1)! = n\Gamma(n). \]  \hspace{1cm} (E.8)

Generalizing (E.6) for any value of \( p \) we have

\[ \Gamma(p + 1) = p! \]  \hspace{1cm} (E.9)

and the corresponding relations

\[ \Gamma(p) = (p - 1)! \]  \hspace{1cm} (E.10)

and

\[ \Gamma(p + 1) = p\Gamma(p). \]  \hspace{1cm} (E.11)

Substituting (E.10) into (E.3) the transformed version of the Beta function is

\[ B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} = \frac{(p - 1)!(q - 1)!}{(p + q - 1)!}, \]  \hspace{1cm} (E.12)

which is suitable for calculation.

References


Appendix F
Inverse Incomplete Beta Function

In this Appendix the Table of the inverse incomplete Beta function $f(\alpha, x)$ for various values of parameter $\alpha$ is given.

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