Chapter A

Appendix. Near-minimizers for Brudnyi and Triebel–Lizorkin spaces

In Chapter 9, near-minimizers were constructed for couples of the form \((L^p, \dot{W}_q^k)\) and \((L^p, \dot{C}^{\alpha,k}_p)\). It is natural to ask whether it is possible to replace \(L^p\) in these results by a more general function space \(X\). Our purpose in this chapter is to supplement the previous constructions by a generalization of the algorithm of Chapter 9. We shall test this general algorithm on couples of the form \((X,Y)\), where \(X\) is a Triebel–Lizorkin space (or Brudnyi’s generalized variation spaces), and \(Y\) is a Morrey space constructed on the basis of \(X\). It will turn out that the algorithm in question really provides near-minimizers for such couples.

An outline of the algorithm in an “abstract” setting will be given in §1. The last two sections (§§2 and 3) are devoted to the specific couples of spaces mentioned above. Fairly bulky formulas occurring in those sections seem to be unavoidable.

A.1 Description of the general algorithm

Let \(X\) be a (Banach or quasi-Banach) function space on \(\mathbb{R}^n\). Suppose that for every cube \(Q\) some space \(X(Q)\) is defined (heuristically, it is obtained by localization of the norm of \(X\) to \(Q\)). In particular, if \(f \in X\), then \(f\chi_Q \in X(Q)\) and

\[
\|f\chi_Q\|_{X(Q)} \leq \|f\|_X.
\]

For example, if \(X = L^p\), then \(X(Q) = L^p(Q)\); if \(X = \dot{W}_p^k\), then \(X(Q)\) is determined by the seminorm

\[
\|f\|_{X(Q)} = \sup_{|m|=k} \|D^m f\|_{L^p(Q)}.
\]
In what follows, we shall present more examples related to the Brudnyi and Triebel–Lizorkin spaces.

Suppose we are given a function \( f \in X \) and a number \( t > 0 \). The algorithm in question depends on two parameters \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( k \in \mathbb{Z}_+ \) (the symbol \( \mathbb{Z}_+ \) stands for the set of nonnegative integers). As before, we denote by \( P_k \) the space of algebraic polynomials of degree strictly smaller than \( k \), in particular, \( P_0 = \{0\} \).

For every \( x \in \mathbb{R}^n \), we consider the following function of \( r > 0 \):
\[
\varphi_x(r) = \inf_{P \in P_k} \frac{1}{|Q(x,r)|^\lambda} \| f - P \|_{X(Q(x,r))} \tag{A.1}
\]
and put
\[
\Omega = \left\{ x \in \mathbb{R}^n : \sup_{r > 0} \varphi_x(r) > t \right\}. \tag{A.2}
\]

First, we describe the algorithm in the case where \( \lambda > 0 \). In this case, since \( \| f \chi_Q \|_{X(Q)} \leq \| f \|_X < \infty \), we see that \( \varphi_x(r) \to 0 \) as \( r \to \infty \). Therefore, for every \( x \in \Omega \), there exists a positive number \( r_x \) with
\[
\sup_{r \geq r_x} \varphi_x(r) \leq t \tag{A.3}
\]
and
\[
\sup_{r \geq \frac{1}{2} r_x} \varphi_x(r) > t. \tag{A.4}
\]
We put
\[
Q_x = Q(x,r_x).
\]
Thus, we have constructed a family \( \{Q_x\}_{x \in \Omega} \) of cubes.

Now, conditions (A.3)—(A.4) imply the existence of \( r \in \left[ \frac{1}{2} r_x, r_x \right] \) such that
\[
t \leq \frac{1}{|Q(x,r)|^\lambda} \inf_{P \in P_k} \| f - P \|_{X(Q(x,r))} \leq \frac{1}{|Q(x,r)|^\lambda} \| f \|_X.
\]
Consequently,
\[
|Q_x| \leq 2^n |Q(x,r)| \leq 2^n \left( \frac{1}{t} \| f \|_X \right)^\frac{1}{\lambda}.
\]
So, the cubes \( \{Q_x\}_{x \in \Omega} \) possess the Besicovitch property
\[
\sup_{x \in \Omega} |Q_x| < \infty,
\]
which makes it possible to apply the controlled extension theorem. As a result, we obtain a family \( \{K_i\}_{i \in I} \) of cubes that is a WB-covering. We take a smooth partition \( \{\psi_i\}_{i \in I} \) of unity adjusted to this WB-covering. Also, let \( f_{K_i}, i \in I \), be polynomials of degree strictly smaller than \( k \) yielding the best approximation on the cubes \( K_i, i \in I \), that is
\[
\| f - f_{K_i} \|_{X(K_i)} = \inf_{P \in P_k} \| f - P \|_{X(K_i)}.
\]
Such polynomials exist because the space $P_k$ is finite-dimensional.

We define a function $f_t$ by the formula

$$f_t = \sum_{i \in I} f_{K_i} \psi_i + f \chi_{\mathbb{R}^n \setminus \bigcup K_i}.$$  \hfill (A.5)

It will be shown below that in many specific cases formula (A.5) defines a near-minimizer for a certain couple $(X, Y)$. Finding the space $Y$ is a nontrivial problem.

Now we describe the construction of $f_t$ for $\lambda < 0$. In this case, the function $\varphi_x(r)$ given by (A.1) tends to zero as $r \to 0$. We shall also assume that the function $f \in X$ is such that $\varphi_x(r)$ is continuous in the variable $r > 0$ (if $X = L^p$, all functions have this property). Therefore, if $x \in \Omega$, where $\Omega$ is defined by (A.2), then there exists $r_x > 0$ such that

$$\sup_{r \leq r_x} \varphi_x(r) \leq t \quad \text{and} \quad \varphi_x(r_x) = \frac{1}{|Q(x, r_x)|} \inf_{P \in P_k} \|f \chi_{Q(x, r)} - P\|_{X(Q(x, r))} \geq \frac{1}{2} t. \quad \hfill (A.6)$$

We put $Q_x = Q(x, r_x)$, obtaining a family $\{Q_x\} \in \Omega$ of cubes. From (A.6) and the relation

$$\inf_{P \in P_k} \|f - P\|_{X(Q)} \leq \|f\|_X$$

we deduce that

$$|Q_x| \geq \left( \frac{t}{2 \|f\|_X} \right)^{-\frac{1}{\lambda}}$$

for every $x \in \Omega$ and, consequently, the assumption of the controlled contraction theorem is fulfilled:

$$\inf_{x \in \Omega} |Q_x| > 0.$$

Application of that theorem yields a family $\{K_i\} \in I$ of cubes that forms a WB-covering of $\mathbb{R}^n$.

The function $f_t$ is defined by the formula

$$f_t = \sum_{i \in I} f_{K_i} \psi_i,$$  \hfill (A.7)

where the $f_{K_i}, i \in I$, are polynomials of degree strictly smaller than $k$ that provide the best approximations on the cubes $K_i, i \in I$. The absence of the term $f \chi_{\mathbb{R}^n \setminus \bigcup K_i}$ in the formula for $f_t$ is related to the fact that $\bigcup K_i = \mathbb{R}^n$ in the controlled contraction theorem.

Below we consider two examples in which the algorithm described above does lead to near-minimizers. The proofs are quite similar in both cases. First, we consider the case where $X$ is Brudnyi’s generalized variation space, see [Br1] and the references therein, because then the proof is slightly less involved than for the more well-known Triebel–Lizorkin spaces. For instance, the Fefferman–Stein maximal theorem can be avoided for Brudnyi spaces.


A.2 Near-minimizers for Morrey spaces built on the basis of Brudnyi spaces

Putting

\[ E_k(f, Q) = \inf_{P \in \mathcal{P}_k} \int_Q |f - P|, \]

we define the generalized variation space \( \dot{V}^{s,k}_p \) (see [Br0]) to be the space of functions on \( \mathbb{R}^n \) for which the following seminorm is finite:

\[
\|f\|_{\dot{V}^{s,k}_p} = \sup_{\pi = \{Q_i\}} \left( \sum_i \left( \frac{1}{|Q_i|^{1+\frac{1}{p}}} E_k(f, Q_i) \right)^p |Q_i| \right)^{\frac{1}{p}}.
\]

(A.8)

Here the supremum is taken over all packings \( \pi = \{Q_i\} \) that consist of cubes.\(^1\) In what follows, we shall assume that

\[ k \geq s \geq 0, \quad p \in (0, \infty) \quad \text{and} \quad 1 + \frac{s}{n} - \frac{1}{p} > 0. \]

(A.9)

Comparison with the seminorm in the Triebel–Lizorkin spaces \( \dot{F}^{s,k}_{p,q} \) to be treated below shows that the parameter \( s \) plays the role of “smoothness”, and the quantity \( |Q_i| \) in each summand in (A.8) replaces integration over \( Q_i \).

The localization of the space \( \dot{V}^{s,k}_p \) (denoted by \( \dot{V}^{s,k}_p(Q) \)) is defined by formula (A.8) with the additional restriction that the supremum is taken over packings \( \pi = \{Q_i\} \) that consist only of cubes included in \( Q \).

The Morrey space built on the basis of the Brudnyi space \( \dot{V}^{s,k}_p \) is defined with the help of the seminorm

\[
\|f\|_{\dot{C}^{\lambda,k}(\dot{V}^{s,k}_p)} = \sup_Q \left( \frac{1}{|Q|^\lambda} \inf_{P \in \mathcal{P}_k} \|f - P\|_{\dot{V}^{s,k}_p(Q)} \right),
\]

(A.10)

where \( \lambda > 0 \) is some positive number.

Remark A.1. The results (with slightly more complicated proofs) can be extended to the case of more general Morrey–Campanato spaces determined by the seminorms

\[
\|f\|_{\dot{C}^{\lambda,k_1}(\dot{V}^{s,k}_p)} = \sup_Q \left( \frac{1}{|Q|^\lambda} \inf_{P \in \mathcal{P}_{k_1}} \|f - P\|_{\dot{V}^{s,k}_p(Q)} \right),
\]

where \( k_1 \geq k \). We have omitted this improvement because it involves even bulkier formulas.

\(^1\)In Chapter 11 we encountered another generalized variation space, also depending on three indices, which were arranged differently. The point is that, in fact, there are four indices in the most general definition of these spaces, see [T]. Two cases in question involve different triples out of the quad.
Remark A.2. If $k = s = 0$, it is known that $\dot{V}^s,k_p = L^p$. Next, taking $\lambda = \frac{1}{p} - \frac{\sigma}{n}$, $\sigma \in \left[0, \frac{2}{p}\right)$, we obtain the Morrey spaces (see (9.26)) built on the basis of $L^p$. This justifies the name given above to the spaces $\dot{C}^{\lambda,k}(\dot{V}^s,k_p)$. If $k = s = 1$, the space $\dot{V}^s,k_p$ coincides with the Sobolev space $\dot{W}^1_p$ for $p > 1$ and with BV for $p = 1$. Thus, in this case the seminorm (A.10) generates what can naturally be called “the Morrey space built on the basis of a Sobolev space or BV”.

In the sequel, a polynomial $P$ of degree strictly smaller than $k$ at which the infimum in (A.10) is attained will be denoted by $f_Q$. So,

$$\|f\|_{\dot{C}^{\lambda,k}(\dot{V}^s,k_p)} = \sup_Q \left(\frac{1}{|Q|^\lambda} \|f - f_Q\|_{\dot{V}^s,k_p(Q)}\right).$$

We take $\dot{V}^s,k_p$ for the role of $X$ and apply the general algorithm of constructing a near-minimizer with parameters $\lambda$ and $k$ to a function $f \in \dot{V}^s,k_p$ and a number $t > 0$. That is, for arbitrary $x \in \mathbb{R}^n$, we consider the function

$$\varphi_x(r) = \frac{1}{|Q(x, r)|^\lambda} \|f - f_Q(x, r)\|_{\dot{V}^s,k_p(Q(x, r))}.$$  

Since $\lambda > 0$, this function tends to zero as $r \to \infty$. Next, we consider the set

$$\Omega = \left\{ x \in \mathbb{R}^n : \sup_{r > 0} \varphi_x(r) > t \right\}$$

and for $x \in \Omega$ choose $r_x > 0$ such that

$$\sup_{r \geq r_x} \varphi_x(r) \leq t \quad \text{and} \quad \sup_{r \geq \frac{1}{2}r_x} \varphi_x(r) > t. \quad (A.11)$$

Putting

$$Q_x = Q(x, r_x),$$

we obtain a family $\{Q_x\}_{x \in \Omega}$ of cubes.

It is important that this family has finite $\alpha$-capacity for $\alpha = \lambda p$. Indeed, let $\pi = \{Q_{x_i}\}$ be a packing consisting of some cubes among $\{Q_x\}_{x \in \Omega}$. By (A.11), for some $r \in \left[\frac{1}{2}r_{x_i}, r_{x_i}\right]$ we have

$$\|f\|_{\dot{V}^s,k_p(Q_{x_i})} \geq \|f\|_{\dot{V}^s,k_p(Q(x_i, r))} \geq \|f - f_Q(x_i, r)\|_{\dot{V}^s,k_p(Q(x_i, r))} \geq t^p |Q(x_i, r)|^{\lambda p} \geq ct^p |Q_{x_i}|^{\lambda p}.$$  

Since the cubes $Q_{x_i}$ are mutually disjoint, the definition of the seminorm in $\dot{V}^s,k_p$ implies

$$\|f\|_{\dot{V}^s,k_p} \geq \sum_{x_i \in \pi} \|f\|_{\dot{V}^s,k_p(Q_{x_i})} \geq c \sum_{x_i \in \pi} t^p |Q_{x_i}|^{\lambda p}.$$
and, consequently,
\[ |\{Q_x\}_{x \in \Omega}|_{\lambda p} \leq c \frac{\|f\|_{\dot{V}^{s,k}_p}}{t^p} < \infty. \]
Since \( \lambda > 0 \), we see that
\[ \sup_{x \in \Omega} |Q_x| < \infty, \]
and the controlled extension theorem can be applied. This results in a WB-covering \( \{K_i\}_{i \in I} \). We define a function \( f_t \) by the formula
\[ f_t = \sum_{i \in I} f_{K_i} \psi_i + f \chi_{\mathbb{R}^n \setminus \cup K_i}, \tag{A.12} \]
where \( \{\psi_i\}_{i \in I} \) is a smooth partition of unity adjusted to the WB-covering mentioned above, and the \( f_{K_i} \in \mathcal{P}_k \) are polynomials of best approximation of \( f \) on the cubes \( K_i, \ i \in I \).

To state the result, we introduce yet another family of cubes:
\[ \Omega_{f,t} = \left\{ Q : \frac{1}{|Q|^\lambda} \inf_{P \in \mathcal{P}_k} \|f - P\|_{\dot{V}^{s,k}_p(Q)} \geq t \right\}. \tag{A.13} \]

**Theorem A.3.** Suppose we are given a function \( f \in \dot{V}^{s,k}_p \) and a number \( t > 0 \). If (A.9) is fulfilled and \( \lambda \leq \frac{1}{p} + \frac{k-s}{n} \), then
\[ \text{dist}_{\dot{V}^{s,k}_p}(f, B_{\dot{V}^{s,k}_p}(\dot{V}^{s,k}_p(t))) \approx t \left( \frac{|\Omega_{f,t}|_{\lambda p}}{t^p} \right)^\frac{1}{p}, \tag{A.14} \]
and the function \( f_t \) (see (A.12)) is a near-minimizer for the distance functional of the couple \( (\dot{V}^{s,k}_p, \dot{V}^{s,k}_p(\dot{V}^{s,k}_p)) \).

**Remark A.4.** For \( k = s = 0 \), the restriction \( \lambda \leq \frac{1}{p} + \frac{k-s}{n} \) turns into \( \lambda \leq \frac{1}{p} \), which coincides with the restriction imposed in the case of classical Morrey spaces. We remind the reader that the equivalence (A.14) means the existence of constants \( c_1, c_2 > 0 \) independent of \( f \in \dot{V}^{s,k}_p \) and \( t > 0 \) such that
\[ c_1 \text{dist}_{\dot{V}^{s,k}_p}(f, B_{\dot{V}^{s,k}_p}(\dot{V}^{s,k}_p)} \left( \frac{t}{c_1} \right) \leq t \left( |\Omega_{f,t}|_{\lambda p} \right)^\frac{1}{p} \leq c_2 \text{dist}_{\dot{V}^{s,k}_p}(f, B_{\dot{V}^{s,k}_p}(\dot{V}^{s,k}_p \left( \frac{t}{c_2} \right). \]

We also remind the reader that the statement that \( f_t \) is a near-minimizer for the above distance functional means that, for some \( c > 0 \) independent of \( f \in \dot{V}^{s,k}_p \) and \( t > 0 \), we have
\[ \|f_t\|_{\dot{V}^{s,k}_p(\dot{V}^{s,k}_p)} \leq ct \quad \text{and} \quad \|f - f_t\|_{\dot{V}^{s,k}_p} \leq c \text{dist}_{\dot{V}^{s,k}_p}(f, B_{\dot{V}^{s,k}_p(\dot{V}^{s,k}_p)} \left( \frac{t}{c} \right). \]
A.2.1 Auxiliary lemmas

In the proof, we shall need three lemmas. The first two will express the following observation. Suppose we are given an arbitrary WB-covering \( \{K_i\}_{i \in I} \), and let \( \{\psi_i\}_{i \in I} \) be a smooth partition of unity adjusted to this covering. For a function \( h \in \dot{V}_p^{s,k} \), consider the decomposition

\[
h = h_{\{K_i\}} + (h - h_{\{K_i\}}),
\]

where

\[
h_{\{K_i\}} = \sum_{i \in I} h_{K_i} \psi_i + h \chi_{\mathbb{R}^n / \cup K_i}
\]  

(A.15)

and the \( h_{K_i} \) are polynomials of degree strictly smaller than \( k \) that provide the best approximation to \( h \) on the cubes \( K_i \) in the metric of the space \( \dot{V}_p^{s,k}(K_i), i \in I \).

It turns out that the function \( h_{\{K_i\}} \) admits a good estimate on cubes included in some cube of the family \( \{K_i\}_{i \in I} \), whereas \( h - h_{\{K_i\}} \) admits a good estimate on cubes that are not included in any cube of this family. We give precise definitions.

**Definition A.5.** A cube \( Q \) is said to be small relative to the family \( \{K_i\}_{i \in I} \) if there exists a cube \( K_i \) such that \( Q \subset K_i \). If a cube \( Q \) is not small, it is said to be big.

In the sequel, packings \( \pi = \{Q_i\} \) composed of small cubes will be denoted by \( \pi_s \), and those composed of big cubes will be denoted by \( \pi_b \).

The next lemma shows that the function \( h_{\{K_i\}} \) admits a good estimate on packings \( \pi_s \).

**Lemma A.6.** For every packing \( \pi_s \) composed of small cubes we have

\[
\left( \sum_{Q \in \pi_s} \left( \frac{1}{|Q|^{1+\frac{n}{p}}} \int_Q \left| h_{\{K_i\}} - (h_{\{K_i\}})_{Q_i} \right|^p \right) \right)^{\frac{1}{p}} 
\leq c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{n}{p}}} \int_{K_i} |h - h_{K_i}|^p \right) \right)^{\frac{1}{p}}.
\]

Proof. Consider a packing \( \pi_s = \{Q_j\} \) as described above. The function \( h_{\{K_i\}} \) defined by (A.15) is infinitely differentiable on \( \cup K_i \). Therefore, if for every \( Q_j \) we replace the best approximation polynomial \( (h_{\{K_i\}})_{Q_j} \) by the Taylor polynomial of order \( k - 1 \) (relative to the center of \( Q_j \)) for \( h_{\{K_i\}} \), we obtain (see Theorem 0.20)

\[
\int_{Q_j} |h_{\{K_i\}} - (h_{\{K_i\}})_{Q_j}| \leq c |Q_j|^{1+\frac{n}{p}} \max_{|m|=k} \|D^m(h_{\{K_i\}})\|_{L^\infty(Q_j)}.
\]

Consequently,

\[
\left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\frac{n}{p}}} \int_{Q_j} \left| h_{\{K_i\}} - (h_{\{K_i\}})_{Q_j} \right|^p \right) \right)^{\frac{1}{p}} 
\leq c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{n}{p}}} \int_{K_i} |h - h_{K_i}|^p \right) \right)^{\frac{1}{p}}.
\]
\[
\sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1 + \frac{k}{n}}} |Q_j|^{1 + \frac{s}{n}} \max_{|m| = k} \|D^m(h_{\{K_i\}})\|_{L^\infty(Q_j)} \right)^p |Q_j|^{\frac{1}{p}} \\
= \left( \sum_{Q_j \in \pi_s} \left( |Q_j|^{\frac{1}{p} + \frac{k}{n} - \frac{s}{n}} \max_{|m| = k} \|D^m(h_{\{K_i\}})\|_{L^\infty(Q_j)} \right)^p \right)^{\frac{1}{p}}.
\]

Since \( k \geq s \) (see (A.9)), we have \( (\frac{1}{p} + \frac{k}{n} - \frac{s}{n})p \geq 1 \), and for the cubes \( Q_j \) included in \( K_i \) we have
\[
\sum_{Q_j \subset K_i} \left( |Q_j|^{\frac{1}{p} + \frac{k}{n} - \frac{s}{n}} \right)^p \leq \left( |K_i|^{\frac{1}{p} + \frac{k}{n} - \frac{s}{n}} \right)^p.
\]

Now, all cubes \( Q_j \) are small, so each of them is included in some \( K_i \). Therefore,
\[
\left( \sum_{Q_j \in \pi_s} \left( |Q_j|^{\frac{1}{p} + \frac{k}{n} - \frac{s}{n}} \max_{|m| = k} \|D^m(h_{\{K_i\}})\|_{L^\infty(Q_j)} \right)^p \right)^{\frac{1}{p}}
\leq \left( \sum_{i \in I} \left( |K_i|^{\frac{1}{p} + \frac{k}{n} - \frac{s}{n}} \max_{|m| = k} \|D^m(h_{\{K_i\}})\|_{L^\infty(K_i)} \right)^p \right)^{\frac{1}{p}}.
\]

Since \(|m| = k\), we have \( D^m(h_{K_i}) = 0 \) and, consequently, on \( K_{i_0} \) we have
\[
D^m(h_{\{K_i\}}) = \sum_{|l| < k} \sum_{i : K_i \cap K_{i_0} \neq \emptyset} c_l D^l(h_{K_i}) D^{m-l} \psi_i,
\]
\[
D^m h_{K_{i_0}} = D^m \left( \sum_i h_{K_{i_0}} \psi_i \right) = \sum_{|l| < k} \sum_{i : K_i \cap K_{i_0} \neq \emptyset} c_l D^l(h_{K_{i_0}}) D^{m-l} \psi_i = 0.
\]

Hence,
\[
D^m(h_{\{K_i\}}) = \sum_{|l| < k} \sum_{i : K_i \cap K_{i_0} \neq \emptyset} c_l \left( (D^l(h_{K_i}) - (D^l(h_{K_{i_0}})) \right) D^{m-l} \psi_i
\]
on \( K_{i_0} \). Applying the Markovinequality (Theorem 0.21) to the right-hand side and using the finite multiplicity and engagement properties for the family \( \{K_i\}_{i \in I} \), as well as Theorem 0.22 and estimates for derivatives of the \( \psi_i \), we obtain
\[
\|D^m(h_{\{K_i\}})\|_{L^\infty(K_{i_0})} \leq c \sum_{|l| < k} \sum_{i : K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_{i_0}|^{\frac{k-l}{n}}} \|h_{K_i} - h_{K_{i_0}}\|_{L^\infty(K_{i_0})} \frac{1}{|K_{i_0}|^{\frac{k-l}{n}}}
\leq c \frac{1}{|K_{i_0}|^{\frac{k}{n}}} \frac{1}{|K_{i_0}|} \sum_{i : K_i \cap K_{i_0} \neq \emptyset} \int_{K_i} |h - h_{K_i}| \leq c \sum_{i : K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1 + \frac{k}{n}}} \int_{K_i} |h - h_{K_i}|.
\]
Since we are dealing with a WB-covering, the number of cubes $K_i$ that intersect $K_{i_0}$ is finite and is controlled by a constant depending only on the dimension $n$. So, finally we arrive at

\[
\left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\frac{s}{n}}} \int_Q |h_{\{K_i\}} - (h_{\{K_i\}})_{Q_j}|^p |Q_j| \right)^{\frac{1}{p}} \right) \leq c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{s}{n}}} \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |h - h_{\{K_i\}}|^p |K_i| \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]

□

To prove the next lemma, we need some geometric properties of big cubes.

**Proposition A.7.** a) If a cube $Q$ is big, then

\[
\bigcup_{i: K_i \cap Q \neq \emptyset} K_i \subset \gamma Q.
\]

with some constant $\gamma > 0$ depending only on the dimension $n$. b) Let a packing $\pi_b = \{Q_j\}$ consist of big cubes. Then the number of cubes in $\pi_b$ that intersect $K_i$ is finite and is controlled in terms of the dimension $n$.

**Proof.** a) Suppose $K_i \cap Q \neq \emptyset$, and take an arbitrary point $x \in K_i \cap Q$. By the Whitney property, there exists a cube $K_j$ such that $x \in 1 + \frac{\delta}{r} K_j$. Since $Q$ is not included in $K_j$, it follows that $2r(Q) \geq \frac{\delta}{1+\delta} r(K_j)$. By the strong engagement property, we have $r(K_i) \leq cr(K_j)$ with a constant $c > 0$ depending only on the dimension $n$. Consequently,

\[
r(K_i) \leq cr(Q),
\]

and, therefore, (A.17) is fulfilled.

b) Consider the cube $2K_i$. If a cube $Q_j \in \pi_b$ intersects $K_i$, formula (A.18) shows that the volume of the intersection of $Q_j$ with $2K_i$ is at least $c|K_i|$, where $c > 0$ is a constant depending only on the dimension $n$. Since the cubes in $\pi_b$ are mutually disjoint, the claim follows.

The next lemma shows that the function $h - h_{\{K_i\}}$ admits a good estimate on packings $\pi_b$ that consist of big cubes.

**Lemma A.8.** If (A.9) is fulfilled, then for every packing $\pi_b = \{Q_j\}$ that consists of big cubes we have

\[
\left( \sum_{Q_j \in \pi} \left( \frac{1}{|Q_j|^{1+\frac{s}{n}}} \int_{Q_j} |h - h_{\{K_i\}}|^p |Q_j| \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq c \left( \sum_{i \in I} \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |h - h_{\{K_i\}}|^p |K_i| \right)^{\frac{1}{p}}.
\]
\[
\leq c \left( \sum_{i: K_i \cap (\bigcup Q_j) \neq \emptyset} \left( \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |h - h_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}}
\]

with a constant \( c > 0 \) depending only on the dimension \( n \).

**Proof.** The arguments are different in the cases where \( p \geq 1 \) and \( p < 1 \). Suppose first that \( p \geq 1 \). If \( Q \) is an arbitrarily big cube, we have

\[
\int_Q |h - h_{\{K_i\}}| \leq \sum_{j: Q \cap K_j \neq \emptyset} \int_{K_j} |h - h_{K_j}|
\]

(A.19)

\[
\leq \sum_{j: Q \cap K_j \neq \emptyset} \left( \frac{|K_j|^{\frac{1}{p}}}{|K_j|^{1+\frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right) |K_j|^{1+\frac{s}{n} - \frac{1}{p}}
\]

\[
\leq \left( \sum_{j: Q \cap K_j \neq \emptyset} \left( \frac{|K_j|^{\frac{1}{p}}}{|K_j|^{1+\frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right)^p \right)^{\frac{1}{p}} \left( \sum_{j: Q \cap K_j \neq \emptyset} |K_j|^{(1+\frac{s}{n} - \frac{1}{p})p'} \right)^{\frac{1}{p'}}.
\]

Now, \((1 + \frac{s}{n} - \frac{1}{p})p' \geq 1\) because \( s \geq 0 \). Next, Proposition A.7 and the finite multiplicity property of the family \( \{K_i\}_{i \in I} \) show that

\[
\sum_{j: Q \cap K_j \neq \emptyset} |K_j| \leq c |Q|,
\]

implying

\[
\sum_{j: Q \cap K_j \neq \emptyset} |K_j|^{(1+\frac{s}{n} - \frac{1}{p})p'} \leq c |Q|^{(1+\frac{s}{n} - \frac{1}{p})p'}.
\]

Therefore, from (A.19) we obtain

\[
\frac{|Q|^{\frac{1}{p}}}{|Q|^{1+\frac{s}{n}}} \int_Q |h - h_{\{K_i\}}| \leq c \left( \sum_{j: Q \cap K_j \neq \emptyset} \left( \frac{|K_j|^{\frac{1}{p}}}{|K_j|^{1+\frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right)^p \right)^{\frac{1}{p}}
\]

and, consequently,

\[
\left( \frac{|Q|^{\frac{1}{p}}}{|Q|^{1+\frac{s}{n}}} \int_Q |h - h_{\{K_i\}}| \right)^p \leq c \sum_{j: Q \cap K_j \neq \emptyset} \left( \frac{|K_j|^{\frac{1}{p}}}{|K_j|^{1+\frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right)^p. \quad \text{(A.20)}
\]

Let \( \pi_b \) be a packing that consists of big cubes. By statement b) of Proposition A.7, the cube \( K_j \) may intersect only a controlled finite number of cubes in \( \pi_b \). Therefore, summation of (A.20) over cubes in \( \pi_b \) implies the required estimate.
Now, we treat the case of $p < 1$. Since $1 + \frac{s}{n} - \frac{1}{p} > 0$ and $p < 1$, statement a) of Proposition A.7 implies

$$
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1 + \frac{s}{n} - \frac{1}{p}}} \int_{Q_j} |h - h_{\{K_i\}}| \right)^p |Q_j| \right)^{\frac{1}{p}} 
\leq c \left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1 + \frac{s}{n} - \frac{1}{p}}} \sum_{i: Q_j \cap K_i \neq \emptyset} \int_{K_i} |h - h_{K_i}| \right)^p \right)^{\frac{1}{p}} 
\leq c \left( \sum_{Q_j \in \pi_b} \left( \sum_{i: Q_j \cap K_i \neq \emptyset} \frac{1}{|K_i|^{1 + \frac{s}{n} - \frac{1}{p}}} \int_{K_i} |h - h_{K_i}| \right)^p \right)^{\frac{1}{p}} 
\leq c \left( \sum_{Q_j \in \pi_b} \sum_{i: Q_j \cap K_i \neq \emptyset} \left( \frac{1}{|K_i|^{1 + \frac{s}{n} - \frac{1}{p}}} \int_{K_i} |h - h_{K_i}| \right)^p \right)^{\frac{1}{p}} .
$$

It remains to observe that, by statement b) of Proposition A.7, the number of occurrences of every cube $K_i$ on the right in this inequality is finite and is controlled in terms of the dimension $n$ only.

Lemmas A.6 and A.8 above imply the following statement.

**Proposition A.9.** If $h \in \dot{V}^{s,k}_p$ and $1 + \frac{s}{n} - \frac{1}{p} > 0$, then

$$
\max(\|h_{\{K_i\}}\|_{\dot{V}^{s,k}_p}, \|h - h_{\{K_i\}}\|_{\dot{V}^{s,k}_p}) \leq c \|h\|_{\dot{V}^{s,k}_p},
$$

where $c > 0$ depends only on the parameters $p, s, k$.

**Proof.** It suffices to show that

$$
\|h_{\{K_i\}}\|_{\dot{V}^{s,k}_p} \leq c \|h\|_{\dot{V}^{s,k}_p} .
$$

Let $\pi_s$ be a packing that consists of small cubes. Then the claim follows from Lemma A.6 and the finite multiplicity property of the family $\{K_i\}_{i \in I}$. For a packing $\pi_b$ of big cubes, we write $h_{\{K_i\}}$ as the difference of $h$ and $h - h_{\{K_i\}}$, and use the triangle inequality and Lemma A.8.

We need yet another lemma.

**Lemma A.10.** There is $c > 0$ independent of $f \in \dot{V}^{s,k}_p$ and $t > 0$ such that

$$
t \left( |\Omega_f|_{\lambda_p} \right)^{\frac{1}{p}} \leq c \dist_{\dot{V}^{s,k}_p} (f, B_{\dot{C}^{s,k}_p}(f, \frac{t}{c})).
$$
Proof. Suppose \( \{Q_j\} \) is a packing that consists of cubes in \( \Omega_{f,t} \). The definition of \( \Omega_{f,t} \) (see (A.13)) shows that

\[
\|f - f_{Q_j}\|_{\dot{V}^{s,k}_p(Q_j)} \geq t.
\]

Now, for every \( j \) the formula for the seminorm in \( \dot{V}^{s,k}_p(Q_j) \) implies the existence of a packing \( \pi_j \) composed of subcubes of \( Q_j \) such that

\[
\left( \sum_{Q \in \pi_j} \left( \frac{1}{|Q|^{1 + \frac{s}{n}}} E_k(f - f_{Q_j}, Q) \right)^p \right)^{\frac{1}{p}} \geq ct |Q_j|^\lambda,
\]

or, equivalently,

\[
\sum_{Q \in \pi_j} \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(f - f_{Q_j}, Q)^p \geq ct^p |Q_j|^\lambda p.
\]

Since the cubes \( Q_j \) are mutually disjoint, the union \( \pi = \bigcup \pi_j \) is a packing. Moreover,

\[
\sum_{Q \in \pi} \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(f - f_{Q_j}, Q)^p \geq ct^p \sum_j |Q_j|^\lambda p.
\]

Now, let \( g \in \dot{C}^{\lambda,k}(\dot{V}^{s,k}_p) \) satisfy \( \|g\|_{\dot{C}^{\lambda,k}(\dot{V}^{s,k}_p)} \leq \varepsilon t \), where \( \varepsilon > 0 \) will be chosen later. For every cube \( Q_j \), there is a polynomial \( g_{Q_j} \) of degree strictly smaller than \( k \) and such that

\[
\sum_{Q \in \pi_j} \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(g - g_{Q_j}, Q)^p \leq \varepsilon^p t^p |Q_j|^\lambda p,
\]

whence

\[
\sum_j \sum_{Q \in \pi_j} \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(g - g_{Q_j}, Q)^p \leq \varepsilon^p t^p \sum_j |Q_j|^\lambda p.
\]

The quasinorm

\[
\|h\| = \left( \sum_j \sum_{Q \in \pi_j} \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(h, Q)^p \right)^{\frac{1}{p}}
\]

obeys the triangle inequality with a constant \( c_p > 0 \). Therefore,

\[
\|f - g\|_{\dot{V}^{s,k}_p} \geq \left( \sum_{Q \in \pi} \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(f - g, Q)^p \right)^{\frac{1}{p}}
\]
\[ \geq c_p^{-1} \left( \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(f - g_{Q_j}, Q)^p \right)^{\frac{1}{p}} - \left( \frac{1}{|Q|^{(1 + \frac{s}{n})p - 1}} E_k(g - g_{Q_j}, Q)^p \right)^{\frac{1}{p}} \]

Since \( g \in B_{\dot{C}^{\lambda,k}(\dot{V}^{s,k}_p)}(\varepsilon t) \) and the packing \( \{Q_j\} \) are arbitrary, for sufficiently small \( \varepsilon > 0 \) we obtain

\[ \text{dist}_{\dot{V}^{s,k}_p}(f, B_{\dot{C}^{\lambda,k}(\dot{V}^{s,k}_p)}(\varepsilon t)) \geq \varepsilon t \left( \frac{1}{|\Omega_{f,t}|^{\lambda_p}} \right)^{\frac{1}{p}}. \]

Putting \( c = \frac{1}{\varepsilon} \), we arrive at the required statement. \( \square \)

### A.2.2 Proof of the main result (Theorem A.3)

**Proof.** First, we show that

\[ \left| \{Q_{x}\}_{x \in \Omega} \right|^{\lambda_p} \leq c \left| \Omega_{f,t} \right|^{\lambda_p}. \quad (A.21) \]

Let \( \pi = \{Q_{x}\} \) be a packing formed by cubes of the family \( \{Q_{x}\}_{x \in \Omega} \). By (A.11), for every cube \( Q_{x_j} \) there exists \( \lambda_j \in \left[ \frac{1}{2}, 1 \right] \) such that \( \lambda_j Q_{x_j} \in \Omega_{f,t} \). Since \( \lambda_j \leq 1 \), the cubes \( \{\lambda_j Q_{x_j}\} \) are mutually disjoint. Therefore,

\[ \sum_j |Q_{x_j}|^{\lambda_p} \leq 2^{n\lambda_p} \sum_j |\lambda_j Q_{x_j}|^{\lambda_p} \leq 2^{n\lambda_p} |\Omega_{f,t}|^{\lambda_p}. \]

This implies (A.21).
Looking at (A.21) and Lemma A.10, we see that the theorem will be proved if we verify the following inequalities:

\[ \| f_t \|_{C^{\lambda,k} (\dot{V}^s \dot{k}_p)} \leq ct \quad \text{and} \quad \| f - f_t \|_{\dot{V}^s \dot{k}_p} \leq ct \left( |\{ Q_x \}_{x \in \Omega} |_{\lambda p} \right)^{\frac{1}{p}}. \]  

(A.22)

We start with the second inequality. Let \( \pi \) be a packing; we split it into the families \( \pi_b \) and \( \pi_s \) of big (respectively, small) cubes. It suffices to show that

\[
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{s}{p}}} E_k(f - f_t, Q_j) \right)^p |Q_j| \right)^{\frac{1}{p}} \leq ct \left( |\{ Q_x \}_{x \in \Omega} |_{\lambda p} \right)^{\frac{1}{p}},
\]

(A.23)

\[
\left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\frac{s}{p}}} E_k(f - f_t, Q_j) \right)^p |Q_j| \right)^{\frac{1}{p}} \leq ct \left( |\{ Q_x \}_{x \in \Omega} |_{\lambda p} \right)^{\frac{1}{p}}.
\]

(A.24)

We prove (A.23). Recall that \( f_t = f_{\{ K_i \}} \). In Lemma A.8, we put \( h = f \), leaving a packing \( \pi_b \) of big cubes arbitrary. This yields

\[
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{s}{p}}} E_k(f - f_t, Q_j) \right)^p |Q_j| \right)^{\frac{1}{p}} \leq \left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{s}{p}}} \int_{Q_j} |f - f_t| \right)^p |Q_j| \right)^{\frac{1}{p}} \leq c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{s}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}}.
\]

To estimate the last quantity, we use the fact that the WB-covering \( \{ K_i \}_{i \in I} \) is controlled by some subfamily \( \{ Q_x \}_{j \in J} \) of \( \{ Q_x \}_{x \in \Omega} \), and this subfamily is of finite multiplicity, the corresponding constant being controlled in terms of the dimension.

So, for every cube \( K_i \) there is a cube \( Q_{x_j} \) in the controlling family such that

\[ K_i \subset \gamma Q_{x_j} \]

with a numerical constant \( \gamma \geq 1 \).

The definition of the quasinorm in the space \( \dot{V}^s \dot{k}_p (\gamma Q_{x_j}) \) implies

\[
\sum_{K_i \subset \gamma Q_{x_j}} \left( \frac{1}{|K_i|^{1+\frac{s}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \leq \left\| f - f_{\gamma Q_{x_j}} \right\|_{\dot{V}^s \dot{k}_p (\gamma Q_{x_j})}.\]

Furthermore, the construction of the family \( \{ Q_x \}_{x \in \Omega} \) (see (A.11)) shows that

\[
\left\| f - f_{\gamma Q_{x_j}} \right\|_{\dot{V}^s \dot{k}_p (\gamma Q_{x_j})} \leq t. \]

(A.25)
Therefore, the finite multiplicity property of the controlling family \( \{Q_{x_j}\}_{j \in J} \) implies

\[
\left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\nu}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}} \leq c \left( \sum_{j \in J} |Q_{x_j}|^{\lambda p} \right)^{\frac{1}{p}} \leq c \left( |\{Q_{x}\}_{x \in \Omega}|_{\lambda p} \right)^{\frac{1}{p}}. \tag{A.26}
\]

This proves (A.23). We pass to (A.24). By the triangle inequality, we have

\[
\left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\nu}} E_k(f - f_{t, Q_j}) \right)^p |Q_j| \right)^{\frac{1}{p}}
\]

\[
\leq c \left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\nu}} E_k(f, Q_j) \right)^p |Q_j| \right)^{\frac{1}{p}}
\]

\[
+ c \left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\nu}} E_k(f_t, Q_j) \right)^p |Q_j| \right)^{\frac{1}{p}}
\]

\[
= c \left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\nu}} \int_{Q_j} |f - f_{Q_j}| \right)^p |Q_j| \right)^{\frac{1}{p}}
\]

\[
+ c \left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\nu}} \int_{Q_j} |f_t - (f_t)_{Q_j}| \right)^p |Q_j| \right)^{\frac{1}{p}}.
\]

Since \( f_t = f_{\{K_i\}} \) (cf. (A.12) and (A.15)), Lemma A.6 shows that the second summand in the last expression is dominated by

\[
\left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\nu}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}},
\]

which has already been estimated (see (A.26)). So, it remains to handle the first summand, that is, the quantity

\[
\left( \sum_{Q_i \in \pi_s} \left( \frac{1}{|Q_i|^{1+\nu}} \int_{Q_i} |f - f_{Q_i}| \right)^p |Q_i| \right)^{\frac{1}{p}}.
\]

Since the cubes \( Q_i \) are small, for each of them there exists a cube \( Q_{x_j} \) of the controlling family \( \{Q_{x_j}\}_{j \in J} \) with \( Q_i \subset \gamma Q_{x_j} \). The definition of the quasinorm in
\(V_{p}^{s,k}(\gamma Q_{x_j})\) yields
\[
\sum_{Q_i \subset \gamma Q_{x_j}} \left( \frac{1}{|Q_i|^{1+\frac{s}{n}}} \int_{Q_i} |f - f_{Q_i}| \right)^{p} |Q_i| \leq \left\| f - f_{\gamma Q_{x_j}} \right\|_{V_{p}^{s,k}(\gamma Q_{x_j})}^{p}.
\]
So, as above, we obtain
\[
\left( \sum_{Q_i \in \pi_s} \left( \frac{1}{|Q_i|^{1+\frac{s}{n}}} \int_{Q_i} |f - f_{Q_i}| \right)^{p} |Q_i| \right)^{\frac{1}{p}} \leq c \left( \left| \{ Q_x \}_{x \in \Omega} \right| |\lambda_p \right)^{\frac{1}{p}},
\]
proving the inequality
\[
\| f - f_t \|_{V_{p}^{s,k}} \leq c \left( \left| \{ Q_x \}_{x \in \Omega} \right| |\lambda_p \right)^{\frac{1}{p}}.
\]
Next, we verify the inequality
\[
\| f_t \|_{C_{\lambda,k}(V_{p}^{s,k})} \leq c,
\]
which means that
\[
\frac{\| f_t - (f_t)_Q \|_{V_{p}^{s,k}(Q)}}{|Q|^\lambda} \leq c \tag{A.27}
\]
for every cube \(Q\).

Now, \(Q\) is either big or small. If it is small, it is included in some \(K_{i_0}\). Then for every packing \(\pi\) composed of subcubes of \(Q\), we have
\[
\left( \sum_{Q_i \in \pi} \left( \frac{1}{|Q_i|^{1+\frac{\lambda}{n}}} E_k(f_t - (f_t)_Q, Q_i) \right)^{p} |Q_i| \right)^{\frac{1}{p}} \leq \left( \sum_{Q_i \in \pi} \left( \frac{1}{|Q_i|^{1+\frac{\lambda}{n}}} \int_{Q_i} |f_t - (f_t)_Q| \right)^{p} |Q_i| \right)^{\frac{1}{p}} \leq c \left( \sum_{Q_i \in \pi} \left( \frac{1}{|Q_i|^{1+\frac{\lambda}{n}}} |Q_i|^{1+\frac{\lambda}{n}} \max_{|m|=k} \| D^{m}(f_t) \|_{L^{\infty}(Q_i)} \right)^{p} |Q_i| \right)^{\frac{1}{p}} \leq c \left( \sum_{Q_i \in \pi} \left( \frac{1}{|Q_i|^{1+\frac{\lambda}{n}}} \cdot \max_{|m|=k} \| D^{m}(f_t) \|_{L^{\infty}(K_{i_0})} \right)^{p} \right)^{\frac{1}{p}} \leq c \left( \sum_{Q_i \in \pi} |Q_i|^{1+\frac{\lambda}{n}} \cdot \max_{|m|=k} \| D^{m}(f_t) \|_{L^{\infty}(K_{i_0})} \right)^{\frac{1}{p}} \leq c \left( \sum_{Q_i \in \pi} |Q_i|^{1+\frac{\lambda}{n}} \cdot \max_{|m|=k} \| D^{m}(f_t) \|_{L^{\infty}(K_{i_0})} \right)^{\frac{1}{p}}
\]
\[
\leq c |Q|^{-\frac{k-s}{p}} \max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})}.
\]

To estimate the last quantity, we observe that the formula for \( f_t \), the strong engagement property of the family \( \{K_i\}_{i \in I} \), and an estimate of the remainder term in the Taylor formula imply

\[
\max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})} \leq c \sum_{K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{k-s}{p}}} \int_{K_i} |f - f_{K_i}|.
\]

Furthermore, the definition of the quasinorm \( \|f - f_{K_i}\|_{V^p_{\lambda,k}(K_i)} \) shows that

\[
\frac{1}{|K_i|^{1+\frac{k-s}{p}}} \int_{K_i} |f - f_{K_i}| \leq \|f - f_{K_i}\|_{V^p_{\lambda,k}(K_i)},
\]

and

\[
\|f - f_{K_i}\|_{V^p_{\lambda,k}(K_i)} \leq t |K_i|^{\lambda}
\]

by the controlled extension theorem. Thus, the last quantity in (A.28) is dominated by

\[
c |Q|^{-\frac{k-s}{p}} \max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})} \leq c |Q|^{-\frac{k-s}{p}} \sum_{K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{k-s}{p}}} |K_i|^{1+\frac{\lambda}{p}} t.
\]

By assumption, \( \frac{1}{p} + \frac{k-s}{n} \geq \lambda \). Since the family \( \{K_i\}_{i \in I} \) is strongly engaged and has finite multiplicity, we arrive at the required inequality

\[
c |Q|^{-\frac{k-s}{p}} \sum_{K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{k-s}{p}}} |K_i|^{1+\frac{\lambda}{p}} t \leq c |Q|^{-\frac{k-s}{p} - \frac{\lambda}{p}} |Q|^{\lambda} t \leq c |Q|^{\lambda} t.
\]

It remains to prove (A.27) in the case where \( Q \) is big. Let \( \pi = \{Q_j\} \) be a packing composed of subcubes of \( Q \). Splitting it into two subpackings \( \pi_b \) and \( \pi_s \) that consist, respectively, of big and small cubes, we see that it suffices to establish the following inequalities:

\[
\left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\frac{k-s}{n}}} \int_{Q_j} |f_t - (f_t)_{Q_j}| \right)^{\frac{p}{p}} |Q_j| \right)^{\frac{1}{p}} \leq c |Q|^{\lambda} t, \tag{A.29}
\]

\[
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{k-s}{n}}} \int_{Q_j} |f_t - (f_t)_{Q_j}| \right)^{\frac{p}{p}} |Q_j| \right)^{\frac{1}{p}} \leq c |Q|^{\lambda} t. \tag{A.30}
\]

We verify (A.29). Lemma A.6 and Proposition A.7 show that

\[
\left( \sum_{Q_j \in \pi_s} \left( \frac{1}{|Q_j|^{1+\frac{k-s}{n}}} \int_{Q_j} |f_t - (f_t)_{Q_j}| \right)^{\frac{p}{p}} |Q_j| \right)^{\frac{1}{p}} \leq c |Q|^{\lambda} t.
\]

\[
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{k-s}{n}}} \int_{Q_j} |f_t - (f_t)_{Q_j}| \right)^{\frac{p}{p}} |Q_j| \right)^{\frac{1}{p}} \leq c |Q|^{\lambda} t.
\]
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\[
\leq c \left( \sum_{K_i \cap Q \neq \emptyset} \left( \frac{1}{|K_i|^{1+\frac{1}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}} \leq c \|f - f_{\gamma Q}\|_{\dot{V}_{\mu,k}^{s,h}(\gamma Q)} \leq c |Q|^\lambda.
\]

The last inequality is a consequence of the controlled extension theorem. To prove (A.30), we write

\[
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{1}{p}}} \int_{Q_j} |f(t) - (f(t))_{Q_j}| \right)^p |Q_j| \right)^{\frac{1}{p}} \leq \left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{1}{p}}} \int_{Q_j} |f(t)| \right)^p |Q_j| \right)^{\frac{1}{p}} \leq \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{1}{p}}} \int_{Q_j} |f(t)| \right)^p |Q_j| \leq \|f - f_{\gamma Q}\|_{\dot{V}_{\mu,k}^{s,h}(\gamma Q)} \leq c |Q|^\lambda.
\]

By Lemma A.8, the first summand in the last expression is controlled by

\[
\left( \sum_{i: K_i \cap (\cup Q_j) \neq \emptyset} \left( \frac{1}{|K_i|^{1+\frac{1}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}} \leq \left( \sum_{i: K_i \subset \gamma Q} \left( \frac{1}{|K_i|^{1+\frac{1}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}} .
\]

Now, Proposition A.7 shows that if $K_i \cap Q \neq \emptyset$, then $K_i \subset \gamma Q$. Thus, since the cube $\gamma Q$ is big, we obtain

\[
\left( \sum_{i: K_i \cap Q \neq \emptyset} \left( \frac{1}{|K_i|^{1+\frac{1}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \right)^{\frac{1}{p}} \leq \sum_{i: K_i \subset \gamma Q} \left( \frac{1}{|K_i|^{1+\frac{1}{p}}} \int_{K_i} |f - f_{K_i}| \right)^p |K_i| \leq \|f - f_{\gamma Q}\|_{\dot{V}_{\mu,k}^{s,h}(\gamma Q)} \leq c |Q|^\lambda.
\]

To estimate the second summand, we use the assumption that $Q$ is big. Therefore,

\[
\left( \sum_{Q_j \in \pi_b} \left( \frac{1}{|Q_j|^{1+\frac{1}{p}}} \int_{Q_j} |f(t) - (f(t))_{Q_j}| \right)^p |Q_j| \right)^{\frac{1}{p}} \leq \|f - f_{\gamma Q}\|_{\dot{V}_{\mu,k}^{s,h}(Q)} \leq c |Q|^\lambda. \quad \square
\]
A.3 Near-minimizers for Morrey spaces built on the basis of Triebel–Lizorkin spaces

The seminorm in the Triebel–Lizorkin space $\dot{F}_{s,k}^{p,q}$ (see [T])\(^2\) is given by the formula

$$
\|f\|_{\dot{F}_{s,k}^{p,q}} = \left\| \left( \sum_{i \in \mathbb{Z}} \left( \frac{1}{|Q(x,2^i)|^{1 + \frac{s}{n}}} E_k(f,Q(x,2^i)) \right)^q \right)^{\frac{1}{q}} \right\|_{L_p}
$$

(A.31)

$$
k > s > 0, \quad 0 < p < \infty, \quad 1 + \frac{s}{n} - \frac{1}{p} > 0;
$$

(A.32)

the localization (denoted by $\dot{F}_{s,k}^{p,q}(Q)$) of this space to a cube $Q$ is defined by formula (A.31) with an additional restriction; namely, the cubes $Q(x,2^i)$ must be included in $Q$.

We introduce a Morrey-type seminorm on the basis of $\dot{F}_{s,k}^{p,q}$ by

$$
\|f\|_{\dot{C}^{\lambda,k}(\dot{F}_{s,k}^{p,q})} = \sup_{Q} \frac{1}{|Q|^{\lambda}} \inf_{P \in P_k} \|f - P\|_{\dot{F}_{s,k}^{p,q}(Q)}, \lambda > 0.
$$

(A.33)

We are interested in construction of a near-minimizer for the couple $(\dot{F}_{s,k}^{p,q}, \dot{C}^{\lambda,k}(\dot{F}_{s,k}^{p,q}))$.

It will turn out that the general algorithm described at the beginning of this chapter can be used for that. In the case in question, this algorithm looks like this.

We take $\dot{F}_{s,k}^{p,q}$ for the role of $X$. Suppose we are given a function $f \in \dot{F}_{s,k}^{p,q}$ and a number $t > 0$. For arbitrary $x \in \mathbb{R}^n$, we consider the function

$$
\varphi_x(r) = \frac{1}{|Q(x,r)|^{\lambda}} \|f - f_{Q(x,r)}\|_{\dot{F}_{s,k}^{p,q}(Q(x,r))},
$$

which tends to zero as $r \to \infty$, because $\lambda > 0$. Also, we consider the set

$$
\Omega = \left\{ x \in \mathbb{R}^n : \sup_{r > 0} \varphi_x(r) > t \right\}.
$$

For $x \in \Omega$, we choose $r_x > 0$ in such a way that

$$
\sup_{r \geq r_x} \varphi_x(r) \leq t \quad \text{and} \quad \sup_{r \geq \frac{1}{2} r_x} \varphi_x(r) > t.
$$

(A.34)

Putting

$$
Q_x = Q(x, r_x),
$$

\(^2\)This seminorm is not quite standard, see Notes and remarks at the end of this appendix.
we obtain a family $\{Q_x\}_{x \in \Omega}$.

Now, by (A.34) we obtain

$$\|f\|_{\dot{F}^{s,k}_{p,q}} \geq \|f\|_{\dot{F}^{s,k}_{p,q}(Q_x)} \geq ct |Q_x|^\lambda$$  \hspace{1cm} (A.35)

and, since $\lambda > 0$, we see that

$$\sup_{x \in \Omega} |Q_x| < \infty.$$  

So, the controlled extension theorem is applicable, yielding a WB-covering $\{K_i\}_{i \in I}$. We define a function $f_t$ by the formula

$$f_t = \sum_{i \in I} f_{K_i} \psi_i + f \chi_{\mathbb{R}^n \setminus \cup K_i},$$  \hspace{1cm} (A.36)

where $\{\psi_i\}_{i \in I}$ is a smooth partition of unity that is adjusted to the WB-covering $\{K_i\}_{i \in I}$, and the $f_{K_i} \in \mathcal{P}_k$ are polynomials of the best approximation of $f$ on the cubes $K_i$, $i \in I$.

**Proposition A.11.** We have

$$|\{Q_x\}_{x \in \Omega}|_{\lambda p} \leq c \frac{\|f\|^p_{\dot{F}^{s,k}_{p,q}}}{t^p} < \infty.$$  \hspace{1cm} (A.37)

**Proof.** Taking into account the fact that the cubes $Q_{x_j}$ are mutually disjoint, for every $x \in \mathbb{R}^n$ we obtain:

$$\left(\sum_{i \in I} \left(\frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} E_k(f, Q(x,2^i))\right)^q \right)^{\frac{1}{q}} \geq \sum_{j \in J} \left(\sum_{i: Q(x,2^i) \subset Q_{x_j}} \left(\frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} E_k(f, Q(x,2^i))\right)^q \right)^{\frac{1}{q}}.$$  

The definition of the seminorm in $\dot{F}^{s,k}_{p,q}$ and (A.35) show that

$$\|f\|^p_{\dot{F}^{s,k}_{p,q}} \geq \sum_{Q_{x_j} \in \pi} \|f\|_{\dot{F}^{s,k}_{p,q}(Q_{x_j})} \geq c \sum_{Q_{x_j} \in \pi} t^p |Q_{x_j}|^{\lambda p},$$

and this is (A.37). \hfill \Box

In order to state the result, we need the following family of cubes:

$$\Omega_{f,t} = \left\{ \frac{1}{|Q|^\lambda} \inf_{P \in \mathcal{P}_k} \|f - P\|_{\dot{F}^{s,k}_{p,q}(Q)} \geq t \right\}.$$
Theorem A.12. Suppose we are given a function \( f \in \dot{F}^{s,k}_{p,q} \) and a number \( t > 0 \). If 
\( 1 \leq q, s > 0, \frac{1}{p} + \frac{k-s}{n} \geq \lambda > 0, \) and (A.32) is fulfilled, then 
\[
\dist_{\dot{F}^{s,k}_{p,q}}(f, B_{\dot{C}^{\lambda,k}_{\dot{F}^{s,k}_{p,q}}}(t)) \approx t \left( |\Omega_{f,t}|_{\lambda p} \right)^{\frac{1}{p}}
\]
and the function \( f_t \) (see (A.36)) is a near-minimizer for the distance functional of 
the couple \((\dot{F}^{s,k}_{p,q}, \dot{C}^{\lambda,k}_{\dot{F}^{s,k}_{p,q}}))\).

A.3.1 Auxiliary lemmas

Below, we denote by \( f_Q \) the polynomial of degree strictly smaller then \( k \) and such 
that 
\[
E_k(f, Q) = \| f - f_Q \|_{\dot{F}^{s,k}_{p,q}(Q)} = \inf_{P \in P_k} \| f - P \|_{\dot{F}^{s,k}_{p,q}(Q)}.
\]
We prove the following analog of Lemma A.10.

Lemma A.13. For some \( c > 0 \) independent of \( f \in \dot{F}^{s,k}_{p,q} \) and \( t > 0 \), we have 
\[
t \left( |\Omega_{f,t}|_{\lambda p} \right)^{\frac{1}{p}} \leq c \dist_{\dot{F}^{s,k}_{p,q}}(f, B_{\dot{C}^{\lambda,k}_{\dot{F}^{s,k}_{p,q}}}(t)) \left( \frac{t}{c} \right).
\]

Proof. Let \( \pi = \{Q_j\} \) be a packing consisting of cubes of the family \( \Omega_{f,t} \). The 
definition of this family shows that 
\[
\| f - f_{Q_j} \|_{\dot{F}^{s,k}_{p,q}(Q_j)} \geq t.
\]
Therefore, the formula for the seminorm in the space \( \dot{F}^{s,k}_{p,q}(Q_j) \) implies 
\[
\| f - f_{Q_j} \|_{\dot{F}^{s,k}_{p,q}(Q_j)}^p = \int_{\mathbb{R}^n} \left( \sum_{i : Q(x, 2^i) \subset Q_j} \left( \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n}}} E_k(f - f_{Q_j}, Q(x, 2^i)) \right)^q \right)^{\frac{p}{q}} \geq t^p |Q_j|^{\lambda p}.
\]
Now, let \( g \in \dot{C}^{\lambda,k}_{\dot{F}^{s,k}_{p,q}} \), and let 
\[
\| g \|_{\dot{C}^{\lambda,k}_{\dot{F}^{s,k}_{p,q}}} \leq \varepsilon t,
\]
where \( \varepsilon > 0 \) will be chosen later. For every cube \( Q_j \) there exists a polynomial \( g_{Q_j} \) of degree strictly smaller than \( k \) such that 
\[
\| g - g_{Q_j} \|_{\dot{F}^{s,k}_{p,q}(Q_j)}^p = \int_{\mathbb{R}^n} \left( \sum_{i : Q(x, 2^i) \subset Q_j} \left( \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n}}} E_k(g - g_{Q_j}, Q(x, 2^i)) \right)^q \right)^{\frac{p}{q}} \leq \varepsilon^p t^p |Q_j|^{\lambda p},
\]
whence
\[
\sum_{Q_j \in \pi} \int_{\mathbb{R}^n} \left( \sum_{i \in \{Q(x, 2^i) \}} \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n} E_k(g - g_{Q_j}, Q(x, 2^i))}} \right)^q 
\leq \varepsilon^p t^p \sum_j |Q_j|^\lambda^p.
\]

Next, since \( f_{Q_j} \) is a best approximation polynomial for \( f \) on \( Q_j \), we have
\[
\| f - f_{Q_j} \|_{\dot{F}_{p, q}^{s, k}(Q_j)}^p \leq \| f - g_{Q_j} \|_{\dot{F}_{p, q}^{s, k}(Q_j)}^p.
\]

Therefore,
\[
\int_{\mathbb{R}^n} \left( \sum_{i \in \{Q(x, 2^i) \}} \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n} E_k(f - g_{Q_j}, Q(x, 2^i))}} \right)^q \leq t |Q_j|^\lambda^p.
\]

Since the cubes of the family \( \{Q_j\} \) are mutually disjoint, we obtain
\[
\sum_{Q_j \in \pi} \int_{\mathbb{R}^n} \left( \sum_{i \in \{Q(x, 2^i) \}} \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n} E_k(f - g_{Q_j}, Q(x, 2^i))}} \right)^q \geq t^p \sum_j |Q_j|^\lambda^p.
\]

Next,
\[
\| f - g \|_{\dot{F}_{p, q}^{s, k}} \geq \left( \int_{\mathbb{R}^n} \left( \sum_{i \in \{Q(x, 2^i) \}} \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n} E_k(f - g, Q(x, 2^i))}} \right)^q \right)^{\frac{1}{q}}
\]

(A.38)
\[
= \left( \sum_{j \in J} \int_{\mathbb{R}^n} \left( \sum_{i \in \{Q(x, 2^i) \} \subset Q_j} \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n} E_k(f - g, Q(x, 2^i))}} \right)^q \right)^{\frac{1}{q}}.
\]

We estimate the last quantity from below with the help of the triangle inequality for the seminorm
\[
\| h \| = \left\| \sum_{j \in J} \int_{\mathbb{R}^n} \left( \sum_{i \in \{Q(x, 2^i) \} \subset Q_j} \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n} E_k(h, Q(x, 2^i))}} \right)^q \right\|_{L^p}.
\]

This leads to the following minorant:
\[ c_p^{-1} \left( \sum_{j \in J} \int_{\mathbb{R}^n} \left( \sum_{i:Q(x,2^i) \subset Q_j} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{q}}} E_k(f-g_{Q_j}, Q(x,2^i)) \langle h - h_{\{K_i\}} \rangle \right)^q \right)^{\frac{1}{q}} \right)^{1/p} \]

\[ - \left( \sum_{j \in J} \int_{\mathbb{R}^n} \left( \sum_{i:Q(x,2^i) \subset Q_j} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{q}}} E_k(g-g_{Q_j}, Q(x,2^i)) \langle h - h_{\{K_i\}} \rangle \right)^q \right)^{\frac{1}{q}} \right)^{1/p} \]

\[ \geq c_p^{-1} \left( \varepsilon t \left( \sum_{j} |Q_j|^{\lambda_p} \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \]

Since \( g \in B_{C^{\lambda,k} \left( \tilde{F}_{p,q}^k \right)}(\varepsilon t) \) and the packing \( \{Q_j\} \) are arbitrary, taking \( \varepsilon > 0 \) sufficiently small (specifically, \( \varepsilon \leq \frac{1}{2} c_p^{-1} \) suffices) we arrive at the inequality

\[ \text{dist}_{\tilde{F}_{p,q}^k} \left( f, B_{C^{\lambda,k} \left( \tilde{F}_{p,q}^k \right)}(\varepsilon t) \right) \geq \varepsilon t \left( |\Omega_{f,t}|^{\lambda_p} \right)^{\frac{1}{p}}. \]

The claim follows if we put \( c = \frac{1}{\varepsilon}. \)

Consider an arbitrary WB-covering \( \{K_i\}_{i \in I} \). Let \( \{\psi_i\}_{i \in I} \) be a smooth partition of unity adjusted to this covering. Taking a function \( h \in \tilde{F}_{p,q}^k \), we decompose it as follows:

\[ h = h_{\{K_i\}} + (h - h_{\{K_i\}}) \]

where

\[ h_{\{K_i\}} = \sum_{i \in I} h_{K_i} \psi_i + h \chi_{\mathbb{R}^n \setminus \cup K_i} \]

(A.39)

and the \( h_{K_i} \) are polynomials of degree strictly smaller than \( k \) that provide the best approximation to \( h \) on the cubes \( K_i, i \in I \), in the metric of \( \tilde{F}_{p,q}^k(K_i) \).

We prove the following analog of Lemma A.8.

\textbf{Lemma A.14.} If \( 1 \leq q \) and \( s > 0 \), then

\[ \left\| \left( \sum_{i:Q(x,2^i) \text{ is big}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{s}}} \int_{Q(x,2^i)} |h - h_{\{K_i\}}|^q \right) \right)^{\frac{1}{q}} \right\|_{L^p} \]

\[ \leq c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{n}{s}}} \int_{K_i} |h - h_{K_i}|^p \right)^{\frac{1}{p}} \right). \]

(A.40)

\textbf{Proof.} For arbitrary cube \( Q \) we have

\[ \int_{Q} |h - h_{\{K_i\}}| \leq \sum_{j:Q \cap K_j \neq \emptyset} \int_{K_j} |h - h_{K_j}|, \]

where \( c \) is a constant depending on \( A, k, p, q, s \).
so that the left-hand side of (A.40) is dominated by the quantity

\[
\left\| \left( \sum_{Q(x,2^i) \text{ is big}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{q}}} \sum_{j:Q(x,2^i) \cap K_j \neq \emptyset} \int_{K_j} |h-h_{K_j}| \right) \right) \right\|_{L^p}^{\frac{q}{q}}.
\]

We denote by \(Q(x,2^{i(j)})\) the smallest of the big cubes of the form \(Q(x,2^i)\) that intersect \(K_j\). Since \(q \geq 1\), we obtain

\[
\left\| \sum_{Q(x,2^i) \text{ is big}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{q}}} \sum_{j:Q(x,2^i) \cap K_j \neq \emptyset} \int_{K_j} |h-h_{K_j}| \right) \right\|_{L^p}
\leq c \left\| \sum_{j} \frac{1}{|Q(x,2^{i(j)})|^{1+\frac{n}{q}}} \int_{K_j} |h-h_{K_j}| \right\|_{L^p}.
\]

Since the cube \(Q(x,2^{i(j)})\) is big, Proposition A.7 (see (A.17)) shows that

\[
K_j \subset \gamma Q(x,2^{i(j)})
\]

for some \(\gamma\) depending only on the dimension \(n\). Consequently (this is an important point of the proof), we have the following estimate in terms of the Hardy–Littlewood maximal function:

\[
\frac{|K_j|}{|Q(x,2^{i(j)})|} = \gamma^n \int_{\gamma Q(x,2^{i(j)})} \chi_{K_j} \, dx \leq c M(\chi_{K_j})(x).
\]

Therefore, the expression to be estimated can be rewritten in a form convenient for application of the Fefferman–Stein maximal theorem (Theorem 0.23):

\[
\left\| \sum_{j} \frac{1}{|Q(x,2^{i(j)})|^{1+\frac{n}{q}}} \int_{K_j} |h-h_{K_j}| \right\|_{L^p}
= \left\| \sum_{j} \frac{|K_j|^{1+\frac{n}{q}}}{|Q(x,2^{i(j)})|^{1+\frac{n}{q}}} \cdot \frac{1}{|K_j|^{1+\frac{n}{q}}} \int_{K_j} |h-h_{K_j}| \right\|_{L^p}
\leq c \left\| \sum_{j} (M\chi_{K_j})^{1+\frac{n}{q}} \frac{1}{|K_j|^{1+\frac{n}{q}}} \int_{K_j} |h-h_{K_j}| \right\|_{L^p}.
\tag{A.41}
\]
A.3. Morrey spaces built on Triebel–Lizorkin spaces

\[ = c \left( \left\| \left( \sum_j \left( M \left[ \frac{1}{|K_j|} \left( \int_{K_j} |h - h_{K_j}| \right)^{\frac{1}{1 + \frac{s}{n}}} \chi_{K_j} \right) \right] \right)^{1 + \frac{s}{n}} \right\|_{L^p(1 + \frac{s}{n})} \right)^{1 + \frac{s}{n}}. \]

By (A.32), we have \( p(1 + \frac{s}{n}) > 1 \). Therefore, the Fefferman–Stein maximal theorem (Theorem 0.23) is applicable, yielding

\[ \left\| \sum_j \frac{1}{|Q(x, 2^{i(j)})|^{1 + \frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right\|_{L^p} \leq c \left( \left\| \left( \sum_j \left( \frac{1}{|K_j|} \left( \int_{K_j} |h - h_{K_j}| \right)^{\frac{1}{1 + \frac{s}{n}}} \chi_{K_j} \right) \right)^{1 + \frac{s}{n}} \right\|_{L^p(1 + \frac{s}{n})} \right)^{1 + \frac{s}{n}}. \]

The family \( \{K_i\} \) being of finite multiplicity, we see that

\[ \left\| \sum_j \frac{1}{|Q(x, 2^{i(j)})|^{1 + \frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right\|_{L^p} \leq c \left\| \sum_j \frac{1}{|K_j|^{1 + \frac{s}{n}}} \left( \int_{K_j} |h - h_{K_j}| \right) \chi_{K_j} \right\|_{L^p}. \]

Again by finite multiplicity, we estimate the right-hand side by the quantity

\[ c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1 + \frac{s}{n}}} \int_{K_j} |h - h_{K_j}| \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}. \]

The same argument yields the following statement, to be used below.

**Proposition A.15.** If a cube \( Q \) is big, then

\[ \left\| \left( \sum_{i: Q(x, 2^i) \text{ is big, } Q(x, 2^i) \subset Q} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} \int_{Q(x, 2^i)} |h - h_{\{K_i\}}| \right)^q \right) \right\|_{L^p} \leq c \left( \sum_{i: K \cap \bar{Q} \neq \emptyset} \left( \frac{1}{|K_i|^{1 + \frac{s}{n}}} \int_{K_i} |h - h_{K_i}| \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}. \]

We have also an analog of Lemma A.6.
Lemma A.16. The following inequality is true:

\[
\left\| \sum_{i:Q(x,2^i)\text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \int_{Q(x,2^i)} \left| h_{\{K_i\}} - (h_{\{K_i\}})_Q(x,2^i) \right| \right)^q \right\|_{L^p} \leq c \sum_{i:K_i \cap K_j \neq \emptyset} \left( \frac{1}{|K_i|^{1+s}} \int_{K_i} \left| h - h_{K_i} \right| \right)^{\frac{1}{q}}.
\]

Proof. As in the proof of Lemma A.6, for every small cube \( Q(x,2^i) \) we can write

\[
\frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \int_{Q(x,2^i)} \left| h_{\{K_i\}} - (h_{\{K_i\}})_Q(x,2^i) \right| \leq c \left| Q(x,2^i) \right|^{\frac{k-s}{n}} \max_{|m| = k} \left\| D^m (h_{\{K_i\}}) \right\|_{L^\infty(Q(x,2^i))}.
\]

We denote by \( i_x \) the largest \( i \) such that the cube \( Q(x,2^i) \) is small, and by \( K_{j_x} \) a cube in the family \( \{K_i\} \) with \( Q(x,2^{i_x}) \subset K_{j_x} \).

Since \( k - s > 0 \) (see the definition (A.31) of the Triebel–Lizorkin spaces), we obtain

\[
\left( \sum_{i \leq i_x} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \int_{Q(x,2^i)} \left| h_{\{K_i\}} - (h_{\{K_i\}})_Q(x,2^i) \right| \right)^q \right)^{\frac{1}{q}} \leq c \left( \sum_{i \leq i_x} \left| Q(x,2^i) \right|^{\frac{k-s}{n}} \max_{|m| = k} \left\| D^m (h_{\{K_i\}}) \right\|_{L^\infty(Q(x,2^i))} \right)^{\frac{1}{q}} \leq c \left| K_{j_x} \right|^{\frac{k-s}{n}} \max_{|m| = k} \left\| D^m (h_{\{K_i\}}) \right\|_{L^\infty(K_{j_x})}.
\]

As above (see (A.16)), we establish the inequality

\[
\left\| D^m (h_{\{K_i\}}) \right\|_{L^\infty(K_{j_x})} \leq c \sum_{i:K_i \cap K_{j_x} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} \left| h - h_{K_i} \right|.
\]

Therefore, the strong engagement property of the family \( \{K_i\} \) implies

\[
\left( \sum_{i \leq i_x} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \int_{Q(x,2^i)} \left| h_{\{K_i\}} - (h_{\{K_i\}})_Q(x,2^i) \right| \right)^q \right)^{\frac{1}{q}} \leq c \sum_{i:K_i \cap K_{j_x} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} \left| h - h_{K_i} \right|.
\]
Invoking the finite multiplicity property of the family \( \{ K_i \} \) once again, we arrive at the required estimate

\[
\left\| \left( \frac{1}{|K_i|^{1+\frac{s}{n}}} \sum_{K_i \cap K_j \neq \emptyset} \int_{K_i} |h - h_{K_j}| \chi_{K_j} \right) \right\|_{L^p} \leq c \left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |h - h_{K_j}|^p \right)^\frac{1}{p} \right).
\]

**Proposition A.17.** Under the assumptions of Theorem A.12, we have

\[
\max(\|f_{\{K_i\}}\|_{\dot{F}^s,k_{p,q}}, \|f - f_{\{K_i\}}\|_{\dot{F}^s,k_{p,q}}) \leq c \|f\|_{\dot{F}^s,k_{p,q}},
\]

where \( c > 0 \) depends only on the dimension and the parameters \( p, q, s, k \).

**Proof.** It suffices to show that

\[
\|f_{\{K_i\}}\|_{\dot{F}^s,k_{p,q}} \leq c \|f\|_{\dot{F}^s,k_{p,q}}.
\]

As in the proof of Proposition A.9, the matter reduces to the estimate

\[
\left( \sum_{i \in I} \left( \frac{1}{|K_i|^{1+\frac{s}{n}}} E_k(f, K_i) \right)^p |K_i|^{\frac{1}{p}} \right) \leq c \|f\|_{\dot{F}^s,k_{p,q}}.
\]

This inequality is proved as follows. Since the family \( \{ K_i \} \) is of finite multiplicity, we may assume that it consists of mutually disjoint cubes. Now, we observe that for \( x \in K_i \) we have

\[
\frac{1}{|K_i|^{1+\frac{s}{n}}} E_k(f, K_i) \leq c \left( \sum_{i \in Z} \left( \frac{1}{|Q(x, 2^i)|^{1+\frac{s}{n}}} E_k(f, Q(x, 2^i)) \right)^q \right)^\frac{1}{q}.
\]

Taking the \( L^p \)-norm of both sides, we arrive at \( \text{(A.42)} \). \( \square \)
A.3.2 Proof of the main result (Theorem A.12)

Largely, the arguments repeat those in the proof of Theorem A.3. For the reader’s convenience, we present the details.

Proof. First, we show that

$$\left|\{Q_x\}_{x \in \Omega}^{\lambda_p}\right| \leq c \left|\Omega_{f,t}^{\lambda_p}\right|.$$  \hspace{1cm} (A.44)

Indeed, consider a packing $\pi = \{Q_{x_j}\}$ composed of cubes of the family $\{Q_x\}_{x \in \Omega}$. By (A.34), for every $Q_{x_j}$ there exists $\lambda_j \in \left[\frac{1}{2}, 1\right]$ with $\lambda_j Q_{x_j} \in \Omega_{f,t}$. As in the proof of Theorem A.3, this implies (A.44).

Now, (A.44) and Lemma A.13 show that the theorem will be proved if we establish the inequalities

$$\|f_t\|_{C^{\lambda,k}(\tilde{F}_{s,k}^{p,q})} \leq c t$$ and $$\|f - f_t\|_{\tilde{F}_{s,k}^{p,q}} \leq c t \left(\left|\{Q_x\}_{x \in \Omega}^{\lambda_p}\right|^{\frac{1}{p}}\right).$$  \hspace{1cm} (A.45)

We start with the second. By Lemma A.14 with $h = f$, we have

$$\left\|\sum_{i:Q(x,2^i)\text{is big}} \left(\frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \int_{Q(x,2^i)} |f - f_t|^{q}\right)^{\frac{1}{q}}\right\|_{L^p} \leq c \sum_{i \in I} \left(\frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |f - f_{K_i}|^p \right)^{\frac{1}{p}}.$$ \hspace{1cm} (A.46)

Then the invocation of a controlling family $\pi = \{Q_{x_j}\}$ of cubes leads to the following (in the same way as in Theorem A.3):

$$\left(\sum_{\delta K_i \subset \gamma Q_{x_j}} \left(\frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |f - f_{K_i}|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq c \left|Q_{x_j}\right|^\lambda.$$ Therefore,

$$\left(\sum_{i \in I} \left(\frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |f - f_{K_i}|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq c \left(\sum_{j \in J} \left|Q_{x_j}\right|^\lambda\right)^{\frac{1}{p}} \leq c \left(\left|\{Q_x\}_{x \in \Omega}^{\lambda_p}\right|^{\frac{1}{p}}\right).$$

The corresponding estimate over small cubes involves the inequality

$$\left\|\sum_{i:Q(x,2^i)\text{ is small}} \left(\frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} E_k(f - f_t, Q(x,2^i))^{-\frac{1}{q}}\right)^{\frac{1}{q}}\right\|_{L^p}$$
A.3. Morrey spaces built on Triebel–Lizorkin spaces

\[ \leq c \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} E_k(f, Q(x,2^i)) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \]

\[ + c \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} E_k(f_t, Q(x,2^i)) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \]

\[ \leq c \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} \int_{Q(x,2^i)} |f - f_{Q(x,2^i)}| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \]

\[ + c \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} \int_{Q(x,2^i)} |f_t - (f_t)_{Q(x,2^i)}| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} . \]

With the help of Lemma A.16 with \( h = f \), the second term in the last expression is dominated by the right-hand side of (A.46), which has already been estimated. So, it remains to control the quantity

\[ \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} \int_{Q(x,2^i)} |f - f_{Q(x,2^i)}| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} . \]

Consider a family \( \{Q_{x,j}\}_{j \in J} \) that controls the family \( \{K_i\}_{i \in I} \). In relation to the former family, the cubes \( Q(x,2^i) \) will be split as follows. For each \( x \in \bigcup K_i \), find \( i = i(x) \) with the property that \( Q(x,2^i) \) is still small, but \( Q(x,2^{i+1}) \) is big. Then \( Q(x,2^i) \) is included in \( \gamma Q_{x,j} \), where \( Q_{x,j} \) is a cube of the controlling family. Then

\[ \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} \int_{Q(x,2^i)} |f - f_{Q(x,2^i)}| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \]

\[ \leq \left\| f - f_{\gamma Q_{x,j}} \right\|_{L^p} \leq c t^p |Q_{x,j}|^{\lambda_p} \]

and therefore

\[ \left\| \left( \sum_{i:Q(x,2^i) \text{ is small}} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{n}{2}}} \int_{Q(x,2^i)} |f - f_{Q(x,2^i)}| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \]

\[ \leq c t^p \sum_j |Q_{x,j}|^{\lambda_p} \leq c t^p \left\| \{Q_x\}_{x \in \Omega} \right\|_{\lambda_p} . \]
This proves the inequality
\[ \|f - f_t\|_{\dot{F}^s_{p,q}} \leq ct \left( |\{Q_x\}_{x \in \Omega}|_{\lambda_p} \right)^{\frac{1}{p}}. \]

Now, we verify the inequality
\[ \|f_t\|_{C^\lambda,k(\dot{F}^s_{p,q})} \leq ct, \]
which means that
\[ \frac{\|f_t - (f_t)Q\|_{\dot{F}^s_{p,q}(Q)}}{|Q|^{\lambda}} \leq ct. \]
for every cube \( Q \).

Two cases are possible: either \( Q \) is small or it is big. In the former case, \( Q \) is a subset of some \( K_{i_0} \).

By assumption, \( \frac{1}{p} + \frac{k-s}{n} \geq \lambda. \) Since
\[ \max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})} \leq c \sum_{K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{s}{n}}} \int_{K_i} |f - f_{K_i}| \]
and
\[ \frac{1}{|K_i|^{1+\frac{s}{n} - \frac{1}{p}}} \int_{K_i} |f - f_{K_i}| \leq |K_i|^{\lambda} t, \]
we obtain
\[
\|f_t - (f_t)Q\|_{\dot{F}^s_{p,q}(Q)} = \left\| \left( \sum_{Q(x,2^i) \subset Q} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} E_k(f_t, Q(x,2^i)) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
\leq c \left\| \left( \sum_{Q(x,2^i) \subset Q} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} \max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})} \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
\leq c |Q|^{\frac{1}{p} + \frac{k-s}{n} - \lambda} \max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})} \\
\leq c |Q|^{\frac{1}{p} + \frac{k-s}{n} - \lambda} \sum_{K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{s}{n}}} |K_i|^{\lambda} t \\
\leq c \frac{|Q|^{\frac{1}{p} + \frac{k-s}{n} - \lambda}}{|K_{i_0}|^{\frac{1}{p} + \frac{k-s}{n} - \lambda}} |Q|^{\lambda} t \leq c |Q|^{\lambda} t. \]

Now, let \( Q \) be big. Then
\[
\|f_t - (f_t)Q\|_{\dot{F}^s_{p,q}(Q)} = \left\| \left( \sum_{Q(x,2^i) \subset Q} \left( \frac{1}{|Q(x,2^i)|^{1+\frac{s}{n}}} E_k(f_t, Q(x,2^i)) \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
\leq c |Q|^{\frac{1}{p} + \frac{k-s}{n} - \lambda} \max_{|m|=k} \|D^m(f_t)\|_{L^\infty(K_{i_0})} \\
\leq c |Q|^{\frac{1}{p} + \frac{k-s}{n} - \lambda} \sum_{K_i \cap K_{i_0} \neq \emptyset} \frac{1}{|K_i|^{1+\frac{s}{n}}} |K_i|^{\lambda} t \\
\leq c \frac{|Q|^{\frac{1}{p} + \frac{k-s}{n} - \lambda}}{|K_{i_0}|^{\frac{1}{p} + \frac{k-s}{n} - \lambda}} |Q|^{\lambda} t \leq c |Q|^{\lambda} t. \]

\[
\begin{align*}
&\leq c \left\| \sum_{Q(x, 2^i) \subset Q, Q(x, 2^i) \text{ is small}} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} E_k(f_t, Q(x, 2^i)) \right)^q \right\|_{L^p}^{\frac{1}{q}} \\
&+ c \left\| \sum_{Q(x, 2^i) \subset Q, Q(x, 2^i) \text{ is big}} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} E_k(f_t, Q(x, 2^i)) \right)^q \right\|_{L^p}^{\frac{1}{q}}.
\end{align*}
\]

With the help of Lemma A.16, the first term on the right is dominated by the right-hand side of (A.46), which has already been estimated. The second term is treated as follows:

\[
\begin{align*}
&\leq c \left\| \sum_{Q(x, 2^i) \subset Q, Q(x, 2^i) \text{ is big}} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} E_k(f_t - f, Q(x, 2^i)) \right)^q \right\|_{L^p}^{\frac{1}{q}} \\
&+ c \left\| \sum_{Q(x, 2^i) \subset Q, Q(x, 2^i) \text{ is big}} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} E_k(f, Q(x, 2^i)) \right)^q \right\|_{L^p}^{\frac{1}{q}}.
\end{align*}
\]

The first norm on the right is estimated with the help of Propositions A.15 and A.7, and the observation that the cube $\gamma Q$ is big for $\gamma \geq 1$:

\[
\begin{align*}
&\leq c \left\| \sum_{Q(x, 2^i) \subset Q, Q(x, 2^i) \text{ is big}} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} E_k(f_t - f, Q(x, 2^i)) \right)^q \right\|_{L^p}^{\frac{1}{q}} \\
&\leq \left\| \sum_{Q(x, 2^i) \subset Q, Q(x, 2^i) \text{ is big}} \left( \frac{1}{|Q(x, 2^i)|^{1 + \frac{s}{n}}} \int_{Q(x, 2^i)} |f - f_t| \right)^q \right\|_{L^p}^{\frac{1}{q}} \\
&\leq c \left( \sum_{i: K_i \cap Q \neq \emptyset} \left( \frac{1}{|K_i|^{1 + \frac{s}{n}}} \int_{K_i} |f - f_{K_i}| \right)^p \right)^{\frac{1}{p}} \\
&\leq c \left( \sum_{i: K_i \subset \gamma Q} \left( \frac{1}{|K_i|^{1 + \frac{s}{n}}} \int_{K_i} |f - f_{K_i}| \right)^p \right)^{\frac{1}{p}} \leq c t |Q|^{\lambda}.
\end{align*}
\]

The second term on the right in (A.47) is controlled directly by $t |Q|^{\lambda}$ because
Chapter A. Near-minimizers for Brudnyi and Triebel–Lizorkin spaces

\[ Q \text{ is big:} \]
\[
\left\| \sum_{Q(x,2^i) \subset Q, \text{Q(x,2^i) is big}} \left( \frac{1}{|Q(x,2^i)|^{1 + \frac{s}{n}}} E_k(f, Q(x, 2^i)) \right)^{\frac{q}{q}} \right\|_{L^p} \leq c \lambda |Q|^\lambda. \]

Notes and remarks

This Appendix is a revision of an unpublished paper by Kruglyak and Kuznetsov. That work was highly influenced by discussions between Kruglyak and Lars-Inge Hedberg in the Fall of 2005 in Oberwolfach, Germany.

Generalized variation spaces were introduced by Brudnyi in [Br0]; see also the survey [Br1]. The seminorm on the Triebel–Lizorkin space employed here is not the usual seminorm (see [T]) but is equivalent to it (see §2.3.3 and especially Theorem (ii) in [T]); a similar seminorm was invoked in [HN].
Bibliography


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SYMBOLS

(X₀, X₁)θ,q, interpolation spaces of the real method, 125
D, the set of dyadic cubes, 13
Dkf, partial derivative, 6
Hp, Hardy space, 153
Hp(w), weighted Hardy space, 154
Hp,(C₊), Hardy space, 154
Hp(w), weighted Hardy space, 156
H₁, 103
L₁(Ω), 15
L₁,∞(E), weak L₁, 15
Lp,q, Lorentz space, ix
L loc(Ω), 15
Mk,p,α(f), 92
Pf,t, 27
Q, a cube with edges parallel to coordinate axes, 13
Q(x, r), the cube centered at x and of radius r, 13, 161
Qx, a cube centered at x, 65, 161
X(Q), localization of the space X to a cube Q, 273
BV, the space of functions of bounded variation, 244
BVq, the space of functions of bounded q-variation, 268
Δh, kth difference, 95
Lip, 38
̂Cp,k, 92
Cp,k, Morrey–Campanato space, 6, 15, 36
̂Cp,k, 95
Fp,q,k, Triebel–Lizorkin space, 276, 291
Vp,k, a generalized variation space, 244, 247
Vp,k, a generalized variation space, 276
Wq, Sobolev space, 4, 15, 92
Wq,(Ω), Sobolev space, 91
|Ω|, the volume of Ω, 13
{Qx}x∈Ω, 65
ln(T), 29
C₊, upper half-plane, 154
Rn, Euclidean space, 13
Z, the set of integers, 13
D(Rn), 48
Pk, 15
S(Rn), 59
πb, a packing of big cubes, 279
πs, a packing of small cubes, 279
f#, 125
r(Q), the radius of a cube Q, 13
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