Appendix A
Conventions and Some Properties of Vector Spaces

A.1 Notation

A group of relations with a single equation number (***)) will be numbered by counting "=" signs. Thus, (***)_5 refers to the relation with the fifth "=" sign. Minor exceptions will be clear from context.

Vectors and tensors are denoted by boldface characters respectively and scalars by ordinary script. The real line is denoted by $\mathbb{R}$, the nonnegative reals by $\mathbb{R}^+$, and the strictly positive reals by $\mathbb{R}^{++}$. Also, $\mathbb{R}^-$ denotes the nonpositive reals and $\mathbb{R}^{--}$ the strictly negative reals. We will be dealing with spaces of scalar quantities with values in $\mathbb{R}$ or $\mathbb{R}^+$, vector quantities in $\mathbb{R}^3$, second-order tensors and symmetric second-order tensors, the latter space being denoted by Sym. Let one of these spaces, or a composite of more than one, be denoted by $V$. The space of linear transformations $V \mapsto V$ is denoted by $\text{Lin}(V)$. If $V$ is omitted, it is understood to be $\mathbb{R}^3$; also $\text{Lin}^+$ denotes the set of linear transformations with positive determinant from $\mathbb{R}^3$ to $\mathbb{R}^3$.

The space of linear transformations $V_1 \mapsto V_2$ is $\text{Lin}(V_1, V_2)$.

We will be considering frequency-space quantities, defined by analytic continuation from integral definitions, as functions on the complex plane $\Omega$, where

$$\Omega^+ = \{ \omega \in \Omega \mid \text{Im}\omega \in \mathbb{R}^+\},$$

$$\Omega^{++} = \{ \omega \in \Omega \mid \text{Im}\omega \in \mathbb{R}^{++}\}.$$ 

Similarly, $\Omega^-$ and $\Omega^{(-)}$ are the lower half-planes including and excluding the real axis, respectively.

In certain contexts, for example where the complex plane is not necessarily related to frequency, we use the above convention but with $\mathbb{C}$ replacing $\Omega$. The latter symbol is in some chapters used to denote the spatial region occupied by the body under discussion, for consistency with usage in the literature.
A.2 Finite-Dimensional Vector Spaces

We deal extensively with quadratic forms in the main text, so that it is worthwhile describing certain notation and concepts relating to finite vector spaces. More in-depth discussion can be found in, for example, [133]. Consider first the case that $V$, of dimension $m$, is a vector space over the reals. For any two vectors $L, M$ in $V$, we denote their inner (scalar or dot) product by $L \cdot M$. Let $C_i, i = 1, 2, \ldots, m$, be an orthonormal basis of $V$, so that

\[ C_i \cdot C_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, m. \]

We have

\[
L = \sum_{i=1}^{m} L_i C_i, \quad M = \sum_{i=1}^{m} M_i C_i. \tag{A.2.1}
\]

Also

\[
L \cdot M = L^\top M = \sum_{i=1}^{m} L_i M_i = M \cdot L = M^\top L, \tag{A.2.2}
\]

and $|M|^2 = M \cdot M$ is the squared norm of $M$. A linear transformation $K \in \text{Lin}(V)$ has the representation (we keep the same notation)

\[
K = \sum_{i,j=1}^{m} K_{ij} C_i \otimes C_j,
\]

where $K_{ij}, i, j = 1, 2, \ldots, m$, are its components in this basis. We will generally refer to linear transformations (with well-defined behavior under a change of reference frame) as tensors and their representations as matrices. The tensor $K^\top$ is the transpose of $K$. Now

\[
(C_i \otimes C_j)C_k = C_j \cdot C_k C_i = \delta_{jk} C_i,
\]

giving

\[
(KM)_i = KM \cdot C_i = \sum_{j=1}^{m} K_{ij} M_j.
\]

We have the quadratic form

\[
L \cdot KM = \sum_{i,j=1}^{m} L_i K_{ij} M_j = \sum_{i,j=1}^{m} (K^\top)_{ji} L_i M_j = K^\top L \cdot M = M \cdot K^\top L. \tag{A.2.3}
\]

A commonly used scalar product on the vector space $\text{Lin}(V)$ is

\[
K \cdot N = \text{tr}(KN^\top) = \sum_{i,j=1}^{m} K_{ij} N_{ij}, \tag{A.2.4}
\]
and the associated squared norm $|\mathbb{K}|^2$ of $\mathbb{K} \in \text{Lin}(\mathcal{V})$ is

$$|\mathbb{K}|^2 = \text{tr}((\mathbb{K}\mathbb{K}^\top)) = \sum_{i,j=1}^{m} K_{ij}K_{ji}. \quad (A.2.5)$$

It it frequently the case in the present work that $\mathbb{K}$ is symmetric, so that $\mathbb{K} = \mathbb{K}^\top$.

If $\mathcal{V}$ is over the complex numbers—which is the case in this work when we are dealing with quantities in the frequency domain—we define the complex conjugate of $L$ by

$$\overline{L} = \sum_{i=1}^{m} \overline{L}_i C_i,$$  \quad (A.2.6)

where $\overline{L}_i$ is the ordinary complex conjugate of $L_i$. The dot product is still defined by (A.2.2). Thus, for example,

$$\overline{L} \cdot M = L^\top M = \sum_{i=1}^{m} \overline{L}_i M_i,$$

giving a real, positive squared norm

$$|L|^2 = \overline{L} \cdot L = \sum_{i=1}^{m} |L_i|^2.$$

If $\mathbb{K} \in \text{Lin}(\mathcal{V})$, now with complex components, we have

$$\overline{L} \cdot \mathbb{K} M = \mathbb{K}^* L \cdot M,$$  \quad (A.2.7)

where $\mathbb{K}^*$ is the Hermitian conjugate of $\mathbb{K}$, defined by

$$\mathbb{K}^* = \overline{\mathbb{K}}^\top,$$  \quad (A.2.8)

where the overhead bar indicates taking the complex conjugate of each element. A Hermitian tensor is one with the property

$$\mathbb{K}^* = \mathbb{K}. \quad (A.2.9)$$

Note that

$$\overline{L} \cdot \mathbb{K} L = \overline{L} \cdot \mathbb{K}^* L,$$

so that $\overline{L} \cdot \mathbb{K} L$ is real if $\mathbb{K}$ is Hermitian. We deal largely with symmetric tensors. A Hermitian symmetric tensor is defined by $\mathbb{K}^* = \overline{\mathbb{K}}$.

A natural choice of orthonormal basis is the eigenvectors of a Hermitian tensor $\mathbb{K} = \mathbb{K}^*$. Denoting these as above by $C_i$, $i = 1, 2, \ldots, m$, the quantities $C_i \otimes C_i$, $i = 1, 2, \ldots, m$, are the projectors on the eigenspaces of this tensor. The spectral form of $\mathbb{K}$ is the representation
\[ K = \sum_{i=1}^{m} \lambda_i C_i \otimes C_i, \quad (A.2.10) \]

where \( \lambda_i, i = 1, 2, \ldots, m, \) are the real eigenvalues of \( K. \) Any tensor \( N \) that can be given by
\[ N = \sum_{i=1}^{m} \mu_i C_i \otimes C_i, \quad (A.2.11) \]

where \( \mu_i, i = 1, 2, \ldots, m, \) are arbitrary complex numbers, commutes with \( K. \)

The tensor \( N \) is, in general, a normal transformation, in the sense that it commutes with its Hermitian conjugate \( N^*. \)

Note that for any \( N \) given by (A.2.11) and \( L, M \) given by (A.2.1),
\[ \overline{L} \cdot NM = \sum_{i=1}^{m} \mu_i \overline{L_i M_i}. \]

### A.2.1 Positive Definite Tensors

A Hermitian tensor \( K \) is positive definite if for every \( L \in \mathcal{V} \) (the relation for vector spaces over the reals is given in parentheses; the tensor \( K \) can be taken to be symmetric, since any antisymmetric portion does not contribute),
\[ \overline{L} \cdot KL \geq 0 \quad (L \cdot KL \geq 0), \quad (A.2.12) \]

where equality is true only if \( L = 0, \) where 0 is the zero in \( \mathcal{V} \) or \( \text{Lin}(\mathcal{V}). \) If equality occurs for \( L \neq 0, \) then \( K \) is positive semidefinite. The relation \( K > 0 \) indicates that \( K \) is positive definite, while \( K \geq 0 \) implies that it is positive semidefinite. The description nonnegative for a tensor is equivalent to positive semidefinite. Negative definiteness and semidefiniteness can be defined analogously. We have an ordering on \( \text{Lin}(\mathcal{V}) \) in that, for example,
\[ K_1 > K_2 \iff K_1 - K_2 > 0. \]

A tensor is positive definite if and only if its eigenvalues are all positive. It is positive semidefinite if all eigenvalues are nonnegative.

**Remark A.2.1.** An example of a positive semidefinite tensor that occurs in the main text is the following. Let \( L \in \mathcal{V} \) be of the form (A.2.1)_1, but with complex components, \( \overline{L} \) being given by (A.2.6). Then consider
\[ K = \overline{L} \otimes L = \sum_{i,j=1}^{m} \overline{L_i L_j} C_i \otimes C_j. \]
Clearly, $\overline{N} \cdot KN \geq 0$ for any $N \in V$, as required by (A.2.12). However, any vector $M$ perpendicular to $L$ in $V$ will obey the relation $KM = 0$, so that it is an eigenvector of $K$ with eigenvalue zero. Thus, $K$ is positive semidefinite.

**A.2.2 Differentiation with Respect to Vector Fields**

Let $\phi$ be a scalar depending on $L$. The quantity

$$\frac{\partial \phi}{\partial L} \in V$$

is a vector with components $\partial \phi / \partial L_i$, $i = 1, 2, \ldots, m$, in a given basis. We take it to be a column vector, the transpose of which is a row vector. The quantity

$$\frac{\partial L}{\partial L} \in \text{Lin}(V)$$

is a tensor with components

$$\frac{\partial L_i}{\partial L_j} = \delta_{ij}, \quad i, j = 1, 2, \ldots, m,$$

so that $\partial L / \partial L$ is the unit tensor. We have

$$\frac{\partial}{\partial L} (L \cdot KM) = KM$$

and

$$\frac{\partial}{\partial M} (L \cdot KM) = \frac{\partial}{\partial M} (K^\top L \cdot M) = K^\top L$$

for a vector space over $\mathbb{R}$. In the complex case, we have

$$\frac{\partial}{\partial L} (\overline{L} \cdot KM) = KM$$

and

$$\frac{\partial}{\partial M} (\overline{L} \cdot KM) = \frac{\partial}{\partial M} (\overline{K^\top L} \cdot M) = \overline{K^\top L}.$$

**A.2.3 The Vector Space Sym**

The space of symmetric second-order tensors acting on $\mathbb{R}^3$ is denoted by $\text{Sym} := \{E \in \text{Lin}(\mathbb{R}^3) : E = E^\top\}$. Operating on $\text{Sym}$ is the space of fourth-order tensors $\text{Lin}($Sym$)$. 

The vector space \text{Sym} is isomorphic to \( \mathbb{R}^6 \). In particular, for every \( \mathbf{E}, \mathbf{F} \in \text{Sym} \), if \( \mathbf{C}_i, i = 1, 2, \ldots, 6, \) is an orthonormal basis of \text{Sym} with respect to the inner product (A.2.4) in \text{Lin}(\mathbb{R}^3), namely \( \text{tr}(\mathbf{E}\mathbf{F}^\top) \), it is clear that the representation
\[
\mathbf{E} = \sum_{i=1}^{6} E_i \mathbf{C}_i, \quad \mathbf{F} = \sum_{i=1}^{6} F_i \mathbf{C}_i, \quad \text{(A.2.13)}
\]
yields that \( \text{tr}(\mathbf{E}\mathbf{F}^\top) = \sum_{i=1}^{6} E_i F_i \). Therefore, we can treat each tensor of \text{Sym} as a vector in \( \mathbb{R}^6 \) and denote by \( \mathbf{E} \cdot \mathbf{F} \) the inner product between elements of \text{Sym}:
\[
\mathbf{E} \cdot \mathbf{F} = \text{tr}(\mathbf{E} \mathbf{F}^\top) = \text{tr}(\mathbf{E} \mathbf{F}) = \sum_{i=1}^{6} E_i F_i \quad \text{(A.2.14)}
\]
and \( |\mathbf{F}|^2 = \mathbf{F} \cdot \mathbf{F} \). Consequently, any fourth-order tensor \( \mathbf{G} \in \text{Lin}(\text{Sym}) \) will be identified with an element of \text{Lin}(\mathbb{R}^6) \) by the representation
\[
\mathbf{G} = \sum_{i,j=1}^{6} K_{ij} \mathbf{C}_i \otimes \mathbf{C}_j, \quad \text{(A.2.15)}
\]
and \( \mathbf{G}^\top \) is the transpose of \( \mathbf{G} \) as an element of \text{Lin}(\mathbb{R}^6). The scalar product and norm in \text{Lin}(\mathbb{R}^3) \) are given by (A.2.4) and (A.2.5) for \( m = 6 \).

For complex-valued tensors, let \text{Sym}(\Omega) \) and \text{Lin}(\text{Sym}(\Omega)) \) be respectively the sets of tensors represented by the forms (A.2.13) and (A.2.15) with \( L_i, M_i, K_{ij} \in \Omega \). Then for \( \mathbf{E}, \mathbf{F} \in \text{Sym}(\Omega) \), we have, instead of (A.2.14),
\[
\mathbf{E} \cdot \overline{\mathbf{F}} = \text{tr}((\mathbf{E}^*)^\top) = \text{tr}(\mathbf{E} \overline{\mathbf{F}}) = \sum_{i=1}^{6} E_i \overline{F_i}. \quad \text{(A.2.16)}
\]

In the present work, we deal with \( \mathcal{V} = \Gamma^+ \) defined by (5.1.10), which has dimension \( m = 10, \) or vector spaces contained in \( \Gamma^+ \). The scalar product between two elements of \( \Gamma^+ \) is understood to mean the sum of (A.2.14) or (A.2.16) on \text{Sym}, the standard scalar product on \( \mathbb{R}^3 \) and the product of quantities in \( \mathbb{R} \).
Appendix B
Some Properties of Functions on the Complex Plane

B.1 Introduction

We describe briefly, for the sake of convenient reference, some properties of analytic functions that are required in various contexts, mainly in Part III. For a more complete treatment of these topics, we refer to the numerous available standard references, for example [203]. A useful now classical reference is [179].

Of all functions defined on the $xy$ plane, there is a very special class, termed analytic functions, that have the property that they are functions only of the combination $z = x + iy$ and have a uniquely defined derivative with respect to $z$ at each point in the region of analyticity. This latter requirement is very restrictive in that it means that the derivative is independent of the infinite number of directions from which the limit may be taken. If we write such a function $F(z)$ in the form

$$F(z) = F(x, y) = u(x, y) + iv(x, y),$$

then the uniqueness of the limit gives the Cauchy–Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$ 

These conditions are necessary consequences of the analyticity assumption. If the derivatives are continuous at $(x, y)$, it may also be shown that they are sufficient to ensure analyticity.

Note that the Cauchy–Riemann equations imply that if the real part of a complex function is known, its imaginary part is determined to within a constant and vice versa.
### B.1.1 Cauchy’s Theorem and Integral Formula

If \( F(z) \) is analytic on an open set \( O \), then for any contour \( C \) in \( O \), we have

\[
\oint_C F(z) \, dz = 0. \tag{B.1.1}
\]

This is a simple statement of Cauchy’s theorem. The term contour is taken to mean a closed contour. Cauchy’s integral formula states that if \( F(z) \) is analytic within and on a contour \( C \), then

\[
F(z) = \frac{1}{2\pi i} \oint_C \frac{F(z')}{z' - z} \, dz', \tag{B.1.2}
\]

if the contour is taken counterclockwise, which is the conventional positive direction, and is a manifestation of a more basic convention, namely that angles are presumed to increase in a counterclockwise direction. If \( C \) is clockwise, then the integral in (B.1.2) is equal to \(-F(z)\). Unless otherwise stated, a contour \( C \) may be taken to be counterclockwise. If \( z \) is outside of the contour \( C \), this integral gives zero. An immediate consequence of (B.1.2) is

\[
F^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{F(z')}{(z' - z)^{n+1}} \, dz', \tag{B.1.3}
\]

where \( F^{(n)}(z) \) is the \( n \)th derivative of \( F \).

Now let \( z \) approach the contour from the inside toward a point \( z_0 \) on \( C \). The integral can be assigned a particular finite value as a result of a limiting process that will now be described. We distort the contour into a small semicircle around \( z_0 \) outside of \( C \). In the limit, as this semicircle gets smaller, it can be shown that it yields a finite contribution of \( F(z_0)/2 \). We define the Cauchy principal value of the integral as the value obtained by means of this limiting process minus the contribution \( F(z_0)/2 \) of the semicircle. Therefore, for \( z \) on the contour,

\[
\frac{1}{\pi i} \text{P} \oint_C \frac{F(z')}{z' - z} \, dz' = F(z), \tag{B.1.4}
\]

where the integral is interpreted as a Cauchy principal value.

We are mainly interested in cases in which \( C \) encloses the upper or lower half-plane \( \Omega^{\pm} \). Let \( F(z) \) be analytic in the upper half-plane and let it go to zero at infinity more strongly than \( z^{-1} \), at least in this half-plane. We take \( C \) in (B.1.4) to be the real axis and the infinite semicircle enclosing the upper half-plane. This contour is counterclockwise, so we obtain from (B.1.4), for \( x \) on the real axis,

\[
\frac{1}{\pi i} \text{P} \int_{-\infty}^{\infty} \frac{F(x')}{x' - x} \, dx' = F(x), \tag{B.1.5}
\]

where the integral is a Cauchy principal value.
If $F$ were analytic in the lower half-plane, the sign on the right-hand side would be negative, since the contour is clockwise.

A basic property of analytic functions is that they can be expressed as infinite (or finite in the case of polynomials) power series about any point $z_0$ in their region of analyticity. This power series has a radius of convergence equal to the distance between $z_0$ and the nearest singular point.

### B.1.2 Analytic Continuation

In different parts of the complex plane, an analytic function may have different representations, as power series about different points, for example. A region of analyticity defined by a circle of convergence of one power series can be extended by considering a power series about another point. Given two representations, the question arises whether they are the same complex function or distinct functions. If they represent the same function, they are said to constitute analytic continuations of each other.

A fundamental result states that if a function is analytic in a region $R$ and zero along any continuous arc in $R$, then it is zero everywhere in $R$. It follows trivially that if two functions, analytic in a region $R$, are equal on a continuous arc contained in this region, they are equal over all of $R$.

More generally, let two functions be analytic on $R$ and let $O \subset R$ be a nonempty open set. If the two representations are equal on $O$, then they are the same analytic function defined on $R$.

It can be shown that if two different analytic continuations are constructed from a set $R$ into $R_1$ and $R_2$, where $R_3 = R_1 \cap R_2$ is nonempty, then the functions are equal on $R_3$. Note that $R \cap R_1, R \cap R_2$ are nonempty.

The most widespread use of this concept in the present work is a simple application of these kinds of results. If we have a formula for a function on a certain part of the complex plane, for definiteness, let us say a part or all of the real axis, given by $G(x)$, then the analytic continuation of this function into the complex plane is given by $G(z)$, for whatever values of $z$ this quantity is meaningful. If $G(x)$ is some combination of elementary functions, for example, then $G(z)$ will exist at all values of $z$ that are not singular points. So, for example, if

$$G(x) = \frac{1}{1 + x^2}, \quad x \in \mathbb{R},$$

its analytic continuation to the complex plane is

$$G(z) = \frac{1}{1 + z^2},$$

(B.1.6)

which is valid everywhere except at the singular points $z = \pm i$. Its uniqueness everywhere except at these points is guaranteed by the above results.
B.1.3 Liouville’s Theorem

A result that is fundamental to the developments of Part III is Liouville’s theorem, which we state in a somewhat generalized form.

Let a function $F(z)$ be analytic at every finite point in the complex plane and let it behave like $z^n$ as $z$ tends to infinity. Then it must be a polynomial of degree $n$. In particular, if its limit at infinity is a constant, it is equal to this constant everywhere. The most important case for our purposes is where the constant is zero, and $F(z)$ is zero everywhere.

B.1.4 Singularities

What makes analytic functions interesting are their singularities, or points where they are not analytic. In fact, the content of Liouville’s theorem is that if they have no singularities, they are trivial.

The simplest singularities are poles, that is to say, behaving at $z_0$ like $(z - z_0)^{-n}$, where $n$ is a positive integer called the order of the pole. These are isolated singularities. A function whose only singularities are poles is known as a meromorphic function. A function behaving like $z^n$ for large $|z|$, where $n$ is a positive integer, is regarded as having a pole of order $n$ at infinity.

Remark B.1.1. If a real function has simple poles on the real axis, then it must have at least one zero between each two poles, because in passing through each pole $(x - a_i)^{-1}$, moving in a positive direction, the function switches from being a large negative number to being a large positive number. Therefore, it must pass through zero on the passage to the next pole.

Poles of infinite order are referred to as essential singularities. For example, the function $\exp(1/z)$ has an essential singularity at the origin, and $\exp(iz)$ has an essential singularity at infinity. Note that on the lower half-plane $\exp(iz)$ diverges exponentially and on the upper half-plane decays exponentially. It can be treated as analytic at infinity on the latter half-plane in the sense that it is analytic at finite points and we can take infinite contours over $\Omega^{(+)}$ for integrands with $\exp(iz)$ as a factor. For simplicity, though with some imprecision, we shall refer to its behavior in this half-plane as analytic.

Functions that are analytic over the whole complex plane are said to be entire or integral functions. They must be constant everywhere or singular at infinity. If the singularity is of finite order, then they are polynomials. This is a restatement of Liouville’s theorem, as given above. Nonpolynomial entire functions must have an essential singularity at infinity. Examples are $e^{z}$, $\sin z$, and $\cos z$.

We assume, except in Chapter 14, that the functions we deal with are analytic at infinity, that is to say, behave as a constant or go to zero at large $|z|$. It is further assumed that essential singularities do not occur at finite points on $\Omega$. 
B.1.5 Branch Points

If one follows an analytic function around a contour to the initial point and it does not return to the same value, then the function is multivalued. This is associated with a branch point within the contour. A branch point is a type of singularity, distinct from a pole or an essential singularity. It is not isolated because, as we shall see below, its effects are not localized at any one point. The function \((z - a)^\gamma\) is, for noninteger values of \(\gamma\), a multivalued function that is of interest in Chapter 13. In the standard polar representation, it becomes

\[
(z - a)^\gamma = |z - a|^{\gamma} e^{i\gamma \theta}, \tag{B.1.7}
\]

where \(\theta\) is the argument of \((z - a)\). Let \(\gamma\) be a real quantity. If it is rational, let us write it as \(p/q\), where \(p, q\) have no common factors. Then if we circle the point \(a\), say \(r\) times, where \(r < q\), the function returns to different values each time. When \(r = q\), the function returns to its original value. We say that \((z - a)^\gamma\) has a branch point at \(z = a\) and has \(q\) distinct branches. If \(\gamma\) is irrational, the function has an infinite number of branches.

Branch points always occur in pairs. The function \((z - a)^\gamma\) also has a branch point at infinity, where it behaves like \(z^\gamma\). We join the point \(a\) to infinity by some convenient line and agree that the function undergoes a discontinuous jump in crossing this line. If \(a\) is real, this line is conventionally chosen to be the \(x\)-axis from the point \(a\) to \(-\infty\). This is, however, an arbitrary choice. It is perfectly possible to choose another line of discontinuity. It would not be the same function, however.

The complex plane, excluding the line of discontinuity, is sometimes referred to as the cut plane and the line itself as a branch cut or simply a cut. A multivalued function with say \(q\) distinct branches can be completely characterized by taking \(q\) cut complex planes and defining a singlevalued branch on each of them.

A process of unique analytic continuation cannot go around a branch point. The branch cut represents a barrier. Thus, if we continue \(R\) into \(R_1\) and \(R_2\) around a branch point, then we cannot form a nonempty overlapping set \(R_3 = R_1 \cap R_2\). However, it can go around an isolated singularity.

Returning to the function \(F(z) = (z - a)^\gamma\) (where \(a\) is real) with a cut along \((-\infty, a]\), let \(F^\pm(x)\) be the limiting values of \(F(z)\) from above and below the real axis, respectively. Using (B.1.7), one can show that

\[
F^-(x) = e^{-2\pi i \gamma} F^+(x).
\]

Note that this applies also if \(\gamma\) is complex, in which case there is a real as well as an imaginary exponential factor.

Another example is

\[
F(z) = (z - a)^\gamma(z - b)^{1-\gamma}, \tag{B.1.8}
\]

where \(a\) and \(b\) are real with \(b > a\). The cut for \(F(z)\) given by (B.1.8) must join \(a\) and \(b\). The simplest choice is to take the portion of the real axis \([a, b]\) as the branch
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cut. The function will in general have many branches, which we can represent as follows. Let \( \arg(z-a) = \theta_a \) and \( \arg(z-b) = \theta_b \). Then

\[
F(z) = |z-a|^\gamma|z-b|^{1-\gamma} \exp\{i[\theta_a \gamma + \theta_b (1 - \gamma) + 2\pi m \gamma + 2\pi n (1 - \gamma)]\},
\]

\( m, n \) integers.

One can show that

\[
F^-(x) = \Gamma F^+(x),
\]

where

\[
\Gamma = \begin{cases} 
1, & x \notin [a, b], \\
e^{2\pi i \gamma}, & x \in [a, b].
\end{cases}
\]

Therefore this function is analytic except in the interval \([a, b]\). It will be different for different choices of \( m \) and \( n \), though these contribute only a constant factor. Unique analytic continuation is possible around \([a, b]\), avoiding the branch cut.

Another multivalued function that has a role in our considerations is

\[
F(z) = \log(z-a) = \log|z-a| + i \arg(z-a),
\]

(B.1.9)

which has a branch point at \( a \). We take the choice \( \arg(z-a) = 0 \) if \( z-a \) is real as the standard branch of the logarithm. This vanishes if \( |z-a| \) is unity. The function \( \arg(z-a) \) has range \([-\pi, \pi]\). If \( a \in \mathbb{R} \), the branch cut is conventionally taken along \((-\infty, a]\). The function

\[
F(z) = \log\left(\frac{z-b}{z-a}\right)
\]

(B.1.10)

has a branch cut on a line joining \( a \) and \( b \), the simplest being the straight line segment \([a, b]\).

### B.1.6 Evaluation of Contour Integrals

Let \( F \) be analytic on a contour \( C \) but with singularities within \( C \). Then (B.1.1) generalizes to

\[
\int_C F(z)dz = 2\pi i \sum \text{residues within } C.
\]

The contour \( C \) can be deformed at will without changing the value of the integral, provided it does not cross any singularities. If \( F \) has a pole of order \( n \) at \( z_0 \), that is to say,

\[
F(z) \xrightarrow{z \to z_0} G(z) \frac{1}{(z-z_0)^n},
\]

where \( G \) is analytic at \( z_0 \), then the residue of that pole can be determined to be \( G^{(n-1)}(z_0)/(n-1)! \), using (B.1.3).
If $F$ has a branch cut $B$ within $C$ joining branch points $a$, $b$, then the residue is obtained by shrinking $C$ tightly around the branch cut, giving

$$
\frac{1}{2\pi i} \int_a^b d(u)du, \quad d(u) = F^-(u) - F^+(u), \quad F^\pm(u) = \lim_{z \to u^\pm} F(z), \quad u \in B, \quad (B.1.11)
$$

the positive side of $B$ being that along which the shrunken contour is going from $b$ to $a$ and the negative side being that along which it is going from $a$ to $b$.

We deal in the main text with integrals of the form

$$
\int_{-\infty}^{\infty} F(u)du,
$$

where $F(z)$ is analytic on an open region, including the real axis, with the behavior

$$
F(z) \sim \frac{A}{z^p}, \quad p \geq 1,
$$

at large $|z|$, where $A$ is a constant. Consider a contour $C$ in $\Omega^+$ consisting of a semi-circle of radius $R$ on a segment $[-R, R]$ of $\mathbb{R}$ that encloses all the singularities of $F$ in $\Omega^+$. Let

$$
I_R = \oint_C F(z)dz.
$$

Now on the circumference of the semicircle,

$$
z = Re^{i\theta}, \quad dz = iRd\theta. \quad (B.1.12)
$$

If $p = 1$ then

$$
\lim_{R \to \infty} I_R = 2\pi i \sum \text{residues of singularities in } \Omega^+ = \int_{-\infty}^{\infty} F(u)du + i\pi A, \quad (B.1.13)
$$

the rightmost term being the contribution of the circumference of the semicircle, obtained using (B.1.12). If $p > 1$, we have

$$
\lim_{R \to \infty} I_R = 2\pi i \sum \text{residues of singularities in } \Omega^+ = \int_{-\infty}^{\infty} F(u)du. \quad (B.1.14)
$$

Closing the contour in $\Omega^-$ gives

$$
\lim_{R \to \infty} I_R = -2\pi i \sum \text{residues of singularities in } \Omega^- = \begin{cases} 
\int_{-\infty}^{\infty} F(u)du - i\pi A, & p = 1, \\
\int_{-\infty}^{\infty} F(u)du, & p > 1,
\end{cases} \quad (B.1.15)
$$

where the sign changes are a result of the fact that the contour direction is now clockwise.
Remark B.1.2. Let \( F(z) \) be a function analytic on \( \Omega \) except at a variety of singularities. The notation \( \overline{F}(z) \) indicates the complex conjugate function, leaving the variable \( z \) untouched. Then \( \overline{F}(z) \) is analytic except at singularities that are a mirror image in the real axis of those of \( F(z) \).

In particular, if \( F \) is analytic in \( \Omega^+ (\Omega^-) \), then \( \overline{F} \) will be analytic in \( \Omega^- (\Omega^+) \).

Proposition B.1.3. Let \( F(z) \) be analytic in \( \Omega^+ \) and \( G(z) \) in \( \Omega^- \). Let both go to zero as \( |z|^{-p} \), \( p > 1/2 \) at large \( |z| \). Then
\[
\int_{-\infty}^{\infty} \overline{F}(s)G(s)ds = \int_{-\infty}^{\infty} F(s)\overline{G}(s)ds = 0,
\]
so that they are orthogonal in an \( L^2 \) scalar product.

Proof. This follows from Cauchy’s theorem by closing the first integral on \( \Omega^- \) and the second on \( \Omega^+ \).

B.2 Cauchy Integrals

We consider integrals of the following type [186]:
\[
F(z) = \frac{1}{2\pi i} \int_L \frac{f(u)}{u-z}du,
\]
where \( L \) is a sectionally smooth curve in the complex plane. By this, we mean a finite number of nonintersecting smooth arcs and contours. The term contour is used to indicate a closed curve, as before, while arc refers to a curve that is not closed and therefore has endpoints. Smoothness indicates that a tangent exists at each point of \( L \) and its slope varies continuously. In other words, each arc or contour, if represented parametrically, has continuous first derivatives with respect to its parameter.

The more interesting developments around Cauchy integrals deal largely with the case in which \( L \) is finite in length. However, we are mainly interested in the cases in which \( L \) is infinite in length, given by \( \mathbb{R} \), discussed below, or \( \mathbb{R}^+ \) for example, or infinite segments of the imaginary axis. Provided convergence issues are taken into account, there is no difficulty in dealing with \( L \) infinite in length.

We need to choose a positive direction along \( L \). For contours, this is generally taken to be the counterclockwise direction, but for arcs, there is no set convention. For an integral along a line segment \([a, b]\) anywhere in the complex plane, written as \( \int_{a}^{b} \), the positive direction is taken to be from \( a \) to \( b \). Thus, if \([a, b]\) is on the real axis and \( a < b \), the positive direction is the positive \( x \) direction. The region of the complex plane to the left, as one moves along \( L \) in the positive direction, is denoted by \( S^+ \) and the region to the right by \( S^- \). These are the upper and lower half-planes respectively, for \( L \) in a positive direction along the real axis.

The function \( f \), referred to as the density function, is assumed to be bounded everywhere, except possibly at endpoints of arcs, denoted by \( c_k, k = 1, 2, \ldots \), where
it may have integrable singular points with

\[ f(u) \sim \frac{f_0}{|u - c_k|^\alpha}, \quad 0 \leq \alpha < 1, \tag{B.2.2} \]

where \( f_0 \) is a constant. Furthermore, it is assumed that \( f \) is Hölder continuous at each point of \( L \) where it is not singular. This property is defined as follows: for any two points \( u_1, u_2 \), there exist positive real constants \( A, \mu \) such that

\[ |f(u_1) - f(u_2)| \leq A|u_1 - u_2|^\mu. \tag{B.2.3} \]

It is easy to show that if \( \mu > 1 \), the derivative of \( f(u) \) is zero, so that it is a constant. This case is not of great interest, so it is always assumed that \( \mu \leq 1 \). For \( \mu = 1 \), the Hölder condition is termed the Lipschitz condition and is obeyed by any differentiable function and others not in this class. For \( \mu < 1 \), the condition implies continuity in the ordinary sense. The case \( \mu = 0 \), which is excluded, is consistent with discontinuity. A function obeying this condition at a point, or on a line, will be described as obeying the \( H(\mu) \) condition on that set if \( \mu \) is specified, or otherwise just the \( H \) condition.

At large \( |z| \), the function \( F \) behaves like

\[ F(z) \sim -\frac{A}{z}, \quad 2\pi i A = -\int_L f(u)du, \]

if the integral is nonzero. If it is zero, \( F \) falls off as some higher power of \( z \). Consider the limiting value as \( z \) approaches a point \( u \) on \( L \) at which \( f \) is nonsingular and that is not an endpoint of an arc. We write

\[ F(z) = \frac{1}{2\pi i} \int_L \frac{f(t) - f(u)}{t - z} dt + \frac{f(u)}{2\pi i} \int_L \frac{1}{t - z} dt. \tag{B.2.4} \]

The Hölder condition (B.2.3) implies that the first term approaches a well-defined integral

\[ \frac{1}{2\pi i} \int_L \frac{f(t) - f(u)}{t - u} dt \]

as \( z \to u \) because the behavior at the singularity is integrable. This step illustrates the importance of the Hölder property. The second term can be assigned a finite value but one that depends on the direction in which the limit is taken. Let

\[ \bar{\psi}(z) = \frac{1}{2\pi i} \int_L \frac{dt}{t - z} \]

and denote by \( \bar{\psi}^+(u) \), \( \bar{\psi}^-(u) \) the limiting values of \( \bar{\psi}(z) \) as \( z \) approaches \( u \) from \( S^+ \) and \( S^- \), respectively. In each of these cases, we deform the contour into a small semicircle around \( u \) and consider the limit as this semicircle shrinks to zero. It is easy to show that
Some Properties of Functions on the Complex Plane

\[ \tilde{\psi}^+(u) = \frac{1}{2} + \frac{1}{2\pi i} \int_L \frac{dt}{t-u}, \quad \tilde{\psi}^-(u) = -\frac{1}{2} + \frac{1}{2\pi i} \int_L \frac{dt}{t-u}, \tag{B.2.5} \]

where the integrals are Cauchy principal values. The more general formulas

\[ F^+(u) = \frac{1}{2} f(u) + \frac{1}{2\pi i} \int_L \frac{f(t)}{t-u} dt, \tag{B.2.6} \]

\[ F^-(u) = -\frac{1}{2} f(u) + \frac{1}{2\pi i} \int_L \frac{f(t)}{t-u} dt, \]

follow from (B.2.4) and (B.2.5), since we can write the first, well-defined, integral in (B.2.4) as its Cauchy principal value, and the two integrals can then be recombined. These are the well-known Plemelj formulas which are of great importance in Part III of the present work. Another form of these relations is given by

\[ F^+(u) - F^-(u) = f(u), \]

\[ F^+(u) + F^-(u) = \frac{1}{\pi i} \int_L \frac{f(t)}{t-u} dt, \]

which show clearly that \( F \), defined by (B.2.1), is discontinuous across \( L \) at all points where \( f \) is nonzero. This implies the existence of branch points on \( L \), resulting in branch cuts along \( L \).

For \( z \notin L \), \( F(z) \), given by (B.2.1), is analytic since it is differentiable. It should be noted that this property requires no assumption on \( f \) other than Hölder continuity. In particular, no analyticity requirements need be imposed.

Consider the Cauchy integral over a single arc \([a, b]\) of finite length:

\[ F(z) = \frac{1}{2\pi i} \int_a^b \frac{f(u)}{u-z} du. \]

This is a function analytic everywhere on \( \Omega \) except on \([a, b]\). If \( f \) is nonzero on this segment, \( F \) has a branch cut between \( a \) and \( b \). It is of interest to determine the behavior of \( F(z) \) as \( z \) approaches the endpoints. Consider \( z \) close to \( a \). Let \( f(a) \) be finite. Then, using the same trick as in (B.2.4), we obtain

\[ F(z) = \frac{f(a)}{2\pi i} \log \left( \frac{b-z}{a-z} \right) + \frac{1}{2\pi i} \int_a^b \frac{f(t) - f(a)}{t-z} dt = \frac{f(a)}{2\pi i} \log \left( \frac{1}{a-z} \right) + F_1(z), \tag{B.2.7} \]

where \( F_1(z) \) has a definite, nonsingular, limit as \( z \to a \). Similarly, near \( z = b \),

\[ F(z) = \frac{f(b)}{2\pi i} \log(b-z) + F_2(z), \tag{B.2.8} \]

where \( F_2(b) \) is nonsingular. Therefore, if \( f(a) \) or \( f(b) \) is finite, there is a logarithmic singularity in \( F(z) \) at that endpoint. If the limit is taken along the branch cut, similar formulas may be given by applying the Plemelj formulas to the singular term. Let us write
$a - z = |a - z|e^{i\theta},$

where $\theta = \theta_0$ gives the limit to the cut from the positive side and $\theta = \theta_0 + 2\pi$ is the limit from the negative side. Then we see that the dominant term has the form

$$\frac{1}{2}(F^+(u) + F^-(u)) \begin{cases} \sim f(a) \frac{1}{2\pi i} \log \frac{1}{|a - u|}, \\ \sim f(b) \frac{1}{2\pi i} \log |b - u|. \end{cases} \quad (B.2.9)$$

If the end value of $f(u)$ is zero, then $F(z)$ approaches a definite, finite, limit at that point. If $f(u)$ has a singularity at an endpoint of the type given by (B.2.2), then $F(z)$ has a singularity of the same type. This may be seen intuitively by considering the dominant term of the integral. Therefore, if we have the behavior (B.2.2) at $a$, then

$$F(z) \sim_a \frac{A}{(z - a)^\alpha} \quad (B.2.10)$$

off the cut and

$$\frac{1}{2} [F^+(u) + F^-(u)] \sim_u \frac{A_1}{(u - a)^\alpha}, \quad (B.2.11)$$

where rigorous arguments and detailed expressions for the constants $A, A_1$ are given by Muskhelishvili [186] and Gakhov [96], for example. Similar formulas apply for such behavior at $b$.

### B.2.1 Cauchy Integrals on the Real Line

A most important special case of the Cauchy integral in the present context is that in which the curve $L$ is the real axis $\mathbb{R}$, so that

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u)}{u - z} du. \quad (B.2.12)$$

If

$$I = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(u)du$$

exists, at least in the sense of a principal value infinite integral where $f(u) \sim u^{-1}$ at large values of $|u|$ and

$$I = \frac{1}{2\pi i} \lim_{L \to \infty} \int_{-L}^{L} f(u)du,$$

then

$$F(z) = -\frac{I}{z} + O\left(\frac{1}{z^2}\right) \quad (B.2.13)$$
at large \( z \). For most examples in the present work, we have \( f(u) \sim u^{-2} \) for large \( u \), so that \( I \) exists as an ordinary integral.

In this case, the complex plane is segmented into the upper half-plane \( \Omega^+ \) and the lower half-plane \( \Omega^- \). For \( z \in \Omega^+ \), \( F(z) \) is analytic in \( \Omega^+ \), while for \( z \in \Omega^- \), it is analytic in \( \Omega^- \).

We write out the Plemelj formulas in this case, for reference purposes:

\[
\begin{align*}
F^+(x) &= \frac{1}{2} f(x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u)}{u-x} \, du, \\
F^-(x) &= -\frac{1}{2} f(x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u)}{u-x} \, du,
\end{align*}
\]

or

\[
\begin{align*}
F^+(x) + F(x^-) &= \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{f(u)}{u-x} \, du, \\
F^+(x) - F^-(x) &= f(x),
\end{align*}
\]

where \( F^\pm(x) \) are the limits of \( F(z) \) as \( z \to x \) from \( \Omega^{\pm} \). Note that \( F^\pm \) correspond to \( F_z \) in the notation used in the main text and referred to after (C.2.2).

**Remark B.2.1.** Thus, any Hölder continuous function defined on \( \mathbb{R} \) can be written as the difference between the limits of two functions, one analytic in \( \Omega^+ \) and the other in \( \Omega^- \).

As noted earlier, there is no requirement that \( f \) be analytic. However, if \( f \) is analytic on an open set containing \( \Omega^+ \), we see by comparing (B.1.5) and (B.2.14), that \( f(x) = F^+(x) \) and indeed \( f(z) = F(z), \, z \in \Omega^+ \). Similarly, if \( f \) is analytic on \( \Omega^- \), we have \( f(z) = -F(z), \, z \in \Omega^- \). Let us introduce the notation

\[
F_i(z) = F(z), \quad z \in \Omega^-, \quad F_u(z) = F(z), \quad z \in \Omega^+,
\]

where \( F \) is defined by (B.2.12) and \( F_i, F_u \) are analytic in \( \Omega^- \) and \( \Omega^+ \), respectively. Let \( f \) be analytic on an open set containing \( \mathbb{R} \) but have singularities in \( \Omega^{\pm} \), away from the real axis. Also, let \( f(z) \) behave like \( z^{-p}, \, p > 0 \), at large \( |z| \). We take \( z \in \Omega^- \) and close the contour in (B.1.15) on \( \Omega^- \). Then by (B.1.14),

\[
F_i(z) = \sum_u \text{residues of } \frac{f(u)}{u-z},
\]

where the sum is over isolated singularities and integrals on branch cuts. Thus, \( F_i(z) \) is analytic in a band in \( \Omega^+ \), \( 0 \leq \text{Im} z < \alpha \), where \( \alpha \) is the position of the singularity nearest to the real axis, which can be the position of an isolated singularity or a point on a branch cut. Therefore, \( F_i \) can be analytically continued into a band parallel to the real axis in \( \Omega^+ \) and indeed into larger regions of this half-plane, avoiding singularities. Note, however, that branch points can cause difficulties, as discussed in Section B.1.5.
Similarly, $F_u$ can be analytically continued into regions of $\Omega^-$. Note, however, that the analytical continuation of $F_i$ into $\Omega^+$ is not equal to $F_u$ in $\Omega^+$, and vice versa.

The following observation follows from (B.2.15).

**Remark B.2.2.** Let $f$ be analytic on an open set containing $\mathbb{R}$ but have singularities in $\Omega^\pm$, away from the real axis. Then it can be expressed on $\mathbb{R}$ as the difference between two functions, one analytic in an open set containing $\Omega^+$ and the other in an open set containing $\Omega^-$. 
Appendix C
Fourier Transforms

We summarize in this appendix various properties of Fourier transforms required in the main text. References include the now classical works [205, 200] and the many modern texts on the topic.

C.1 Definitions

For any function $f : \mathbb{R} \rightarrow \mathcal{V}$, where $\mathcal{V}$ is a finite-dimensional vector space, its Fourier transform $f_F : \mathbb{R} \rightarrow \mathcal{V}$ is defined by

$$ f_F(\omega) = \int_{-\infty}^{\infty} f(s)e^{-i\omega s} \, ds. \quad (C.1.1) $$

This formula and each of the properties noted below apply to each component of $f$ and $f_F$. If $f \in L^1(\mathbb{R})$, then $f_F$ exists on $\mathbb{R}$. The inverse transform is defined by

$$ g(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_F(\omega)e^{i\omega s} \, d\omega. \quad (C.1.2) $$

If $f \in L^1(\mathbb{R})$ and its first derivatives are piecewise continuous on $\mathbb{R}$, then

$$ g(s) = \frac{1}{2} [f(s^+) + f(s^-)], \quad s \in \mathbb{R}. $$

This is one version of Fourier’s integral theorem. Another is the following. Let $f \in L^2(\mathbb{R})$. Then $f_F \in L^2(\mathbb{R})$ and $g = f$ almost everywhere on $\mathbb{R}$. The existence of the transform and inverse transform for functions in $L^2(\mathbb{R})$ is at first sight unclear. However [205], convergent forms can be given as follows:
\[ f_F(\omega) = -\frac{d}{d\omega} \int_{-\infty}^{\infty} f(s) \frac{e^{-i\omega s} - 1}{is} ds, \]
\[ f(s) = \frac{1}{2\pi} \frac{d}{ds} \int_{-\infty}^{\infty} f(\omega) \frac{e^{i\omega s} - 1}{i\omega} d\omega. \]

Let us define
\[ f_+(\omega) = \int_{0}^{\infty} f(s) e^{-i\omega s} ds, \quad f_-(\omega) = \int_{-\infty}^{0} f(s) e^{-i\omega s} ds, \]
\[ f_s(\omega) = \int_{0}^{\infty} f(s) \sin \omega s ds, \quad f_c(\omega) = \int_{0}^{\infty} f(s) \cos \omega s ds. \] (C.1.3)

We have
\[ f_F(\omega) = f_+(\omega) + f_-(\omega). \] (C.1.4)

Assuming that \( f \) is even, we obtain from (C.1.1) and (C.1.2) the form of \( f_F \) and the inverse cosine transform, given by (taking \( f = g \))
\[ f_F(\omega) = 2f_c(\omega), \quad f(s) = \frac{2}{\pi} \int_{0}^{\infty} f_c(\omega) \cos(\omega s) d\omega, \quad s \in \mathbb{R}^+. \] (C.1.5)

Also, if \( f \) is odd, we have the form of \( f_F \) and the inverse sine transform,
\[ f_F(\omega) = -2if_s(\omega), \quad f(s) = \frac{2}{\pi} \int_{0}^{\infty} f_s(\omega) \sin(\omega s) d\omega, \quad s \in \mathbb{R}^+. \] (C.1.6)

For these, the statements of Fourier’s integral theorem also apply but modified by replacing \( \mathbb{R} \) with \( \mathbb{R}^+ \). For functions nonzero only on \( \mathbb{R}^+ \) or \( \mathbb{R}^- \), the properties of \( f \) may be stated on \( \mathbb{R}^\pm \) as appropriate.

We shall generally assume that \( f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) (or \( f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \)) as appropriate), ensuring the existence of the transform and the property that \( f_F \in L^2(\mathbb{R}) \) (or \( f_F \in L^2(\mathbb{R}^+) \)), though in certain cases discussed below, we need to consider certain functions not in this category. Membership of these function spaces imposes restrictions on the behavior at infinity of \( f \). For example, if a piecewise continuous function \( f : \mathbb{R}^+ \mapsto \mathbb{R} \) belongs to \( L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \), then at large positive \( s \), we must have
\[ |f(s)| \leq As^{-p}, \quad p > 1, \]
where \( A \) is a constant and, since \( f_+ \in L^2(\mathbb{R}^+) \),
\[ |f_+(\omega)| \leq B|\omega|^{-q}, \quad q > \frac{1}{2}, \]
where \( B \) is a constant, for large real \( \omega \).

It is easily shown that
\[ f'_F(\omega) = i\omega f_F(\omega), \]
where \( f' \) is the derivative of \( f \).
If \( f \) is real, which is almost always the case, then
\[
\overline{f_F(\omega)} = f_F(-\omega), \tag{C.1.7}
\]
where the bar denotes the complex conjugate. If \( h(s) = f(s + u) \), where \( u \) is a constant, then we have
\[
h_F(\omega) = e^{iu\omega} f_F(\omega). \tag{C.1.8}
\]

### C.2 Fourier Transforms on the Complex Plane

It is central to the considerations of the present work, most particularly Part III, that we consider various quantities defined on the frequency domain over the complex plane \( \Omega \). This is effectively an analytic continuation from their definition on \( \mathbb{R} \). Consider the quantity \( f_+ \) extended to the complex plane. Its integral definition allows us to carry out this extension to \( \Omega^{(-)} \) without difficulty in that the quantity
\[
f_+(\omega) = \int_0^\infty f(s) e^{-i\omega s} ds, \quad \omega = \omega_r - i\omega_i, \quad \omega_r, \omega_i \in \mathbb{R}, \quad \omega_i \in \mathbb{R}^{++}, \tag{C.2.1}
\]
exists everywhere on \( \Omega^{(-)} \) if it exists on \( \mathbb{R} \). It is also uniquely differentiable everywhere on the open set \( \Omega^{(-)} \) with respect to \( \omega = \omega_r - i\omega_i \) and therefore analytic on this set. It goes to zero as \( \text{Im}\omega \to -\infty \).

Also, if \( f_+ \) is free of singularities in \( \Omega^{(-)} \), then
\[
f(s) = \frac{1}{2\pi} \int_0^{\infty} f_+(\omega) e^{i\omega s} d\omega = 0, \quad s \in \mathbb{R}^{--},
\]
by (B.1.2). Thus, we have the following result.

**Proposition C.2.1.** The function \( f \) is zero on \( \mathbb{R}^{--} \) if and only if \( f_+ \) is analytic on \( \Omega^{(-)} \).

A similar result holds for \( f \) zero on \( \mathbb{R}^{++} \), where the analyticity of \( f_- \) is on \( \Omega^{(+)} \).

As noted in Section B.1.4, we exclude the possibility of essential singularities in the extension of \( f_r \) to the complex plane, at finite points, and except in the context of Chapter 14, assume analytic behavior at infinity, given in fact by (C.2.16) below.

**Proposition C.2.2.** The function \( f_+(\omega) \) is analytic on a band in \( \Omega^+ \), \( 0 \leq \text{Im}\omega < \alpha \) (but not in a band \( \text{Im}\omega < \beta, \beta > \alpha \)), if and only if \( f(s) \) decays like \( \exp(-\alpha s) \) for large \( s \).

**Proof.** If \( f(s) \) decays like \( \exp(-\alpha s) \) for large \( s \), then putting \( \omega = \omega_r + i\omega_i \), where \( \omega_r, \omega_i \) are real, we have that
\[
f_+(\omega_r + i\omega_i) = \int_0^{\infty} f(s) e^{-i(\omega_r + i\omega_i)s} ds.
\]
exists and is analytic for $\omega_i < \alpha$.

Let $f_F(\omega)$ be analytic for $\text{Im}\omega < \alpha$ but not in a band $\text{Im}\omega < \beta, \beta > \alpha$. We can evaluate

$$f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_+(\omega) e^{i\omega s} d\omega,$$

where $f_+$ is analytic at infinity, as a contour integral over $\Omega^+$ with no contribution from the infinite semicircle. Then, using (B.1.14), we have

$$f(s) = 2\pi i \sum_{\omega_i} \text{residues of } f_F(\omega) e^{i\omega s} \text{ at singular points } \omega_i \text{ in } \Omega^+.$$

The position $\omega_n$ of the singularity (or more than one) nearest the real axis, whether this is an isolated singularity or a point on a branch cut, must be such that $\text{Im}\omega_n = \alpha$. All other singular points will yield more strongly decaying exponentials. \(\square\)

Thus, the integral definition of $f_+$ will typically not exist on $\Omega^+$ except perhaps on a band of finite thickness parallel to the real axis. Outside of $\Omega^-$ and this region, we must define $f_+(\omega)$ by analytic continuation from the region of analyticity, though not using the integral definition. If we can obtain an explicit formula for the transform, then the analytic continuation is very easy, as can be seen from (B.1.6). It will certainly have singularities in the region $\text{Im}\omega \geq \alpha$ unless it is a constant.

Similarly, $f_-$ is analytic in $\Omega^-$ and perhaps in certain regions of $\Omega^±$.

**Hypothesis C.2.3.** For crucial manipulations in Part III, we will always assume that the regions of analyticity of $f_±$ (this quantity being an independent field variable or a relaxation function derivative) is extended to include an open region containing $\mathbb{R}$.

This assumption is, for brevity, sometimes stated in the main text as that $f_±$ is analytic on $\Omega^±$ and $\mathbb{R}$, or on $\Omega^\pm$. It is a restrictive assumption in that it means, by virtue of Proposition C.2.2, that $f(s)$ decays exponentially at large $s$. For relaxation functions determined by branch-cut singularities, this is particularly important in that if the cuts are allowed to touch the real axis, interesting nonexponential behaviors are possible. This raises the issue whether one can take the limit of the cut approaching the real axis after final results have been obtained, which is discussed in Chapter 13. Isolated singularities off the real axis are always associated with exponential decay, though by taking poles sufficiently close to the real axis, slow decay can be simulated.

For $f : \mathbb{R}^+ \to \mathcal{V}$ we can always extend the domain of $f$ to $\mathbb{R}$, by considering its *causal* extension

$$f(s) = \begin{cases} f(s) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0, \end{cases}$$

in which case

$$f_F(\omega) = f_+(\omega) = f_+(\omega) - if_-(\omega). \quad (C.2.2)$$

The quantities $f_±$ provide an example of the notation used in Part III whereby the subscript $±$ indicates that the function is analytic in $\Omega^{±}$. 

Let $f$ be zero on $\mathbb{R}^-$ and let it diverge like $\exp(\lambda_1 s)$ at large $s$. Then we consider the function

$$g(s) = f(s) e^{-\lambda s}, \quad \lambda > \lambda_1, \quad s \in \mathbb{R}^+.$$  

Its transform is given by

$$g_+(\omega) = \int_0^{\infty} g(s) e^{-i\omega s} ds = \int_0^{\infty} f(s) e^{-i(\omega s - i\lambda)} ds = f_+(\omega - i\lambda),$$

so that $f_+$ exists and is analytic below the line $z = -i\lambda$. Taking the inverse transform of $g_+$, we obtain

$$g(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_+(\omega) e^{i\omega s} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_+(\omega - i\lambda) e^{i\omega s} d\omega,$$

so that

$$f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_+(\omega - i\lambda) e^{i(\omega s - i\lambda)} ds d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_+(\xi) e^{i\xi s} d\xi. \quad (C.2.3)$$

### C.2.1 Laplace Transforms

Let $f$ be zero on $\mathbb{R}^-$. Then

$$f_L(\alpha) = \int_0^{\infty} f(s) e^{-\alpha s} ds = f_+(-i\alpha), \quad \alpha \in \Omega, \quad (C.2.4)$$

is the Laplace transform of $f$. It is analytic for $\text{Re} \alpha > 0$. The imaginary axis is included if hypothesis C.2.3 is introduced. If $f$ diverges like $\exp(\alpha_0 s)$ for large $s$, then $f_L$ exists and is analytic for $\text{Re} \alpha > \alpha_0$. Allowing for this possibility, we use (C.2.3) to determine the (unique) inverse Laplace transform. Making a change of variable $\alpha = i\xi$, we obtain

$$f(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} f_L(\alpha) e^{\alpha s} d\alpha, \quad \lambda > \alpha_0. \quad (C.2.5)$$

This gives zero if $\text{Re} s < 0$, allowing the contour to be closed in the right-hand half-plane.

### C.2.2 The Fourier Transform of Functions with Compact Support

**Proposition C.2.4.** Consider the case that $f(s) = 0, s \notin [0, d]$, and $f$ is continuous for $s \in [0, d], d > 0$, with $f(0)$ and $f(d)$ nonzero. Then
The quantity \( f_F \) is an entire function with an essential singularity at infinity and dominant behavior given by

\[
f_+(\omega) \xrightarrow{\text{Im} \omega \to +\infty} \frac{f(d)}{i\omega} e^{-i\omega d}.
\]  

(C.2.7)

Also, for \(|\text{Re} \omega|\) large and \(\text{Im} \omega\) fixed,

\[
f_+(\omega) \sim \frac{f(0)}{i\omega} \left(1 - e^{-i\omega d}\right).
\]  

(C.2.8)

For \(\text{Im} \omega \to -\infty\),

\[
f_+(\omega) \sim \frac{f(0)}{i\omega}.
\]  

(C.2.9)

Conversely, if \(f_+\) is an entire function that has an essential singularity of the form (C.2.7), then \(f(s) = 0\), \(s > d\).

**Proof.** The analyticity of \(f_+\) at all finite points in the complex plane follows from the fact that the integral exists and is uniquely differentiable everywhere. The analyticity of \(f_+\) on \(\Omega^+\) follows from the observation after (C.2.1). By taking \(|\omega| \to \infty\) along the real axis and changing integration variables, we obtain (C.2.8). This formula can be analytically continued onto \(\Omega^+\), so that (C.2.9) follows. Relation (C.2.8) cannot be analytically continued into \(\Omega^+\) because of the presence of a divergence at infinity. By changing the integration variable in (C.2.6), we obtain

\[
f_+(\omega) = e^{-i\omega d} g(\omega), \quad g(\omega) = \int_{-d}^{0} f(s + d) e^{-i\omega s} ds.
\]

It follows from its definition that \(g(\omega)\) is analytic on \(\Omega^+\). As before, we find that

\[
g(\omega) \sim -\frac{f(d)}{i\omega} \left(1 - e^{i\omega d}\right)
\]

for \(|\text{Re} \omega|\) large and \(\text{Im} \omega\) fixed, giving

\[
g(\omega) \xrightarrow{\text{Im} \omega \to +\infty} \frac{f(d)}{i\omega},
\]

and (C.2.7) follows.

**Remark C.2.5.** If \(f(d)\) vanishes and \(f'(d) \neq 0\), a slightly different version of (C.2.7) emerges.

Conversely, we assume that \(f_+\) is analytic at all finite points of \(\Omega\) and diverges as indicated by (C.2.7) on \(\Omega^+\). Then \(g\) is an entire function that goes to zero as \(\text{Im} \omega \to +\infty\). We write

\[
f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega(s - d)} d\omega.
\]
If $s > d$, the contour can be closed in $\Omega^(+)$ with the contribution from the infinite portion exponentially attenuated. The analyticity of $g(\omega)$ in $\Omega^(+)$ ensures that the result is zero. □

Proposition C.2.4 is closely related to the Paley–Wiener theorem [194].

### C.2.3 Functions That Do Not Belong to $L^1 \cap L^2$

It is necessary to include cases of functions that do not belong to $L^1 \cap L^2$. Consider the case that $f(\infty) \neq 0$ but with $f_0 \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ defined by

$$f_0(s) = f(s) - f(\infty).$$

In this case, we write

$$f_+(\omega) = \int_0^\infty e^{-i\omega s} f(s) ds = \int_0^\infty e^{-i\omega s} f_0(s) ds + f(\infty) \int_0^\infty e^{-i\omega s} ds,$$

(C.2.10)

$$= f_0(\omega) + \frac{f(\infty)}{i\omega}; \quad \omega^- = \lim_{\alpha \to 0^+} (\omega - i\alpha),$$

where we have moved $\omega$ in the rightmost integral of the second relation into $\Omega^(-)$ to give a finite result. The limit in the definition of $\omega^-$ is taken after any integrations in frequency space are carried out. This is a well-known device for handling such functions—effectively as a limit of $L^1 \cap L^2$ functions—which avoids the use of distribution theory. It is, in the present context, largely redundant, since $f_+(\omega)$ will generally multiply functions that vanish at $\omega = 0$ in such a way as to cancel the pole.

Similarly, if $f(-\infty) \neq 0$ but if $f_0 \in L^1(\mathbb{R}^-) \cap L^2(\mathbb{R}^-)$, where

$$f_0(s) = f(s) - f(-\infty),$$

then

$$f_-(\omega) = \int_{-\infty}^0 e^{-i\omega s} f(s) ds = f_0(\omega) - \frac{f(-\infty)}{i\omega^+}, \quad \omega^+ = \lim_{\alpha \to 0^+} (\omega + i\alpha).$$

(C.2.11)

Finally, we note the formal relation, referred to on occasion in the main text,

$$\int_{-\infty}^\infty e^{\pm i\omega s} ds = 2\pi \delta(\omega),$$

(C.2.12)

where $\delta$ is the singular delta function.
C.2.4 The Form of \( f_{\pm} \) at Large Frequencies

The Riemann–Lebesgue lemma states that if \( f \in L^1(\mathbb{R}) \), then

\[
\lim_{\omega \to \infty} \int_{-\infty}^{\infty} f(s)e^{\pm i\omega s} \, ds = 0. \tag{C.2.13}
\]

Similar statements apply to \( f \) defined on \( \mathbb{R}^\pm \). It is of interest to determine in more detail, however, the behavior of Fourier transforms at large \( \omega \). The results apply to inverse transforms with minor changes of sign. Consider the relations

\[
\int_{0}^{\infty} e^{-i\omega s} \, ds = \frac{1}{i\omega}, \quad \int_{-\infty}^{0} e^{-i\omega s} \, ds = -\frac{1}{i\omega^+}, \tag{C.2.14}
\]

obtained by the device introduced in (C.2.10) and (C.2.11). Differentiating \( n \) times yields

\[
\int_{0}^{\infty} s^n e^{-i\omega s} \, ds = \frac{n!}{(i\omega)^{n+1}}, \quad \int_{-\infty}^{0} s^n e^{-i\omega s} \, ds = -\frac{n!}{(i\omega^+)^{n+1}}. \tag{C.2.15}
\]

If \( f_{\pm}(\omega) \) is analytic at infinity and if the first \( N \) right and left derivatives of \( f \) exist at the origin, then we obtain, by Taylor expansion and (C.2.15), the asymptotic behavior

\[
f_{\pm}(\omega) \xrightarrow{\omega \to \infty} \pm \sum_{n=0}^{N} f^{(n)}(0)_{\pm} \left(\frac{1}{\omega^{n+2}} + O\left(\frac{1}{\omega^{N+2}}\right)\right), \tag{C.2.16}
\]

where \( f^{(n)}(0+)_{\pm} = f^{(n)}(0-)_{\pm} \) is the \( n \)th right (left) derivative of \( f \) at the origin. Thus

\[
f_{\even}(\omega) \xrightarrow{\omega \to \infty} \sum_{n \text{ even}}^{N} f^{(n)}(0)_{\pm} \left(\frac{1}{i\omega^{n+1}} + O\left(\frac{1}{\omega^{N+2}}\right)\right), \tag{C.2.17}
\]

\[
f_{\odd}(\omega) \xrightarrow{\omega \to \infty} i \sum_{n \text{ odd}}^{N} f^{(n)}(0)_{\pm} \left(\frac{1}{i\omega^{n+1}} + O\left(\frac{1}{\omega^{N+2}}\right)\right) + i\delta(\omega). \]

If

\[
f^{(n)}(0+) = f^{(n)}(0-) = f^{(n)}(0), \quad n = 0, 1, 2, \ldots, m,
\]

in other words, if \( f \) is differentiable \( n \) times at the origin, then it follows from (C.2.16) that

\[
f_{F}(\omega) \sim \omega^{-(m+2)} \tag{C.2.18}
\]

at large \( \omega \).

Note that combining (C.2.12) and (C.2.14) yields

\[
\frac{1}{\omega^-} - \frac{1}{\omega^+} = 2\pi i\delta(\omega). \tag{C.2.19}
\]
### C.2.5 Expressions for $f_{\pm}$ in Terms of $f_F$

Using the inverse transform to express $f$ in terms of $f_F$, together with (C.1.3), we obtain

\[
f_+(\omega) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_F(\omega')}{\omega' - \omega} d\omega',
\]
\[
f_-(\omega) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_F(\omega')}{\omega' + \omega} d\omega',
\]
\[
\omega^\pm = \lim_{\alpha \to 0^+} (\omega \pm i\alpha).
\]

Thus, we move $\omega$ slightly into the half-plane of analyticity of $f_{\pm}$ to achieve convergence in the time integration, as in (C.2.10) and (C.2.11). This also ensures that the integrals on the right-hand side of (C.2.20) have a well-defined meaning. The limit is taken after the integration is carried out. The forms of the analytic functions $f_{\pm}$, $\omega \in \Omega^{(\pm)}$, are given by

\[
f_{\pm}(\omega) = \mp \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_F(\omega')}{\omega' - \omega} d\omega'.
\]

Note that (C.1.4) follows from (C.2.20) and the Plemelj formula (B.2.15). Using (C.1.4) in (C.2.21) and (C.2.16), we see that in the formula for $f_+$, the contribution from $f_-$ in the integral vanishes by Cauchy’s theorem (closing the contour on $\Omega^{(\pm)}$). Thus, we obtain

\[
f_+(\omega) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f_+(\omega')}{\omega' - \omega} d\omega', \quad \omega \in \Omega^{(-)},
\]

with a similar relation for $f_-$ (no minus sign on the integral). These relations and their limit as $\omega$ approaches the real axis provides an example of the properties noted after Remark B.2.1, where $F^\pm$ corresponds to $f_{\mp}$.

### C.3 Parseval’s Formula and the Convolution Theorem

Parseval’s formula states that

\[
\int_{-\infty}^{\infty} \overline{f(u)}g(u)du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f_F(\omega)}g_F(\omega)d\omega,
\]

so that the $L^2$ scalar products in the time and frequency domains are proportional. We have allowed the possibility of complex functions in the time domain, which occurs only rarely in the main text. The convolution, or faltung, theorem gives that if $h$ is the convolution product $f \ast g$, namely
\[ h(s) = \int_{-\infty}^{\infty} f(s - u)g(u)du = \int_{-\infty}^{\infty} f(u)g(s - u)du, \quad (C.3.2) \]

then
\[ h_F(\omega) = f_F(\omega)g_F(\omega). \quad (C.3.3) \]

The converse also holds. Note that if \( f \) and \( g \) are causal, then \( h \) also has this property and is given by
\[ h(s) = \int_{0}^{s} f(s - u)g(u)du, \quad s \in \mathbb{R}^+. \quad (C.3.4) \]

Remark C.3.1. From (C.3.2), we see that the convolution product is commutative if \( f \) and \( g \) commute. It can also be shown to be associative.


48. C.M. Dafermos and J.A. Nohel, Energy methods for nonlinear hyperbolic Volterra integro-
49. P.L. Davis, On the hyperbolicity of the equations of the linear theory of heat conduction for
160.
1–34.
52. W.A. Day, Reversibility, recoverable work and free energy in linear viscoelasticity, *Quart. J.
55. G. Del Piero and L. Deseri, Monotonic, completely monotonic and exponential relaxation
56. G. Del Piero and L. Deseri, On the analytic expression of the free energy in linear viscoelastic-
57. G. Del Piero and L. Deseri, On the concepts of state and free energy in linear viscoelasticity,
58. L. Deseri, M. Fabrizio, and J.M. Golden, On the concept of a minimal state in viscoelasticity:
96.
59. L. Deseri, G. Gentili, and J.M. Golden, An explicit formula for the minimum free energy in
60. L. Deseri, G. Gentili, and J.M. Golden, On the minimum free energy and the Saint-Venant
principle in linear viscoelasticity, in *Mathematical Models and Methods for Smart Materials*,
61. L. Deseri and J.M. Golden, The minimum free energy for continuous spectrum materials,
62. E.D. Dill, Simple materials with fading memory, in *Continuum Physics II*, Academic Press,
64. G. Duvant and J. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin,
1976.
69. M. Fabrizio, C. Giorgi, and A. Morro, Free energies and dissipation properties for systems
70. M. Fabrizio, C. Giorgi, and A. Morro, Internal dissipation, relaxation property and free en-
71. M. Fabrizio, C. Giorgi, and V. Pata, A new approach to equations with memory, *Arch. Ratio-
72. M. Fabrizio and J.M. Golden, Maximum and minimum free energies and the concept of a
73. M. Fabrizio and J.M. Golden, Maximum and minimum free energies for a linear viscoelastic


References


179. P.M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York, 1953.


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