Appendix A

A.1 Proof of Proposition 7.2.1

Proof
Given $P_F = P_{F_i}$ and $P_M = P_{M_i}$ for all $i \in \{1, \ldots, K\}$, false alarm and miss detection probabilities resulting from $\gamma^K$ are $P_{F_0} = 1 - B(K/2; K, P_F)$ and $P_{M_0} = B(K/2; K, 1 - P_M)$, respectively, where $B(t_0; K, P)$ is a binomial cumulative distribution function with at most $t_0$ successes out of $K$ trials each having a success probability $P$. Let $X \sim B(K, P)$ and $Y \sim B(K, 1 - P)$ be two Binomial r.v.s with $K$ trials each having a success probability $P$ and $1 - P$, respectively. Then, for two disjoint events $E_1 = X \leq K/2$ and $E_2 = (K - X) \leq K/2$,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) = 1.$$ 

Now, by noting that $Y = K - X$ in distribution, we have

$$P(E_1) + P(E_2) = P(X \leq K/2) + P(Y \leq K/2) = B(K/2; K, P) + B(K/2; K, 1 - P) = 1$$

which implies that $P_{F_0}$ and $P_{M_0}$ own the same polynomial function $f$ s.t. $P_{F_0} = f(P_F)$ and $P_{M_0} = f(P_M)$. From Proposition 7.4.1, $f$ is monotonically increasing, hence $P_{M_0} = P_{F_0}$ iff $P_M = P_F$. □

A.2 Proof of Proposition 7.2.2

Proof
Using the substitution $j = i + 1$, we have

© Springer International Publishing AG 2017
G. Gül, Robust and Distributed Hypothesis Testing, Lecture Notes in Electrical Engineering 414, DOI 10.1007/978-3-319-49286-5

135
\[ P_F P_{F_0}^{2K-1} = \sum_{i=K}^{2K-1} \binom{2K-1}{i} P_F^{i+1} (1 - P_F)^{2K-1-i} \]

\[ = \sum_{j=K+1}^{2K} \binom{2K-1}{j-1} P_F^j (1 - P_F)^{2K-j} \]

\[ = \left( \sum_{j=K+1}^{2K-1} \binom{2K-1}{j-1} P_F^j (1 - P_F)^{2K-j} \right) + P_F^{2K} \quad (A.1) \]

and

\[ (1 - P_F) P_{F_0}^{2K-1} = \sum_{i=K}^{2K-1} \binom{2K-1}{i} P_F^i (1 - P_F)^{2K-i} \]

\[ = \binom{2K-1}{K} P_F^K (1 - P_F)^K \]

\[ + \sum_{i=K+1}^{2K-1} \left( \binom{2K-1}{i-1} + \binom{2K-1}{i} \right) P_F^i (1 - P_F)^{2K-i} + P_F^{2K} \]

\[ = \frac{1}{2} \binom{2K}{K} P_F^K (1 - P_F)^K + \sum_{i=K+1}^{2K} \binom{2K}{i} P_F^i (1 - P_F)^{2K-i} \]

\[ = P_F^{2K} \quad (A.2) \]

Adding up (A.1) and (A.2), we get

\[ (1 - P_F) P_{F_0}^{2K-1} + P_F P_{F_0}^{2K-1} = \left( \frac{2K - 1}{K} \right) P_F^K (1 - P_F)^K \]

\[ + \sum_{i=K+1}^{2K-1} \left( \binom{2K-1}{i-1} + \binom{2K-1}{i} \right) P_F^i (1 - P_F)^{2K-i} + P_F^{2K} \]

\[ = \frac{1}{2} \binom{2K}{K} P_F^K (1 - P_F)^K + \sum_{i=K+1}^{2K} \binom{2K}{i} P_F^i (1 - P_F)^{2K-i} \]

\[ = P_F^{2K} \]

using the identities

\[ \binom{2K-1}{K} = \frac{1}{2} \binom{2K}{K}, \quad \binom{2K-1}{i-1} + \binom{2K-1}{i} = \binom{2K}{i} \]

and

\[ P_F^{2K} = \binom{2K}{2K} P_F^{2K} (1 - P_F)^{2K-2K}. \]

□
A.3 Proof of Lemma 7.3.2

Proof By definition, $P_F$ and $P_M$ are probabilities, hence $(P_F, P_M) \in [0, 1]^2$. Evaluating $P_F = 1 - P_0[l(Y) \leq t]$ and $P_M = P_1[l(Y) \leq t]$ for $\lim_{t \to 0}$ and $\lim_{t \to \infty}$ shows that $r_t$ passes through the points $(1, 0)$ and $(0, 1)$. Let $p_{0,l}$ and $p_{1,l}$ be the density functions of $l(Y)$ for $Y \sim P_0$ and $Y \sim P_1$, respectively. Since $r_t$ is differentiable for every $t$, i.e.

$$\frac{dP_M}{dP_F} = \frac{dP_M}{dt} \frac{dt}{dP_F} = -\frac{p_{1,l}(t)}{p_{0,l}(t)}$$

(A.3)

exists, $r_t$ is continuous. The miss detection probability can also be written as

$$P_M = \int_{\{y : l(y) \leq t\}} p_1(y)dy = \int_{\{y : l(y) \leq t\}} l(y)p_0(y)dy = \int_0^t x p_{0,l}(x)dx,$$

where the last equality follows from

$$p_{0,l}(x) = \left| \frac{dl^{-1}(x)}{dx} \right| p_0(l^{-1}(x))$$

with the change of variable $x = l(y)$. Hence,

$$\frac{dP_M}{dt} = tp_{0,l}(t) \overset{(A.3)}{\Rightarrow} \frac{dP_M}{dP_F} = -t.$$

As a result,

$$\frac{d^2P_M}{dP_F^2} = \frac{d}{dP_F} \left( \frac{dP_M}{dP_F} \right) = -\frac{dt}{dP_F} = \frac{1}{p_{0,l}(t)} \geq 0$$

proves that $r_t$ is convex. \hfill \square

A.4 Proof of Proposition 7.4.1

Letting $p = p(\theta) = 1 - \theta$ and $q = q(\theta) = \frac{1-2\theta}{1-\theta}$, showing that $L^K_\infty$ is negative for sufficiently large $K$ is equivalent to showing that

$$\sum_{i=0}^{K/2} \binom{K}{i} p^i (1-p)^{K-i} < \frac{1}{2} \sum_{i=0}^{K/2} \binom{K}{i} q^i (1-q)^{K-i}$$

(A.4)

for sufficiently large $K$. There are two possible cases:
• Trivial case: For $1/3 \leq \theta < 1/2$, the sum on the left converges to zero and the sum on the right converges to a positive number, so the inequality (A.4) is true for large $K$.

• Remaining case: Suppose $0 < \theta < 1/3$. The inequality of the sums can be proven working term by term. It suffices to show that

$$p^i (1 - p)^{K-i} < \frac{1}{2} q^i (1 - q)^{K-i} \quad (A.5)$$

for all $0 \leq i \leq K/2$, when $K$ is large enough. Note that \(\frac{p(1-q)}{q(1-p)} = \frac{1-\theta}{1-2\theta} > 1\) and \(\frac{(1-\theta)^3}{1-2\theta} < 1\). Therefore,

\[
\left(\frac{1-p}{1-q}\right)^K \left(\frac{p(1-q)}{q(1-p)}\right)^i \leq \left(\frac{1-p}{1-q}\right)^K \left(\frac{p(1-q)}{q(1-p)}\right)^{K/2} = \left(\frac{p(1-p)}{q(1-q)}\right)^{K/2}.
\]

The right hand side of (A.6) can be made less than $1/2$ by taking $K$ sufficiently large, giving the inequality (A.5) and hence the inequality (A.4).

### A.5 Proof of Proposition 7.5.1

**Proof** To prove that $P_{F_0}(P_F, K, t_0)$ and $P_{M_0}(P_M, K, t_0)$ are increasing functions of $P_F$ and $P_M$, respectively, it is sufficient to prove it only for $P_{F_0}(P_F, K, t_0)$. Because

\[
\frac{\partial P_{M_0}(P_M, K, t_0)}{\partial P_M} = \frac{\partial P_{F_0}(P_F, K, t_0)}{\partial P_F} \bigg|_{P_F:=1-P_M}.
\]

Noting that (A.7) is zero for $P_F = 0$, we have

$$P_{F_0} = \sum_{i=t_0}^{K} \binom{K}{i} P_F^i (1 - P_F)^{K-i} = \sum_{i=t_0}^{K} \binom{K-1}{i-1} P_F^i (1 - P_F)^{K-i}$$

$$+ \sum_{i=t_0}^{K} \binom{K-1}{i} P_F^i (1 - P_F)^{K-i}. \quad (A.8)$$

Since in the second sum, the term is zero when $i = K$, we get
\[ \sum_{i=t_0}^{K} \binom{K-1}{i} P_F^i (1 - P_F)^{K-i} = \sum_{i=t_0}^{K-1} \binom{K-1}{i} P_F^i (1 - P_F)^{K-i} \]

\[ < \sum_{i=t_0-1}^{K-1} \binom{K-1}{i} P_F^i (1 - P_F)^{K-i}. \quad (A.9) \]

Changing the variable \( j = i + 1 \),

\[ \sum_{i=t_0-1}^{K-1} \binom{K-1}{i} P_F^i (1 - P_F)^{K-i} = \sum_{j=t_0}^{K} \binom{K-1}{j-1} P_F^{j-1} (1 - P_F)^{K-j+1} \quad (A.10) \]

and writing (A.10) in (A.8) with (A.9), it follows that

\[ \sum_{i=t_0}^{K} \binom{K}{i} P_F^i (1 - P_F)^{K-i} < \sum_{i=t_0}^{K} \binom{K-1}{i} P_F^i (1 - P_F)^{K-i} \]

\[ + \sum_{j=t_0}^{K} \binom{K-1}{j-1} P_F^{j-1} (1 - P_F)^{K-j+1}. \quad (A.11) \]

Using

\[ P_F^i (1 - P_F)^{K-i} + P_F^{i-1} (1 - P_F)^{K-i+1} = P_F^{i-1} (1 - P_F)^{K-i} \]

rewrite (A.11),

\[ \sum_{i=t_0}^{K} \binom{K}{i} P_F^i (1 - P_F)^{K-i} < \sum_{i=t_0}^{K} \binom{K-1}{i-1} P_F^{i-1} (1 - P_F)^{K-i}. \quad (A.12) \]

Multiplying (A.12) with \( K/(1 - P_F) \) and noting that

\[ i \binom{K}{i} = K \binom{K-1}{i-1} \]

we finally get

\[ \sum_{i=t_0}^{K} \binom{K}{i} P_F^{i-1} (1 - P_F)^{K-i-1} (i - KP_F) = \frac{\partial P_{F_0}(P_F, K, t_0)}{\partial P_F} > 0. \]
A.6 Proof of Proposition 7.5.2

Proof The claim will be proven for odd $t_0$, while its extension to even $t_0$ can be accomplished following the same line of arguments. Let the threshold be $t_0 \in \{0, \lfloor K/2 \rfloor - 1\}$ for some $K$. If $t_0 = \lfloor K/2 \rfloor$, then clearly

$$P_{F_0}(x, K, \lfloor K/2 \rfloor) = P_{M_0}(x, K, \lfloor K/2 \rfloor), \quad \forall x \in [0, 1].$$

One can also see that, cf. Remark 7.4.1,

$$P_{F_0}(x, K, \lfloor K/2 \rfloor - 1) > P_{F_0}(x, K, \lfloor K/2 \rfloor), \quad \forall x \in (0, 1),$$

and

$$P_{M_0}(x, K, \lfloor K/2 \rfloor - 1) < P_{M_0}(x, K, \lfloor K/2 \rfloor), \quad \forall x \in (0, 1).$$

Hence,

$$P_{F_0}(x, K, \lfloor K/2 \rfloor - 1) > P_{M_0}(x, K, \lfloor K/2 \rfloor - 1), \quad \forall x \in (0, 1). \quad (A.13)$$

For a pair $(P_F, P_M)$ to be valid, it should be in $(\mathcal{F} \times \mathcal{M})_{t_0}^K$, i.e.

$$P_{F_0}(P_F, K, \lfloor K/2 \rfloor - 1) = P_{M_0}(P_M, K, \lfloor K/2 \rfloor - 1). \quad (A.14)$$

Assume that (A.14) holds for some $(P_F^*, P_M^*)$ with $P_M^* = P_F^* < P_M^*$ or with $P_M^* < P_F^*$. Then, both cases are obviously a contradiction with (A.13), since both $P_{F_0}$ and $P_{M_0}$ are monotonically increasing functions of $P_F$ and $P_M$, respectively, cf. Proposition 7.4.1. Therefore, $P_M^* > P_F^*$ must be true for all pairs $(P_F^*, P_M^*) \in (\mathcal{F} \times \mathcal{M})_{t_0}^K$. This proves that $h_{t_0}^K(P_F) > P_F$ for all $P_F \in (0, 1)$. Clearly, when $t_0 \in \{\lfloor K/2 \rfloor - 1, K\}$, due to symmetry, e.g., $P_{M_0}(x, K, \lfloor K/2 \rfloor + 1) = P_{F_0}(x, K, \lfloor K/2 \rfloor - 1)$, the inequalities above change direction and we get $h_{t_0}^K(P_F) < P_F$ for all $P_F \in (0, 1)$. Next, assume that $(P_F^*, P_M^*) \neq (1, 1)$ is a valid pair that satisfies (A.14) and fix a small positive number $\delta$. Since $P_{F_0}$ is increasing,

$$P_{F_0}(P_F^* + \delta, K, \lfloor K/2 \rfloor - 1) > P_{F_0}(P_F^*, K, \lfloor K/2 \rfloor - 1), \quad \forall P_F^* \in (0, 1).$$

This suggests that the left hand side of (A.14) increases by adding $\delta$ to $P_F^*$. In order (A.14) to hold, its right hand side must also increase, which implies an increase of $P_M^*$ by some positive number $\epsilon$, since $P_{M_0}$ is also an increasing function. Then, $(P_F^* + \delta, P_M^* + \epsilon) \in (\mathcal{F} \times \mathcal{M})_{t_0}^K$ for all $(P_F, P_M) \neq (1, 1)$ implies that $h_{t_0}^K$ is a monotonically increasing function. \qed
Proof of (7.18)

Proof Introducing a random variable $X_K$ with a binomial distribution $B(K, \theta)$, it can be shown that

$$L^K(\theta) = P[X_K > \lfloor K/2 \rfloor] - \frac{1}{1 + \hat{\theta}(\theta)^{-K}}.$$

For every $\theta \leq \frac{1}{2}$, $P[X_K > \lfloor K/2 \rfloor] \leq \frac{1}{2}$ hence $P[X_K > \lfloor K/2 \rfloor] < \frac{1}{2}$. Assume that $\theta = \theta_K(x)$ where $\theta_K(x) = \frac{1}{2} \left(1 - \frac{x}{\sqrt{K}}\right)$, for some fixed positive $x$. Then, $\lfloor K/2 \rfloor = E[X_K] + x_K \sigma(X_K)$ with $x_K = x/\sqrt{4\theta_K(x)(1 - \theta_K(x))} \sim x$. The central limit theorem implies that

$$P[X_K > \lfloor K/2 \rfloor] = P[X_K > E[X_K] + x_K \sigma(X_K)]$$

$$= P \left[ \frac{X_K - E[X_K]}{\sigma(X_K)} > x_K \right]$$

$$= P[X'_K > x] = 1 - F(x) \text{ when } K \to \infty$$

where $X'_K \sim N(0, \sigma^2)$. Since $\hat{\theta}(\theta_K(x))^{-K} \to \infty$ when $K \to \infty$, we get,

$$\lim_{k \to \infty} \sup_{\theta \leq 1/2} L^K(\theta) \geq \lim_{k \to \infty} L^K(\theta_K(x)) = 1 - F(x).$$

As $F(x) \to \frac{1}{2}$ when $x \to 0^+$, this proves the claim. \qed