Appendix A
Mathematical Background

In this appendix, several fundamental mathematical tools are presented in the form of definitions and lemmas that supplement the control development and closed-loop stability analyses presented in the previous chapters. The proofs of most of the following lemmas are omitted, but can be found in the cited references.

Definition A.1 [9]

Consider a function $f(t) : \mathbb{R}_+ \rightarrow \mathbb{R}$. Let the 2-norm (denoted by $\| \cdot \|_2$) of a scalar function $f(t)$ be defined as

$$
\| f(t) \|_2 = \sqrt{\int_0^\infty f^2(\tau) \, d\tau}.
$$

(A.1)

If $\| f(t) \|_2 < \infty$, then we say that the function $f(t)$ belongs to the subspace $L_2$ of the space of all possible functions (i.e., $f(t) \in L_2$). Let the $\infty$-norm (denoted by $\| \cdot \|_\infty$) of $f(t)$ be defined as

$$
\| f(t) \|_\infty = \sup_t |f(t)|.
$$

(A.2)

If $\| f(t) \|_\infty < \infty$, then we say that the function $f(t)$ belongs to the subspace $L_\infty$ of the space of all possible functions (i.e., $f(t) \in L_\infty$).
Definition A.2 [9]

The induced 2-norm of matrix $A(t) \in \mathbb{R}^{n \times n}$ is defined as follows

$$\|A(t)\|_2 = \sqrt{\lambda_{\text{max}} \{A^T(t)A(t)\}}. \quad (A.3)$$

Lemma A.1 [4]

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set $S \subset \mathbb{R}^n$ and given points $(x_{10}, ..., x_{n0})$ and $(x_1, ..., x_n)$ in $S$ that are joined by a straight line that lies entirely in $\mathbb{R}^n$, then there exists a point $(\xi_1, ..., \xi_n)$ on the line between the endpoints, such that

$$f(x_1, ..., x_n) = f(x_{10}, ..., x_{n0}) + \sum_{j=1}^n \frac{\partial}{\partial x_j} f(\xi_1, ..., \xi_n) (x_j - x_{j0}). \quad (A.4)$$

This lemma is often referred to as the Mean Value Theorem.

Lemma A.2 [6]

Given a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ that is continuously differentiable at every point $(x, y)$ on an open set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, if there is a point $(x_0, y_0)$ on $S$ where

$$f(x_0, y_0) = 0 \quad (A.5)$$

and

$$\frac{\partial f}{\partial x} (x_0, y_0) \neq 0, \quad (A.6)$$

then there are neighborhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ of $x_0$ and $y_0$, respectively, such that for all $y \in V$ the expression in (A.5) has a unique solution $x \in U$. This unique solution can be written as $x = g(y)$ where $g$ is continuously differentiable at $y = y_0$. This lemma is often referred to as the Implicit Function Theorem.

Lemma A.3 [3]

Given $a, b, c \in \mathbb{R}^n$, any of the following cyclic permutations leaves the scalar triple product invariant

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \quad (A.7)$$

and the following interchange of the inner and vector product

$$a \cdot (b \times c) = (a \times b) \cdot c \quad (A.8)$$

leaves the scalar triple product invariant where the notation $a \cdot b$ represents the dot product of $a$ and $b$ and the notation $a \times b$ represents the cross product of $a$ and $b$. 
Lemma A.4 [3]

Given \( a, b, c \in \mathbb{R}^n \), the vector triple products satisfy the following expressions

\[
a \times (b \times c) = (a \cdot c) b - (a \cdot b) c \tag{A.9}
\]

\[
(a \times b) \times c (a \cdot c) b - (b \cdot c) a \tag{A.10}
\]

where the notation \( a \cdot b \) represents the dot product of \( a \) and \( b \) and the notation \( a \times b \) represents the cross product of \( a \) and \( b \).

Lemma A.5 [3]

Given \( a, b \in \mathbb{R}^n \), the vector product satisfies the following skew-symmetric property

\[
a \times b = -b \times a \tag{A.11}
\]

where the notation \( a \times b \) represents the cross product of \( a \) and \( b \).

Lemma A.6 [3]

Given \( a = [ a_1 \ a_2 \ a_3 ]^T \in \mathbb{R}^3 \) and \( a^x \in \mathbb{R}^{3 \times 3} \) which is defined as follows

\[
a^x = \begin{bmatrix}
0 & -a_3 & a_2 \\
 a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix} \tag{A.12}
\]

then the product \( a^T a^x \) satisfies the following property

\[
a^T a^x = [ 0 \ 0 \ 0 ]^T. \tag{A.13}
\]

Lemma A.7 [8] (Theorems 9–11)

Given the symmetric matrix \( A \in \mathbb{R}^{n \times n} \) and the diagonal matrix \( D \in \mathbb{R}^{n \times n} \), then \( A \) is orthogonally similar to \( D \) and the diagonal elements of \( D \) are necessarily the eigenvalues of \( A \).

Lemma A.8 [9]

If \( w(t) : \mathbb{R}_+ \rightarrow \mathbb{R} \) is persistently exciting and \( w(t), \dot{w}(t) \in L_\infty \), then the stable minimum-phase rational transfer function \( \bar{H}(w) \) is also persistently exciting.
Lemma A.9 \[10\]  
If \( \dot{f}(t) \triangleq \frac{d}{dt} f(t) \) is bounded for \( t \in [0, \infty) \), then \( f(t) \) is uniformly continuous for \( t \in [0, \infty) \).

Lemma A.10 \[1\]  
Let \( V(t) \) be a nonnegative scalar function of time on \( [0, \infty) \) which satisfies the differential inequality

\[
\dot{V}(t) \leq -\gamma V(t)
\]  
(A.14)

where \( \gamma \) is a positive constant. Given (A.14), then

\[
V(t) \leq V(0) \exp(-\gamma t) \quad \forall t \in [0, \infty)
\]  
(A.15)

where \( \exp(\cdot) \) denotes the base of the natural logarithm.

Lemma A.11  
Given a nonnegative function denoted by \( V(t) \in \mathbb{R} \) as follows

\[
V = \frac{1}{2} x^2
\]  
(A.16)

with the following time derivative

\[
\dot{V} = -k_1 x^2,
\]  
(A.17)

then \( x(t) \in \mathbb{R} \) is square integrable (i.e., \( x(t) \in L_2 \)).

Proof: To prove Lemma A.11, we integrate both sides of (A.17) as follows

\[
-\int_0^\infty \dot{V}(t) dt = k_1 \int_0^\infty x^2(t) dt.
\]  
(A.18)

After evaluating the left side of (A.18), we can conclude that

\[
k_1 \int_0^\infty x^2(t) dt = V(0) - V(\infty) \leq V(0) < \infty
\]  
(A.19)

where we used the fact that \( V(0) \geq V(\infty) \geq 0 \) (see (A.16) and (A.17)). Since the inequality given in (A.19) can be rewritten as follows

\[
\sqrt{\int_0^\infty x^2(t) dt} \leq \sqrt{\frac{V(0)}{k_1}} < \infty
\]  
(A.20)

we can use Definition A.1 to conclude that \( x(t) \in L_2 \). \( \square \)
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Lemma A.12 [5]

Let \( A \in \mathbb{R}^{n \times n} \) be a real symmetric positive-definite matrix; therefore, all of the eigenvalues of \( A \) are real and positive. Let \( \lambda_{\min}\{A\} \) and \( \lambda_{\max}\{A\} \) denote the minimum and maximum eigenvalues of \( A \), respectively, then for \( \forall x \in \mathbb{R}^n \)
\[
\lambda_{\min}\{A\} \|x\|^2 \leq x^T Ax \leq \lambda_{\max}\{A\} \|x\|^2
\] (A.21)
where \( \|\cdot\| \) denotes the standard Euclidean norm. This lemma is often referred to as the Rayleigh-Ritz Theorem.

Lemma A.13 [1]

Given a scalar function \( r(t) \) and the following differential equation
\[
r = \dot{e} + \alpha e
\] (A.22)
where \( \dot{e}(t) \in \mathbb{R} \) represents the time derivative \( e(t) \in \mathbb{R} \) and \( \alpha \in \mathbb{R} \) is a positive constant, if \( r(t) \in L_\infty \), then \( e(t) \) and \( \dot{e}(t) \in L_\infty \).

Lemma A.14 [1]

Given the differential equation in (A.22), if \( r(t) \) is exponentially stable in the sense that
\[
|r(t)| \leq \beta_0 \exp(-\beta_1 t)
\] (A.23)
where \( \beta_0 \) and \( \beta_1 \in \mathbb{R} \) are positive constants, then \( e(t) \) and \( \dot{e}(t) \) are exponentially stable in the sense that
\[
|e(t)| \leq |e(0)| \exp(-\alpha t) + \frac{\beta_0}{\alpha - \beta_1} (\exp(-\beta_1 t) - \exp(-\alpha t))
\] (A.24)
and
\[
|\dot{e}(t)| \leq \alpha |e(0)| \exp(-\alpha t) + \frac{\alpha \beta_0}{\alpha - \beta_1} (\exp(-\beta_1 t) - \exp(-\alpha t)) + \beta_0 \exp(-\beta_1 t)
\] (A.25)
where \( \alpha \) was defined in (A.22).

Lemma A.15 [1]

Given the differential equation in (A.22), if \( r(t) \in L_\infty \), \( r(t) \in L_2 \), and \( r(t) \) converges asymptotically in the sense that
\[
\lim_{t \to \infty} r(t) = 0,
\] (A.26)
then \( e(t) \) and \( \dot{e}(t) \) converge asymptotically in the sense that
\[
\lim_{t \to \infty} e(t), \dot{e}(t) = 0.
\] (A.27)
Lemma A.16 [9]

Consider a function \( f(t) : \mathbb{R}_+ \to \mathbb{R} \). If \( f(t) \in \mathcal{L}_\infty \), \( \dot{f}(t) \in \mathcal{L}_\infty \), and \( f(t) \in \mathcal{L}_2 \), then

\[
\lim_{t \to \infty} f(t) = 0. \tag{A.28}
\]

This lemma is often referred to as Barbalat’s Lemma.

Lemma A.17 [1, 7]

If a scalar function \( N_d(x, y) \) is given by

\[
N_d = \Omega(x)xy - k_n \Omega^2(x)x^2 \tag{A.29}
\]

where \( x, y \in \mathbb{R}, \Omega(x) \in \mathbb{R} \) is a function dependent only on \( x \), and \( k_n \) is a positive constant, then \( N_d(x, y) \) can be upper bounded as follows

\[
N_d \leq \frac{y^2}{k_n}. \tag{A.30}
\]

The bounding of \( N_d(x, y) \) in the above manner is often referred to as nonlinear damping [7] since a nonlinear control function (e.g., \( k_n \Omega^2(x)x^2 \)) can be used to “damp-out” an unmeasurable quantity (e.g., \( y \)) multiplied by a known measurable nonlinear function, (e.g., \( \Omega(x) \)).

Lemma A.18 [1]

Let \( V(t) \) be a nonnegative scalar function of time on \([0, \infty)\) which satisfies the differential inequality

\[
\dot{V} \leq -\gamma V + \varepsilon \tag{A.31}
\]

where \( \gamma \) and \( \varepsilon \) are positive constants. Given (A.31), then

\[
V(t) \leq V(0) \exp(-\gamma t) + \frac{\varepsilon}{\gamma} \left(1 - \exp(-\gamma t)\right) \quad \forall t \in [0, \infty). \tag{A.32}
\]

Lemma A.19 [1]

If the differential equation in (A.22) can be bounded as follows

\[
|r(t)| \leq \sqrt{A + B \exp(-kt)} \tag{A.33}
\]

where \( k, A, \) and \( B \in \mathbb{R} \) and \( A + B \geq 0 \), then \( e(t) \) given in (A.22) can be bounded as follows

\[
|e(t)| \leq |e(0)| \exp(-\alpha t) + \frac{a}{\alpha} (1 - \exp(-\alpha t)) + \frac{2b}{2\alpha - k} \left( \exp\left(-\frac{1}{2}kt\right) - \exp(-\alpha t) \right) \tag{A.34}
\]

where

\[
a = \sqrt{A} \quad \text{and} \quad b = \sqrt{B}. \tag{A.35}
\]
Lemma A.20 [6]

If a function $f(t) : \mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous and if the integral

$$\lim_{t \to \infty} \int_0^t |f(\tau)| \, d\tau$$

exists and is finite, then

$$\lim_{t \to \infty} |f(t)| = 0.$$  \hspace{1cm} (A.37)

This lemma is often referred to as the integral form of Barbalat’s Lemma.

Lemma A.21 [2]

If a given differentiable function $f(t) : \mathbb{R}_+ \to \mathbb{R}$ has a finite limit as $t \to \infty$ and if $f(t)$ has a time derivative, defined as $\dot{f}(t)$, that can be written as the sum of two functions, denoted by $g_1(t)$ and $g_2(t)$, as follows

$$\dot{f}(t) = g_1 + g_2$$  \hspace{1cm} (A.38)

where $g_1(t)$ is a uniformly continuous function and

$$\lim_{t \to \infty} g_2(t) = 0$$  \hspace{1cm} (A.39)

then

$$\lim_{t \to \infty} \dot{f}(t) = 0 \quad \lim_{t \to \infty} g_1(t) = 0.$$  \hspace{1cm} (A.40)

This lemma is often referred to as the Extended Barbalat’s Lemma.

Lemma A.22 [6]

Let the origin of the following autonomous system

$$\dot{x} = f(x)$$  \hspace{1cm} (A.41)

be an equilibrium point $x(t) = 0$ where $f(\cdot) : D \to \mathbb{R}^n$ is a map from the domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Consider a continuously differentiable positive definite function $V(\cdot) : D \to \mathbb{R}^n$ containing the origin $x(t) = 0$ where

$$\dot{V}(x) \leq 0 \quad \text{in } D.$$  \hspace{1cm} (A.42)

Let $\Gamma$ be defined as the set of all points where $\{x \in D | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in $\Gamma$ other than the trivial solution $x(t) = 0$. Then the origin is globally asymptotically stable. This Lemma is a corollary to LaSalle’s Invariance Theorem.
References


In this appendix, supplementary lemmas and definitions are provided to support the mathematical development in the previous chapters. Proofs are provided for most of the lemmas.

B.1 Chapter 2 Lemmas

B.1.1 Convolution Operations for Torque Filtering

Lemma B.1 The filtered control input signal $u_f(t)$ defined in (2.32) can be expressed as the linear parameterization given in (2.34).

Proof: To rewrite (2.32) in terms of the linear parameterization given in (2.34), the expression given in (2.10) is written in the following form [2]

$$u = \dot{h} + g \quad \text{(B.1)}$$

where

$$\dot{h} = \frac{d}{dt}(M(q)\dot{q}) \quad \text{(B.2)}$$

and

$$g = -\ddot{M}(q)\dot{q} + V_m(q, \dot{q}). \quad \text{(B.3)}$$
After substituting (B.1) into (2.32), the following expression can be obtained

\[ u_f = \dot{f}(t) * Y_A(q, \dot{q}) \phi + Y_B(q, \dot{q}) \phi + f(t) * Y_C(q, \dot{q}) \phi \]  
\[ (B.4) \]

where the standard convolution properties

\[ f * \{ \dot{h} + g \} = f * \dot{h} + f * g \]  
\[ (B.5) \]

and

\[ f * c = \dot{h} * h + f(0)h - fh(0) \]  
\[ (B.6) \]

have been used. The linear regression matrices \( Y_A(q, \dot{q}) \phi \), \( Y_B(q, \dot{q}) \phi \), and \( Y_C(q, \dot{q}) \phi \) in (B.4) are defined as follows

\[ Y_A(q, \dot{q}) \phi = M(q(t)) \dot{q}(t) \]  
\[ (B.7) \]

\[ Y_B(q, \dot{q}) \phi = f(0) M(q(t)) \dot{q}(t) - f(t) M(q(0)) \dot{q}(0) \]

\[ Y_C(q, \dot{q}) \phi = -\dot{M}(q(t)) \dot{q}(t) + V_m(q, \dot{q}). \]

By utilizing standard Laplace Transform techniques, the expression in (B.4) can be rewritten as follows

\[ u_f = (\tilde{Y}_A(q, \dot{q}) + Y_B(q, \dot{q}) + \tilde{Y}_C(q, \dot{q})) \phi \]  
\[ (B.8) \]

where the regression matrices \( \tilde{Y}_A(q, \dot{q}) \) and \( \tilde{Y}_C(q, \dot{q}) \) are generated by the following differential expressions

\[ \dot{\tilde{Y}}_A(q, \dot{q}) + \gamma \tilde{Y}_A(q, \dot{q}) = -\gamma^2 Y_A(q, \dot{q}) \]  
\[ (B.9) \]

\[ \dot{\tilde{Y}}_C(q, \dot{q}) + \gamma \tilde{Y}_C(q, \dot{q}) = \gamma Y_C(q, \dot{q}) \]

where (2.33) and (2.36) were utilized. Hence, based on (B.8) and (B.9), it is straightforward to conclude that (2.32) can be rewritten as the linear parameterization given in (2.34). □

### B.1.2 Control Signal Bound

**Lemma B.2** The term \( \chi(q, \dot{q}, t) \) defined in (2.48) can be upper bounded by the following inequality

\[ \| \chi \| \leq \zeta_1 \| z \| \]  
\[ (B.10) \]
where \( \zeta_1 \in \mathbb{R} \) is a known positive bounding constant and \( z(t) \in \mathbb{R}^6 \) is defined as follows

\[
z(t) = \begin{bmatrix} x(t) & y(t) & e(t) & r^T(t) \end{bmatrix}^T.
\] (B.11)

**Proof:** To prove that (B.10) is a valid bound for \( \chi(\cdot) = [\chi_1 \ \chi_2 \ \chi_3]^T \), (2.11), (2.12), (2.22), (2.38), and (2.39) are used to compute the difference given in (2.48) as follows

\[
\chi_1 = -\alpha_1 m \dot{x} - mL \left[ \sin(\theta + \beta) \dot{\theta} + \cos(\theta + \beta) \ddot{\theta} \right]
- \cos \beta \left( \ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta \right) + \sin \beta \left( -\ddot{\theta} \cos \theta + \dot{\theta}^2 \sin \theta \right).
\] (B.12)

\[
\chi_2 = -\alpha_2 m \dot{y} - mL \left[ -\cos(\theta + \beta) \ddot{\theta} + \sin(\theta + \beta) \dot{\theta} \dot{\theta} \right]
+ \cos \beta \left( \ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta \right) - \sin \beta \left( \ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta \right).
\] (B.13)

\[
\chi_3 = mL (\alpha_1 \dot{x} \sin(\theta + \beta) - \alpha_2 \dot{y} \cos(\theta + \beta)).
\] (B.14)

After utilizing (2.18) and (2.19), exploiting several trigonometric identities, and cancelling common terms, the expressions given in (B.12–B.14) can be rewritten as follows

\[
\chi_1 = \alpha_1 m (r_1 + \alpha_1 x) + mL \cos(\theta + \beta) \dot{\theta} \dot{e}
\] (B.15)

\[
\chi_2 = \alpha_2 m (r_2 + \alpha_2 y) + mL \sin(\theta + \beta) \dot{\theta} \dot{e}
\] (B.16)

\[
\chi_3 = mL (-\alpha_1 (r_1 + \alpha_1 x) \sin(\theta + \beta) + \alpha_2 (r_2 + \alpha_2 y) \cos(\theta + \beta)).
\] (B.17)

Based on the expressions given in (B.15–B.17) and the definition of \( z(t) \) given in (B.11), the bound given in (B.10) can now be directly obtained. \( \square \)

### B.1.3 Control Signal Bound

**Lemma B.3** The auxiliary signal \( \Omega(q, \dot{q}, t) \) defined in (2.45) can be upper bounded by the following inequality

\[
\| \Omega \| \leq \gamma \zeta \| \psi \|
\] (B.18)

where \( \gamma \) was defined in (2.33), \( \zeta \in \mathbb{R} \) is a known positive bounding constant, and \( \psi(t) \in \mathbb{R}^9 \) is defined as follows

\[
\psi(t) = \begin{bmatrix} z^T(t) & \dot{z}^T(t) \end{bmatrix}^T.
\] (B.19)
Proof: To prove that (B.18) is a valid bound for $\Omega(q, \dot{q}, t)$, the expression given in (2.47) is first rewritten as

$$u = Y\phi = \chi + Y_d\phi_2 + \psi_1$$  \hspace{1cm} (B.20)

where $\psi_1(q, \dot{q}, t) \in \mathbb{R}$ is an auxiliary function defined as follows

$$\psi_1 = -M\dot{r} - V_m r$$  \hspace{1cm} (B.21)

and $\chi(q, \dot{q}, t)$ was defined in (2.130). After substituting the closed-loop dynamics for $r(t)$ given by (2.52) into (B.21), and cancelling common terms, the following expression is obtained

$$\psi_1 = -Y_d\ddot{\phi}_2 - \chi + K_s r - [ k_{p1} x \quad k_{p2} y \quad 0 ]^T + k_n \zeta_1^2 r.$$  \hspace{1cm} (B.22)

Since $\dot{\theta}_d(t)$, $\ddot{\theta}_d(t)$, and the trigonometric terms in (2.38) always remain bounded, an upper bound for $Y_d(q, t)\ddot{\phi}_2$ can be formulated as follows

$$\|Y_d\ddot{\phi}_2\| \leq \zeta_y \|\ddot{\phi}_2\|$$  \hspace{1cm} (B.23)

where $\zeta_y \in \mathbb{R}$ is a positive scalar constant. By utilizing (2.49), (2.50), and the bound given in (B.23), an upper bound for (B.22) can be formulated as follows

$$\|\psi_1\| \leq \zeta_y \|\ddot{\phi}_2\| + \zeta_1 \|z\| + K_s \|r\| + k_{p1} |x| + k_{p2} |y| + k_n \zeta_1^2 \|r\|.$$  \hspace{1cm} (B.24)

Given the definitions for $\psi(t)$ and $z(t)$ in (2.56) and (2.50), respectively, the upper bound given in (B.24) can be rewritten in the following compact form

$$\|\psi_1\| \leq \zeta_2 \|\psi\|$$  \hspace{1cm} (B.25)

where $\zeta_2 \in \mathbb{R}$ is a positive bounding constant. After rewriting (B.20) as follows

$$Y\phi - Y_d\phi_2 = \chi + \psi_1$$  \hspace{1cm} (B.26)

and then utilizing (2.49) and (B.25), the following inequality can be formulated

$$\|Y\phi - Y_d\phi_2\| \leq \|\chi\| + \|\psi_1\| \leq \zeta_1 \|z\| + \zeta_2 \|\psi\| \leq \zeta \|\psi\|$$  \hspace{1cm} (B.27)

where $\zeta \in \mathbb{R}$ is a positive bounding constant, and $\psi(t)$ and $z(t)$ were defined in (2.56) and (2.50), respectively.

Based on (2.32), (2.37), and (2.45), $\Omega(q, \dot{q}, t)$ can be expressed as follows

$$\Omega = f \ast (Y\phi - Y_d\phi_2)$$  \hspace{1cm} (B.28)
where \( f(t) \) was defined in (2.33). An upper bound for (B.28) can now be formulated as follows

\[
\|\Omega\| \leq \gamma \|Y\phi - Y_d\phi_2\| \tag{B.29}
\]

where the standard inequality property of convolution in [4] and the upper bound \( \|f(t)\| \leq \gamma \) have been applied (see (2.33)). After substituting (B.27) into (B.29), the inequality given by (B.18) can be obtained. \( \Box \)

### B.1.4 Control Signal Bound

Before developing lower and upper bounds for \( P^{-1}(t) \) given in (2.43), the following preliminary lemma is presented.

**Lemma B.4** Given the definition of \( P^{-1}(t) \) in (2.43), it can be shown that

\[
\|P\|_2 P^{-1} \geq I_3 \tag{B.30}
\]

where \( I_3 \) is the \( 3 \times 3 \) identity matrix and the matrix inequality \( A > B \) for \( A, B \in \mathbb{R}^{p \times p} \) is a shorthand notation to denote \( \xi^T A \xi > \xi^T B \xi, \forall \xi \in \mathbb{R}^p \).

**Proof:** Given the definition of \( P^{-1}(t) \) in (2.43) and the fact that \( P(0) \) is selected to be positive-definite and symmetric, \( P^{-1}(t) \) is positive-definite and symmetric for all time. Hence, it follows that \( P(t) \) is also positive-definite and symmetric for all time. Since \( P(t) \) is a positive-definite and symmetric gain matrix, the Rayleigh-Ritz Theorem (see Lemma A.12 of Appendix A) can be used to prove that

\[
x^T \lambda_{\max} \{ P \} x \geq x^T P x \tag{B.31}
\]

where \( \lambda_{\max} \{ \cdot \} \) denotes the maximum eigenvalue of \( \{ \cdot \} \). Based on Lemma A.7 and Definition A.2 of Appendix A, the following property can be proven

\[
\lambda_{\max} \{ P \} = \sqrt{\lambda_{\max} \{ P^T P \} = \|P\|_2} \tag{B.32}
\]

By utilizing (B.32), the expression given in (B.31) can be rewritten as follows

\[
x^T \|P\|_2 x \geq x^T P x. \tag{B.33}
\]

To facilitate the remainder of the proof, \( y \in \mathbb{R} \) is defined as follows

\[
y = \sqrt{P} x. \tag{B.34}
\]

After rewriting the right-hand side of (B.33) as follows

\[
x^T P x = x^T \sqrt{P} \sqrt{P} x \tag{B.35}
\]
and then utilizing the expression given in (B.34), the following expression can be obtained

\[ y^T \left( \sqrt{P} \right)^{-1} \sqrt{P} \sqrt{P} \left( \sqrt{P} \right)^{-1} y = y^T I_3 y. \]  

(B.36)

In a similar manner, (B.34) can be used to rewrite the left-hand side of (B.33) as follows

\[ x^T \|P\|_{i2} x = \|P\|_{i2} y^T \left( \sqrt{P} \right)^{-1} \left( \sqrt{P} \right)^{-1} y = y^T \|P\|_{i2} P^{-1} y. \]  

(B.37)

The expressions given in (B.33–B.37) can now be used to prove the result given in (B.30).

Lemma B.5 The gain forgetting factor \( \lambda(t) \) and the gain matrix \( P^{-1}(t) \) defined in (2.58) and (2.43), respectively, can be upper and lower bounded by the following inequalities

\[ 0 < \frac{\lambda_1 k_1 \kappa}{1 + k_1 \kappa} \leq \lambda(t) \]  

(B.38)

\[ \frac{1 + k_1 \kappa}{k_1} I_3 \leq P^{-1}(t) \leq k_2 I_3 \]  

(B.39)

where \( \kappa \) is defined in (2.63).

Proof: To prove Lemma B.5, the lower bound for \( P^{-1}(t) \) is obtained and then the upper and lower bounds on the gain adjusted forgetting factor \( \lambda(t) \) are developed, which are used to derive the upper bound for \( P^{-1}(t) \).

Lower bound for \( P^{-1}(t) \):

By substituting (2.58) into (2.42), the following expression can be obtained

\[ \dot{P}^{-1} = -\lambda_1 P^{-1} + \left[ \frac{\lambda_1}{k_1} \|P\|_{i2} P^{-1} + Y_{df}^T Y_{df} \right]. \]  

(B.40)

The solution to the differential equation given in (B.40) can be determined as follows

\[ P^{-1}(t) = P^{-1}(0) \exp(-\lambda_1 t) + \int_0^t \exp(-\lambda_1 (t - \sigma)) \left[ \frac{\lambda_1}{k_1} \|P(t)\|_{i2} P^{-1}(t) + Y_{df}^T(\cdot) Y_{df}(\cdot) \right] d\sigma. \]  

(B.41)
Based on the result of Lemma B.4, the following inequality can be developed

\[ P^{-1}(t) \geq P^{-1}(0) \exp(-\lambda_1 t) + \frac{\lambda_1}{k_1} I_3 \exp(-\lambda_1 t) \left[ \int_0^t \exp(\lambda_1 \sigma) \, d\sigma \right] \]

\[ + \int_0^t \exp(-\lambda_1 (t - \sigma)) Y_d^T(\cdot) Y_d(\cdot) \, d\sigma. \]  

(B.42)

After evaluating the bracketed integral given in (B.42), the resulting expression can be simplified as follows

\[ P^{-1}(t) \geq \left( P^{-1}(0) - \frac{1}{k_1} I_3 \right) \exp(-\lambda_1 t) + \frac{1}{k_1} I_3 \]

\[ + \left[ \int_0^t \exp(-\lambda_1 (t - \sigma)) Y_d^T(\cdot) Y_d(\cdot) \, d\sigma \right]. \]

(B.43)

To facilitate the remaining analysis, let the time interval \([0, t]\) be divided into \(n\) different intervals where the length of the interval from \(t_i\) to \(t_{i+1}\) is denoted by \(\delta_i\) (i.e., length of the interval from \(t_0\) to \(t_1\) is \(\delta_0\), \(t_1\) to \(t_2\) is \(\delta_1\), ..., \(t_n\) to \(t_{n+1}\) is \(\delta_n\)).

**For the interval** \(0 \leq t \leq t_1\) **where** \(t_1 = \delta_0\): Since the bracketed term in (B.43) is always positive, the following lower bound can be developed

\[ P^{-1}(t) \geq \left( P^{-1}(0) - \frac{1}{k_1} I_3 \right) \exp(-\lambda_1 t) + \frac{1}{k_1} I_3. \]  

(B.44)

Provided that the gain condition given in (2.64) is satisfied, the following fact can be utilized

\[ \exp(-\lambda_1 t) \geq \exp(-\lambda_1 \delta_0) \quad \forall t \in [0, t_1] \]

To rewrite (B.44) as

\[ P^{-1}(t) \geq \left( P^{-1}(0) - \frac{1}{k_1} I_3 \right) \exp(-\lambda_1 \delta_0) + \frac{1}{k_1} I_3. \]  

(B.45)

**For the interval** \(t_1 \leq t \leq t_2\) **where** \(t_2 = (t_1 + \delta_1)\): Provided that the gain condition given in (2.64) is satisfied, the first term on the right-hand side of (B.43) will always be positive, and hence, the following lower bound can be formulated for (B.43)

\[ P^{-1}(t) \geq \frac{1}{k_1} I_3 + \left[ \int_{t_1}^{t_1+\delta_1} \exp(-\lambda_1 (t - \sigma)) Y_d^T(\cdot) Y_d(\cdot) \, d\sigma \right]. \]  

(B.46)
Since $\sigma$ used in (B.46) satisfies the inequality $0 \leq \sigma \leq t$, and $t \leq \delta_0 + \delta_1$ in the interval $t_1 \leq t \leq t_2$, it follows that $t - \sigma \leq \delta_0 + \delta_1$. Since $t - \sigma \leq \delta_0 + \delta_1$, the following inequality can be developed

$$\exp(-\lambda_1(t - \sigma)) \geq \exp(-\lambda_1(\delta_0 + \delta_1)). \quad (B.47)$$

Hence, (B.46) can be rewritten as

$$P^{-1}(t) \geq \frac{1}{k_1} I_3 + \exp(-\lambda_1(\delta_0 + \delta_1)) \left[ \int_{t_0}^{t_0 + \delta_0} Y_{df}(\cdot) Y_{df}(\cdot) d\sigma \right]. \quad (B.48)$$

For interval $t_n \leq t \leq t_{n+1}$ where $t_{n+1} = (t_n + \delta_n)$: After extending the results of (B.46) and (B.47) to the interval $t_n \leq t \leq t_{n+1}$, the following expression is obtained

$$P^{-1}(t) \geq \frac{1}{k_1} I_3 + \left[ \int_{t_{n-1}}^{t_{n-1} + \delta_{n-1}} \exp(-\lambda_1(t - \sigma)) Y_{df}^T(\cdot) Y_{df}(\cdot) d\sigma \right]. \quad (B.49)$$

and

$$\exp(-\lambda_1(t - \sigma)) \geq \exp(-\lambda_1(\delta_{n-1} + \delta_n)). \quad (B.50)$$

Hence, (B.49) can be rewritten as

$$P^{-1}(t) \geq \frac{1}{k_1} I_3 + \exp(-\lambda_1(\delta_{n-1} + \delta_n)) \left[ \int_{t_{n-1}}^{t_{n-1} + \delta_{n-1}} Y_{df}^T(\cdot) Y_{df}(\cdot) d\sigma \right]. \quad (B.51)$$

If the PE condition given in (2.57) is satisfied, then (B.51) can be lower bounded as follows

$$P^{-1}(t) \geq \frac{1}{k_1} I_3 + \exp(-\lambda_1(\delta_{n-1} + \delta_n)) \mu I_3 \geq \left( \frac{1}{k_1 + \kappa} \right) I_3 \quad (B.52)$$

where $\kappa$ is defined in (2.63). It should be noted that to obtain the expression given in (2.63), the interval $0 \leq t \leq t_1$ and the minimum from the other intervals $t_{i-1} \leq t \leq t_i$ were used where $i = 2, 3, ..., n$. The lower bound on $P^{-1}(t)$ can now be formulated as

$$P^{-1}(t) \geq \left( \frac{1 + k_1 \kappa}{k_1} \right) I_3. \quad (B.53)$$

Finally, by invoking Lemma B.4 the following expression is obtained

$$P(t) \leq \lambda_{\max} \{ P(t) \} I_3 = \| P(t) \|_{\mathcal{L}_2} I_3 \leq \frac{k_1}{1 + k_1 \kappa} I_3. \quad (B.54)$$
**Bounds for \( \lambda(t) \):**

After applying the result given in (B.54) to (2.58), lower and upper bounds for \( \lambda(t) \) can be formulated as follows

\[
\lambda_1 \geq \lambda(t) \geq \lambda_1 \left( \frac{k_1 \kappa}{1 + k_1 \kappa} \right). \tag{B.55}
\]

**Upper bound for \( P^{-1}(t) \):**

After applying the lower bound in (B.55) to (2.43), the following expression is obtained

\[
P^{-1}(t) \leq P^{-1}(0) \exp \left(-\lambda_1 \left( \frac{k_1 \kappa}{1 + k_1 \kappa} \right) t \right)
+ \int_0^t \exp \left(-\lambda_1 \left( \frac{k_1 \kappa}{1 + k_1 \kappa} \right) (t - \sigma) \right) Y_{df}^T (\cdot) Y_{df} (\cdot) \, d\sigma,
\]

which can be further upper bounded by the following expression

\[
P^{-1}(t) \leq P^{-1}(0) + \left\| Y_{df}^T (\cdot) Y_{df} (\cdot) \right\|_{i\infty} I_3
\]

\[
\frac{\left( 1 - \exp \left(-\lambda_1 \left( \frac{k_1 \kappa}{1 + k_1 \kappa} \right) t \right) \right) (1 + k_1 \kappa)}{\lambda_1 k_1 \kappa}
\]

where the definition for the induced-infinity norm of a matrix is given in Definition A.1 in Appendix A. The bracketed term in (B.57) can be upper bounded as

\[
\frac{\left( 1 - \exp \left(-\lambda_1 \left( \frac{k_1 \kappa}{1 + k_1 \kappa} \right) t \right) \right) (1 + k_1 \kappa)}{\lambda_1 k_1 \kappa} \leq \frac{(1 + k_1 \kappa)}{\lambda_1 k_1 \kappa}; \tag{B.58}
\]

thus, (B.57) can be further bounded as

\[
P^{-1}(t) \leq P^{-1}(0) + \frac{\left\| Y_{df}^T (\cdot) Y_{df} (\cdot) \right\|_{i\infty} (1 + k_1 \kappa)}{\lambda_1 k_1 \kappa} I_3. \tag{B.59}
\]

Given that \( \tilde{\theta}_d(t) \) and \( \tilde{\theta}_d(t) \) are upper bounded by some known positive constants, it can be shown that \( \| Y_d(\cdot) \|_{i\infty} \) defined in (2.38) can also be upper bounded by some known positive constant. Given the definitions in (2.37) and (2.33), the standard inequality property of convolution in [4] can be used to prove that

\[
\| Y_{df} (\cdot) \|_{i\infty} \leq \gamma \| Y_d (\cdot) \|_{i\infty} \tag{B.60}
\]
where the upper bound $\|f(t)\| \leq \gamma$ for (2.33) has been used. Since $\|Y_{df}(\cdot)\|_{i\infty}$ in (B.60) is bounded, the following inequality can be applied to show that $\|Y_{df}(\cdot)^T Y_{df}(\cdot)\|_{i\infty}$ is bounded:

$$\|AB\|_i \leq \|A\|_i \|B\|_i \quad \forall A \in \mathbb{R}^{n \times m}, \forall B \in \mathbb{R}^{m \times p} \quad \text{(B.61)}$$

where $\|\cdot\|_i$ denotes any induced norm of a matrix. Since $\|Y_{df}(\cdot)^T Y_{df}(\cdot)\|_{i\infty}$ is bounded and $P^{-1}(0)$ is a positive-definite symmetric constant matrix, there exists a positive bounding constant $k_2 \in \mathbb{R}$ such that

$$P^{-1}(t) \leq k_2 I_3. \quad \text{(B.62)}$$

The result given in (2.75) can now be obtained from (B.53) and (B.62). $\square$

### B.1.5 Inequality Proofs

In this section, proofs for the inequalities given in Property 2.7 are provided. The following facts are exploited to facilitate these proofs

$$\begin{align*}
|\cos a - \cos b| &\leq 8 |\tanh (a - b)| & \forall a, b \in \mathbb{R}, \\
|\sin a - \sin b| &\leq 8 |\tanh (a - b)| & \forall a, b \in \mathbb{R}.
\end{align*} \quad \text{(B.63)}$$

**Lemma B.6** The following inequality is valid for the transformed ship dynamics of (2.95)

$$\|M^*(u) - M^*(w)\|_{i\infty} \leq \zeta_m \|\tanh (u - w)\| \quad \forall u, w \in \mathbb{R}^3. \quad \text{(B.64)}$$

**Proof:** After substituting the definition of $M^*(\cdot)$ given in (2.96) into (2.102), the following bound can be obtained

$$\|M^*(u) - M^*(w)\|_{i\infty} \leq \max_j \left\{ \tilde{M}_j^*(u, w) \right\} \quad \forall j = 1, 2, 3 \quad \text{(B.65)}$$

where $\tilde{M}^*(t) = [\tilde{M}_1^*(t) \quad \tilde{M}_2^*(t) \quad \tilde{M}_3^*(t)] \in \mathbb{R}^3$ is an auxiliary expression defined as follows

$$\tilde{M}_1^* = |(m_{11} - m_{22}) (\sin u_3 \cos u_3 - \sin w_3 \cos w_3)|$$

$$+ m_{23} |\sin u_3 - \sin w_3|$$

$$+ |(m_{11} (\cos^2 u_3 - \cos^2 w_3) + m_{22} (\sin^2 u_3 - \sin^2 w_3))|,$$

$$\tilde{M}_2^* = \tilde{M}_3^*.$$
\[ \tilde{M}_2^* = \left| (m_{11} - m_{22}) (\sin u_3 \cos w_3 - \sin w_3 \cos u_3) \right| \]
\[ + m_{23} |\cos u_3 - \cos w_3| \]
\[ + \left| (m_{11} (\sin^2 u_3 - \sin^2 w_3) + m_{22} (\cos^2 u_3 - \cos^2 w_3)) \right|, \]
\[ (B.67) \]
\[ \tilde{M}_3^* = |m_{23} (\cos u_3 - \cos w_3)| + |m_{23} (\sin u_3 - \sin w_3)|, \]
\[ (B.68) \]

and \( u_3(t), w_3(t) \) are the third elements of the vectors \( u(t), \ w(t) \in \mathbb{R}^3 \), respectively. After some algebraic manipulation, the expression given in (B.66) can be rewritten as follows

\[ \tilde{M}_1^* = \left| (m_{11} - m_{22}) ((\cos u_3 - \cos w_3) \sin u_3 + (\sin u_3 - \sin w_3) \cos w_3) \right| \]
\[ + m_{23} |\sin u_3 - \sin w_3| \]
\[ + \left| (m_{11} ((\cos u_3 - \cos w_3) \cos u_3 + (\cos u_3 - \cos w_3) \cos w_3) \right| \]
\[ + m_{22} ((\sin u_3 - \sin w_3) \sin u_3 + (\sin u_3 - \sin w_3) \sin w_3)) \right| \]
\[ (B.69) \]

which can be upper bounded as follows

\[ \tilde{M}_1^* \leq (|m_{11} - m_{22}| + 2m_{11}) |\cos u_3 - \cos w_3| \]
\[ + (m_{23} + 2m_{22} + |m_{11} - m_{22}|) |\sin u_3 - \sin w_3|. \]
\[ (B.70) \]

After making use of (B.63), \( \tilde{M}_1^*(t) \) can be further upper bounded by the following expression

\[ \tilde{M}_1^* \leq \zeta_{m_1} |\tanh (u_3 - w_3)| \leq \zeta_{m_1} \|Tanh (u - w)\| \]
\[ (B.71) \]

where \( \zeta_{m_1} \in \mathbb{R} \) is a positive bounding constant. Likewise, similar bounds can be obtained for \( \tilde{M}_2^*(t) \) and \( \tilde{M}_3^*(t) \) of (B.67) and (B.68), respectively. These bounds can then be used in (B.65) to obtain the result given in (B.64). \( \square \)

**Lemma B.7** The following inequality is valid for the transformed ship dynamics of (2.95)

\[ \|V_m(u, \dot{\eta}) - V_m(w, \dot{\eta})\|_{i\infty} \leq \zeta_{v_2} ||\dot{\eta}|| \|Tanh(u - w)\| \quad \forall u, w \in \mathbb{R}^3. \]
\[ (B.72) \]

**Proof:** After substituting the definition of \( V_m(\cdot) \) given in (2.96) into (2.102), the following bound can be obtained

\[ \|V_m(u, \dot{\eta}) - V_m(w, \dot{\eta})\|_{i\infty} \leq \max_j \left\{ \tilde{V}_{m_j}(u, w, \dot{\eta}) \right\} \quad \forall j = 1, 2, 3 \]
\[ (B.73) \]
where the elements of the vector $\tilde{V}_m(t) = [\tilde{V}_{m1}(t) \quad \tilde{V}_{m2}(t) \quad \tilde{V}_{m3}(t)] \in \mathbb{R}^3$ are defined as

\[
\tilde{V}_{m1} = \left| \dot{\psi} (m_{22} - m_{11}) (\sin u_3 \cos u_3 - \sin w_3 \cos w_3) \right|
+ \left| \dot{\psi} (m_{11} (\cos^2 u_3 - \cos^2 w_3) + m_{22} (\sin^2 u_3 - \sin^2 w_3)) \right|,
\]

(B.74)

\[
\tilde{V}_{m2} = \left| \dot{\psi} (m_{11} (\sin^2 u_3 - \sin^2 w_3) + m_{22} (\cos^2 u_3 - \cos^2 w_3)) \right|
+ \left| \dot{\psi} (m_{22} - m_{11}) (\sin u_3 \cos u_3 - \sin w_3 \cos w_3) \right|,
\]

(B.75)

\[
\tilde{V}_{m3} = \left| \dot{\psi} m_{23} (\cos u_3 - \cos w_3) \right| + \left| \dot{\psi} m_{22} (\sin u_3 - \sin w_3) \right|,
\]

(B.76)

and $u_3(t)$, $w_3(t)$ are the third elements of the vectors $u(t)$, $w(t) \in \mathbb{R}^3$, respectively. After some algebraic manipulation, the expression given in (B.74) can be rewritten as

\[
\tilde{V}_{m1} = \left| \dot{\psi} (m_{22} - m_{11})
((\cos u_3 - \cos w_3) \sin u_3 + (\sin u_3 - \sin w_3) \cos w_3) \right|
+ \left| \dot{\psi} (m_{11} ((\cos u_3 - \cos w_3) \cos u_3 + (\cos u_3 - \cos w_3) \cos w_3)
+ m_{22} ((\sin u_3 - \sin w_3) \sin u_3 + (\sin u_3 - \sin w_3) \sin w_3)) \right|
\]

(B.77)

which can be upper bounded as

\[
\tilde{V}_{m1} \leq \left| \dot{\psi} \right| |m_{22} - m_{11}| (|\cos u_3 - \cos w_3| + |\sin u_3 - \sin w_3|)
+ 2 \left| \dot{\psi} \right| m_{11} |\cos u_3 - \cos w_3| + 2 \left| \dot{\psi} \right| m_{22} |\sin u_3 - \sin w_3|.
\]

(B.78)

After making use of (B.63), $\tilde{V}_{m1}(t)$ can be further upper bounded by the following expression

\[
\tilde{V}_{m1} \leq \zeta_{\rho 1} \left| \dot{\psi} \right| |\tanh (u_3 - w_3)| \leq \zeta_{\rho 1} \left\| \dot{\eta} \right\| \left\| \tanh (u - w) \right\|.
\]

(B.79)

where $\zeta_{\rho 1} \in \mathbb{R}$ is a positive bounding constant. Likewise, similar bounds can be obtained for $\tilde{V}_{m2}(t)$ and $\tilde{V}_{m3}(t)$ of (B.75) and (B.76), respectively. These bounds can then be used in (B.73) to obtain the result given in (B.72). □
Lemma B.8 The following inequality is valid for the ship dynamics of (2.95)

\[ \| F_1(u) - F_1(w) \| \leq \zeta_{f2} \| \tanh(u - w) \| \quad \forall u, w \in \mathbb{R}^3. \]  

**Proof:** The proof of (B.80) is straightforward from (2.92), (2.96), and the proof of Lemmas B.6 and B.7. □

### B.1.6 Control Signal Bounds

**Lemma B.9** The auxiliary control signal \( \chi(t) \) given in (2.130) can be upper bounded by the following expression

\[ \| \chi \| \leq \zeta_1 \| \varrho \| + \zeta_2 \| z \|^2 + \zeta_3 \| z \|^3 + \zeta_4 \| z \|^4 + \zeta_5 \| z \|^5 + \zeta_6 \| r \| \| z \| \]  

where the composite state vector \( \varrho(t) \) was defined in (2.137) and \( \zeta_i \in \mathbb{R}, i = 1, \ldots, 6 \) are some positive bounding constants that depend on the system parameters and the desired trajectory.

**Proof:** To prove Lemma B.9, the norm of (2.130) is used to obtain the following upper bound

\[ \| \chi \| \leq \| M^*(\eta) \| \| \cosh^{-2}(e) \| \| (r - \tanh(e) - z) \| + \| M^*(\eta) \| \| T \| \| (z - \tanh(e)) \| 
+ \| V_m(\eta, \dot{n}_d + \tanh(e) + z) \| \| (\tanh(e) + z) \| \]  

(B.82)

\[ + \| V_m(\eta, \dot{n}_d) \| \| (\tanh(e) + z) \| 
+ \| V_m(\eta, r) \| \| (\dot{n}_d + \tanh(e) + z) \|. \]

After utilizing (2.97), (2.100), and the fact that \( |\cosh^{-2}(\cdot)| \leq 1 \), (B.82) can be upper bounded as follows

\[ \| \chi \| \leq m_2 (\| r \| + \| \tanh(e) \| + \| z \|) 
+ m_2 \| T \| (\| z \| + \| \tanh(e) \|) 
+ \zeta_{v1} (\| \dot{n}_d \| + \| \tanh(e) \| + \| z \|) (\| \tanh(e) \| + \| z \|) \]  

(B.83)

\[ + \zeta_{v1} \| \dot{n}_d \| (\| \tanh(e) \| + \| z \|) 
+ \zeta_{v1} \| r \| (\| \dot{n}_d \| + \| \tanh(e) \| + \| z \|). \]
To facilitate further analysis, the definition of $T(\cdot)$ given in (2.132) can be used to prove that the following inequality is valid

$$\|T\| \leq (1 + \|z\|^2)^2. \quad (B.84)$$

The result given in (B.81) is now straightforward after substituting the inequality given in (B.84) into (B.83) for $\|T(\cdot)\|$ and then using (2.137) and the facts that $|\tanh(\cdot)| \leq 1$ and $\dot{\eta}_d(t)$ is bounded. □

**Lemma B.10** The auxiliary control signal $\hat{Y}(\cdot)$ given in (2.129) can be upper bounded as follows

$$\|\hat{Y}\| \leq \zeta_7 \|\theta\| \quad (B.85)$$

where $\zeta_7$ is some positive bounding constant that depends on the system parameters and the desired trajectory.

**Proof:** To prove the result given in (B.85), we substitute (2.127) into (2.129) for $Y_d(\cdot)\phi$ to obtain the following expression

$$\hat{Y} = [M^*(\eta) - M^*(\eta_d)] \ddot{\eta}_d + \left[V_m(\eta, \dot{\eta}_d) - V_m(\eta_d, \dot{\eta}_d)\right] \dot{\eta}_d$$

$$+ [F_1(\eta)\dot{\eta} - F_1(\eta)\dot{\eta}_d] + [F_1(\eta)\dot{\eta}_d - F_1(\eta_d)\dot{\eta}_d]$$

where the term $F_1(\eta)\dot{\eta}_d$ has been added and subtracted. After utilizing (2.101), (2.102), (2.106), and (2.120), an upper bound can be formulated for (B.86) as follows

$$\|\hat{Y}\| \leq \zeta_m \|\tanh(e)\| \|\dot{\eta}_d\| + \zeta_{v_2} \|\tanh(e)\| \|\ddot{\eta}_d\|^2$$

$$\zeta_{f_1} (\|r\| + \|\tanh(e)\| + \|z\|) + \zeta_f \|\tanh(e)\| \|\dddot{\eta}_d\|. \quad (B.87)$$

The result given in (B.85) follows directly from (B.87), given (2.137) and the fact that $\dot{\eta}_d(t)$ and $\ddot{\eta}_d(t)$ are bounded. □

**Remark B.1** Note that if the mooring effects are included in the dynamic model of the surface ship, the expressions given in (2.138) and (B.85) could not be used to upper bound $\hat{Y}(t)$.

**B.1.7 Matrix Property**

**Lemma B.11** Given the definition for the Jacobian-type matrix $B(q)$ given in (2.177), the following property holds

$$B^T B = I_3 \quad (B.88)$$

where $I_3$ is the $3 \times 3$ identity matrix.
**Proof:** The definition for $B(q)$ given in (2.177) can be written in the following form

$$B = \begin{bmatrix}
-qv_1 & -qv_2 & -qv_3 \\
q_0 & -qv_3 & qu_2 \\
v_3 & q_0 & -qv_1 \\
-qv_2 & qv_1 & q_0
\end{bmatrix}. \quad (B.89)$$

Based on the structure of (B.89), we can utilize (2.170) to prove (B.88). □

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**B.2 Chapter 3 Definitions and Lemmas**

**B.2.1 Supplemental Definitions**

**Definition B.1** The measurable functions $\Omega_{ij}(\cdot) \forall i = 1, 2, 3, 4$ introduced in (3.75) are defined as follows

$$\begin{align*}
\Omega_{1j} &= \frac{dI_{dj}}{dq} \ddot{q} + \frac{B_j(q, I_j)}{L_j(q, I_j)} \ddot{q} + \frac{dI_{dj}}{d\tau_d} \left(\dddot{M}_m \dot{q}_d + (\dddot{M}_m + k_s)q_d\right) \\
&\quad + k_s \alpha \dot{q}_d + \frac{\partial \tau_d}{\partial \dot{M}_m} \dddot{M}_m + \frac{\partial \tau_d}{\partial \theta_m} \dot{\theta}_m + \frac{\partial \tau_d}{\partial \dot{q}} \dot{q}
\end{align*}$$

$$\Omega_{2j} = -\frac{dI_{dj}}{d\tau_d} \frac{\partial \tau_d}{\partial \dot{q}}. \quad (B.90)$$

$$\Omega_{3j} = \frac{I_j}{L_j(q, I_j)}$$

$$\Omega_{4j} = \frac{dI_{dj}}{d\tau_d} \frac{\partial \tau_d}{\partial q} \sum_{j=1}^{m} \tau_j(q, I_j).$$

---

**B.2.2 Stability Analysis for Projection Cases**

**Lemma B.12** Given the mechanical and electrical closed-loop error systems in (3.69) and (3.78) as well as the adaptive update laws given in (3.80-3.82), an upper bound for the expression given in (3.89) can be formulated as follows

$$\dot{V} \leq -\lambda_3 \|x\|^2. \quad (B.91)$$
Proof: After utilizing (3.65), (3.77), and (3.80), the expression in (3.89) can be rewritten as follows

\[ \dot{V} = -K_s r^2 - \sum_{j=1}^{m} K_{ej} M_m n_j^2 \]

\[ + \tilde{M}_m \left( \sum_{j=1}^{m} \left( -\tilde{M}_m^{-1} \left( \Omega_{4j} + \Omega_{2j} W_m \theta_m + u_j \right) n_j \right) \right) \]

\[ + \tilde{M}_m \left( (\tilde{q}_d + \alpha \dot{\theta}) r - \Gamma_1^{-1} \dot{\tilde{M}}_m \right) . \]  

(B.92)

To substitute (3.81) and (3.82) into (B.126) for \( \dot{M}_m (t) \), the three following cases must be considered.

Case 1: \( \dot{M}_m (t) > M_m \)

When \( \dot{M}_m (t) > M_m \), the first equation in (3.81) can be used to express \( V (t) \) as follows

\[ V (t) \leq -K_s r^2 - \sum_{j=1}^{m} K_{ej} M_m n_j^2 \leq -\lambda_3 \| x \|^2 \]  

(B.93)

where (3.91) and (3.92) have been utilized. Thus, for Case 1, we can conclude that (3.89) reduces to the expression given in (B.91). In addition, the direction in which the estimate \( \dot{M}_m (t) \) is updated for Case 1 is irrelevant, since the worse case scenario is that \( \dot{M}_m (t) \) will move toward \( M_m \) which will be covered in Cases 2 and 3.

Case 2: \( \dot{M}_m = M_m \) and \( \Omega_m \geq 0 \)

When \( \dot{M}_m (t) = M_m \) (i.e., \( \dot{M}_m (t) \) equals the lower bound) and \( \Omega_m \) is nonnegative, the second equation in (3.81) alongside (3.82) can be used to express \( \dot{V} (t) \) as follows

\[ \dot{V} \leq -K_s r^2 - \sum_{j=1}^{m} K_{ej} M_m n_j^2 \leq -\lambda_3 \| x \|^2 . \]  

(B.94)

where (3.91) and (3.92) have been utilized. Thus, for Case 2, we can conclude that (3.89) reduces to the expression given in (B.91). Geometrically, \( \dot{M}_m (t) \) is updated such that it moves away from the boundary case of \( \dot{M}_m (t) = M_m \) and into the region given by \( \dot{M}_m (t) > M_m \).

Case 3: \( \dot{M}_m = M_m \) and \( \Omega_m < 0 \)

When \( \dot{M}_m (t) = M_m \) (i.e., \( \dot{M}_m (t) \) equals the lower bound) and \( \Omega_m \) is negative, the third equation in (3.81) alongside (3.82) can be utilized in
order to express \( \dot{V}(t) \) as follows

\[
\dot{V} = -K_s r^2 - \sum_{j=1}^{m} K_{ej} M_m \eta_j^2 \\
+ \mathcal{M}_m \left( \sum_{j=1}^{m} \left( -\mathcal{M}_m^{-1} \left( \Omega_{4j} + \Omega_{2j} W_m \dot{\theta}_m + u_j \right) \eta_j \right) + (\ddot{q}_d + \alpha \dot{r}) r \right).
\]

(B.95)

By definition, the following relationships must hold for this case

\[
\mathcal{M}_m(t) = M_m - \mathcal{M}_m \geq 0.
\]

(B.96)

Moreover, the parenthesized term in (B.95) can be written as

\[
\sum_{j=1}^{m} \left( -\mathcal{M}_m^{-1} \left( \Omega_{4j} + \Omega_{2j} W_m \dot{\theta}_m + u_j \right) \eta_j \right) + (\ddot{q}_d + \alpha \dot{r}) r = \Gamma_1^{-1} \Omega_m < 0
\]

where we have utilized (3.82) and the fact that \( \Gamma_1 > 0 \) and \( \Omega_m < 0 \). From (B.96) and (B.97), it is clear that the second line of (B.95) is negative, and hence, (B.95) can be lower bounded as follows

\[
\dot{V} \leq -K_s r^2 - \sum_{j=1}^{m} K_{ej} M_m \eta_j^2 \leq -\lambda_3 \|x\|^2
\]

(B.98)

where (3.91) and (3.92) have been utilized. Thus, for Case 3, we can conclude that (3.89) reduces to the expression in given in (B.91). Geometrically, this case ensures that \( \mathcal{M}_m(t) \) stays on the boundary defined by \( \mathcal{M}_m(t) = M_m \) as long as it is not directed to move into the prescribed region for \( M_m \) as defined in Remark 3.4 of Chapter 3. □

### B.2.3 Dynamic Terms for a 6-DOF AMB System

The dynamic equation for the center of mass of the circular cylinder rotor with respect to the fixed coordinate frame \((x_b, y_b, z_b)\) can be written as follows [1], [3]

\[
M^*(q^*) \ddot{q}^* + V^*_{m}(q^*, \dot{q}^*) \dot{q}^* + G^* = F^*
\]

(B.99)

where \( q^* = [x_b^o, y_b^o, z_b^o, \phi, \theta, \psi]^T \in \mathbb{R}^6 \), \((x_b^o, y_b^o, z_b^o)\) denotes the position of \( o_b \) with respect to \( o_b \) (see Figure 3.11), \((\phi, \theta, \psi)\) denote the Euler angles,\(^1\) and \( F^* \in \mathbb{R}^6 \) is the force/torque vector. The dynamic terms

---

\(^1\)The orientation of the rotor is given by a series of three rotations. Assuming that the rotor is initially oriented so that the moving coordinate frame \((x_o, y_o, z_o)\) is aligned
in (B.99) are defined as follows [1], [3]

\[ M^* (q^*) = \text{diag} \{ mI_3, \Gamma^{-T} H \Gamma^{-1} \} \in \mathbb{R}^{6 \times 6} \]  
(B.100)

\[ G^* = [0, 0, mg, 0, 0, 0]^T \in \mathbb{R}^6 \]

\[ V_m^* (q^*, \dot{q}^*) = \text{diag} \left\{ mR \left( -R^{-1} \dot{R} R^{-1} + \Gamma^{-1} S \left( \hat{\Omega} \right) R^{-1} \right), \right. \]
\[ \Gamma^{-T} \left( -H \Gamma^{-1} \dot{\Gamma} \Gamma^{-1} + \Gamma^{-1} S \left( \hat{\Omega} \right) H \Gamma^{-1} \right) \} \in \mathbb{R}^{6 \times 6} \]  
(B.101)

where \( m \) is the mass of the rotor, \( g \) is the gravitational constant, \( I_3 \) represents the \( 3 \times 3 \) identity matrix, \( S \left( \hat{\Omega} \right) \in \mathbb{R}^{3 \times 3} \) is a skew-symmetric matrix with \( \hat{\Omega} = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T \), \( H \in \mathbb{R}^{3 \times 3} \) denotes the diagonal inertia matrix of the rotor with respect to the \( (x_o, y_o, z_o) \) axes, and \( \Gamma, R \in \mathbb{R}^{3 \times 3} \) are defined as follows

\[ \Gamma = \begin{bmatrix} 1 & \sin(\phi) \tan(\theta) & \cos(\phi) \tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) \sec(\theta) & \cos(\phi) \sec(\theta) \end{bmatrix} \]  
(B.102)

\[ R = \begin{bmatrix} c(\psi) c(\theta) & c(\psi) s(\theta) s(\phi) - s(\psi) c(\phi) & c(\psi) s(\theta) c(\phi) + s(\psi) s(\phi) \\ c(\psi) s(\theta) s(\phi) + s(\psi) c(\phi) & s(\psi) s(\theta) \cos(\phi) - c(\psi) s(\phi) & \cos(\phi) \sin(\phi) \\ -s(\theta) & c(\theta) s(\phi) & c(\theta) c(\phi) \end{bmatrix} \]  
(B.103)

with \( c(\cdot), s(\cdot) \) denoting the cosine and sine of the argument, respectively. The kinematic relationship between the position vector \( q(t) \) defined in Section 3.4.1 of Chapter 3 and \( q^* \) of (B.99) is defined as follows

\[ q = \begin{bmatrix} x_b^o + (d - z_b^o) \left( c(\psi) \tan(\theta) + \frac{s(\psi) \tan(\phi)}{c(\theta)} \right) \\ y_b^o + (d - z_b^o) \left( s(\psi) \tan(\theta) + \frac{c(\psi) \tan(\phi)}{c(\theta)} \right) \\ x_b^o - z_b^o \left( c(\psi) \tan(\theta) + \frac{s(\psi) \tan(\phi)}{c(\theta)} \right) \\ y_b^o - z_b^o \left( s(\psi) \tan(\theta) + \frac{c(\psi) \tan(\phi)}{c(\theta)} \right) \\ z_b^o - \frac{L}{2} c(\theta) c(\phi) \end{bmatrix} = f(q^*) \]  
(B.104)

with the fixed coordinate frame \( (x_b, y_b, z_b) \), the rotations are defined as follows [1]: (1) a rotation \( \psi \) about the \( z_o \) axis, (2) a rotation \( \theta \) about the current \( y_o \) axis, and (3) a rotation \( \phi \) about the current \( x_o \) axis.
where \( d \in \mathbb{R} \) denotes the distance between the origins \( o_b \) and \( o_t \) measured along the \( z_b \) axis (see Figure 3.11), and \( L \in \mathbb{R} \) is the length of the rotor. From (B.104), the dynamic quantities defined in (3.60) can be explicitly calculated as follows

\[
M(q) = J^{-T}(q)M^*(q)J^{-1}(q)
\]

\[
G(q) = J^{-T}(q)G^*
\]

\[
V_m(q, \dot{q}) = J^{-T}(q) \left[ -M^*(q)J^{-1}(q)\dot{J}(q) + V_m^*(q, \dot{q}) \right] J^{-1}(q)
\]

\[
\tilde{F}(q) = J^{-T}(q)F^*
\]

\[
J(q) = \frac{\partial f(q)}{\partial q}
\]

\[
q^*(q) = f^{-1}(q)
\]

where \( f(\cdot) \) was defined in (B.104).

**B.2.4 Partial Derivative Definitions**

**Definition B.2** The partial derivative expressions required to complete (3.143) are defined as follows

\[
\frac{\partial f_d}{\partial q_d} = M(q)
\]

\[
\frac{\partial f_d}{\partial \dot{q}_d} = M(q)\alpha + V_m(q, \dot{q}) + k_s\lambda_2(q)I_6
\]

\[
\frac{\partial f_d}{\partial q_d} = V_m(q, \dot{q})\alpha + k_s\lambda_2(q)\alpha I_6
\]

\[
\frac{\partial f_d}{\partial q} = \frac{\partial w}{\partial q} + k_s \frac{\partial \lambda_2}{\partial q} r - k_s\lambda_2(q)\alpha
\]

\[
\frac{\partial f_d}{\partial \dot{q}} = \frac{\partial w}{\partial \dot{q}} - k_s\lambda_2(q)I_6
\]

where \( I_6 \) denotes the \( 6 \times 6 \) identity matrix. In (B.106), the partial derivative terms \( \frac{\partial w(\cdot)}{\partial q} \) and \( \frac{\partial w(\cdot)}{\partial \dot{q}} \) are defined as follows

\[
\frac{\partial w}{\partial q} = A_1(q, \zeta_1) + A_2(q, \zeta_2) - V_m(q, \dot{q})\alpha + \frac{\partial G(q)}{\partial q}
\]
where the auxiliary terms $A_1(q, \zeta_1)$, $A_2(q, \zeta_2)$, and $A_3(q, \zeta_3) \in \mathbb{R}$ are defined as follows

\[
A_1(q, \zeta_1) = \begin{bmatrix}
\frac{\partial M(q)}{\partial q_1} \zeta_1, & \ldots, & \frac{\partial M(q)}{\partial q_6} \zeta_1,
\end{bmatrix}, \quad \zeta_1 = \dot{q}_d + \alpha \dot{e} \tag{B.109}
\]

\[
A_2(q, \zeta_2) = \begin{bmatrix}
\frac{\partial V_m(q, \dot{q})}{\partial q_1} \zeta_2, & \ldots, & \frac{\partial V_m(q, \dot{q})}{\partial q_6} \zeta_2,
\end{bmatrix}, \quad \zeta_2 = \dot{q}_d + \alpha e \tag{B.110}
\]

\[
A_3(q, \zeta_3) = \begin{bmatrix}
\frac{\partial V_m(q, \dot{q})}{\partial \dot{q}_1} \zeta_3, & \ldots, & \frac{\partial V_m(q, \dot{q})}{\partial \dot{q}_6} \zeta_3,
\end{bmatrix}, \quad \zeta_3 = \dot{q}_d + \alpha e. \tag{B.111}
\]

\section*{B.3 Chapter 4 Lemmas}

\subsection*{B.3.1 Inequality Lemma}

\textbf{Lemma B.13} Based on the definition of $sat_\beta(\cdot)$ given in (4.13), the following inequality can be obtained

\[
(\xi_{1i} - \xi_{2i})^2 \geq (sat_{\beta_i}(\xi_{1i}) - sat_{\beta_i}(\xi_{2i}))^2 \quad \forall |\xi_{1i}| \leq \beta_i, |\xi_{2i}| \leq \beta_i, i = 1, 2, \ldots, m. \tag{B.112}
\]

\textbf{Proof:} To prove the inequality given in (B.112), the proof is divided into three possible cases.

\textbf{Case 1:} $|\xi_{1i}| \leq \beta_i, |\xi_{2i}| \leq \beta_i$

For this case, the definition of $sat_{\beta_i}(\cdot)$ given in (4.13) can be used to prove that

\[
sat_{\beta_i}(\xi_{1i}) = \xi_{1i}, \quad sat_{\beta_i}(\xi_{2i}) = \xi_{2i}. \tag{B.113}
\]

After substituting (B.113) into (B.112), the following expression can be obtained

\[
(\xi_{1i} - \xi_{2i})^2 = (sat_{\beta_i}(\xi_{1i}) - sat_{\beta_i}(\xi_{2i}))^2 \quad \forall |\xi_{1i}| \leq \beta_i, |\xi_{2i}| \leq \beta_i, \tag{B.114}
\]

and hence, the inequality given in (B.112) is true for Case 1.

\textbf{Case 2:} $|\xi_{1i}| \leq \beta_i, |\xi_{2i}| > \beta_i$

For this case, the definition of $sat_{\beta_i}(\cdot)$ given in (4.13) can be used to prove that

\[
(\xi_{2i} + \beta_i) \geq 2\xi_{1i} \quad \forall |\xi_{1i}| \leq \beta_i, |\xi_{2i}| > \beta_i. \tag{B.115}
\]
After multiplying (B.115) by \((\xi_{2i} - \beta_i)\) and then simplifying the left-hand side of the inequality, (B.115) can be rewritten as

\[
\xi_{2i}^2 - \beta_i^2 \geq 2 (\xi_{2i} - \beta_i) \xi_{1i}
\] (B.116)

where the fact that \(\xi_{2i} - \beta_i > 0\) has been used for this case. After adding \(\xi_{1i}^2\) to (B.116) and then rearranging the resulting expression, the following inequality can be obtained

\[
\xi_{1i}^2 - 2\xi_{1i} \xi_{2i} + \xi_{2i}^2 \geq \xi_{1i}^2 - 2\beta_i \xi_{1i} + \beta_i^2.
\] (B.117)

Based on the expression given in (B.117), the following facts

\[
sat_{\beta_i}(\xi_{1i}) = \xi_{1i} \quad sat_{\beta_i}(\xi_{2i}) = \beta_i
\] (B.118)

can be used to prove that

\[
(\xi_{1i} - \xi_{2i})^2 \geq (sat_{\beta_i}(\xi_{1i}) - sat_{\beta_i}(\xi_{2i}))^2 \quad \forall |\xi_{1i}| \leq \beta_i, \xi_{2i} > \beta_i.
\] (B.119)

**Case 3:** \(|\xi_{1i}| \leq \beta_i, \xi_{2i} < -\beta_i\)

For this case, the definition of \(sat_{\beta_i}(\cdot)\) given in (4.13) can be used to prove that

\[
(\xi_{2i} - \beta_i) \leq 2\xi_{1i} \quad \forall |\xi_{1i}| \leq \beta_i, \xi_{2i} < -\beta_i.
\] (B.120)

After multiplying both sides of (B.120) by \((\xi_{2i} + \beta_i)\) and then simplifying the left-hand side of the inequality, the following inequality can be obtained

\[
\xi_{2i}^2 - \beta_i^2 \geq 2 (\xi_{2i} + \beta_i) \xi_{1i}
\] (B.121)

where the fact that \(\xi_{2i} + \beta_i < 0\) has been utilized for this case. After adding \(\xi_{1i}^2\) to (B.121) and then rearranging the resulting expression, the following inequality can be obtained

\[
\xi_{1i}^2 - 2\xi_{1i} \xi_{2i} + \xi_{2i}^2 \geq \xi_{1i}^2 + 2\beta_i \xi_{1i} + \beta_i^2.
\] (B.122)

Based on the inequality given in (B.122), the following facts

\[
sat_{\beta_i}(\xi_{1i}) = \xi_{1i} \quad sat_{\beta_i}(\xi_{2i}) = -\beta_i
\] (B.123)

can be utilized to prove that

\[
(\xi_{1i} - \xi_{2i})^2 \geq (sat_{\beta_i}(\xi_{1i}) - sat_{\beta_i}(\xi_{2i}))^2 \quad \forall |\xi_{1i}| \leq \beta_i, \xi_{2i} < -\beta_i;
\] (B.124)

hence, (B.112) is true for all possible cases. \(\square\)
B.3.2 Stability Analysis for Projection Cases

Lemma B.14 Given the closed-loop dynamics in (4.102), (4.113), and (4.114), if \( \hat{\theta}_m(0) \in \text{int}(\Lambda) \), then \( \hat{\theta}_m(t) \) never leaves the region \( \Lambda \) described in Property 4.6, \( \forall t \geq 0 \), and an upper bound for the expression given in (4.117) can be formulated as follows

\[
\dot{V} \leq -kr^T r - k\lambda^T e_{\lambda}.
\]  

(B.125)

Proof: To prove Lemma B.14, the closed-loop dynamics given in (4.102) and (4.113) can be substituted into (4.117) for \( \dot{\hat{\theta}}_1(t) \) and \( \dot{\hat{\theta}}_2(t) \), respectively, to obtain the following expression

\[
\dot{V} \leq -k (r^T r + e_{\lambda}^T e_{\lambda}) - e_{\lambda}^T \frac{h^T \hat{\theta}_m}{h^T \theta_m} Y_2 \hat{\theta}_2 + \hat{\theta}_m^T \frac{\dot{h}}{h^T \theta_m} \hat{\theta}_m.
\]  

(B.126)

To substitute (4.114) and (4.112) into (B.126) for \( \hat{\theta}_m(t) \), the following three cases must be considered.

Case 1: \( \hat{\theta}_m(t) \in \text{int}(\Lambda) \)

When the parameter estimate \( \hat{\theta}_m(t) \) lies in the interior of the convex region \( \Lambda \), described in Property 4.6, (B.126) can be expressed as follows

\[
\dot{V} \leq -k (r^T r + e_{\lambda}^T e_{\lambda}) - e_{\lambda}^T \frac{h^T \hat{\theta}_m}{h^T \theta_m} Y_2 \hat{\theta}_2 + \hat{\theta}_m \left( \frac{h}{h^T \theta_m} [Y_2 \hat{\theta}_2]^T e_{\lambda} \right).
\]  

(B.127)

Thus, for Case 1, (4.117) reduces to the expression in given in (B.125). In addition, the direction in which the estimate \( \hat{\theta}_m(t) \) is updated for Case 1 is irrelevant, since the worst case scenario is that \( \hat{\theta}_m(t) \) will move towards the boundary of the convex region denoted by \( \partial(\Lambda) \).

Case 2: \( \hat{\theta}_m(t) \in \partial(\Lambda) \) and \( \mu_1^T(t) \hat{\theta}_m(t) \leq 0 \)

When the parameter estimate \( \hat{\theta}_m(t) \) lies on the boundary of the convex region \( \Lambda \) described in Property 4.6 and \( \mu_1^T(t) \hat{\theta}_m(t) \leq 0 \), then (4.117) can be concluded that (4.117) reduces to the expression in given in (B.125). In addition, the vector \( \mu_1(t) \) has a zero or nonzero component perpendicular to the boundary \( \partial(\Lambda) \) at \( \hat{\theta}_m(t) \) that points in the direction towards the \( \text{int}(\Lambda) \). Geometrically, this means that \( \hat{\theta}_m(t) \) is updated such that it either moves towards the \( \text{int}(\Lambda) \) or remains on the boundary. Hence, \( \hat{\theta}_m(t) \) never leaves \( \Lambda \).

Case 3: \( \hat{\theta}_m(t) \in \partial(\Lambda) \) and \( \mu_1^T(t) \hat{\theta}_m(t) > 0 \)

When the parameter estimate \( \hat{\theta}_m(t) \) lies on the boundary of the convex region \( \Lambda \) described in Property 4.6 and \( \mu_1^T(t) \hat{\theta}_m(t) > 0 \), then (4.117) can be expressed as

\[
\dot{V} \leq -k (r^T r + e_{\lambda}^T e_{\lambda}) - \hat{\theta}_m^T (-\mu_1 + P_r^t(\mu_1)).
\]  

(B.128)
where (4.112) was utilized. Based on (B.128), Property 4.6 can be used to conclude that

\[
\dot{V}_1 \leq -k \left( r^T r + e^T_\lambda e_\lambda \right) - \tilde{\theta}_m^T \left( - (P_r^\perp(\mu_1) + P_r^t(\mu_1)) + P_r^t(\mu_1) \right) \tag{B.129}
\]

\[
\leq -k \left( r^T r + e^T_\lambda e_\lambda \right) + \tilde{\theta}_m^T P_r^\perp(\mu_1).
\]

Because \( \hat{\theta}_m(t) \in \partial(\Lambda) \), and \( \theta_m \) must lie either on the boundary or in the interior of \( \Lambda \), then the convexity of \( \Lambda \) implies that \( \hat{\theta}_m(t) \) will either point tangent to \( \partial(\Lambda) \) or towards \( \text{int}(\Lambda) \) at \( \hat{\theta}_m(t) \). That is, \( \hat{\theta}_m(t) \) will have a component in the direction of \( \hat{\theta}_m^\perp(t) \) that is either zero or negative. In addition, since \( P_r^\perp(\mu_1) \) points away from \( \text{int}(\Lambda) \), the following inequality can be determined

\[
\tilde{\theta}_m^T P_r^\perp(\mu_1) \leq 0. \tag{B.130}
\]

The inequality given in (B.130) can now be used to simplify the expression given in (B.129) to the expression given in (B.125). Furthermore, since \( \hat{\theta}_m(t) = P_r^t(\mu_1) \), the parameter estimate \( \hat{\theta}_m(t) \) is ensured to be updated such that it moves tangent to \( \partial(\Lambda) \). Hence, \( \hat{\theta}_m(t) \) never leaves \( \Lambda \). □

### B.3.3 Boundedness Lemma

**Lemma B.15** *The second time derivative of the force tracking error \( e_\lambda(t) \) defined in (4.133) is bounded (i.e., \( \ddot{e}_\lambda \in L_\infty \)).*

**Proof:** To prove Lemma B.15, the open-loop dynamics for the force tracking error \( e_\lambda(t) \) of (4.133) are rewritten as follows

\[
\dot{e}_\lambda = \lambda_d - \Pi^{-T} \tau_2 + \Pi^{-T} \left( \tilde{V}_{m21} \dot{u}_1 + \tilde{N}_2 \right)
+ \Pi^{-T} \tilde{M}_{21} \tilde{M}_{11}^{-1} \left[ \tau_1 - \tilde{V}_{m11} \dot{u}_1 - \tilde{N}_1 \right]. \tag{B.131}
\]
After taking the time derivative of (B.131), the following expression for \( \ddot{e}_\lambda(t) \) can be obtained

\[
\ddot{e}_\lambda = \dot{\lambda}_d - \frac{d}{dt} \left( \Pi^{-T} \right) \ddot{\tau}_2 - \Pi^{-T} \ddot{\tau}_2 + \frac{d}{dt} \left( \Pi^{-T} \right) \left( \dot{V}_{m21} \ddot{u}_1 + \ddot{N}_2 \right) \\
+ \Pi^{-T} \left( \dot{\dot{V}}_{m21} \ddot{u}_1 + \ddot{V}_{m21} \dot{u}_1 + \dddot{N}_2 \right) \\
+ \frac{d}{dt} \left( \Pi^{-T} \right) \dot{M}_{21} \dot{M}^{-1}_{11} \left( \ddot{\tau}_1 - \dot{V}_{m11} \ddot{u}_1 - \ddot{N}_1 \right) \\
+ \Pi^{-T} \dot{M}_{21} \dot{M}^{-1}_{11} \left( \ddot{\tau}_1 - \dot{V}_{m11} \ddot{u}_1 - \ddot{N}_1 \right) \\
+ \Pi^{-T} \dot{M}_{21} \dot{M}^{-1}_{11} \left( \dddot{\tau}_1 - \dddot{V}_{m11} \dddot{u}_1 - \dddot{N}_1 \right).
\]

(B.132)

Since \( u_1(t), \dot{u}_1(t), \) and \( \ddot{u}_1(t) \in L_\infty \) from the proof of Theorem 4.3, standard signal chasing arguments can be used to prove that all of the signals on the right-hand side of (B.132) are bounded, however, special care must be directed towards proving that \( \dddot{\tau}_1(t), \dddot{\tau}_2(t) \in L_\infty \). To this end, the time derivative of \( \dddot{\tau}_1(t) \) given in (4.107) can be obtained as follows

\[
\dddot{\tau}_1 = \dot{Y}_1 \dot{\theta}_1 + Y_1 \dddot{\theta}_1 - k \dot{\tau}.
\]

(B.133)

Since the desired position trajectory is assumed to be bounded up to its third time derivative and \( e(t), \dot{e}(t), \ddot{e}(t) \in L_\infty \), the regression matrices \( Y_1(\cdot), \dot{Y}_1(\cdot) \in L_\infty \). Since \( \dot{\theta}_1(t), \dot{\tau}(t) \in L_\infty \), (B.133) can be used to prove that \( \dddot{\tau}_1(t) \in L_\infty \). After taking the time derivative of \( \dddot{\tau}_2(t) \) given in (4.107), the following expression can be obtained

\[
\dddot{\tau}_2 = \frac{d}{dt} \left( \frac{1}{h^T \dot{\theta}_m} \right) \Pi^T \left( Y_2 \dot{\theta}_2 \right) + \frac{1}{h^T \dot{\theta}_m} \Pi^T \left( Y_2 \dddot{\theta}_2 \right) \\
+ \frac{1}{h^T \dot{\theta}_m} \Pi^T \frac{d}{dt} \left( Y_2 \dot{\theta}_2 \right) + \Pi^T \frac{k}{m_d} \dddot{e}_\lambda + \Pi^T \frac{k}{m_d} \dddot{e}_\lambda.
\]

(B.134)

From the proof of Theorem 4.3, \( e_\lambda(t), \dot{e}_\lambda(t), \Pi^T(x), \Pi^T(x), Y_2(\cdot) \dot{\theta}_2(t), \) and \( \frac{d}{dt} \left( Y_2 \dot{\theta}_2 \right) \) are all bounded. Since the projection-based parameter update law for \( \dot{\theta}_m(t) \) given in (4.111) ensures that \( h^T(u) \dot{\theta}_m(t) > 0 \) (see the development in Lemma B.14), \( \frac{d}{dt} \left[ (h^T \dot{\theta}_m) \right]^{-1} \) also remains bounded; therefore,
(B.134) can be used to prove that \( \hat{r}_2(t) \in L_\infty \). The fact that all of the terms on the right-hand side of (B.132) are bounded can now be used to prove that \( \hat{e}_\lambda(t) \in L_\infty \). \( \square \)

### B.3.4 State-Dependent Disturbance Bound

**Lemma B.16** Given a function \( F(x) \in \mathbb{R}^{m \times n} \) and a variable \( x(t) \in \mathbb{R}^k \) such that

\[
\frac{\partial F_{ij}(x)}{\partial x} \in L_\infty \quad \text{if} \quad x(t) \in L_\infty, \quad \forall i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{B.135}
\]

where \( F_{ij}(x) \) represents the \( ij^{th} \) element of \( F(x) \), then

\[
\| \hat{F} \| = \| F(x_d) - F(x) \| \leq \rho_f (\| x_d \|, \| \bar{x} \|) \| \bar{x} \| \tag{B.136}
\]

where \( \bar{x}(t) \in \mathbb{R}^k \) represents the mismatch between \( x_d(t) \in \mathbb{R}^k \) and \( x(t) \) as follows

\[
\bar{x} = x_d - x, \tag{B.137}
\]

and \( \rho_f (\cdot) \in \mathbb{R} \) is a positive nondecreasing bounding function.

**Proof:** To prove Lemma B.16, the \( ij^{th} \) element of \( \hat{F}(x) \) is first defined as follows

\[
\hat{F}_{ij} = F_{ij} (x_{d1}, x_{d2}, \ldots, x_{dk}) - F_{ij} (x_1, x_2, \ldots, x_k). \tag{B.138}
\]

After adding and subtracting the terms \( F_{ij}(x_1, x_{d2}, \ldots, x_{dk}), F_{ij}(x_1, x_2, x_{d3}, \ldots, x_{dk}), \ldots, F_{ij}(x_1, x_2, \ldots, x_{k-1}, x_{dk}) \) to the right-hand side of (B.138), the following expression can be obtained

\[
\hat{F}_{ij} = \left[ F_{ij} (x_{d1}, x_{d2}, \ldots, x_{dk}) - F_{ij} (x_1, x_{d2}, \ldots, x_{dk}) \right] + \left[ F_{ij} (x_1, x_{d2}, \ldots, x_{dk}) - F_{ij} (x_1, x_2, x_{d3}, \ldots, x_{dk}) \right] + \ldots + \left[ F_{ij} (x_1, x_2, \ldots, x_{k-1}, x_{dk}) - F_{ij} (x_1, x_2, \ldots, x_{k-1}, x_k) \right]. \tag{B.139}
\]

After applying the Mean Value Theorem (given in Lemma A.1 of Appendix A) to each bracketed term of (B.139), the following expression can be obtained

\[
\hat{F}_{ij} = \frac{\partial F_{ij} (\sigma_1, x_{d2}, \ldots, x_{dk})}{\partial \sigma_1} \big|_{\sigma_1 = v_1} (x_{d1} - x_1) + \frac{\partial F_{ij} (x_1, \sigma_2, x_{d3}, \ldots, x_{dk})}{\partial \sigma_2} \big|_{\sigma_2 = v_2} (x_{d2} - x_2) + \ldots + \frac{\partial F_{ij} (x_1, x_2, \ldots, x_{k-1}, \sigma_k)}{\partial \sigma_k} \big|_{\sigma_k = v_k} (x_{dk} - x_k) \tag{B.140}
\]

where $v_i(t)$ has a value in between $x_i(t)$ and $x_{di}(t)$ for $i = 1, 2, ..., k$. To simplify the notation, (B.140) is rewritten in the following matrix form

$$
\tilde{F}_{ij} = \frac{\partial F_{ij}(\sigma, x, x_d)}{\partial \sigma} \bigg|_{\sigma = v} (x_d - x) \tag{B.141}
$$

where

$$x = [x_1, ..., x_k]^T, \quad v = [v_1, ..., v_k]^T, \quad \sigma = [\sigma_1, ..., \sigma_k]^T, \quad \frac{\partial F_{ij}(\cdot)}{\partial \sigma} = \left[ \frac{\partial F_{ij}}{\partial \sigma_1}, ..., \frac{\partial F_{ij}}{\partial \sigma_k} \right].
$$

From (B.141), the following upper bound for $|\tilde{F}_{ij}(x)|$ can be obtained

$$
|\tilde{F}_{ij}| \leq \left\| \frac{\partial F_{ij}(\sigma, x, x_d)}{\partial \sigma} \bigg|_{\sigma = v} \right\| |x_d - x|.
\tag{B.142}
$$

By noting that

$$v_i = x_{di} - c_i \bar{x}_i \quad \forall c_i \in (0, 1), i = 1, 2, ... k, \tag{B.143}
$$

an upper bound for $|\tilde{F}_{ij}(x)|$ can be developed from (B.135), (B.137), and (B.142) as follows

$$
|\tilde{F}_{ij}| \leq \rho_{ij} \left( \|x_d\|, \|\bar{x}\| \right) \|\tilde{x}\| \quad \text{for} \quad i = 1, ..., m, j = 1, ..., n \tag{B.144}
$$

where $\rho_{ij}(\cdot)$ is some positive nondecreasing scalar function. The result given in (B.136) can now be obtained. \(\square\)

**Lemma B.17** Given the expressions in (4.130) and (4.134), Lemma B.16 can be utilized to construct the following inequality

$$
\rho(\zeta_{dp}, \zeta_{dv}, \zeta_{da}, \|y\|, \|y\|) \geq \max \left\{ \|\chi\|, \|\tilde{Y}_v\| \right\} \tag{B.145}
$$

where $\zeta_{dp}$, $\zeta_{dv}$, and $\zeta_{da}$ were defined in (4.86), $y(t)$ is given in (4.142), and $\rho(\cdot) \in \mathbb{R}$ is a positive nondecreasing function.

**Proof:** To prove Lemma B.17, the result in (B.145) is first developed for $\tilde{Y}_v(\cdot)$. Based on the proof for the result in (B.145) for $\tilde{Y}_v(\cdot)$, the proof for $\chi(\cdot)$ can then be obtained. To this end, the expression given in (4.135) can
be substituted into (B.146) for $\tilde{Y}_v(\cdot)\theta_v$ to obtain the following expression

$$
\tilde{Y}_v = -\frac{1}{2} \frac{\partial (\det(\bar{M}_{11}))}{\partial u_1} \dot{e} + \Pi^{-T} \bar{M}_{21} \text{adj}(\bar{M}_{11}) \bar{V}_{m11}(u_1, \dot{u}_1) \dot{e}
$$

$$
+ \Pi^{-T} \bar{M}_{21} \text{adj}(\bar{M}_{11}) [\bar{V}_{m11}(u_1, \dot{u}_{d1}) - \bar{V}_{m11}(u_1, \dot{u}_1)] \dot{u}_{d1}
$$

$$
+ \Pi^{-T} \bar{M}_{21} \text{adj}(\bar{M}_{11}) [\bar{N}_1(u_1, \dot{u}_{d1}) - \bar{N}_1(u_1, \dot{u}_1)]
$$

$$
- \det(\bar{M}_{11}) \Pi^{-T} \bar{V}_{m21}(u_1, \dot{u}_1) \dot{e}
$$

$$
- \det(\bar{M}_{11}) \Pi^{-T} [\bar{V}_{m21}(u_1, \dot{u}_{d1}) - \bar{V}_{m21}(u_1, \dot{u}_1)] \dot{u}_{d1}
$$

$$
- \det(\bar{M}_{11}) \Pi^{-T} [\bar{N}_2(u_1, \dot{u}_{d1}) - \bar{N}_2(u_1, \dot{u}_1)]
$$

(B.146)

where common terms have been cancelled, and several terms have been added and subtracted from the resulting expression. After invoking Lemma B.16, the following upper bound can be formulated

$$
\|\tilde{Y}_v\| \leq \frac{1}{2} \left\| \frac{\partial (\det(\bar{M}_{11}))}{\partial u_1} \right\| \|\dot{e}\| \|e_\lambda\|
$$

$$
+ \left\| \Pi^{-T} \bar{M}_{21} \text{adj}(\bar{M}_{11}) \right\| \left\| \bar{V}_{m11}(u_1, \dot{u}_1) \right\| \|\dot{e}\|
$$

$$
+ \left\| \Pi^{-T} \bar{M}_{21} \text{adj}(\bar{M}_{11}) \right\|
$$

$$
\cdot (\rho_{v1}(\|u_1\|, \zeta_{dv}, \|\dot{e}\|) \|\dot{e}\| \zeta_{dv} + \rho_{n1}(\|u_1\|, \zeta_{dv}, \|\dot{e}\|) \|\dot{e}\|)
$$

$$
+ \left\| \det(\bar{M}_{11}) \Pi^{-T} \right\|
$$

$$
\cdot (\|\bar{V}_{m21}(u_1, \dot{u}_1)\| \|\dot{e}\| + \rho_{v2}(\|u_1\|, \zeta_{dv}, \|\dot{e}\|) \|\dot{e}\| \zeta_{dv})
$$

$$
+ \left\| \det(\bar{M}_{11}) \Pi^{-T} \right\| \rho_{n2}(\|u_1\|, \zeta_{dv}, \|\dot{e}\|) \|\dot{e}\|
$$

(B.147)

where (4.86) has been utilized and $\rho_{v1}(\cdot), \rho_{v2}(\cdot), \rho_{n1}(\cdot), \rho_{n2}(\cdot) \in \mathbb{R}$ are some positive nondecreasing bounding functions. The remaining matrix norms of (B.147) can also be bounded by some positive function, hence

$$
\|\tilde{Y}_v\| \leq \rho_1(\|u_1\|, \|\dot{e}\|) \|e_\lambda\| + \rho_2(\|u_1\|, \|\dot{u}_1\|, \zeta_{dv}, \|\dot{e}\|) \|\dot{e}\|
$$

(B.148)

where $\rho_1(\cdot), \rho_2(\cdot) \in \mathbb{R}$ are some positive nondecreasing bounding functions. To express (B.148) in terms of $y(t)$ and the constant motion trajectory
bounds of (4.86), the expressions given in (4.92), (4.86), and (4.128) can be used along with the following inequality

\[ \|y\| \geq \|e\|, \|e_f\|, \|\eta\|, \|e_\lambda\| \]

to formulate an upper bound for \( \|\tilde{Y}_v(\cdot)\| \) as follows

\[ \|\tilde{Y}_v\| \leq \rho_1(\zeta_{dp}, \|y\|) \|y\| + \rho_2(\zeta_{dp}, \zeta_{dv}, \|y\|) \|y\| \leq \rho_w(\zeta_{dp}, \zeta_{dv}, \|y\|) \|y\| \]

(B.149)

where \( \rho_w(\cdot) \in \mathbb{R} \) is some positive nondecreasing bounding function. Using a similar procedure, the following upper bound can be formulated for \( \|\chi(\cdot)\| \)

\[ \|\chi\| \leq \rho_\chi(\zeta_{dp}, \zeta_{dv}, \zeta_{da}, \|y\|) \|y\| \]

(B.150)

where \( \rho_\chi(\cdot) \in \mathbb{R} \) is some positive nondecreasing bounding function. The result given in (B.145) can now be directly obtained from (B.149) and (B.150). □

### B.3.5 Matrix Property

**Lemma B.18** Providing the following condition is satisfied

\[ \cos(0) > \left| \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \right| \]

(B.151)

the matrix \( BR + (BR)^T \) will be positive-definite, where \( B \) and \( R \) are defined in (4.157) and (4.158), respectively.

**Proof:** To prove Lemma B.18, the definitions for \( B \) and \( R \) given in (4.157) and (4.158), respectively, can be used to obtain the following symmetric matrix

\[ \frac{1}{2}(BR + (BR)^T) = \frac{\lambda}{z} \begin{bmatrix} \beta_1 \cos \theta & -\frac{1}{2} \beta_1 \sin \theta + \frac{1}{2} \beta_2 \sin \theta \\ -\frac{1}{2} \beta_1 \sin \theta + \frac{1}{2} \beta_2 \sin \theta & \beta_2 \cos \theta \end{bmatrix} \]

(B.152)

Using standard linear algebra techniques, the eigenvalues of (B.152) can be determined as follows

\[ \lambda_1 = \frac{\lambda}{2z} ((\beta_1 + \beta_2) \cos \theta_0 + (\beta_1 - \beta_2)), \]

\[ \lambda_2 = \frac{\lambda}{2z} ((\beta_1 + \beta_2) \cos \theta_0 - (\beta_1 - \beta_2)) \]

(B.153)

where \( \lambda_i \) denotes the \( i \)th eigenvalue. For the symmetric matrix of (B.152) to be positive-definite, the eigenvalues given in (B.153) must be positive.
From (B.153) and the fact that $\lambda, z, \beta_1, \beta_2 > 0$, the following eigenvalues can be proven to be positive provided

$$\lambda_1 > 0 \quad \Rightarrow \quad \cos(\theta_0) > \frac{\beta_2 - \beta_1}{\beta_1 + \beta_2}$$

(B.154)

$$\lambda_2 > 0 \quad \Rightarrow \quad \cos(\theta_0) > \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}.$$  

Hence, if the condition given in (B.151) is satisfied, the matrix $BR + (BR)^T$ will be positive-definite.

### B.4 Chapter 5 Lemmas

#### B.4.1 Skew-Symmetry Property

**Lemma B.19** *The transformed inertia and centripetal-Coriolis matrices introduced in the dynamic model given in (5.26) satisfy the following skew-symmetric relationship*

$$\xi^T \left( \frac{1}{2} j^* - C^* \right) \xi = 0 \quad \forall \xi \in \mathbb{R}^3.$$  

(B.155)

**Proof:** In order to prove (B.155), we first take the time derivative of (5.25) and then utilize (5.28) to obtain the following expression

$$\xi^T \left( j^* - 2C^* \right) \xi = \xi^T \left( \dot{T}^{-T} JT^{-1} + T^{-T} J\dot{T}^{-1} \right) \xi$$

$$+ 2\xi^T \left( j^* \ddot{T} T^{-1} + 2T^{-T} (JT^{-1} \dot{e}_v)^\times T^{-1} \right) \xi$$  

(B.156)

$$= 2\xi^T \left( T^{-T} J\dot{T}^{-1} + T^{-T} JT^{-1} \dot{T} T^{-1} \right) \xi$$

where the fact that

$$\xi^T T^{-T} (JT^{-1} \dot{e}_v)^\times T^{-1} \xi = 0$$  

(B.157)

has been utilized. Based on the expression given in (B.156), the following property can be used to prove the result given in (B.155)

$$\dot{T} = -T\ddot{T}^{-1}T.$$  

(B.158)
B.4.2 Control Signal Bound

**Lemma B.20** Given the definitions in (5.23), (5.66), and (5.67), the following inequality can be developed

\[ \bar{X} \leq \rho (\|z\|) \|z\| \]  
\[ \text{(B.159)} \]

where \( z(t) \) is defined in (5.69).

**Proof:** After substituting (5.64) and (5.66) into (5.67), the following expression can be obtained

\[ \bar{X} = T^T C^* (e_{vf} + e_v) + T^T J^* \left( \eta - e_v + \frac{e_v}{(1 - e_v^T e_v)^2} \right) \]  
\[ \text{(B.160)} \]

\[ -2T^T J^* e_{vf} + T^T J^* \omega_d + \frac{1}{2} T^T \omega_d^\times J \omega_d - T^T N^* . \]

By taking the norm of (B.160), the following inequality can be developed

\[ \|\bar{X}\| \leq \|T^T C^*\| (\|e_{vf}\| + \|e_v\|) \]
\[ + \|T^T J^*\| \left( \|\eta\| + \|e_v\| + \left\| \frac{e_v}{(1 - e_v^T e_v)^2} \right\| \right) \]  
\[ \text{(B.161)} \]

\[ + 2 \|T^T J^*\| \|e_{vf}\| + \|\psi\| \]

where \( \psi(t) \in \mathbb{R}^3 \) is defined as follows

\[ \psi = T^T J^* \dot{\omega}_d + \frac{1}{2} T^T \omega_d^\times J \omega_d - T^T N^* . \]  
\[ \text{(B.162)} \]

After substituting (5.29) into (B.162) for \( N^* (\cdot) \) and utilizing (5.60) and the fact that \( a^X b = -b^X a \) \( \forall a, b \in \mathbb{R}^3 \), the following expression can be obtained

\[ \psi = \left[ (\hat{J} \hat{\omega}_d)^X - (\hat{\omega}_d)^X J - J (\hat{\omega}_d)^X \right] T^{-1} (\eta - e_v - e_{vf}) \]
\[ + \left( T^T J - \frac{1}{2} J \hat{\omega}_d \right) \dot{\omega}_d + \frac{1}{2} \left( T^T \omega_d^\times J \omega_d - (\hat{\omega}_d)^X J \hat{\omega}_d \right) . \]  
\[ \text{(B.163)} \]

By utilizing (5.12), (5.34), and (5.23) the following inequality can be developed

\[ \|\psi\| \leq \rho_1 (e_v) (\|\eta\| + \|e_v\| + \|e_{vf}\|) + \rho_2 (e_v) \left( \left\| \frac{e_v}{\sqrt{1 - e_v^T e_v}} \right\| \right) \]  
\[ \text{(B.164)} \]
where \( \rho_1(\cdot), \rho_2(\cdot) \) are some positive nondecreasing functions. Since (5.34) ensures that \( \sqrt{1 - e_v^T e_v} \) is always positive and real, multiplying and dividing the last term in (B.163) by \( \sqrt{1 - e_v^T e_v} \) facilitates the structure of (B.164). Based upon the definition of (5.23), (5.25), (5.28), (5.34), (5.69), and (B.164), the expression in (B.161) can now be simplified to yield the result found in (5.68).

### B.5 Chapter 6 Definitions and Lemmas

#### B.5.1 Definitions for Dynamic Terms

**Definition B.3** The components of the dynamic model given in (6.1) \( M(q) \in \mathbb{R}^{4 \times 4}, V_m(q, \dot{q}) \in \mathbb{R}^{4 \times 4} \), and \( G(q) \in \mathbb{R}^4 \) are defined as follows

\[
M = \begin{bmatrix}
  m_p + m_r + m_c & 0 \\
  0 & m_p + m_c \\
  m_p L \cos \theta \sin \phi & m_p L \cos \theta \cos \phi \\
  m_p L \sin \theta \cos \phi & -m_p L \sin \theta \sin \phi \\
\end{bmatrix}
\]  

\[ (B.165) \]

\[
V_m = \begin{bmatrix}
  0 & 0 & -m_p L \left( \sin \theta \sin \phi \dot{\theta} - \cos \theta \cos \phi \dot{\phi} \right) \\
  0 & 0 & -m_p L \left( \sin \theta \cos \phi \dot{\theta} + \cos \theta \sin \phi \dot{\phi} \right) \\
  0 & 0 & m_p L^2 \sin \theta \cos \phi \\
  0 & 0 & m_p L \left( \cos \theta \cos \phi \dot{\theta} - \sin \theta \sin \phi \dot{\phi} \right) \\
  & & -m_p L \left( \cos \theta \sin \phi \dot{\theta} + \sin \theta \cos \phi \dot{\phi} \right) \\
  & & -m_p L^2 \sin \theta \cos \phi \dot{\theta} \\
  & & m_p L^2 \sin \theta \cos \theta \dot{\phi} \\
\end{bmatrix}
\]  

\[ (B.166) \]

and

\[
G = \begin{bmatrix}
  0 & 0 & m_p g L \sin \theta & 0 \\
\end{bmatrix}^T.
\]  

\[ (B.167) \]
B.5.2 Linear Control Law Analysis

Lemma B.21 Given the linear controller of (6.25), the time derivative of the following function

\[ V = k_E E + \frac{1}{2} k_p e T e \quad (B.168) \]

is given by the following expression

\[ \dot{V} = 0 \quad (B.169) \]

only when

\[ E(q, \dot{q}) = e(t) = 0 \quad (B.170) \]

where \( E(q, \dot{q}) \) and \( e(t) \) are defined in (6.21) and (6.24), respectively.

Proof: To prove Lemma B.21, we define the set of all points where (B.169) is satisfied as \( \Gamma \). In the set \( \Gamma \), it is clear from (6.29) and (B.169) that

\[ \dot{r}(t) = 0 \quad \ddot{r}(t) = 0, \quad (B.171) \]

and hence, we can conclude from (6.16), (B.168), (B.169), and (B.171) that \( x(t), y(t), \) and \( V_1(t) \) are constant, and that

\[ \ddot{x}(t) = 0 \quad \ddot{y}(t) = 0. \quad (B.172) \]

Furthermore, (6.22), (6.24), and (B.171) can be used to prove that

\[ \dot{E}(q, \dot{q}) = \dot{e}(t) = 0. \quad (B.173) \]

Based on (B.173), \( E(q, \dot{q}) \) and \( e(t) \) are constant, and hence, from (6.25) and (B.171), we can prove that \( F(t) \) is constant. To complete the proof, the stability of the system must be analyzed for the cases when \( \dot{\theta} = 0 \) and when \( \dot{\theta} \neq 0. \)

Case 1a: \( \dot{\theta}(t) = 0 \) and \( \dot{\phi}(t) = 0 \)

Based on the proposition that \( \dot{\theta}(t) = 0 \) and \( \dot{\phi}(t) = 0 \), it is straightforward to prove that

\[ \ddot{\theta}(t) = 0 \quad \ddot{\phi}(t) = 0. \quad (B.174) \]

By rearranging the first two rows of the expression given in (6.1), the following expressions can be obtained

\[ \frac{F_x}{m_p L} = \frac{m_p + m_r + m_c}{m_p L} \dot{x} + \dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi \quad (B.175) \]

\[ -\left( \dot{\theta}^2 + \dot{\phi}^2 \right) \sin \theta \sin \phi + 2\dot{\theta} \dot{\phi} \cos \theta \cos \phi \]
\[ \frac{F_y}{m_p L} = \frac{m_p + m_c}{m_p} \dot{y} + \ddot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi \]  

\[ - (\dot{\theta}^2 + \dot{\phi}^2) \sin \theta \cos \phi - 2 \dot{\theta} \dot{\phi} \cos \theta \sin \phi. \]  

(B.176)

Based on the expression given in (B.172), (B.174–B.176), and the proposition that \( \dot{\theta}(t) = 0 \) and \( \dot{\phi}(t) = 0 \), the following expression is obtained

\[ F_x(t) = F_y(t) = 0. \]  

(B.177)

From (6.19), (6.25), (B.171), and (B.177), it is clear that

\[ e(t) = 0. \]  

(B.178)

Furthermore, by rearranging the third row of the vector given in (6.1), the following expression can be obtained

\[ \ddot{\theta} = \gamma_3 \dot{\phi}^2 \sin \theta \cos \theta - \gamma_2 \sin \theta - \frac{m_p L}{m_p L^2 + I} (\ddot{x} \cos \theta \sin \phi + \dot{y} \cos \theta \cos \phi) \]  

(B.179)

where (6.2–6.3) were utilized, and \( \gamma_2, \gamma_3 \in \mathbb{R} \) are positive constants defined as follows

\[ \gamma_2 = \frac{m_p g L}{m_p L^2 + I}, \quad \gamma_3 = \frac{m_p L^2}{m_p L^2 + I}. \]  

(B.180)

Based on (B.172), (B.174), and the proposition that \( \dot{\phi}(t) = 0 \), (B.179) can be used to prove that

\[ \sin \theta = 0, \]  

(B.181)

and hence, from (6.6), it is clear that

\[ \theta(t) = 0. \]  

(B.182)

Given (B.178) and (B.182), the expressions given in (6.16), (6.21), (6.23), and (6.24) can be used to prove Lemma (B.21) under the proposition that \( \dot{\theta}(t) = 0 \) and \( \dot{\phi}(t) = 0. \) \( \square \)

**Case 1b:** \( \dot{\theta}(t) = 0 \) and \( \dot{\phi}(t) \neq 0 \)

By rearranging the fourth row of the vector given in (6.1), the following expression can be obtained

\[ \gamma_1(\theta) \ddot{\phi} = -2 \dot{\theta} \dot{\phi} \sin \theta \cos \theta - \left( \frac{\ddot{x} \sin \theta \cos \phi + \dot{y} \sin \theta \sin \phi}{L} \right) \]  

(B.183)

where \( \gamma_1(\theta) \in \mathbb{R} \) is defined as follows

\[ \gamma_1(\theta) = \left( \sin^2 \theta + \frac{I}{m_p L^2} \right). \]  

(B.184)
Based on (B.172), (B.183), and the proposition that $\dot{\theta}(t) = 0$, it is clear that
\begin{align*}
\ddot{\phi} &= 0 \quad \ddot{\theta} = 0, \quad \text{(B.185)}
\end{align*}
and hence, $\theta(t)$ and $\dot{\phi}(t)$ are constant. From (B.172), (B.175), (B.176), and (B.185), the fact that $F(t)$ remains constant, and the proposition that $\dot{\phi}(t) \neq 0$, the following expression can be obtained
\begin{align*}
F_x &= -m_p L \phi^2 \sin \theta \sin \phi \quad \text{(B.186)} \\
F_y &= -m_p L \phi^2 \sin \theta \cos \phi. \quad \text{(B.187)}
\end{align*}

To continue the analysis, the cases of $\sin \theta = 0$ and $\sin \theta \neq 0$ are considered. Under the additional proposition that $\sin \theta \neq 0$, (B.186) and (B.187) can be used to prove that $\sin \phi$, $\cos \phi$, and $\phi(t)$ must be constant since $F_x(t)$ and $F_y(t)$ are constant. However, the conclusion that $\phi(t)$ is constant contradicts the proposition that $\dot{\phi}(t) \neq 0$. Under the additional proposition that $\sin \theta = 0$, (B.186) and (B.187) can be used to prove that
\begin{align*}
F_x = F_y = \sin \theta = 0. \quad \text{(B.188)}
\end{align*}

Given (6.6), (6.19), (6.25), (B.171), and (B.188), it is clear that
\begin{align*}
e(t) = \theta(t) = 0. \quad \text{(B.189)}
\end{align*}

From (B.189), the expressions given in (6.16), (6.21), (6.23), and (6.24) can be used to prove Lemma (B.21) under the propositions that $\dot{\theta}(t) = 0$, $\dot{\phi}(t) \neq 0$, and $\sin \theta = 0$. \( \Box \)

**Case 2:** $\dot{\theta}(t) \neq 0$

If either $\sin \theta = 0$ or $\cos \theta = 0$, then $\theta(t)$ would be constant. Hence, the proposition that $\sin \theta = 0$ or $\cos \theta = 0$ would lead to a contradiction with the proposition that $\dot{\theta}(t) \neq 0$. Since $\theta(t)$ is a continuous function (i.e., since $\dot{\theta}(t) \in L_\infty$), it is clear that both $\sin \theta \neq 0$ and $\cos \theta \neq 0$ cannot be satisfied at the same instant in time. This fact will be used in the subsequent analysis.

To facilitate the stability analysis under the proposition that $\dot{\theta}(t) \neq 0$, each row of (6.1) can be rewritten as follows
\begin{align*}
P_1 \sin \phi + P_2 \cos \phi &= S_1 \quad \text{(B.190)} \\
P_1 \cos \phi - P_2 \sin \phi &= S_2 \quad \text{(B.191)} \\
\ddot{\theta} &= -\gamma_2 \sin \theta + \gamma_3 \dot{\phi}^2 \sin \theta \cos \theta + \gamma_3 S_3 \quad \text{(B.192)}
\end{align*}
\[ \gamma_1 \ddot{\phi} = -2 \dot{\theta} \phi \sin \theta \cos \theta + S_4. \]  

(B.193)

The signals \( P_1(t), P_2(t), S_1(t), S_2(t), S_3(t), S_4(t) \in \mathbb{R} \) introduced in (B.190–B.193) are defined as follows:

\[
P_1(t) = \sin \theta \left( \gamma_3 \dot{\phi}^2 \cos^2 \theta - \gamma_2 \cos \theta - \left( \dot{\theta}^2 + \dot{\phi}^2 \right) \right) + S_3 \gamma_3 \cos \theta \quad \text{(B.194)}
\]

\[
P_2(t) = \dot{\phi} \sin \theta + 2 \dot{\theta} \phi \cos \theta \quad \text{(B.195)}
\]

\[
S_1(t) = \frac{1}{m_p L} \left( F_x - (m_p + m_r + m_c) \ddot{x} \right) \quad \text{(B.196)}
\]

\[
S_2(t) = \frac{1}{m_p L} \left( F_y - (m_p + m_c) \ddot{y} \right) \quad \text{(B.197)}
\]

\[
S_3(t) = - \left[ \frac{\ddot{x} \cos \theta \sin \phi + \ddot{y} \cos \theta \cos \phi}{L} \right] \quad \text{(B.198)}
\]

\[
S_4(t) = - \left[ \frac{\ddot{x} \sin \theta \cos \phi - \ddot{y} \sin \theta \sin \phi}{L} \right] \quad \text{(B.199)}
\]

where (B.192) has been substituted into (B.194) for \( \ddot{\theta}(t), \gamma_1(\theta) \) was defined in (B.184), and \( \gamma_2, \gamma_3 \) were defined in (B.180). After taking the time derivative of the expressions given in (B.190) and (B.191), the following expressions can be obtained:

\[
\dot{\phi} P_1 \cos \phi + \dot{P}_1 \sin \phi - \dot{\phi} P_2 \sin \phi + \dot{P}_2 \cos \phi = \dot{S}_1 \quad \text{(B.200)}
\]

\[
-\dot{\phi} P_1 \sin \phi + \dot{P}_1 \cos \phi - \dot{\phi} P_2 \cos \phi - \dot{P}_2 \sin \phi = \dot{S}_2. \quad \text{(B.201)}
\]

In (B.200) and (B.201), the terms \( \dot{P}_1(t), \dot{P}_2(t), \dot{S}_1(t), \text{ and } \dot{S}_2(t) \) can be written as follows:

\[
\dot{P}_1 = \dot{\theta} \left( -\gamma_2 \left( 1 - 2 \sin^2 \theta \right) + \gamma_3 \dot{\phi}^2 \cos^3 \theta - \left( \dot{\theta}^2 + \dot{\phi}^2 \right) \cos \theta \right)
\]

\[
-2 \gamma_3 \dot{\phi} \left( \gamma_1 \ddot{\phi} \right) \sin \theta - 2 \dot{\theta} \ddot{\phi} \sin \theta - 2 \gamma_3 \ddot{\phi}^2 \cos \theta \sin^2 \theta \quad \text{(B.202)}
\]

\[
- \dot{S}_3 \gamma_3 \dot{\theta} \sin \theta + \dot{S}_3 \gamma_3 \cos \theta
\]

\[
\dot{P}_2 = \dot{\theta} \dot{\phi} \cos \theta + \phi^{(3)} \sin \theta - 2 \left( \dot{\phi} \dot{\phi} \sin \theta \cos \theta - \ddot{\phi} \cos \theta \right) \quad \text{(B.203)}
\]

\[
\dot{S}_1 = \frac{1}{m_p L} \left( \dot{F}_x - (m_p + m_r + m_c) x^{(3)} \right) \quad \text{(B.204)}
\]

\[
\dot{S}_2 = \frac{1}{m_p L} \left( \dot{F}_y - (m_p + m_c) y^{(3)} \right) \quad \text{(B.205)}
\]
where the expression for $\dot{S}_3(t)$ is given as follows

$$
\dot{S}_3 = \frac{1}{L} \left( \dot{\theta} \dot{x} \sin \theta \sin \phi - \dot{\phi} \dot{x} \cos \theta \cos \phi - x^{(3)} \cos \theta \sin \phi \right) + \frac{1}{L} \left( \dot{\theta} \dot{y} \sin \theta \cos \phi + \dot{\phi} \dot{y} \cos \theta \sin \phi - y^{(3)} \cos \theta \cos \phi \right).
$$

(B.206)

After substituting (B.192) and (B.193) into (B.202) for $\dot{\theta}(t)$ and $\dot{\phi}(t)$, respectively, and then performing some algebraic manipulation, the following expression can be obtained

$$
\dot{P}_1 = \dot{\theta} \left( -\gamma_2 (1 - 4 \sin^2 \theta) + \gamma_3 \dot{\phi}^2 \cos^3 \theta - \left( \dot{\theta}^2 + \dot{\phi}^2 \right) \cos \theta \right) - \gamma_3 \left( \left( 2\dot{\phi} S_4 + 3S_3 \dot{\theta} \right) \sin \theta - \dot{S}_3 \cos \theta \right).
$$

(B.207)

After multiplying both sides of the expression given in (B.200) by $\sin \phi$, multiplying both sides of the expression given in (B.201) by $\cos \phi$, and then adding the resulting expressions, the following expression is obtained

$$
\dot{P}_1 - \dot{\phi} P_2 = \dot{S}_1 \sin \phi + \dot{S}_2 \cos \phi.
$$

(B.208)

By multiplying both sides of (B.208) by $\gamma_1(\theta)$, substituting (B.195) and (B.207) into the resulting expression for $P_2(t)$ and $\dot{P}_1(t)$, and then dividing the resulting expression by $\dot{\theta}(t)$, the following expression can be obtained

$$
\gamma_1 \dot{P}_3 - 2\frac{I}{m_p L^2} \dot{\phi}^2 \cos \theta = \frac{S_5}{\dot{\theta}}
$$

(B.209)

where (B.193) was used. In (B.209), the terms $P_3(t)$, $S_5(t) \in \mathbb{R}$ are defined as follows

$$
P_3 = -\gamma_2 (1 - 4 \sin^2 \theta) + \gamma_3 \dot{\phi}^2 \cos^3 \theta - \left( \dot{\theta}^2 + \dot{\phi}^2 \right) \cos \theta
$$

(B.210)

$$
S_5 = \gamma_1 \left( \dot{S}_1 \sin \phi + \dot{S}_2 \cos \phi + \gamma_3 \left( \left( 3S_3 \dot{\theta} + 2\dot{\phi} S_4 \right) \sin \theta - \dot{S}_3 \cos \theta \right) \right) + S_4 \dot{\phi} \sin \theta.
$$

(B.211)

The time derivative of (B.209) is given by the following expression

$$
\gamma_1 \ddot{P}_3 + 2\dot{\theta} P_3 \sin \theta \cos \theta + 2\frac{I}{m_p L^2} \left( \dot{\theta} \dot{\phi}^2 \sin \theta - 2\dot{\phi} \ddot{\phi} \cos \theta \right) = \frac{\dot{S}_5 \dot{\theta} - S_5 \ddot{\theta}}{\dot{\theta}^2}
$$

(B.212)

where the expressions for $\dot{P}_3(t)$ and $\dot{S}_5(t)$ are given as follows

$$
\dot{P}_3 = \dot{\theta} P_4 \sin \theta - 2\gamma_3 \left( S_3 \dot{\theta} \cos \theta + S_4 \dot{\phi} \cos \theta \right)
$$

(B.213)
\[ \dot{S}_5 = 2\dot{\theta} \sin \theta \cos \theta \]

\[ \cdot \left( \dot{S}_1 \sin \phi + \dot{S}_2 \cos \phi + \gamma_3 \left( (3S_3 \dot{\theta} + 2\dot{\phi} S_4) \sin \theta - \dot{S}_3 \cos \theta \right) \right) \]

\[ + \gamma_1 \dot{S}_1 \sin \phi + \gamma_1 \dot{S}_1 \phi \cos \phi + \gamma_1 \dot{S}_2 \cos \phi - \gamma_1 \dot{S}_2 \phi \sin \phi \]

\[ + \gamma_1 \gamma_3 \left( (3S_3 \dot{\theta} + 3S_3 \dot{\theta} + 2\dot{\phi} S_4 + 2\dot{\phi} S_4) \sin \theta \right) \]

\[ + \left( 3S_3 \dot{\theta} + 2\dot{\phi} S_4 \right) \dot{\theta} \cos \theta \]

\[ - \gamma_1 \gamma_3 \dot{S}_3 \cos \theta + \gamma_1 \gamma_3 \dot{S}_3 \sin \theta + \dot{S}_4 \dot{\phi} \sin \theta \]

\[ + S_4 \dot{\phi} \sin \theta + S_4 \dot{\phi} \cos \theta \]

where (B.192) and (B.193) were used. In (B.213), the term \( P_4(t) \in \mathbb{R} \) is defined as follows

\[ P_4 = 10\gamma_2 \cos \theta - \gamma_3 \dot{\phi}^2 \cos^2 \theta + \left( \dot{\theta}^2 + \dot{\phi}^2 \right), \quad (B.215) \]

and in (B.213) and (B.214), the expressions for \( \ddot{S}_1(t) \), \( \ddot{S}_2(t) \), \( \ddot{S}_3(t) \), and \( \ddot{S}_4(t) \) can be determined as

\[ \ddot{S}_1 = \frac{1}{m_p L} \left( \ddot{F}_x - (m_p + m_r + m_c) x^{(4)} \right) \quad (B.216) \]

\[ \ddot{S}_2 = \frac{1}{m_p L} \left( \ddot{F}_y - (m_p + m_c) y^{(4)} \right) \quad (B.217) \]

\[ \ddot{S}_3 = \frac{1}{L} \left( \ddot{\phi} \dot{x} + 2\ddot{\phi} x^{(3)} - 2\dot{\theta} \dot{\phi} \dot{y} \right) \sin \theta \sin \phi \]

\[ + \frac{1}{L} \left( 2\ddot{\phi} \dot{x} + \ddot{\phi} \dot{y} + 2\ddot{\phi} y^{(3)} \right) \sin \theta \cos \phi \]

\[ + \frac{1}{L} \left( \dot{\phi}^2 \dot{x} + \dot{\theta}^2 \dot{x} + \ddot{\phi} \dot{y} + 2\dot{\phi} y^{(3)} - x^{(4)} \right) \cos \theta \sin \phi \]

\[ + \frac{1}{L} \left( \ddot{\phi} \dot{x} - 2\dot{\phi} x^{(3)} + \ddot{\phi} y^{(3)} + \ddot{\phi} \dot{y} - y^{(4)} \right) \cos \theta \cos \phi \]

\[ \ddot{S}_4 = -\frac{1}{L} \left( \ddot{\phi} \cos \theta \cos \phi - \dot{\phi} \dot{x} \sin \theta \sin \phi + x^{(3)} \sin \phi \right) \]

\[ + \frac{1}{L} \left( \ddot{\phi} \cos \theta \sin \phi + \dot{\phi} \dot{y} \sin \theta \cos \phi + y^{(3)} \sin \theta \sin \phi \right). \quad (B.219) \]
By substituting (B.213) into (B.212) for \( \dot{P}_3(t) \) and then multiplying the resulting expression by \( \gamma_1(\theta) \), the following expression can be obtained

\[
\dot{\sin} \theta \left( \gamma_1^2 P_4 + 2 \gamma_1 P_3 \cos \theta + \frac{2I}{m_p L^2} \dot{\phi}^2 (\gamma_1 + 4 \cos^2 \theta) \right) = S_6 \quad (B.220)
\]

where (B.193) was used. In (B.220), \( S_6(t) \in \mathbb{R} \) is defined as

\[
S_6 = \gamma_1 \left( \frac{\dot{S}_5 - S_5 \ddot{\theta}}{\dot{\theta}^2} + \frac{4I \dot{\phi} \cos \theta}{m_p L^2} S_4 \right) + \gamma_1^2 \left( 2\gamma_3 \left( S_3 \dot{\theta} \cos \theta + S_4 \dot{\phi} \cos \theta \right) \right) \quad (B.221)
\]

After dividing (B.220) by \( \dot{\theta}(t) \) sin \( \theta \) and then substituting (B.209) into the resulting expression for \( \gamma_1(\theta)P_3(t) \), the following expression is obtained

\[
\gamma_1^2 P_4 + \frac{2I}{m_p L^2} \dot{\phi}^2 (\gamma_1 + 6 \cos^2 \theta) = \frac{S_6}{\dot{\theta} \sin \theta} - \frac{2S_5 \cos \theta}{\dot{\theta}}. \quad (B.222)
\]

After multiplying (B.209) by \( \gamma_1(\theta) \), multiplying (B.222) by \( \cos \theta \), and then adding the resulting products, the following expression is obtained

\[
\gamma_1^2 (P_3 + P_4 \cos \theta) + \frac{12I}{m_p L^2} \dot{\phi}^2 \cos^3 \theta = S_7. \quad (B.223)
\]

In (B.223), \( S_7(t) \in \mathbb{R} \) is defined as

\[
S_7 = \left( \frac{S_6}{\dot{\theta} \sin \theta} - \frac{2S_5 \cos \theta}{\dot{\theta}} \right) \cos \theta + \frac{S_5}{\dot{\theta}} \gamma_1. \quad (B.224)
\]

The expression for (B.223) can be rewritten by using (B.210) and (B.215) as follows

\[
\gamma_2 \gamma_1^2 (9 - 6 \sin^2 \theta) + \frac{12I}{m_p L^2} \dot{\phi}^2 \cos^3 \theta = S_7. \quad (B.225)
\]

To continue the analysis, the time derivative of (B.225) is determined as follows

\[
4\gamma_2 \dot{\gamma}_1 \left( (9 - 6 \sin^2 \theta) - 3\gamma_1 \right) \sin \theta \cos \theta
+ \frac{12I}{m_p L^2} \dot{\phi} \left( -3\dot{\theta} \sin \theta + 2\dot{\phi} \cos \theta \right) \cos^2 \theta = \dot{S}_7 \quad (B.226)
\]
where $\dot{S}_7(t)$ is given by the following expression

$$\dot{S}_7 = -\left(\frac{S_6}{\sin \theta} - 2S_5 \cos \theta\right) \sin \theta$$

$$+ \left(\frac{\dot{S}_6}{\theta \sin \theta} - \frac{S_6 \cos \theta}{\sin^2 \theta} - \frac{S_6 \ddot{\theta}}{\theta^2 \sin \theta}\right) \cos \theta$$

$$- \left(\frac{2\dot{S}_5 \cos \theta}{\theta} - 2S_5 \sin \theta - \frac{2S_5 \ddot{\theta} \cos \theta}{\theta^2}\right) \cos \theta$$

$$+ \frac{\dot{S}_5}{\theta} \gamma_1 - \frac{S_5 \ddot{\theta}}{\theta^2} \gamma_1 + 2S_5 \sin \theta \cos \theta. \quad (B.227)$$

In $(B.227)$, the expression for $\dot{S}_6(t)$ can be determined as follows

$$\dot{S}_6 = 2\dot{\theta} \sin \theta \cos \theta \left(\frac{\dot{S}_5 \dot{\theta} - S_5 \ddot{\theta}}{\ddot{\theta}^2} + \frac{4I \ddot{\phi} \cos \theta}{m_p L^2} S_4\right)$$

$$+ \gamma_1 \left(\frac{\ddot{S}_5 \dot{\theta} - S_5 \dddot{\theta}}{\ddot{\theta}^2} - 2\ddot{\theta} \left(\frac{\ddot{S}_5 \dot{\theta} - S_5 \dddot{\theta}}{\ddot{\theta}^3}\right)\right)$$

$$+ \frac{4I \ddot{\phi} \sin \theta}{m_p L^2} S_4 + \frac{4I \ddot{\phi} \cos \theta}{m_p L^2} S_4 + \frac{4I \dddot{\phi} \cos \theta}{m_p L^2} \dot{S}_4\right) \quad (B.228)$$

$$+ 4\gamma_1 \dot{\theta} \left(2\gamma_3 \left(S_3 \dot{\theta} \cos \theta + S_4 \ddot{\phi} \cos \theta\right)\right) \sin \theta \cos \theta$$

$$+ 2\gamma_1^2 \gamma_3 \left(\dot{S}_3 \dot{\theta} \cos \theta - S_3 \dddot{\theta} \sin \theta + S_3 \dddot{\theta} \cos \theta\right)$$

$$+ 2\gamma_1^2 \gamma_3 \left(\dot{S}_4 \dddot{\phi} \cos \theta - S_4 \dddot{\phi} \sin \theta + S_4 \dddot{\phi} \cos \theta\right)$$
where \(\ddot{S}_5(t)\) can be determined as

\[
\ddot{S}_5 = \left[ 2 (\cos^2 \theta - \sin^2 \theta) \dot{\theta}^2 + 2 \ddot{\theta} \sin \theta \cos \theta \right] \\
\left( (\dot{S}_1 \sin \phi + \dot{S}_2 \cos \phi) + \gamma_3 \left( 3S_3 \dot{\theta} + 2\ddot{\phi}S_4 \sin \theta - \dot{S}_3 \cos \theta \right) \right) \\
+ \gamma_1 \left( \dot{S}_1^{(3)} \sin \phi + 2\dot{S}_1 \dot{\phi} \cos \phi + \dot{S}_1 \left( \ddot{\phi} \cos \phi - \dot{\phi}^2 \sin \phi \right) \right) \\
+ \gamma_1 \left( \dot{S}_2^{(3)} \cos \phi - 2\ddot{S}_2 \sin \phi + \dot{S}_2 \left( -\ddot{\phi} \sin \phi + \ddot{\phi}^2 \cos \phi \right) \right) \\
+ 3\gamma_1 \gamma_3 \left( \ddot{S}_3 \dot{\theta} \sin \theta + 2\dot{S}_3 \left( \dot{\theta}^2 \cos \theta + \ddot{\theta} \sin \theta \right) \right) \\
+ S_3 \left( -\dot{\theta}^3 \sin \theta + 3\ddot{\theta} \cos \theta + \theta^{(3)} \sin \theta \right) \right) \\
+ 2\gamma_1 \gamma_3 \left( \ddot{S}_4 \sin \theta + 2\dot{S}_4 \left( \dot{\phi} \cos \theta + \ddot{\phi} \sin \theta \right) \right) \\
+ 2\gamma_1 \gamma_3 \left( S_4 \left( -\dot{\phi}^2 \sin \theta + \ddot{\phi} \cos \theta + 2\ddot{\phi} \cos \theta + \phi^{(3)} \sin \theta \right) \right) \\
- \gamma_1 \gamma_3 \left( \dot{S}_3^{(3)} \cos \theta - 2\ddot{S}_3 \dot{\theta} \sin \theta + \ddot{S}_3 \left( -\dot{\theta}^2 \cos \theta - \ddot{\theta} \sin \theta \right) \right) \\
+ 4\dot{\theta} \sin \theta \cos \theta \left( \dot{S}_1 \sin \phi + \dot{S}_1 \dot{\phi} \cos \phi + \dot{S}_2 \cos \phi - \ddot{S}_2 \dot{\phi} \sin \phi \right) \\
+ 4\gamma_3 \dot{\theta} \sin \theta \cos \theta \left( 3\dot{S}_3 \dot{\theta} + 3S_3 \ddot{\theta} + 2\ddot{\phi}S_4 + 2\dot{\phi} \dot{S}_4 \right) \sin \theta \\
+ \left( 3S_3 \dot{\theta} + 2\ddot{\phi}S_4 \right) \dot{\theta} \cos \theta \\
+ 4\gamma_3 \dot{\theta} \sin \theta \cos \theta \left( -\ddot{S}_3 \cos \theta + \ddot{S}_3 \dot{\theta} \sin \theta \right) \\
+ \ddot{S}_4 \dot{\phi} \sin \theta + 2\ddot{S}_4 \left( \dot{\phi} \sin \theta + \ddot{\phi} \cos \theta \right) \\
+ S_4 \left( \phi^{(3)} \sin \theta + 2\ddot{\phi} \cos \theta - \ddot{\phi}^2 \sin \theta + \ddot{\phi} \cos \theta \right).
\] 

(B.229)

In (B.229), the expressions for \(S_1^{(3)}(t), S_2^{(3)}(t), S_3^{(3)}(t),\) and \(\ddot{S}_4(t)\) can be developed as follows

\[
S_1^{(3)} = \frac{1}{m_p L} \left( F_x^{(3)} - (m_p + m_r + m_c) x^{(5)} \right)
\] 

(B.230)
\[ S_2^{(3)} = \frac{1}{m_p L} \left( P_y^{(3)} - (m_p + m_c) y^{(5)} \right) \]  
(B.231)

\[ S_3^{(3)} = \frac{1}{L} \left( \dot{\theta} \cos \theta \sin \phi + \dot{\phi} \sin \theta \cos \phi \right) \left( \ddot{\theta} x + 2\dot{\theta} x^{(3)} - 2\ddot{\phi} y \right) \]

\[ + \frac{1}{L} \left( \theta^{(3)} x + \dot{\theta} x^{(3)} \right) \]

\[ + 2 \left( \ddot{\theta} x^{(4)} + \dot{\theta} x^{(3)} - \ddot{\phi} y - \ddot{\phi} y^{(3)} \right) \sin \theta \sin \phi \]

\[ + \frac{1}{L} \left( \dot{\theta} \cos \theta \cos \phi - \dot{\phi} \sin \theta \sin \phi \right) \left( 2 \ddot{\phi} x + \dot{\phi} y + 2 \dot{\phi} y^{(3)} \right) \]

\[ + \frac{1}{L} \left( 2 \ddot{\phi} x + 2 \ddot{\phi} x^{(3)} + \phi^{(3)} y \right) \]

\[ + 3 \ddot{\phi} y^{(3)} + 2 \dot{\phi} y^{(4)} \right) \sin \theta \cos \phi \]

\[ + \frac{1}{L} \left( -\ddot{\theta} \sin \theta \sin \phi + \dddot{\phi} \cos \theta \cos \phi \right) \cdot \left( \phi^2 x + \dddot{\phi} x + 2 \phi y^{(3)} - x^{(4)} \right) \]

\[ + \frac{1}{L} \left( 2 \dddot{\phi} x + 2 \dddot{\phi} x^{(3)} + \phi^2 x^{(3)} + \phi^{(3)} y \right) \]

\[ + 3 \phi y^{(3)} + 2 \phi y^{(4)} - x^{(5)} \right) \cos \theta \sin \phi \]

\[- \frac{1}{L} \left( \theta \sin \theta \cos \phi + \dot{\phi} \cos \theta \sin \phi \right) \cdot \left( -\phi x - 2 \phi x^{(3)} + \phi^2 y + \phi^2 y - y^{(4)} \right) \]

\[ + \frac{1}{L} \left( -\phi^{(3)} x - 3 \phi x^{(3)} - 2 \phi x^{(4)} + 2 \ddot{\phi} y + 2 \phi \dddot{\phi} y \right) \]

\[ + \phi^2 y^{(3)} + \phi^2 y^{(3)} - y^{(5)} \right) \cos \theta \cos \phi \]

(B.232)
\[
\ddot{S}_4 = \frac{1}{L} \left( \ddot{\phi} x + 2 \ddot{\phi} x^{(3)} - \dot{\phi} \dot{y} - \dot{\phi} \dot{y} + y^{(4)} \right) \sin \theta \sin \phi \\
+ \frac{1}{L} \left( 2 \ddot{\phi} \ddot{\phi} x + \ddot{\phi} \ddot{y} + 2 \ddot{\phi} y^{(3)} \right) \cos \theta \sin \phi \\
+ \frac{1}{L} \left( \dot{\phi} \dot{x} + \ddot{\phi} \ddot{x} + \dot{\phi} \dot{y} + 2 \ddot{\phi} y^{(3)} - x^{(4)} \right) \sin \theta \cos \phi \\
- \frac{1}{L} \left( \ddot{\phi} x + 2 \ddot{\phi} x^{(3)} - 2 \dot{\phi} \dot{y} \right) \cos \theta \cos \phi.
\]

(B.233)

After multiplying (B.226) by \( \gamma_1(\theta) \), the following expression can be obtained

\[
4 \gamma_2 \ddot{\theta} \gamma_1^2 \left( (9 - 6 \sin^2 \theta) - 3 \gamma_1 \right) \sin \theta \cos \theta \\
- \frac{36i}{m_p L^2} \gamma_1 \dot{\phi}^2 \cos^2 \theta \sin \theta - \frac{48i}{m_p L^2} \dot{\phi}^2 \sin \theta \cos^4 \theta
\]

(B.234)

\[
= \gamma_1 \ddot{S}_7 - \frac{24i}{m_p L^2} \dot{\phi} S_4 \cos^3 \theta
\]

where (B.193) was utilized. After dividing (B.234) by \( \dot{\theta}(t) \sin \theta \cos \theta \) and using (B.225), the following expression can be obtained

\[
- \frac{12i \phi^2 \cos \theta}{m_p L^2} \left( 8 \cos^2 \theta + 3 \gamma_1 \right) - 12 \gamma_2 \gamma_1^3 = S_8 - 4S_7
\]

(B.235)

where \( S_8(t) \in \mathbb{R} \) is defined as

\[
S_8 = \left( \frac{1}{\theta \sin \theta \cos \theta} \right) \left( \ddot{S}_7 \gamma_1 - \frac{24i}{m_p L^2} \dot{\phi} S_4 \cos^3 \theta \right).
\]

(B.236)

By multiplying (B.235) by \( \cos^2 \theta \) and then utilizing (B.225), (B.235) can be rewritten as

\[
(9 - 6 \sin^2 \theta) \left( 8 \cos^2 \theta + 3 \gamma_1 \right) - 12 \gamma_1 \cos^2 \theta = \left( \frac{S_9}{\gamma_2 \gamma_1^2} \right)
\]

(B.237)

where \( S_9(t) \in \mathbb{R} \) is defined as

\[
S_9 = (S_8 - 4S_7) \cos^2 \theta + S_7 \left( 8 \cos^2 \theta + 3 \gamma_1 \right).
\]

(B.238)

Given the facts that \( F_x(t) \) and \( F_y(t) \) are constant and \( \ddot{x} = 0 \) and \( \ddot{y} = 0 \), it is clear that

\[
F_x^{(k)} = 0 \quad F_y^{(k)} = 0 \quad k \geq 1
\]

(B.239)
\[ x^{(k)} = 0 \quad y^{(k)} = 0 \quad k \geq 2. \]  

(H.B.240)

Hence, the expressions in (B.196–B.199), (B.204–B.206), (B.211), (B.214), (B.216–B.219), (B.221), (B.224), (B.227–B.233), (B.236), and (B.238) can be used to prove that

\[ S_i = 0, \quad i = 3, 4, \ldots 9; \]
\[ \dot{S}_i = 0, \quad i = 1, 2, \ldots 9. \]  

(B.241)

After some algebraic manipulation, (B.241) can be used to rewrite (B.237) as follows

\[
(1 + 2 \cos^2 \theta) \left( 3 \left( 1 + \frac{I}{m_p L^2} \right) + 5 \cos^2 \theta \right) 
- 4 \cos^2 \theta \left( 1 + \frac{I}{m_p L^2} - \cos^2 \theta \right) = 0. 
\]  

(B.242)

After some further algebraic manipulation, the expression given in (B.242) can be rewritten as follows

\[
\cos^4 \theta + \alpha_1 \cos^2 \theta + \alpha_2 = 0 
\]  

(B.243)

where \( \alpha_1, \alpha_2 \in \mathbb{R} \) are positive constants defined as

\[
\alpha_1 = \frac{1}{2} + \frac{1}{7} \frac{I}{m_p L^2} \quad \alpha_2 = \frac{3}{14} \left( 1 + \frac{I}{m_p L^2} \right). 
\]  

(B.244)

Since the expression given in (B.243) is clearly invalid, the proposition that \( \dot{\theta}(t) \neq 0 \) must be invalid, and hence, \( \dot{\theta}(t) = 0 \). The analysis given in the previous cases can now be used to prove Lemma B.21.

\( \square \)

### B.5.3 Coupling Control Law Analysis

**Lemma B.22** Given the \( E^2 \) coupling control law of (6.33), the time derivative of the following function

\[ V = \frac{1}{2} k_E E^2 + \frac{1}{2} k_p e^T e + \frac{1}{2} k_v \dot{r}^T \dot{r} \]  

(B.245)

is given by the following expression

\[ \dot{V} = 0 \]  

(B.246)

only when

\[ E(q, \dot{q}) = e(t) = \dot{r}(t) = 0 \]  

(B.247)

where \( E(q, \dot{q}), e(t), \) and \( r(t) \) are defined in (6.21), (6.24), and (6.16), respectively.
Proof: To prove Lemma B.22, the set of all points where (B.246) is satisfied is defined as \( \Gamma \). In the set \( \Gamma \), it is clear from (6.38) and (B.246) that
\[
\dot{r}(t) = 0 \quad \ddot{r}(t) = 0
\] (B.248)
and hence, (6.16) and (B.246) can be used to prove that \( x(t), y(t), \) and \( V(t) \) are constant, and that
\[
\ddot{x}(t) = 0 \quad \ddot{y}(t) = 0.
\] (B.249)
Furthermore, from (6.22) and (B.248), it is clear that
\[
\dot{E}(q, \dot{q}) = 0 \quad \dot{e}(t) = 0
\] (B.250)
and hence, \( E(q, \dot{q}) \) and \( e(t) \) are constant.

Similar to the proof of Lemma B.21, the remainder of the analysis can be divided into two cases. For the case of \( \dot{\theta} = 0 \) and \( \dot{\phi} = 0 \), Case 1a in the proof of Lemma B.21 can be used to prove Lemma B.22. For the other cases, (6.17) can be used to rewrite (6.33) in the following equivalent form\(^2\)
\[
F = - k_d \ddot{r} - k_p e - k_v \ddot{r} \quad \frac{k_E}{k_E E}.
\] (B.251)
Based on the structure of (B.251), it is clear from (B.248), (B.250), and (B.251) that \( F(t) \) is constant, and similar arguments as in the proof of Lemma B.21 can be used to prove Lemma B.22. \( \square \)

Lemma B.23  Given the gantry kinetic energy coupling control law (6.41), the time derivative of the following function
\[
V = k_E E + \frac{1}{2} k_v \dot{r}^T (\det(M)P^{-1}) \dot{r} + \frac{1}{2} k_p e^T e
\] (B.252)
is given by the following expression
\[
\dot{V} = 0
\] (B.253)
only when
\[
E(q, \dot{q}) = e(t) = \dot{r}(t) = 0
\] (B.254)
where \( E(q, \dot{q}) \), \( e(t) \), and \( r(t) \) are defined in (6.21), (6.24), and (6.16), respectively.

\(^2\)Since either \( \dot{\theta}(t) \neq 0 \) or \( \dot{\phi}(t) \neq 0 \), (6.21) can be used to show that \( E(q, \dot{q}) > 0 \), and hence, the denominator of (B.251) does not go to zero for the remaining cases.
**Proof:** To prove Lemma B.23, the set of all points where (B.253) is satisfied is defined as $\Gamma$. In the set $\Gamma$, it is clear from (6.46) and (B.253) that

$$\dot{r}(t) = 0 \quad \ddot{r}(t) = 0 \quad (B.255)$$

and hence, (6.16) can be used to prove that $x(t)$, $y(t)$, and $V(t)$ are constant, and that

$$\dot{x}(t) = 0 \quad \ddot{y}(t) = 0. \quad (B.256)$$

Furthermore, from (6.22) and (B.255), it is clear that

$$\dot{E}(q, \dot{q}) = 0 \quad \dot{e}(t) = 0 \quad (B.257)$$

and hence, $E(q, \dot{q})$ and $e(t)$ are constant.

To complete the remaining analysis, (6.17) can be used to rewrite (6.41) in the following equivalent form

$$F = \frac{-kd \dot{r} - k_p e - k_v (\det(M) P^{-1}) \dot{r} - \frac{1}{2} k_v (\frac{d}{dt} (\det(M) P^{-1}) \dot{r}}{k_E} \quad (B.258)$$

Based on the structure of (B.258), it is clear from (B.255) and (B.257) that $F(t)$ is constant, and similar arguments as given in the proof of Lemma B.21 can be used to prove Lemma B.23. □

**B.5.4 Matrix Property**

**Lemma B.24** Given the definition for the matrix $P(q)$ given in (6.18), the following property holds

$$\frac{d}{dt} P^{-1} = -P^{-1} \left( \frac{d}{dt} P \right) P^{-1}. \quad (B.259)$$

**Proof:** To prove Lemma B.24, the time derivative of $P(q) P^{-1}(q)$ is determined as follows

$$\frac{d}{dt} (P P^{-1}) = \left( \frac{d}{dt} P \right) P^{-1} + P \left( \frac{d}{dt} P^{-1} \right). \quad (B.260)$$

After noting that the left-side of (B.260) is equal to zero, the expression given in (B.259) can be directly obtained by premultiplying (B.260) by $P^{-1}(q)$. □

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