Appendix A
Proofs of Uncountability of the Reals

In this appendix, we summarize and review the proofs of uncountability of the reals given in the main text, and indicate how the methods of these proofs generalize and connect to other areas of mathematics. (This appendix is not an exhaustive list of such proofs.)

There were essentially three distinct proofs of uncountability of the reals given in the text. All proofs depend, in the end, on some form of order completeness of \( \mathbb{R} \), but they take very different forms and generalize in different ways to give other significant results in mathematics.

A.1 Order-Theoretic Proofs

Section 8.5 presented a proof of uncountability of the reals which follows immediately from Cantor’s powerful theorem characterizing the order type \( \eta \) (which says any countable dense order without endpoints has order type \( \eta \)). That theorem also implies \( \eta + \eta = \eta \), and so any countable dense order must have Dedekind gaps. Hence any dense linear order without Dedekind gaps, such as \( \mathbb{R} \), must be uncountable.

This proof is so short because it exploits a very powerful result of order theory. It is related to Cantor’s first proof of uncountability of \( \mathbb{R} \), which directly shows that a countable dense order cannot be complete:

Proof (Cantor’s first proof of uncountability of \( \mathbb{R} \)). To get a contradiction, suppose that the set of real numbers can be enumerated as \( p_1, p_2, \ldots \) (without repetition). Recursively define two sequences of reals \( \langle a_n \rangle \) and \( \langle b_n \rangle \) with

\[
a_1 < a_2 < \cdots < a_n < \cdots < b_n < \cdots < b_2 < b_1,
\]

in the following manner. Let \( a_1 = p_1 \), and \( b_1 = p_m \) where \( m \) is the least index such that \( a_1 < p_m \). Having defined \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) with \( a_n < b_n \), define...
\[ a_{n+1} = p_j \] where \( j \) is the least index such that \( a_n < p_j < b_n \) and \( b_{n+1} = p_k \) where \( k \) is the least index such that \( a_{n+1} < p_k < b_n \). Then we have \( a_n < a_{n+1} < b_{n+1} < b_n \), and the recursive definition is complete. In particular, for each \( n \) we have \( a_n = p_{j_n} \) and \( b_n = p_{k_n} \) for some indices \( j_n \) and \( k_n \). Now, by completeness of \( \mathbb{R} \), there must be a real number \( p \) such that \( a_n < p < b_n \) for all \( n \), and so \( p = p_i \) for some \( i \). Since the indices \( j_n \) are all distinct, we can fix \( n \) with \( j_{n+1} > i \). Note that \( a_n < p_i < b_n \) and by definition of \( a_{n+1} = p_{j_{n+1}} \), we see that \( j_{n+1} \) equals the least index \( j \) such that \( p_j \) lies between \( a_n \) and \( b_n \), and so \( j_{n+1} \leq i \), a contradiction. \( \square \)

This was Cantor’s first published proof of the uncountability of \( \mathbb{R} \). Given any enumeration of a countable dense order, it effectively produces a gap in it.

Both the proof of Sect. 8.5 based on Cantor’s theorem characterizing the order type \( \eta \) and Cantor’s first proof given above appeal to order completeness, but note that full completeness is not necessary. For both proofs, it suffices to assume that there are no \( (\omega, *\omega) \) gap in the ordering.

**Proposition 1323.** A dense order without \( (\omega, *\omega) \) gaps has cardinality \( > \aleph_0 \).

In this form, the proof generalizes to \( \eta_1 \) orders without \( (\omega_1, *\omega_1) \) gaps:

**Proposition 1324.** Any \( \eta_1 \) order without \( (\omega_1, *\omega_1) \) gaps has cardinality \( > \aleph_1 \).

**Proof.** Recall that any two \( \eta_1 \) orders of cardinality \( \aleph_1 \) must be isomorphic to each other. If there were an \( \eta_1 \) order \( X \) of cardinality \( \aleph_1 \) without \( (\omega_1, *\omega_1) \) gaps, then any suborder \( Y \) of \( X \) obtained by removing a single point of \( X \) would also be an \( \eta_1 \) order of cardinality \( \aleph_1 \) and so must be isomorphic to \( X \). But \( Y \) has a \( (\omega_1, *\omega_1) \) gap, and so \( X \) has such a gap, a contradiction. \( \square \)

Another related generalization is this: Any dense-in-itself complete order contains an isomorphic copy of the real line and so has cardinality \( \geq \mathfrak{c} \).

**Connected spaces and their uncountability.** As mentioned in the text, the notion of connectedness in topology is a direct generalization of Dedekind’s definition of linear continuum: An order is a continuum if and only if in any Dedekind partition of the order at least one of the sets contains a point which is a limit point of the other. A metric or topological space is connected if and only if for any partition of the space into two nonempty sets, at least one set contains a limit point of the other. Under certain regularity conditions, the uncountability of linear continua carries over to connected spaces. To see this, note that the Intermediate Value Theorem generalizes: The range of any continuous function from a connected space to an order must be a linear continuum. Since the distance function on a metric space is continuous, any connected metric space with at least two points is uncountable.\(^1\)

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\(^1\)By a basic topological result known as Urysohn’s Lemma, this generalizes to any \( T_4 \) (normal Hausdorff) topological space, and in fact to any \( T_3 \) (regular Hausdorff) space: Any connected \( T_3 \) space with at least two points must be uncountable. All these generalizations are thus related to the order-based proof of uncountability of \( \mathbb{R} \).
A.2 Proof Using Cantor’s Diagonal Method

Cantor discovered his “diagonal method” for proving uncountability several years after he obtained his order-based proof given above where he first discovered that $\mathbb{R}$ is uncountable. Unlike the order-theoretic proofs, the diagonal method is applicable in much more general situations where no order may be present.

In a sense, diagonalization means that given an infinite list of conditions, we construct a “counterexample” real number which refutes all those conditions. The nested intervals theorem gives a direct version of this form of diagonalization: Given a sequence of reals $(x_1, x_2, \ldots)$, one builds nested closed intervals of shrinking length $I_1 \supseteq I_2 \supseteq \cdots$ such that $x_1 \notin I_1$ (“$I_1$ avoids $x_1$”), $x_2 \notin I_2$, and so on. The unique real $x$ in their intersection then differs from all the given reals $x_1, x_2, \ldots$. Here the $n$-th given condition is “$x = x_n$,” and the above method of diagonalization via nested intervals produces the real $x$ which satisfies $x \neq x_n$ for all $n$. Therefore, we call $x$ the diagonal counterexample for the given sequence $(x_1, x_2, \ldots)$ of reals.

In this proof, we could, for definiteness, use the specific scheme for building nested closed intervals where the initial interval $I_0$ is the unit interval $I_0 = [0, 1]$, and each $I_n$ is either the left-third or the right-third subinterval of $I_{n-1}$ (whichever avoids the real $x_n$ first). The diagonal counterexample will then always be a member of the Cantor set, and conversely, any member of the Cantor set can be seen to be a diagonal counterexample for a suitably given sequence of reals $(x_1, x_2, \ldots)$. It follows that with this scheme of building nested intervals, the Cantor set is the set of all possible diagonal counterexamples to various given sequences of real numbers.

With a little modification, the above proof of uncountability of $\mathbb{R}$ yields the Baire Category Theorem, where the $n$-th condition to be met is to be inside an arbitrary given dense open set $G_n$ (instead of the special dense open set of the form $\{x \mid x \neq x_n\}$). The Baire category theorem holds in complete metric spaces as well as in locally compact Hausdorff spaces, and thus any such space without isolated points must be uncountable (and in fact of cardinality at least $\mathfrak{c}$). This illustrates how Cantor’s diagonal method leads to a powerful general theorem of very wide applicability.

In a more literal form of diagonalization we regard a family $\langle E_i \mid i \in E \rangle$ of subsets of a set $E$ indexed by $E$ itself as the following relation on $E$:

$$\{(i, j) \in E \times E \mid j \in E_i\},$$

(or, using the identification via characteristic functions, as a binary array $\langle a_{i,j} \mid i, j \in E \rangle$ where each $a_{i,j}$ is 0 or 1). We then form the diagonal set $D := \{i \in E \mid i \in E_i\}$, and finally take its complement to get the “anti-diagonal” set $A := E \setminus D = \{i \in E \mid i \notin E_i\}$, which must differ from all the sets $E_i$. In other words, it shows that $\mathcal{P}(E)$ cannot be listed as a family of sets indexed by $E$. This is Cantor’s theorem that $|E| < |\mathcal{P}(E)|$, another far reaching generalization (of the uncountability of $\mathbb{R}$) which ensures existence of sets of arbitrarily large infinite cardinality.
This last version is a more abstract form of diagonalization which is usually referred to as the Cantor diagonal method.

The Cantor set establishes a close connection between these two forms of the diagonal method: It is constructed by a “binary tree of nested intervals” in which infinite branches (of nested intervals) through the tree correspond, on the one hand, to the points of the Cantor set, and, on the other hand, to infinite binary sequences, i.e., to members of \( \{0, 1\}^\mathbb{N} \) or to subsets of \( \mathbb{N} \).

One thus obtains a variant of the diagonal proof of uncountability of \( \mathbb{R} \) by identifying the Cantor set with \( \mathcal{P}(\mathbb{N}) \) (or with \( \{0, 1\}^\mathbb{N} \)) and then appealing to the abstract Cantor diagonal theorem that \( |\mathcal{P}(\mathbb{N})| > |\mathbb{N}| \).

The more abstract version of the Cantor diagonal method has quite wide ramifications. It not only gives (via Cantor’s theorem that \( |\mathcal{P}(X)| > |X| \)) sets of larger and larger infinite cardinalities by iterating the power set operation, but also is a method used in the proofs of many important theorems of logic and computability, such as Gödel’s incompleteness theorem, the unsolvability of the Halting problem, and Tarski’s undefinability theorem.

### A.3 Proof Using Borel’s Theorem on Interval Lengths

In Corollary 1018 it was shown that the interval \([a, b]\) is uncountable using properties of lengths of intervals. The length of a bounded interval in \( \mathbb{R} \) is defined by

\[
\text{len}([a, b]) = \text{len}((a, b)) = \text{len}([a, b)) = \text{len}((a, b)) = b - a \quad (a \leq b).
\]

The length function thus defined on the intervals has several natural properties (which are essential in obtaining the Lebesgue measure on \( \mathbb{R} \)). For example, the lengths of intervals are easily seen to satisfy the condition of finite additivity, which says that if an interval \( I \) is partitioned into finitely many pairwise disjoint intervals \( I_1, I_2, \ldots, I_n \), then

\[
\text{len}(I) = \text{len}(I_1) + \text{len}(I_2) + \cdots + \text{len}(I_n).
\]

However, the key fact about lengths of intervals used in the uncountability proof mentioned above was Borel’s theorem, which says that the interval \([a, b]\), which has length \( b - a \), cannot be covered by countably many intervals of smaller total length. This important condition is known as countable subadditivity of length, which was established (in Borel’s theorem) using the powerful Heine–Borel theorem. Since any countable set of reals can be covered by countably many intervals having arbitrarily small total length, countable subadditivity immediately implies that a proper interval must be uncountable.

The proof also readily generalizes to more abstract setups as follows. Let \( X \) be a fixed set. A nonempty collection \( S \) of subsets of \( X \) is called a semiring on \( X \) if for
any $A, B \in S$ the intersection $A \cap B$ is in $S$ and the difference $A \setminus B$ can be expressed as the union of finitely many pairwise disjoint sets from $S$. By a set-function on a semiring $S$ we mean a function $\mu$ defined on $S$ which takes nonnegative extended real values (i.e., we allow $\mu(A)$ to be $+\infty$). A set-function $\mu$ on a semiring $S$ on $X$ is said to be continuous if for every $p \in X$ and every $\epsilon > 0$ there is a set $E \in S$ with $p \in E$ and $\mu(E) < \epsilon$, and $\mu$ is said to be countably subadditive on $S$ if whenever $E \in S$ is covered by countably many sets $E_1, E_2, \cdots \in S$, we have $\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$. Essentially the same proof that a countable set has measure zero now immediately gives:

**Proposition 1325.** Suppose that $\mu$ is a nonnegative continuous set function on a semiring $S$ of subsets of a fixed set $X$. If $\mu$ is countably subadditive on $S$, then $E$ is uncountable for any $E \in S$ for which $\mu(E) \neq 0$.

Countable subadditivity is necessary here. For example, let $X$ be the set $Q$ of rational numbers. By a rational half-open interval we mean a set of the form $[a, b) \cap Q$ with $a, b \in Q$. The set of half-open rational intervals forms a semiring on $Q$ on which the length function (defined as before) is continuous and finitely additive. But countable subadditivity fails and every rational interval is countable.

We conclude by noting that under finite additivity, the condition of countable subadditivity (as in Borel’s theorem) actually entails a much stronger and important result known as the measure extension theorem, whose proof can be found in any standard textbook of measure theory. By a measure we mean a nonnegative extended real valued set-function $\nu$ defined on a sigma-algebra which vanishes on the empty set ($\nu(\emptyset) = 0$) and which satisfies the condition that if $\langle A_n \rangle$ is a pairwise disjoint sequence of sets from the sigma-algebra then $\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$ (countable additivity).

**Theorem 1326 (The Measure Extension Theorem).** Let $\mu$ be a finitely additive nonnegative extended real valued set-function on a semiring $S$ of subsets of a fixed set $X$. Assume that $X = \bigcup_n A_n$ for some sets $A_n \in S$ with $\mu(A_n) < \infty$ for all $n$. If $\mu$ is countably subadditive on $S$, then there is a unique measure defined on the sigma-algebra generated by $S$ which extends $\mu$.

Taking $S$ to be the semiring of all real intervals of the form $[a, b)$ and $\mu$ to be the length function on such intervals, we get the following immediate corollary of the theorem: There is a unique measure defined on the Borel subsets of $\mathbb{R}$ for which the measure of any interval is its length. This measure is known as the Lebesgue measure, and it also uniquely extends as a measure to the collection of all Lebesgue measurable sets (the sigma-algebra generated by the Borel sets together with the measure zero sets).
Appendix B
Existence of Lebesgue Measure

This appendix gives a proof of the existence of Lebesgue measure. That is, we prove
Theorem 1028 whose statement is as below. Recall that $E \in L$, or $E$ is measurable,
if for all $\epsilon > 0$ there exist closed $F$ and open $G$ with $F \subseteq E \subseteq G$ and intervals
$I_1, I_2, \ldots$ covering $G \setminus F$ with $\sum_n \text{len}(I_n) < \epsilon$.

Theorem (Lebesgue). There is $m: L \rightarrow [0, \infty]$ such that
1. $m$ is countably additive: If $A_1, A_2, \ldots$ are pairwise disjoint measurable sets, then
   $m(\bigcup_n A_n) = \sum_n m(A_n)$.
2. $m(I) = \text{len}(I)$ for any interval $I$ (thus $m(\emptyset) = 0$).

To prove the theorem, we first define the outer measure $m^*(E)$ of any set $E \subseteq R$
(not necessarily measurable), and then restrict $m^*$ to $L$ to get $m$.

Definition 1327 (Outer Measure). For any $E \subseteq R$, we define:

$$m^*(E) := \inf \{\sum_{n=1}^\infty \text{len}(I_n) | \{I_n\} \text{ is a sequence of intervals covering } E\}.$$ 

$m$ is $m^*$ restricted to $L$, so if $E \in L$, then $m^*(E)$ is denoted by $m(E)$.

Recall Borel’s theorem (Theorem 1011) which says $\text{len}(I) \leq m^*(I)$ for any interval $I$. The following facts are now immediate.

Problem 1328 (Monotonicity). If $A \subseteq B$ then $m^*(A) \leq m^*(B)$.

Proposition 1329. For any interval $I$, $m^*(I) = \text{len}(I)$.

Proof. $m^*(I) \leq \text{len}(I)$ is trivial and Borel’s theorem says $m^*(I) \geq \text{len}(I)$. □

Proposition 1330 (Countable Subadditivity of Outer Measure). For any sequence $E_1, E_2, \ldots$ of sets, $m^*(\bigcup_n E_n) \leq \sum_n m^*(E_n)$.

Proof. Given $\epsilon > 0$, choose, for each $n$, a sequence of intervals $\{I_{n,k} | k \in \mathbb{N}\}$
covering $E_n$ and with $\sum_k \text{len}(I_{n,k}) \leq m^*(E_n) + \frac{\epsilon}{2^n}$. Combining all these sequences
of intervals into a single sequence, we get a covering of \( \bigcup_n E_n \) with total length
\[
\sum_n (m^*(E_n) + \varepsilon / 2^n) = \sum_n m^*(E_n) + \varepsilon.
\]
So we can (and will) prove equalities of the form \( m^*\left( \bigcup_n E_n \right) = \sum_n m^*(E_n) \) by only showing \( m^*\left( \bigcup_n E_n \right) \geq \sum_n m^*(E_n) \) (by countable subadditivity).

**Corollary 1331.** If \( A \) is measurable and \( \varepsilon > 0 \) then there are closed \( F \) and open \( G \) such that \( F \subseteq A \subseteq G \), \( m^*(A) \geq m^*(G) - \varepsilon \), and \( m^*(F) \geq m^*(A) - \varepsilon \).

**Proof.** Let \( \varepsilon > 0 \). Fix closed \( F \) and open \( G \) such that \( F \subseteq A \subseteq G \) and \( m^*(G \setminus F) < \varepsilon \). Then by countable subadditivity and monotonicity, \( m^*(G) \leq m^*(A) + m^*(G \setminus A) \leq m^*(A) + m^*(G \setminus F) \leq m^*(A) + \varepsilon \), so \( m^*(A) \geq m^*(G) - \varepsilon \). Similarly \( m^*(F) \geq m^*(A) - \varepsilon \). \qed

**Proposition 1332.** Let \( G \) be an open set expressed as a disjoint union of open intervals \( \bigcup_n J_n = G \). Then \( m^*(G) = \sum_n \text{len}(J_n) \).

**Proof.** E easily \( m^*(G) \leq \sum_n \text{len}(J_n) \) (since the \( J_n \)'s cover \( G \)).

For the other direction, let \( \{I_n\} \) be any sequence of open intervals covering \( G \). Then for each \( n \), \( \{I_n \cap J_m \mid m \in \mathbb{N}\} \) is a sequence of pairwise disjoint intervals all contained in \( I_n \) and so \( \text{len}(I_n) \geq \sum_m \text{len}(I_n \cap J_m) \). Hence
\[
\sum_n \text{len}(I_n) \geq \sum_n \sum_m \text{len}(I_n \cap J_m) = \sum_m \sum_n \text{len}(I_n \cap J_m) \geq \sum_m \text{len}(J_m),
\]
where the last inequality follows by Borel's theorem since for each \( m \), the intervals \( \{I_n \cap J_m \mid n \in \mathbb{N}\} \) cover \( J_m \). \qed

**Corollary 1333.** If \( G_1 \) and \( G_2 \) are disjoint open sets then \( m^*(G_1 \cup G_2) = m^*(G_1) + m^*(G_2) \).

**Proposition 1334.** If \( F_1 \) and \( F_2 \) are disjoint closed sets then \( m^*(F_1 \cup F_2) = m^*(F_1) + m^*(F_2) \).

**Proof.** Let \( \varepsilon > 0 \). Fix open \( G \) with \( F_1 \cup F_2 \subseteq G \) and \( m^*(F_1 \cup F_2) \geq m^*(G) - \varepsilon \). Fix disjoint open \( G_1 \) and \( G_2 \) containing \( F_1 \) and \( F_2 \) respectively (Problem 938). Then we have \( m^*(F_1 \cup F_2) \geq m^*(G) - \varepsilon \geq m^*(\bigcup (G \cap G_1) \cup (G \cap G_2)) - \varepsilon = m^*(G \cap G_1) + m^*(G \cap G_2) - \varepsilon \geq m^*(F_1) + m^*(F_2) - \varepsilon \). \qed

**Proposition 1335 (Finite Additivity).** Let \( A \) and \( B \) be disjoint measurable sets. Then \( m^*(A \cup B) = m^*(A) + m^*(B) \).

**Proof.** Let \( \varepsilon > 0 \). Fix closed sets \( F_A \subseteq A \) and \( F_B \subseteq B \) with \( m^*(F_A) \geq m^*(A) - \frac{\varepsilon}{2} \) and \( m^*(F_B) \geq m^*(B) - \frac{\varepsilon}{2} \). Then \( m^*(A \cup B) \geq m^*(F_A \cup F_B) = m^*(F_A) + m^*(F_B) \geq m^*(A) + m^*(B) \). \qed

**Proposition 1336 (Countable Additivity of Lebesgue Measure).** If \( A_1, A_2, \ldots \) are disjoint measurable sets then \( m^*(\bigcup_n A_n) = \sum_n m^*(A_n) \).

**Proof.** \( \sum_n m^*(A_n) = \sup_n \sum_k m^*(A_k) = \sup_n m^*(\bigcup_{k=1}^n A_k) \leq m^*(\bigcup_n A_n) \). \qed

The main theorem now follows from and Propositions 1329 and 1336.
Appendix C
List of ZF Axioms

ZF 1 (Extensionality). \( \forall x \forall y (\forall z (z \in x \iff z \in y) \rightarrow x = y) \).

ZF 2 (Empty Set). \( \exists x \forall y (y \notin x) \).

ZF 3 (Separation Scheme). If \( \varphi(x, t_1, t_2, \ldots, t_n) \) is a ZF formula in which the free variables are among \( x, t_1, t_2, \ldots, t_n \), then the following is an axiom:

\[
\forall t_1 \forall t_2 \ldots \forall t_n \forall a \exists b \forall x (x \in b \iff x \in a \land \varphi(x, t_1, t_2, \ldots, t_n)) \).
\]

ZF 4 (Power Set). \( \forall x \exists y \forall z (z \in y \iff \forall w (w \in z \rightarrow w \in x)) \).

ZF 5 (Union). \( \forall x \exists y \forall z (z \in y \iff \exists w (w \in x \land z \in w)) \).

ZF 6 (Unordered Pairs). \( \forall x \forall y \exists z \forall w (w \in z \iff w = x \lor w = y) \).

ZF 7 (Replacement Scheme). If \( \varphi(x, y, t_1, t_2, \ldots, t_n) \) is a ZF formula with free variables among the ones shown, then we have the axiom:

\[
\forall t_1 \forall t_2 \ldots \forall t_n (\forall x \forall y \exists z (\varphi(x, y, t_1, \ldots, t_n) \land \varphi(x, z, t_1, \ldots, t_n) \rightarrow y = z)
\rightarrow \forall a \exists b \forall u \forall v (u \in a \land \varphi(u, v, t_1, \ldots, t_n) \rightarrow v \in b)) \).
\]

ZF 8 (Infinity). \( \exists b (\exists y (y \in b \land \forall z (z \notin y)) \land
\forall x (x \in b \rightarrow \exists y (y \in b \land \forall z (z \in y \iff z \in x \lor z = x))) \).

ZF 9 (Foundation). \( \forall x (\exists y (y \in x) \rightarrow \exists y (y \in x \land \neg \exists z (z \in y \land z \in x))) \).

ZFC is obtained by adding to ZF the Axiom of Choice, which says:

\[
\forall x ((\forall y (y \in x \rightarrow \exists z (z \in y)) \land
\forall u \forall v (u \in x \land v \in x \land u \neq v \rightarrow \neg \exists y (y \in u \land y \in v))
\rightarrow \exists w \forall y (y \in x \rightarrow \exists ! z (z \in y \land z \in w))) \).
\]

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**Online Reference**

List of Symbols and Notations

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