Appendix A
Useful Mathematical Identities

A.1 3-Vector Cross-Product Identities

The following useful cross-product identities hold for arbitrary 3-vectors \( \mathbf{p}, \mathbf{q}, \) and \( \mathbf{r} \):

\[
\begin{align*}
\mathbf{\hat{p}}\mathbf{\hat{p}} &= 0 \\
\mathbf{\hat{p}}^* &= -\mathbf{\hat{p}} \\
\mathbf{\hat{p}}\mathbf{\hat{q}} &= -\mathbf{\hat{q}}\mathbf{\hat{p}} \\
(\mathbf{\hat{p}}\mathbf{\hat{q}}) &= \mathbf{\hat{p}}\mathbf{\hat{q}} - \mathbf{\hat{q}}\mathbf{\hat{p}} \\
\mathbf{\hat{p}}\mathbf{\hat{q}} &= \mathbf{\hat{q}}^* - \mathbf{\hat{p}}^* \mathbf{I} \\
\mathbf{\hat{p}}\mathbf{\hat{r}} + \mathbf{\hat{r}}\mathbf{\hat{p}} + \mathbf{\hat{q}}\mathbf{\hat{r}} &= 0 \\
\mathbf{\hat{p}}\mathbf{\hat{q}}\mathbf{\hat{r}} &= -\mathbf{\hat{q}}\mathbf{\hat{r}}\mathbf{\hat{p}} \\
\mathbf{\hat{p}}^*\mathbf{\hat{r}} &= \mathbf{\hat{r}}^*\mathbf{\hat{p}} \\
\mathbf{\hat{p}}\mathbf{\hat{r}} &= \mathbf{\hat{r}}\mathbf{\hat{p}}
\end{align*}
\]  (A.1)

\( \mathbb{R} \) denotes a rotation matrix.

A.2 Matrix and Vector Norms

The sup norm (or 2-norm) of a matrix \( \mathbf{A} \) is defined as

\[
\|\mathbf{A}\| \triangleq \sup_{\mathbf{x}} \|\mathbf{Ax}\|/\|\mathbf{x}\| \quad (A.2)
\]

where the norm of a vector \( \mathbf{x} \) is defined as \( \|\mathbf{x}\| \triangleq \sqrt{\mathbf{x}^*\mathbf{x}} \).

**Exercise A.1**  **Norm of \( \mathbf{\hat{s}} \).**

Show that \( \|\mathbf{\hat{s}}\| = \|\mathbf{s}\| \).

### A.3 Schur Complement and Matrix Inverse Identities

Assume that we have a block-partitioned matrix, \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), with square A and D sub-matrices. When D is invertible, the **Schur complement** of the matrix is defined as \( A - BD^{-1}C \). The following describe factorizations and solutions of equations involving such block partitioned matrices.

1. If D is invertible, then

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \tag{A.3}
\]

Hence,

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \tag{A.4}
\]

Using this factorization, the solution to the following matrix equation

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{A.5}
\]

is given by

\[
\begin{align*}
\begin{bmatrix} y \\ x_1 \end{bmatrix} &= D^{-1}b_2 \\
x_1 &= (A - BD^{-1}C)^{-1} \left( b_1 - By \right) \\
x_2 &= y - D^{-1}Cx_1
\end{align*} \tag{A.6}
\]

2. If A is invertible, then,

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \tag{A.7}
\]

Hence,

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \tag{A.8}
\]

Using this factorization, the solution to the following matrix equation

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{A.9}
\]
is given by
\[
y = A^{-1}b_1 \\
x_2 = (D - CA^{-1}B)^{-1}(b_2 - Cy) \\
x_1 = y - A^{-1}Bx_2
\] (A.10)

3. If \( A, B \) and \( C \) are invertible, and \( D = 0 \), then (A.8) simplifies to:
\[
\begin{pmatrix}
A & B \\
C & 0
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & C^{-1} \\
B^{-1} & -B^{-1}AC^{-1}
\end{pmatrix}
\] (A.11)

4. Now let us assume that \( D \) is invertible, and further that there are constraints \( Qb_2 = 0 \) and \( x_2 = Q^*\lambda \) for some full-rank matrix \( Q \) and vector \( \lambda \). Then (A.9) can be re-expressed as:
\[
\begin{pmatrix}
I & 0 \\
mQ & Q
\end{pmatrix}\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}\begin{pmatrix}
I & 0 \\
mQ & Q^*
\end{pmatrix}\begin{pmatrix}
x_1 \\
\lambda
\end{pmatrix} = \begin{pmatrix}
b_1 \\
0
\end{pmatrix}
\] (A.12)

That is,
\[
\begin{pmatrix}
A & BQ^* \\
QC & QDQ^*
\end{pmatrix}\begin{pmatrix}
x_1 \\
\lambda
\end{pmatrix} = \begin{pmatrix}
b_1 \\
0
\end{pmatrix}
\implies (A - BQ^*(QDQ^*)^{-1}Q) x_1 = A_6 b_1
\] (A.13)

\( QDQ^* \) is invertible because \( D \) is invertible and \( Q \) is full-rank.

5. Let us assume the same conditions as in (4), except that the original block-matrix has an additional \([c_1, c_2]^*\) term to take the form:
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
\] (A.14)

In this case, (A.13) generalizes to:
\[
\begin{pmatrix}
A & BQ^* \\
QC & QDQ^*
\end{pmatrix}\begin{pmatrix}
x_1 \\
\lambda
\end{pmatrix} = \begin{pmatrix}
b_1 - c_1 \\
-Qc_2
\end{pmatrix}
\implies (A - ZC)x_1 + (c_1 - Zc_2) = A_6 b_1 \text{ where } Z \triangleq BQ^*(QDQ^*)^{-1}Q \\
\quad \text{with } \lambda = -(QDQ^*)^{-1}Q(c_2 + Cx_1) \quad \text{and} \quad x_2 = Q^*\lambda
\] (A.15)

6. Let us assume the same conditions as in (5), except that now the constraint has the form \( x_2 = Q^*\lambda + \gamma \). In this case, (A.13) generalizes to:
\[
\begin{pmatrix}
A & BQ^* \\
QC & QDQ^*
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\lambda
\end{pmatrix} =
\begin{pmatrix}
b_1 - (c_1 + B\gamma) \\
-Q(c_2 + D\gamma)
\end{pmatrix}
(A.16)
\]

\[
\Rightarrow (A - ZC)x_1 + (c_1 + B\gamma - Z(c_2 + D\gamma)) = b_1
\]

where \( Z \triangleq BQ^*(QDQ^*)^{-1}Q \)

with \( \lambda = -(QDQ^*)^{-1}Q(c_2 + D\gamma + Cx_1) \) and \( x_2 = Q^*\lambda + \gamma \)

**A.4 Matrix Inversion Identities**

1. For any matrix \( A \) such that \( (I - A) \) is invertible, the following matrix identity holds:
\[
A(I - A)^{-1} = (I - A)^{-1}A = (I - A)^{-1} - I
\]

(A.17)

2. For a pair of matrices \( A \) and \( B \) such that \( (I + AB) \) is invertible, we have
\[
(I + AB)^{-1} = I - A(I + BA)^{-1}B
\]

(A.18)

3. For matrices \( A, X, R \) and \( Y \), such that \( A, R \) and \( [A + XRY^*] \) are invertible, we have
\[
[A + XRY^*]^{-1} = A^{-1} - A^{-1}X[R^{-1} + Y^*A^{-1}X]^{-1}Y^*A^{-1}
\]

(A.19)

**Lemma A.1** The 1-resolvent of a nilpotent matrix.

If \( U \) is a nilpotent matrix, such that \( U^n = 0 \), then its 1-resolvent, \( W \triangleq (I - U)^{-1} \), is given by
\[
W = I + U + U^2 + \cdots + U^{n-1}
\]

(A.20)

**Proof:** For \( W \) as defined in (A.20)
\[
UW = WU = U + U^2 + \cdots + U^n = U + U^2 + \cdots + U^{n-1} = W - I
\]

Rearranging terms, we have
\[
I = W - UW \implies (I - U)W \implies (I - U)^{-1} = W
\]

**A.5 Matrix Trace Identities**

For a square matrix \( A \)
\[
\text{Trace}\{A\} = -\text{Trace}\{-A\}
\]

(A.21)
For matrices $A$ and $B$ such that $AB$ is square, we have

$$\text{Trace}\{AB\} = \text{Trace}\{BA\} \quad (A.22)$$

### A.6 Derivative and Gradient Identities

#### A.6.1 Function Derivatives

The following identities hold for function derivatives of a smooth function $g(\theta, \dot{\theta})$:

$$\frac{\partial g}{\partial \ddot{\theta}} = \frac{\partial g}{\partial \theta} \quad (A.23)$$

#### A.6.2 Vector Gradients

We now define the notation for gradients and derivatives of vector functions with respect to a vector. Let $f(\theta) \in \mathbb{R}^m$ be a smooth function of a vector, $\theta \in \mathbb{R}^n$. Then the gradient, $\nabla_\theta f(\theta)$, is an $m \times n$ matrix defined as follows:

$$\nabla_\theta f(\theta) \triangleq \begin{pmatrix}
\frac{\partial f_1}{\partial \theta_1} & \cdots & \frac{\partial f_1}{\partial \theta_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial \theta_1} & \cdots & \frac{\partial f_m}{\partial \theta_n}
\end{pmatrix} \in \mathbb{R}^{m \times n} \quad (A.24)$$

### Exercise A.2 Product gradient and chain rules.

1. Given a vector $x \in \mathbb{R}^p$, a smooth, vector-valued function $f(x) \in \mathbb{R}^m$, and a smooth, scalar valued function $g(x)$, show that the following product rule holds:

$$\nabla_x [f(x)g(x)] = \nabla_x f(x) \cdot g(x) + f(x) \cdot \nabla_x g(x) \quad (A.25)$$

2. Given a smooth vector-valued function $f(y) \in \mathbb{R}^m$ of another vector-valued function $y(x) \in \mathbb{R}^n$, which in turn is a smooth function of a vector $x \in \mathbb{R}^p$, show that the following chain rule holds:

$$\nabla_x f(x) = \nabla_y f(y) \cdot \nabla_x y(x) \quad (A.26)$$
At times, we need to differentiate a row-vector valued functions $f(\theta)$ with respect to the variables vector $\theta \in \mathbb{R}^m$. We use the following notation for this purpose:

$$\frac{df(\theta)}{d\theta} \triangleq \left[ \nabla_{\theta}(f^*) \right]^* = \begin{bmatrix} \frac{\partial f}{\partial \theta_1} \\ \vdots \\ \frac{\partial f}{\partial \theta_m} \end{bmatrix}$$ (A.27)

The notational convention in (A.27) also applies to partial derivatives such as $\frac{\partial f}{\partial \theta}$.

### A.6.3 Matrix Derivatives

Let $A(t)$ be a differentiable and invertible matrix. We derive an expression for the time derivative of the inverse of $A(t)$ in terms of the time derivative of $A(t)$. Since $A(t)A^{-1}(t) = I$, differentiating both sides with respect to $t$ yields

$$\frac{dA(t)}{dt}A^{-1}(t) + A(t)\frac{dA^{-1}(t)}{dt} = 0$$

Rearranging terms, we obtain

$$\frac{dA^{-1}(t)}{dt} = -A^{-1}(t)\frac{dA(t)}{dt}A^{-1}(t)$$ (A.28)

Moreover, we have the following identity from Graham [61]:

$$\frac{d\log \{ \det \{ A(t) \} \}}{dt} = \text{Trace} \left\{ A^{-1} \frac{dA(t)}{dt} \right\}$$ (A.29)
Appendix B
Attitude Representations

This appendix summarizes some attitude representation schemes that are alternatives to the direction-cosine matrix attitude representation scheme. References [43, 165] contains a detailed discussion on this topic. Since different conventions exist, a key to working with attitude representations is a clear definition of how they compose with each other, and how they transform vector representations. The \( I^B \) direction-cosine matrix representations for attitude is the least ambiguous, because its composition operation is simply the normal matrix/vector multiplication operation, i.e.,

\[
\hat{p} = I^B \hat{B} \hat{p}
\]

for a 3-vector \( p \). We refer to direction-cosine matrices as rotation matrices.

Exercise B.1  Time derivative of a rotation matrix.

Let \( R(t) \) denote a rotation matrix that is a smooth function of time. Show that

\[
\frac{dI^B}{dt} = \tilde{w} I^B
\]

for some 3-vector \( w \). As discussed in (1.8) on page 5, \( w = I^I \omega(I, B) \), the angular velocity of the \( B \) frame with respect to the \( I \) frame expressed in the \( I \) frame.

In this appendix, we work exclusively with the \( I \) and \( B \) pair of frames while studying the properties of attitude representations. With this in mind, we use the more compact \( \omega \) notation for \( \omega(I, B) \) to simplify the expressions.

B.1 Euler Angles

Euler angles, \( \theta = (\psi, \gamma, \phi) \) are an example of a minimal, 3-parameter attitude representation. This representation is based on expressing rotations as consisting of a sequence of three principal axis rotations. There are several possible options for Euler
angle representations based on the specific choice of the principal axes [43, 69]. For instance, the ZXZ Euler angles representation is defined as a sequence of principal axis rotations about the z axis, the x axis, and, once again, the z axis. The expression for $^B\mathcal{R}_B$ for the ZXZ Euler angle representation is as follows:

$$^B\mathcal{R}_B(\theta) = \begin{pmatrix} c_{\psi} & -s_{\psi} & 0 \\ s_{\psi} & c_{\psi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{\gamma} & -s_{\gamma} \\ 0 & s_{\gamma} & c_{\gamma} \end{pmatrix} \begin{pmatrix} c_{\phi} & -s_{\phi} & 0 \\ s_{\phi} & c_{\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B.1)

The orientation of frame $B$ is obtained through a sequence of rotations. We begin with $B$ aligned with frame $I$. Then, we rotate $B$ about its z axis through the angle $\psi$. Next, we rotate it about its x axis by the angle $\gamma$, and, finally, about its z axis by the angle $\phi$, to obtain the final orientation of frame $B$.

For the ZXZ Euler angle representation, the $B\omega$ angular velocity is related to the time derivatives of the Euler angles, $\dot{\theta}$, as follows:

$$B\omega = \begin{pmatrix} s_{\gamma} c_{\phi} & c_{\phi} & 0 \\ s_{\gamma} c_{\phi} & -s_{\phi} & 0 \\ c_{\gamma} & 0 & 1 \end{pmatrix} \dot{\theta}$$

(B.2)

When $\sin(\gamma) \neq 0$, the matrix in (B.2) is invertible, and the inverse relationship is given by

$$\dot{\theta} = \frac{1}{s_{\gamma}} \begin{pmatrix} s_{\phi} & c_{\phi} & 0 \\ s_{\gamma} c_{\phi} & -s_{\gamma} s_{\phi} & 0 \\ -c_{\gamma} s_{\phi} & -c_{\gamma} c_{\phi} & s_{\gamma} \end{pmatrix} B\omega$$

(B.3)

The relationship in (B.3) is undefined when $\sin(\gamma) = 0$. All 3-parameter attitude representations have such singularities. A representation of size 4 or more is required to avoid singularities.

### B.2 Angle/Axis Parameters

Euler’s theorem states that every rotation is equivalent to a rotation about a fixed vector. With $\mathbf{n}$ denoting the axis of rotation unit vector, and $\theta$ the angle of rotation, the following exponential formula defines the expression for the rotation matrix:

$$^B\mathcal{R}_B(\mathbf{n}, \theta) = \exp[\mathbf{n}\theta]$$

(B.4)
Thus, the four parameters, \( \rho_B \triangleq (n, \theta) \), can be used to represent the relative attitude of the two frames. This representation is known as the angle/axis representation of attitude.

**Exercise B.2  Derivation of the Euler–Rodrigues formula.**

1. Verify that the characteristic polynomial\(^1\) of a \( 3 \times 3 \) skew-symmetric matrix \( \tilde{s} \) is

\[
\lambda^3 + \sigma^2 \lambda \quad (B.5)
\]

where \( \sigma \triangleq \|s\| \) is the vector norm of \( s \).

2. Use this to derive the following Euler–Rodrigues formula for a rotation matrix in terms of its angle/axis representation coordinates:

\[
^B I R_B (^B \rho_B) = \cos(\theta) I_3 + [1 - \cos(\theta)] \tilde{n} \tilde{n}^* + \sin(\theta) \tilde{n} \quad (B.6)
\]

**Exercise B.3  Trace and characteristic polynomial of a rotation matrix.**

Let \( \mathcal{R}(n, \theta) \) denote a rotation matrix.

1. Show that

\[
\gamma \triangleq \text{Trace}\{\mathcal{R}(n, \theta)\} = 1 + 2 \cos(\theta) \quad (B.7)
\]

2. Show that the characteristic polynomial of \( \mathcal{R}(n, \theta) \) is

\[
\lambda^3 - \gamma \lambda^2 + \gamma \lambda - 1 \quad (B.8)
\]

3. Verify that \( n \) is the eigen-vector of \( \mathcal{R}(n, \theta) \) with eigen-value of 1.

4. Show that when \( \sin(\theta) \neq 0 \), \( n \) can be obtained from the \( \mathcal{R}(n, \theta) \) rotation matrix using the following relationship:

\[
\tilde{n} = (\mathcal{R} - \mathcal{R}^*) / (2 \sin(\theta)) \quad (B.9)
\]

**Remark B.1  Eigen-values of a rotation matrix.**

The axis of rotation \( n \) for a rotation matrix \( ^B I R_B \) is its eigen-vector corresponding to the eigenvalue 1. The angle of rotation, \( \theta \), appears in the other complex eigenvalues of the matrix, which, using (B.8) can be shown to be of the form \( \exp[j \theta] \) and

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\(^1\) The characteristic polynomial of a matrix, \( A \), is defined as the polynomial of \( \lambda \), defined by the matrix determinant, \( \det(\lambda I - A) \).
exp[−jθ]. Given a rotation matrix $\mathcal{R}$ instead of these relationships, it is simpler to obtain $\theta$ by using the (B.7) expression for the trace of a $\mathcal{R}$ and $n$ from the expression in (B.9).

Exercise B.4 Angular velocity from the angle/axis rates.

Derive the following expression for the $^B\omega$ angular velocity in terms of $\dot{n}$ and $\dot{\theta}$ angle/axis rates:

$$
^B\omega = \dot{n} - (1 - \cos \theta) \ddot{n} + \sin(\theta) \dot{n}
$$

$$
= \left[ \sin(\theta) - (1 - \cos \theta) \dot{n}, \quad n \right] \left[ \begin{array}{c} \dot{n} \\ \dot{\theta} \end{array} \right]
$$

(B.10)

Exercise B.5 Angle/axis rates from the angular velocity.

Show that the expression for the $\dot{n}$ and $\dot{\theta}$ time derivatives in terms of the $^B\omega$ angular velocity is:

$$
\left[ \begin{array}{c} \dot{n} \\ \dot{\theta} \end{array} \right] = \left[ \begin{array}{c} \frac{1}{2} \left[ \ddot{n} - \cot(\theta/2) \ddot{n} \right] \\ n^* \end{array} \right] ^B\omega
$$

(B.11)

B.3 Unit Quaternions/Euler Parameters

Unit quaternions (also known as Euler parameters) are closely related to the angle/axis parameters and also consist of four scalar parameters. Given an angle/axis representation of the attitude $^p\rho_B = (n, \theta)$, the corresponding quaternion representation, denoted $^q_B = (q, q_0)$, is defined as:

$$
q_0 \triangleq \cos(\theta/2); \quad q \triangleq \left[ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right] = \sin(\theta/2) n
$$

(B.12)

It is easy to verify that quaternions have unit magnitude, that is,

$$
q^* q = q_0^2 + q^* q = 1
$$

(B.13)

A description of quaternions and their transformations can be found in [30, 43, 69].

Exercise B.6 Quaternion expression for a rotation matrix.

Show that the $^i\mathcal{R}_B$ rotation matrix can be obtained from the corresponding $^i\mathcal{q}_B = (q, q_0)$ quaternion representation by any of the following expressions:
\[ \mathcal{R}_B \left( \mathcal{I} \mathbf{q}_B \right) = (q_0^2 - q^* q) \mathbf{I}_3 + 2qq^* + 2q_0 \bar{q} \]
\[ = (2q_0^2 - 1) \mathbf{I}_3 + 2qq^* + 2q_0 \bar{q} \]
\[ = \mathbf{I}_3 + 2(\bar{q}_0 \mathbf{I}_3 + \bar{q}) \bar{q} \]
\[ = (q_0 \mathbf{I}_3 + \bar{q})^2 + qq^* \]
\[ = \begin{pmatrix}
2[q_0^2 + q_1^2] - 1 & 2[q_1 q_2 - q_0 q_3] & 2[q_1 q_3 + q_0 q_2] \\
2[q_1 q_2 + q_0 q_3] & 2[q_0^2 + q_2^2] - 1 & 2[q_2 q_3 - q_0 q_1] \\
2[q_1 q_3 - q_0 q_2] & 2[q_2 q_3 + q_0 q_1] & 2[q_0^2 + q_3^2] - 1
\end{pmatrix} \tag{B.14} \]

Observe that, in contrast to (B.6), the (B.14) expression for \( \mathcal{R}_B \left( \mathcal{I} \mathbf{q}_B \right) \) does not involve trigonometric quantities. Indeed, one of the key advantages of quaternions is that quaternion transformations typically involve algebraic expressions instead of the computationally expensive trigonometric functions typically encountered for other types of attitude representations.

**Exercise B.7 Basic properties of unit quaternions.**

Let \( \mathbf{q} = (q, q_0) \) denote a unit quaternion:

1. Show that the orthogonality of the matrix \( \mathcal{R}(\mathbf{q}) \) in (B.14) follows from the unit magnitude of \( \mathbf{q} \).
2. Verify that the unit quaternions \( \mathbf{q} \) and \( -\mathbf{q} \) are equivalent, in the sense that they both map to the same rotation matrix, i.e., \( \mathcal{R}(\mathbf{q}) = \mathcal{R}(-\mathbf{q}) \).
3. Show that the inverse of a quaternion \( \mathbf{q} \), denoted \( \mathbf{q}^{-1} \), is the unit quaternion given by
   \[ \mathbf{q}^{-1} = \begin{bmatrix}
-q \\
q_0
\end{bmatrix} \tag{B.15} \]

An equivalent representation for \( \mathbf{q}^{-1} \) is \( (q, -q_0) \).
4. Show that the trace of the \( \mathcal{R}(\mathbf{q}) \) rotation matrix is given by
   \[ \text{Trace} \left\{ \mathcal{R}(\mathbf{q}) \right\} = 4q_0^2 - 1 \tag{B.16} \]
5. Show that \( \mathbf{q} \) is the eigen-vector of \( \mathcal{R}(\mathbf{q}) \) with eigenvalue of 1, i.e.,
   \[ \mathcal{R}\mathbf{q} = \mathbf{q} \tag{B.17} \]
6. Show that when \( q_0 \neq 0 \), \( \mathbf{q} \) can be obtained from the \( \mathcal{R} \) rotation matrix using the following relationship:
   \[ \tilde{\mathbf{q}} = (\mathcal{R} - \mathcal{R}^*)/(4q_0) \tag{B.18} \]
B.3.1 The $E_+(q)$ and $E_-(q)$ Matrices

For a unit quaternion $q = (q, q_0)$, define the $4 \times 4$ matrices, $E_+(q)$ and $E_-(q)$, as

$$
E_+(q) \triangleq \begin{pmatrix}
q_0 I_3 + \tilde{q} & q \\
-q^* & q_0
\end{pmatrix}
$$

and

$$
E_-(q) \triangleq \begin{pmatrix}
q_0 I_3 - \tilde{q} & q \\
-q^* & q_0
\end{pmatrix}
$$

(B.19)

Exercise B.8 Properties of $E_-$ and $E_+$ matrices.

Let $q = (q, q_0)$ denote a unit quaternion:

1. Show that $E_+(q)$ and $E_-(q)$ are orthogonal matrices, i.e.,

$$
E_+^*(q)E_-(q) = E_-(q)E_+^*(q) = I
$$

$$
E_+^*(q)E_+(q) = E_+(q)E_+^*(q) = I
$$

(B.20)

2. Show that

$$
E_-(q)E_+(q) = E_+(q)E_-(q) = \begin{pmatrix}
\mathcal{R}(q) & 0 \\
0 & 1
\end{pmatrix} \triangleq T(q)
$$

$$
E_+^*(q)E_-(q) = E_-(q)E_+^*(q) = \begin{pmatrix}
\mathcal{R}^*(q) & 0 \\
0 & 1
\end{pmatrix} = T(q^{-1})
$$

(B.21)

The $T(.)$ homogeneous transforms defined above have zero translation vectors and only rotational components.

3. Verify that

$$
E_+(q^{-1}) = E_+^*(q) \quad \text{and} \quad E_-(q^{-1}) = E_-^*(q)
$$

(B.22)

4. With $p$ denoting another unit quaternion, show that

$$
E_+(p)q = E_-(q)p
$$

(B.23)

5. With $p$ denoting another unit quaternion, show that each of the $E_-(q)$ and $E_-^*(q)$ pair of matrices commutes with each of the $E_+(p)$ and $E_+^*(p)$ matrix pair, i.e.,

$$
E_+(q)E_+(p) = E_+(p)E_-(q)
$$

(B.24a)

$$
E_-(q)E_+^*(p) = E_+^*(p)E_-(q)
$$

(B.24b)

$$
E_+^*(q)E_+(p) = E_+(p)E_+^*(q)
$$

(B.24c)

$$
E_-^*(q)E_+^*(p) = E_+^*(p)E_-^*(q)
$$

(B.24d)
6. Show that $\mathbf{q}$ is an eigen-vector of $T(\mathbf{q})$ with eigenvalue 1, i.e.,

$$T(\mathbf{q})\mathbf{q} = \mathbf{q} \tag{B.25}$$

**B.3.2 Quaternion Transformations**

We denote the composition operation between two quaternions by the symbol “$\otimes$”. The composition of two quaternions $\mathbf{p}$ and $\mathbf{q}$ is defined as

$$\mathbf{p} \otimes \mathbf{q} \triangleq E_+ (\mathbf{p}) \mathbf{q} \quad \text{and} \quad E_- (\mathbf{q}) \mathbf{p} \tag{B.26}$$

**Exercise B.9 Composition of unit quaternions.**

Let $\mathbf{q} = (q, q_0)$ and $\mathbf{p} = (p, p_0)$ denote unit quaternions:

1. Verify that,

$$\mathbf{p} \otimes \mathbf{q} = \begin{bmatrix} p_0 q_0 + p q_0 + \bar{p} q \\ p_0 q_0 - p^* \mathbf{q} \end{bmatrix} \tag{B.27}$$

Verify that $\mathbf{p} \otimes \mathbf{q}$ is of unit norm and is hence also a unit quaternion. Thus, the composition of two quaternions yields another quaternion.

2. For quaternion composition to be consistent with the composition of rotation matrices, we must have

$$R(\mathbf{p} \otimes \mathbf{q}) = R(\mathbf{p}) R(\mathbf{q}) \tag{B.28}$$

Show that this identity holds.

3. Show that the composition of quaternions is an associative operation. That is, with $\mathbf{r}$ denoting another quaternion, we have,

$$\mathbf{p} \otimes (\mathbf{q} \otimes \mathbf{r}) = (\mathbf{p} \otimes \mathbf{q}) \otimes \mathbf{r} \tag{B.29}$$

4. Show that

$$\mathbf{p}^{-1} \otimes \mathbf{q} = E_+^* (\mathbf{p}) \mathbf{q} \quad \text{and} \quad \mathbf{p} \otimes \mathbf{q}^{-1} = E_-^* (\mathbf{q}) \mathbf{p} \tag{B.30}$$

**Exercise B.10 The quaternion identity element.**

Show that the $\mathbf{e}_q = (0,0,0,1)^*$ unit quaternion is the identity element for unit quaternions by establishing the following identities for an arbitrary quaternion $\mathbf{q}$:

$$\mathbf{q} \otimes \mathbf{e}_q = \mathbf{q} \quad \text{and} \quad \mathbf{q} \otimes \mathbf{q}^{-1} = \mathbf{e}_q \tag{B.31}$$
Exercise B.11  Unit quaternion products and inverses.

For a pair of quaternions $\mathbf{p}$ and $\mathbf{q}$, show that:

1. $(\mathbf{p} \otimes \mathbf{q})^{-1} = \mathbf{q}^{-1} \otimes \mathbf{p}^{-1}$
2. With $\mathbf{r} \triangleq \mathbf{p} \otimes \mathbf{q}$, show that,

$$\mathbf{q} = \mathbf{p}^{-1} \otimes \mathbf{r} \quad \text{and} \quad \mathbf{p} = \mathbf{r} \otimes \mathbf{q}^{-1} \quad \text{(B.32)}$$

Exercise B.12  Transforming vectors with unit quaternions.

Given a vector $\mathbf{x} \in \mathbb{R}^3$, and a quaternion $\mathbf{q} = \mathbb{I} \mathbf{q}_B$ defining the attitude of frame $\mathbb{B}$ with respect to frame $\mathbb{I}$, show that

$$\left[ \begin{array}{c} \mathbb{I} \mathbf{x} \\ 0 \end{array} \right] = \mathbf{q} \otimes \left[ \begin{array}{c} \mathbb{B} \mathbf{x} \\ 0 \end{array} \right] \otimes \mathbf{q}^{-1} \quad \text{(B.33)}$$

B.3.3 Quaternion Differential Kinematics

Exercise B.13  Quaternion rates from the angular velocity.

With $\mathbb{I} \mathbf{q}_B = \mathbf{q} = (\mathbf{q}, \mathbf{q}_0)$, Show that the time derivative of the quaternion is related to the $\mathbb{B} \omega$ angular velocity as follows:

$$\frac{\dot{\mathbb{I} \mathbf{q}}_B}{\mathbb{I} \mathbf{q}_B} = \frac{1}{2} \left[ (\mathbf{q}_0 \mathbf{I}_3 + \overline{\mathbf{q}}) \mathbb{B} \omega \right] = \frac{1}{2} \left( \begin{array}{c} -\mathbb{B} \bar{\omega} \mathbb{B} \omega \\ \mathbf{q} \end{array} \right) \mathbf{q}$$

$$= \frac{1}{2} \mathcal{E}_+(\mathbf{q}) \left[ \begin{array}{c} \mathbb{B} \omega \\ 0 \end{array} \right] = \frac{1}{2} \mathbf{q} \otimes \left[ \begin{array}{c} \mathbb{B} \omega \\ 0 \end{array} \right] \quad \text{(B.34)}$$

Exercise B.14  Constant unit quaternion norms.

1. Show that the solution to the (B.34) ordinary differential equation has unit norm for all time $t$.
2. Show that when $\mathbf{q}_0 \neq 0$,

$$\dot{\mathbf{q}}_0 = -\mathbf{q}^* \dot{\mathbf{q}} / \mathbf{q}_0 \quad \text{(B.35)}$$
Exercise B.15  Unit quaternion rate to angular velocity.

Let \( \mathbf{q}_B = \mathbf{q} = (q, q_0) \) denote a unit quaternion:

1. Show that the mapping from quaternion rates to the \( \mathbf{B} \omega \) angular velocity representation is given by the following expression:

\[
\begin{bmatrix}
\mathbf{B} \omega \\
0
\end{bmatrix} = 2 \mathbf{E}^*(q) \mathbf{q} = 2q^{-1} \otimes \mathbf{q}
\]  

(B.36)

Alternatively,

\[
\mathbf{B} \omega = 2 [q_0 \mathbf{I}_3 - \tilde{q}] \mathbf{q} - q \dot{q}_0
\]  

(B.37)

When \( q_0 \neq 0 \),

\[
\mathbf{B} \omega = 2 [q_0 \mathbf{I}_3 - \tilde{q} + qq^*/q_0] \mathbf{q}
\]  

(B.38)

2. Show that the mapping from quaternion rates to the \( \mathbf{I} \omega \) angular velocity representation is as follows:

\[
\begin{bmatrix}
\mathbf{I} \omega \\
0
\end{bmatrix} = 2 \mathbf{E}^*(q) \mathbf{q} = 2 \dot{q} \otimes q^{-1}
\]  

(B.39)

Alternatively,

\[
\mathbf{I} \omega = 2 [q_0 \mathbf{I}_3 + \tilde{q}] \mathbf{q} - q \dot{q}_0
\]  

(B.40)

When \( q_0 \neq 0 \),

\[
\mathbf{I} \omega = 2 [q_0 \mathbf{I}_3 + \tilde{q} + qq^*/q_0] \mathbf{q}
\]  

(B.41)

Exercise B.16  Quaternion double time derivatives.

Let \( \mathbf{q}_B = \mathbf{q} = (q, q_0) \) denote a unit quaternion. Show that

\[
\ddot{\mathbf{q}} = \frac{1}{2} \mathbf{q} \otimes \begin{bmatrix}
\mathbf{B} \omega \\
0
\end{bmatrix} + \frac{1}{4} \| \mathbf{B} \omega \|^2 \mathbf{q}
\]  

(B.42)

Establish the converse relationship

\[
\begin{bmatrix}
\mathbf{B} \omega \\
0
\end{bmatrix} = 2q^{-1} \otimes \ddot{\mathbf{q}} - \frac{1}{2} \| \mathbf{B} \omega \|^2 e_{\mathbf{q}}
\]  

(B.43)
B.4 Gibbs Vector Attitude Representations

Now we use the connection between rotation matrices and skew-symmetric matrices to define the Gibbs vector attitude representation [172].

**Exercise B.17  Gibbs vector attitude representation.**

Let $s$ denote a 3-vector with magnitude $\sigma$:

1. Use the characteristic polynomial for skew-symmetric matrices in Exercise B.2 to show that

$$[I - \tilde{s}]^{-1} = I + (\tilde{s}^2 + \tilde{s})/(1 + \sigma^2)$$

$= I + \tilde{s}(I + \tilde{s})/(1 + \sigma^2)$ \hspace{1cm} (B.44)

2. Use this identity to show that

$$[I - \tilde{s}]^{-1}[I + \tilde{s}] = I + 2(\tilde{s}^2 + \tilde{s})/(1 + \sigma^2)$$ \hspace{1cm} (B.45)

Verify that $\mathcal{R}(s) \triangleq [I - \tilde{s}]^{-1}[I + \tilde{s}]$ is a rotation matrix. Such a representation of attitude matrices using a 3-vector, $s$, is also known as the Rodrigues or Gibbs vector representation for attitude matrices.

3. Establish the converse relationship

$$\tilde{s} = -[I - \mathcal{R}(s)] [I + \mathcal{R}(s)]^{-1}$$

$= -[\gamma I - (1 + \gamma)\mathcal{R}(s) + \mathcal{R}^2(s)]/(1 + \gamma)$ \hspace{1cm} (B.46)

where $\gamma \triangleq \text{Trace(}\mathcal{R}(s))$.

4. Show that the Gibb’s vector, $s$, is the axis of rotation for the $\mathcal{R}(s)$ rotation matrix.

5. Show that the Gibb’s vector, $s$, for a quaternion $q = (q, q_0)$, with $q_0 \neq 0$, is given by

$$s = q/q_0 = \tan(\theta/2)n$$ \hspace{1cm} (B.47)

so that $\mathcal{R}(s) = \mathcal{R}(q) = \mathcal{R}(n, \theta)$.

We now look at the problem of finding the family of rotation matrices that will transform one vector into another [15]. Let $a$ and $b$ denote two vectors of identical norm, i.e., $\|a\| = \|b\|$. The goal is to find a parametrization of all rotation matrices $\mathcal{R}$ such that $\mathcal{R}a = b$.

**Exercise B.18  Rotation of vectors.**

For a pair of 3-vectors $a$ and $b$ of the same norm, let $s$ denote a Gibb’s vector attitude representation such that $\mathcal{R}(s)a = b$:

1. Show that $s$ must satisfy the following relationship:

$$(a + b)s = (a - b)$$ \hspace{1cm} (B.48)
2. Now show that the general solution for \( s \) satisfying \( \mathfrak{R}(s) a = b \) is given by

\[
    s = \lambda \left( \widehat{a - b} \right) (a + b) + \alpha (a + b)
\]

(B.49)

where \( \lambda = 1/\| (a + b) \|^2 \), and \( \alpha \) is an arbitrary constant.

This exercise shows how to compute a rotation matrix that transforms a given vector, \( a \), into another given vector, \( b \). This rotation matrix is not unique, as reflected by the arbitrary parameter \( \alpha \) in (B.49). After being rotated to coincide with \( b \), any additional rotation of \( a \) about the axis \( b \) leaves the vector unchanged, leading to the non-uniqueness of the rotation. A similar circumstance holds if vector \( a \) is rotated about itself prior to rotating it to coincide with \( b \).
Solutions for Chapter 1

Solution 1.1 (pp. 8): Position and velocity vector derivatives

1. \[ \frac{d_B \omega(B)}{dt} \overset{1.12}{=} \alpha(B) + \tilde{\omega}(B) \omega(B) = \alpha(B) \]

This proves the first part of (1.17). The second expression follows since the angular velocity of \( I \) with respect to \( B \) is \(-\omega(B)\) in (1.12).

2. Equation (1.18) follows by differentiating (1.15) with respect to the inertial frame and using (1.12) for the inertial derivative of \( l(x, y) \).

3. Equation (1.19a) follows from noting that \( \delta_v = \frac{d_B l(y, P)}{dt} \) and thus,

\[ v(P) = v(y) + \frac{d_B l(y, P)}{dt} \overset{1.12}{=} v(y) + \delta_v + \tilde{\omega}(B) l(y, P) \quad (C.1) \]

The last term in (1.19a) vanishes because \( l(y, P) \) is instantaneously zero. Equation (1.19b) follows by differentiating (C.1) with respect to the inertial frame.

Solution 1.2 (pp. 9): Spatial vector cross-product identities

The identities can be established by direct verification using the 3-vector components of the \( A \) and \( B \) spatial vectors.

Solution 1.3 (pp. 10): Relationship between the \( \hat{\omega} \), \( \tilde{\omega} \) and \( \hat{\omega} \) operators

These (1.27) equations can be established by direct verification for arbitrary spatial vectors \( A \), \( B \) and \( C \).
Solution 1.4 (pp. 10): Time derivative relationship for spatial vectors

With $X = \begin{bmatrix} x \\ y \end{bmatrix}$, we have

$$\frac{dF}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dF_x}{dt} + \tilde{\omega}x \\ \frac{dF_y}{dt} + \tilde{\omega}y \end{bmatrix} = \frac{dG}{dt} + \tilde{\nu}^\omega (F, G) X$$

Solution 1.5 (pp. 12): Identities involving $\tilde{\cdot}(\cdot), \hat{\cdot}(\cdot)$ and $\phi(\cdot, \cdot)$

The first two identities can be established by direct verification using arbitrary 3-vectors $l(x, y)$ and $X$. The last one is a simple rearrangement of the second identity.

Solution 1.6 (pp. 12): Rigid body transformation of $\tilde{V}(x)$

From (1.33) on page 12 we know that $V(y) = \phi^*(x, y)V(x)$. Thus,

$$\tilde{V}(y) \overset{1.34}{=} \phi^*(x, y) \tilde{V}(x) \phi^{-*}(x, y)$$

and the first equation follows. The latter equation is merely a transposed version of the first equation, followed by the use of (1.26).

Solution 1.7 (pp. 13): Relationships involving $X^\omega$ and $X^\nu$

The first pair of identities can be verified directly by expanding out $X$ and $Y$ into their component 3-vectors and evaluating them.

Equation (1.36b) follows directly from (1.36a).
Equation (1.36c) follows by direct verification.
Equation (1.36d) follows from (1.36c).
Equation (1.36e) can be verified directly by additionally using $\phi(x, y)$ for an arbitrary $l(x, y)$ 3-vector.

Solution 1.8 (pp. 13): The inertial frame derivative of $\phi(x, y)$

For the inertial frame derivative, we use the inertial frame representation of $\phi(x, y)$ from (1.30). Now,

$$\frac{dI}{dt} = \nu(y) - \nu(x)$$

Hence,

$$\frac{dI}{dt} \phi(x, y) = \begin{bmatrix} 0 & \tilde{v}(y) - \tilde{v}(x) \\ 0 & 0 \end{bmatrix} \overset{1.25, 1.21}{=} \tilde{V}^\nu(y) - \tilde{V}^\nu(x)$$
Solution 1.9 (pp. 14): Local time derivative of $\phi^*(x,y)$

1. We have

$$\nu_F(F, G) \overset{1.11}{=} \nu_G(F, G) + \tilde{\omega}(F, G)I(F, G)$$  \hspace{1cm} (C.2)

Equation (1.39) follows from using the above in (1.40).

2. The latter equality in (1.42) is a direct application of the last identity in (1.34) to (1.39).

For the first equality, differentiate (1.41) and use the chain rule to get

$$\frac{d\phi^*(F, G)}{dt} \overset{1.41,1.12}{=} \begin{pmatrix} \tilde{\omega}(G,F) & 0_3 \\ 0_3 & \tilde{\omega}(G,F) \end{pmatrix} \phi^*(F, G) + \begin{pmatrix} G \mathcal{R}_F & 0_3 \\ 0_3 & G \mathcal{R}_F \end{pmatrix} \begin{pmatrix} 0_3 \\ -\tilde{\nu}_F(F, G) \end{pmatrix} \begin{pmatrix} 0_3 \\ 0_3 \end{pmatrix} \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix}$$

$$\overset{1.41}{=} \begin{pmatrix} \tilde{\omega}(G,F) & 0_3 \\ 0_3 & \tilde{\omega}(G,F) \end{pmatrix} \phi^*(F, G) + \begin{pmatrix} 0_3 \\ -\tilde{\nu}_F(F, G) \end{pmatrix} \begin{pmatrix} 0_3 \\ 0_3 \end{pmatrix} \begin{pmatrix} I_3 \\ 0_3 \end{pmatrix}$$

$$\overset{1.41}{=} \begin{pmatrix} \tilde{\omega}(G,F) \\ \tilde{\nu}_F(F, G) \end{pmatrix} \begin{pmatrix} 0_3 \\ \tilde{\omega}(G,F) \end{pmatrix} \phi^*(F, G)$$

$$= \phi^*(G,F) \tilde{V}_F(G,F) \phi^*(F, G)$$

3. We have

$$\phi^*(G,F) \frac{d\phi^*(F, G)}{dt} \phi^*(G,F) \overset{1.42}{=} \phi^*(G,F) \tilde{V}_F(G,F) \phi^*(F, G) \phi^*(G,F)$$

$$= \phi^*(G,F) \tilde{V}_F(G,F) = -\phi^*(G,F) \tilde{V}_F(F,G)$$

This expression agrees with the one in (1.43) and establishes the result.
4. We have that
\[
\frac{d\phi^*(G, H)}{dt} \phi^*(F, G) + \phi^*(G, H) \frac{d\phi^*(F, G)}{dt} = 1.42 \tilde{V}_G(H, G) \phi^*(F, G) + \phi^*(G, H) \phi^*(F, G) \tilde{V}_G(G, F)
\]
\[
= \tilde{V}_G(H, G) \phi^*(F, H) + \phi^*(F, H) \tilde{V}_G(G, F)
\]
\[
= \left( \begin{array}{c}
\bar{\omega}(H, F) \\
\tilde{\nu}_G(H, G) - \bar{\omega}(H, G) \tilde{l}(F, H) + \tilde{\nu}_G(G, F) - \tilde{l}(F, H) \bar{\omega}(G, F)
\end{array} \right)
\]
\[
= \left( \begin{array}{c}
\bar{\omega}(H, F) \\
\tilde{\nu}_G(H, F) - \bar{\omega}(H, G) \tilde{l}(F, H) - \tilde{\nu}_G(G, F) - \tilde{l}(F, H) \bar{\omega}(G, F)
\end{array} \right)
\]
Using (C.2) in the lower left expression, for \( v_G(H, F) \) and simplifying we obtain
\[
\frac{d\phi^*(G, H)}{dt} \phi^*(F, G) + \phi^*(G, H) \frac{d\phi^*(F, G)}{dt} = \left( \begin{array}{c}
\bar{\omega}(H, F) \\
\tilde{\nu}_G(H, F) - \bar{\omega}(H, G) \tilde{l}(F, H) - \tilde{\nu}_G(G, F) - \tilde{l}(F, H) \bar{\omega}(G, F)
\end{array} \right) 1.42 \frac{d\phi^*(F, H)}{dt}
\]
establishing the result.

---

**Solutions for Chapter 2**

**Solution 2.1 (pp. 19): Rigid body center of mass**

We have \( l(x, a) = l(x, y) + l(y, a) \). Therefore,
\[
p(x) = \frac{1}{m} \int_{\Omega} l(x, a) \rho(a) d\theta(a)
\]
\[
= \frac{1}{m} \int_{\Omega} l(x, y) \rho(a) d\theta(a) + \frac{1}{m} \int_{\Omega} l(y, a) \rho(a) d\theta(a) = l(x, y) + p(y)
\]
Solution 2.2 (pp. 19): Parallel-axis theorem for rotational inertias

1. We have

\[ J(x) = -\int_{\Omega} \tilde{I}(x, a) \tilde{l}(x, a) \rho(a) d\theta(a) \]

\[ = -\int_{\Omega} \tilde{I}(x, C) \tilde{l}(x, C) \rho(a) d\theta(a) - \int_{\Omega} \tilde{I}(C, a) \tilde{l}(C, a) \rho(a) d\theta(a) \]

\[ = \frac{2.8}{2} J(C) - m \tilde{p}(x) \tilde{p}(x) - m \tilde{l}(x, C) \tilde{p}(C) - m \tilde{p}(C) \tilde{l}(x, C) \]

The last step used the fact that \( p(C) = 0 \).

2. The symmetry of \( J(x) \) is easy to verify from its definition. Its positive semi-definiteness follows from the positive semi-definiteness of the integrand

\[ -m \tilde{p}(x) \tilde{p}(x) = m [\tilde{p}(x)]^* \tilde{p}(x) \]

is always positive semi-definite implies that \( J(x) \geq J(C) \) for all points \( x \).

3. For \( J(x) \) to fail to be positive definite, there must exist a non-zero vector \( y \) such that \( y^* J(x) y = 0 \). That is,

\[ 0 = y^* J(x) y = -\int_{\Omega} y^* \tilde{l}(x, a) \tilde{l}(x, a) y \rho(a) d\theta(a) \]

\[ = \int_{\Omega} z^*(a) z(a) \rho(a) d\theta(a) \text{ where } z(a) \triangleq \tilde{l}(x, a) y \]

The above integral can vanish in the following cases:

**Point mass:** Here \( \rho(a) = 0 \) for all \( l(x, a) \neq 0 \) and \( J(x) \equiv 0 \).

**Infinitely thin rod along y:** Here \( \rho(a) = 0 \) for all \( l(x, a) \neq ky \) for some scalar \( k \neq 0 \).

Solution 2.3 (pp. 20): Positive semi-definiteness of spatial inertias

\( J(C) \) being positive definite or positive semi-definite implies the same property for \( M(C) \). The same property also applies to \( M(x) = \phi(x, C) M(C) \phi^*(x, C) \) in (2.12) since \( \phi(x, y) \) is always non-singular.
Solution 2.4 (pp. 21): Invariance of the kinetic energy
From (2.12), we have
\[ K_e = \frac{1}{2} V^*(x) M(x) V(x) \]
\[ = \frac{1}{2} V^*(x) \phi(x, y) M(y) \phi^*(x, y) V(y) \]
\[ = \frac{1}{2} V^*(y) M(y) V(y) \]

Since the points \( x \) and \( y \) are arbitrary, this establishes the invariance of the kinetic energy.

Solution 2.5 (pp. 22): Relationship of spatial momenta about points \( x \) and \( y \)
\[ \mathfrak{h}(x) \]
\[ = M(x) V(x) \]
\[ = \phi(x, y) M(y) \phi^*(x, y) V(x) \]
\[ = \phi(x, y) M(y) V(y) \]

Solution 2.6 (pp. 25): Time derivative of rigid body spatial inertia
We have
\[ \dot{I} M(z) = \begin{pmatrix} \mathfrak{R}^B & 0 \\ 0 & \mathfrak{R}^B \end{pmatrix} B M(z) \begin{pmatrix} B \mathfrak{R} & 0 \\ 0 & B \mathfrak{R} \end{pmatrix} \]
Differentiating this equation while noting that \( B M \) is constant leads to:
\[ \dot{M}(z) = \begin{pmatrix} \ddot{\omega} & 0 \\ 0 & \ddot{\omega} \end{pmatrix} B M(z) - M(z) \begin{pmatrix} \ddot{\omega} & 0 \\ 0 & \ddot{\omega} \end{pmatrix} \]
\[ = \ddot{\omega} M(z) - M(z) \ddot{\omega} \]

Solution 2.7 (pp. 26): \( b_{\dot{z}}(C) \) gyroscopic spatial force does no work
We can verify that \( b_{\dot{z}}(z) \) does no work by taking its dot product with the spatial velocity vector to obtain:
\[ V^*(C) b_{\dot{z}}(C) \]
\[ = \begin{pmatrix} \omega^* \left( \ddot{\omega} \mathcal{J}(C) \omega \right) \\ 0 \end{pmatrix} = 0 \]
**Solution 2.8 (pp. 27): Inertial generalized accelerations at two points**

\[
\dot{\beta}_j(y) \triangleq \frac{d}{dt} \dot{\mathcal{V}}(y) = \frac{d}{dt} \left[ \phi^*(x, y) \mathcal{V}(x) \right] = \phi^*(x, y) \dot{\beta}_j(x) + \frac{d}{dt} \left[ \phi^*(x, y) \right] \beta_j(x)
\]

\[
\overset{1.37}{=} \phi^*(x, y) \dot{\beta}_j(x) + \left[ -\ddot{\mathcal{V}}(y) + \ddot{\mathcal{V}}(x) \right] \mathcal{V}(x) = \phi^*(x, y) \dot{\beta}_j(x) + \left[ \begin{array}{c} 0 \\ \tilde{\omega} [\mathcal{V}(y) - \mathcal{V}(x)] \end{array} \right]
\]

\[
\overset{1.15}{=} \phi^*(x, y) \dot{\beta}_j(x) + \left[ \begin{array}{c} 0 \\ \tilde{\omega} \tilde{\omega} [\mathcal{V}(x)] \end{array} \right]
\]

**Solution 2.9 (pp. 28): \( b_j(z) \) gyroscopic spatial force does work**

We can verify that \( b_j(z) \) does work by taking its dot product with the spatial velocity vector as follows:

\[
\mathcal{V}^*(z) b_j(z) \overset{2.26}{=} \mathcal{V}^*(z) \dddot{\gamma}^\omega(z) M(z) \mathcal{V}^\omega(z)
\]

This expression is non-zero in general and hence, the gyroscopic force does work. Assuming that the spatial force is zero, i.e., \( f(z) \equiv 0 \), the time derivative of the kinetic energy of the body is:

\[
\frac{d\mathcal{K}_e}{dt} \overset{2.26, 2.22}{=} \frac{1}{2} \beta_j^*(z) M(z) \dot{\beta}_j(z) + \frac{1}{2} \beta_j^*(z) M(z) \beta_j(z) = \beta_j^*(z) M(z) \dot{\beta}_j(z) + \frac{1}{2} \beta_j^*(z) M(z) \beta_j(z)
\]

\[
\overset{2.26}{=} -m \mathcal{V}^*(z) \tilde{\omega} \tilde{\omega} \mathcal{V}(z) - m \mathcal{V}^* \tilde{\omega} \mathcal{V}(z) = 0
\]

This proves that the kinetic energy is conserved in the absence of external forces.

**Solution 2.10 (pp. 28): Non-conservation of spatial momentum**

For any point \( x \) on the rigid body, from (2.17), \( \mathcal{h}(x) = \phi(x, C) \mathcal{h}(C) \). Differentiating this equation with respect to time in the inertial frame leads to

\[
\frac{d}{dt} \mathcal{h}(x) = \frac{d}{dt} \phi(x, C) \mathcal{h}(C) + \phi(x, C) \frac{d}{dt} \mathcal{h}(C) \overset{2.21}{=} \frac{d}{dt} \phi(x, C) \mathcal{h}(C) + \phi(x, C) f(C)
\]

\[
\overset{1.15, 1.37}{=} \left[ \begin{array}{c} m [\ddot{\mathcal{V}}(C) - \ddot{\mathcal{V}}(x)] \mathcal{V}(C) \\ 0 \end{array} \right] + f(x) = \left[ \begin{array}{c} m \dddot{\mathcal{V}}(C) \mathcal{V}(x) \\ 0 \end{array} \right] + f(x)
\]

\[
= -\dddot{\mathcal{V}}(x) \mathcal{h}(C) + f(x)
\]

(C.3)
The $\nabla^V(x)\hat{h}(C)$ quantity vanishes if and only if either: (a) the point $x$ coincides with $C$; (b) $v(C)$ or $v(x)$ is zero, i.e., the body is spinning about the $l(x,C)$ vector. This establishes (2.27).

Solution 2.11 (pp. 30): Equations of motion using spatial momentum

We have

$$\frac{d}{dt}\hat{h}(x) \overset{1.28}{=} \nabla^\omega(x)\hat{h}(x) + \frac{d}{dt}M(x)\hat{\beta}_{B}(x) = \nabla^\omega(x)\hat{h}(x) + \frac{d}{dt}M(x)\hat{\beta}_{B}(x)$$

Combining this with (C.3) leads to

$$J(x) = M(x)\dot{\hat{\beta}}_{B}(x) + \left[\nabla^\omega(x) + \nabla^V(x)\right]h(x) \overset{1.22}{=} M(x)\dot{\hat{\beta}}_{B}(x) + J(x)h(x)$$

This agrees with (2.28) and establishes the result.

Solution 2.12 (pp. 32): Invariance of $V_I$ to velocity reference point

Now

$$\phi^*(x,\bar{\pi})V(x) \overset{1.33}{=} \phi^*(x,\bar{\pi})\phi^*(\bar{\pi},C)V(C) \overset{1.32}{=} \phi^*(\bar{\pi},C)V(C) \overset{2.30}{=} V_I$$

establishing (2.31).

Solution 2.13 (pp. 32): Time derivative of $M_I$

Differentiating (2.32) we have

$$\frac{d}{dt}M_I \overset{2.32}{=} \phi(\bar{\pi},C)\frac{d}{dt}M(C)\phi^*(\bar{\pi},C) + \frac{d}{dt}\phi(\bar{\pi},C)M(C)\phi^*(\bar{\pi},C)$$

$$+ \phi(\bar{\pi},C)M_I \frac{d}{dt}\phi^*(\bar{\pi},C) \overset{2.22,1.37}{=} \phi(\bar{\pi},C)\left[\nabla^\omega M(C) - M(C)\nabla^\omega\right]\phi^*(\bar{\pi},C)$$

$$+ \nabla^V(C)M(C)\phi^*(\bar{\pi},C) - \phi(\bar{\pi},C)M(C)\nabla^V(C)$$

However, from (1.36) we have

$$\nabla^V(C) = \phi(\bar{\pi},C)\nabla^V(C)$$
and obtain
\[
\frac{d_I M_{II}}{dt} = C^4 \phi(I, C) \left[ \nabla^V(C) + \nabla^\omega(C) \right] M(C) \phi^*(I, C)
- \phi(I, C) M(C) \left[ \nabla^V(C) + \nabla^\omega(C) \right] \phi^*(I, C)
\]
\[
\overset{1.25}{=} \phi(I, C) \nabla(C) M(C) \phi^*(I, C) - \phi(I, C) M(C) \tilde{V}(C) \phi^*(I, C)
\]
\[
\overset{1.35, 2.30}{=} \nabla_I \phi(I, C) M(C) \phi^*(I, C) - \phi(I, C) M(C) \phi^*(I, C) \tilde{V}_I
\]
\[
\overset{2.32}{=} \nabla_I M_{II} - M_{II} \tilde{V}_I
\]

**Solution 2.14 (pp. 33): Equations of motion about a fixed velocity reference point**

1. We have
\[
\frac{d_I h_{II}}{dt} = 2.34 \frac{d_I \phi(I, C)}{dt} h(C) + \phi(I, C) \frac{d_I h(C)}{dt}
\]
\[
\overset{2.21, 1.37}{=} \nabla^V(C) h(C) + \phi(I, C) f(C) \overset{2.35}{=} 0 + f_I = f_I
\]

The last equality above used the \( \nabla^V(C) h(C) = 0 \) identity.

2. From (2.36) we have
\[
f_I = 2.36 \frac{d_I h_{II}}{dt} = 2.34 \frac{d_I h_{II}}{dt} + \frac{d_I M_{II}}{dt} V_{II} \overset{2.33}{=} M_{II} \dot{\beta}_{II} - \nabla_I M_{II} V_{II}
\]

3. We can verify that \( b_I \) is non-working by taking its dot product with \( V_{II} \) and noting that \( V_{II} \nabla_I = 0 \). Recognizing that the kinetic energy can also be written as \( \frac{1}{2} \beta_{II}^T M_{II} \beta_{II} \) and differentiating it with respect to time leads to the conclusion that it is conserved in the absence of external forces.

**Solutions for Chapter 3**

**Solution 3.1 (pp. 41): Hinge map matrix for a universal joint**

The orientation of the second axis in a universal joint depends on the general coordinate of the first axis. The hinge map matrix for the universal joint is
\[
H^* = \begin{pmatrix}
1 & 0 \\
0 & \cos \theta_1 \\
0 & \sin \theta_1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]
Solution 3.2 (pp. 41): Hinge map matrix for a sphere rolling on a surface

Let $\Delta v$ denote the spatial velocity of the frame fixed to the center of the sphere. Also, let $l$ denote the vector from the center of the sphere to the instantaneous point of contact. The orientation of this vector is constant in the inertial frame but is changing in the body frame. The zero linear velocity of the contact point translates to the following constraint expression:

$$\Delta v + \Delta \omega l = 0 \quad \Rightarrow \quad [-\tilde{l}, \ I_3] \Delta v = 0,$$

where $\Delta \omega$ and $\Delta v$ are the angular and linear velocity components of $\Delta v$ at the center of the sphere. This implicit 3-dimensional constraint implies that $\Delta v$ must be of the following form:

$$\Delta v = [I_3 \tilde{l}] \beta$$

where $\beta \in \mathbb{R}^3$ is the vector of generalized velocities for this contact hinge. Thus, with $\beta = \Delta \omega$ as the generalized velocity coordinates, the hinge map matrix for the rolling contact is given by

$$H^* = [I_3 \tilde{l}] \in \mathbb{R}^{6 \times 3}$$

Solution 3.3 (pp. 46): Velocity recursion with $B_k \neq O_k$

In this context, the recursion in (3.19b) can be more precisely expressed as:

$$V(O_k) = \phi^*(O_{k+1}, O_k) V(O_{k+1}) + H^*(k) \dot{\theta}(k)$$

Now use the following versions of (3.20) and (3.23) in the equation above to establish the result:

$$V(k) \triangleq V(B_k) = \phi^*(O_k, B_k) V(O_k)$$

and

$$V(k+1) \triangleq V(B_{k+1}) = \phi^*(O_{k+1}, B_{k+1}) V(O_{k+1})$$

Solution 3.4 (pp. 47): Velocity recursion with inertially fixed reference point

We have

$$V[I](k) \overset{3.24}{=} \phi^*[O_k, I] V(O_k)$$

$$\overset{3.19b}{=} \phi^*[O_k, I] \{ \phi^*[O_{k+1}, O_k] V(O_{k+1}) + H^*(k) \beta(k) \}$$
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\[ \begin{align*}
\phi^*(\mathbb{O}_k,\mathbb{I})\{\phi^*(\mathbb{O}_{k+1},\mathbb{O}_k)\phi^*(\mathbb{O}_{k+1},\mathbb{I})\mathcal{V}_I((k+1)) + H^*(k)\beta(k)\} \\
\mathcal{V}_I(k+1) + H^*_I(k)\beta(k) \equiv \mathcal{V}_I(k+1) + \Delta^I_V(k)\beta(k)
\end{align*} \]

establishing (3.26).

Solution 3.5 (pp. 50): Internal structure of the \( \phi \) operator

Since \((I - E_\phi)\) is lower-triangular, and has identity matrices along the diagonal, the same holds true for its inverse, \( \phi \), as well establishing (3.37). Now, let us examine the lower-triangular component elements of the matrices on both sides of the operator identity:

\[ \phi - E_\phi \phi = I \]

This identity implies that for \( i > j \),

\[ \phi(i,j) - \phi(i,i-1)\phi(i-1,j) = 0 \Rightarrow \phi(i,j) = \phi(i,i-1)\phi(i-1,j) \]

Applying this next to \( \phi(i-1,j) \), and continuing on in this vein, leads to the first half of (3.38). Since the product of a sequence of rigid body transformation matrices is also a rigid body transformation matrix, the latter equality in (3.38) follows as well.

Solution 3.6 (pp. 51): The \( \tilde{\phi} \) spatial operator

1. For a matrix \( A \) such that \((I - A)\) is invertible, (A.17) on page 400 states that:

\[ A(I - A)^{-1} = (I - A)^{-1}A = (I - A)^{-1} - I \]

(3.41) is a consequence of this identity with \( A = E_\phi \) and \( \phi = (I - E_\phi)^{-1} \).

2. From (3.15) it follows that

\[ \mathcal{V}^+ \equiv E_\phi \mathcal{V} \equiv E_\phi^* \phi^* H^* \dot{\theta} \equiv \tilde{\phi}^* H^* \dot{\theta} \]

Solution 3.7 (pp. 53): Recursive evaluation of \( \tilde{\phi}x \) and \( \tilde{\phi}^*x \)

With \( y = \phi x \), we have \( \tilde{y} \triangleq \tilde{\phi}x = E_\phi y \). This implies that \( \tilde{y}(k) = \phi(k,k-1)y(k) \). Thus, the computational algorithm for \( \tilde{y} \) is the one in (3.43) except that the computational step in the loop is now given by

\[ \tilde{y}(k) = \phi(k,k-1)[\tilde{y}(k-1) + x(k-1)] \]

with initial condition \( \tilde{y}(0) = 0 \).

Similarly, with \( y = \phi^*x \), we have \( \tilde{y} \triangleq \tilde{\phi}^*x = E_\phi^* y \). Thus,

\[ \tilde{y}(k) = \phi^*(k+1,k)y(k+1) \]
Thus, the computational algorithm for \( \bar{y} \) is the same as in (3.45) except that the computational step within the loop has the form:

\[
\bar{y}(k) = \phi^*(k+1,k)[\bar{y}(k+1) + x(k+1)]
\]

with initial condition \( \bar{y}(n+1) = 0 \).

---

**Solutions for Chapter 4**

**Solution 4.1 (pp. 63): System center of mass**

1. The overall spatial inertia of the system referenced to the base-body frame is obtained by using the parallel axis theorem for spatial inertias to reference all the link spatial inertias to \( B_n \) and summing them up. That is,

\[
M_S = \sum_{k=1}^{n} \phi(n,k)M(k)\phi^*(n,k) = [\phi(n,1), \cdots \phi(n,n)]M \begin{bmatrix} \phi^*(n,1) \\ \vdots \\ \phi^*(n,n) \end{bmatrix}
\]

establishing the result. We have used the following facts in the above derivation:

\[ E\phi^{3.37} = [\phi(n,1), \cdots \phi(n,n)] \quad \text{and} \quad \tilde{\phi}RE^* = 0 \quad (C.5) \]

2. The base-body frame referenced spatial momentum of the system is obtained by referencing the spatial momentum for each of the bodies to the \( B_n \) frame and summing them up. This leads to:

\[
\mathbf{h}_S \overset{2.17}{=} \sum_{k=1}^{n} \phi(n,k)\mathbf{h}(k) \overset{2.16}{=} \sum_{k=1}^{n} \phi(n,k)M(k)\mathbf{V}(k) \overset{C.5}{=} E\phi M\mathbf{V}
\]

\[
\overset{3.39}{=} E\phi M\phi^*H^*\dot{\theta} \overset{4.10}{=} E[R + \tilde{\phi}R + R\tilde{\phi}^*]H^*\dot{\theta} \overset{C.5}{=} E\phi RH^*\dot{\theta}
\]

For a free flying system, \( H^*(n) = I \) and \( \dot{\theta}(n) = \mathbf{V}(n) \), and hence \( \mathbf{h}_S \) can be re-expressed as

\[
\mathbf{h}_S = R(n)\mathbf{V}(n) + \sum_{k=1}^{n-1} \phi(n,k)R(k)H^*(k)\dot{\theta}(k)
\]

3. The system level spatial momentum \( \mathbf{h}_S \), the system level spatial inertia \( M_S \) and the center of mass velocity \( \mathbf{V}_C \) (referenced about \( B_n \)) are related together by
\[ h_S = M_S V_C \overset{4.13}{=} R(n)V_C \]  
(C.6)

4. For a free-flying system, we have,

\[ V_C = R^{-1}(n) \left[ R(n)V(n) + \sum_{k=1}^{n-1} \phi(n,k)R(k)H^*(k)\dot{\theta}(k) \right] \]

\[ = V(n) + R^{-1}(n) \sum_{k=1}^{n-1} \phi(n,k)R(k)H^*(k)\dot{\theta}(k) \]

establishing (4.18).

From (4.16) it follows directly that adding \( \delta_V \) to the \( V(n) \) base-body spatial velocity will result in an additional \( R(n)\delta_V \) of spatial momentum system to the system. Now for the spatial momentum to be zero, we must have

\[ 0 = h_S + R(n)\delta_V \implies \delta_V = -R^{-1}(n)h_S \overset{C.6}{=} -V_C \]

---

**Solution 4.2 (pp. 65): Trace of the mass matrix**

From the decomposition of the mass matrix in (4.19), and noting that the trace of strictly lower- and upper-triangular matrices is zero, we have

\[ \text{Trace} \{ M(\theta) \} = \text{Trace} \{ H\dot{R}H^* + H\ddot{\phi}R + HR\ddot{\phi}^*H^* \} \]

\[ = \sum_{i=1}^{n} \text{Trace} \{ H(k)R(k)H^*(k) \} \]

In the above we have used the zero trace property of \( H\ddot{\phi}R \) which is a consequence of its strictly lower-triangular structure.

---

**Solution 4.3 (pp. 68): The \( M_D(\theta, \dot{\theta}) \) matrix**

1. We have

\[ M_D^*\dot{\theta} = \nabla_\theta (M(\dot{\theta}))^*\dot{\theta} \overset{A.24}{=} \left[ \frac{\partial M(\dot{\theta})}{\partial \theta(1)} \right]' \cdots \left[ \frac{\partial M(\dot{\theta})}{\partial \theta(n)} \right]' \dot{\theta} \]

\[ = \begin{bmatrix} \frac{\partial \dot{\theta}_1^* M}{\partial \theta(1)} \\ \vdots \\ \frac{\partial \dot{\theta}_n^* M}{\partial \theta(n)} \end{bmatrix} \dot{\theta} = \begin{bmatrix} \frac{\partial \hat{\theta}_1^* M}{\partial \theta(1)} \\ \vdots \\ \frac{\partial \hat{\theta}_n^* M}{\partial \theta(n)} \end{bmatrix} \overset{A.27}{=} \frac{\partial}{\partial \dot{\theta}} (\dot{\theta}^* M \dot{\theta}) \]
2. \[ M_D(\theta, \dot{\theta}) \dot{\theta}^{4.29} = \left[ \frac{\partial M}{\partial \theta(1)} \dot{\theta}, \ldots, \frac{\partial M}{\partial \theta(n)} \dot{\theta} \right]^* \dot{\theta} = \sum_{k=1}^{N} \frac{\partial M}{\partial \theta(k)} \dot{\theta}(k) \]
\[ = \left[ \sum_{k=1}^{N} \frac{\partial M}{\partial \theta(k)} \dot{\theta}(k) \right] \dot{\theta} = \mathcal{N}(\theta) \dot{\theta} \]

Solution 4.4 (pp. 68): \( \mathcal{N} \dot{\theta} - 2\mathcal{C} \) is a non-working force

1. Equation (4.33) follows from combining (4.31) and (4.32).
2. The work done by a generalized force is given by its dot product with the \( \dot{\theta} \) generalized velocities vector. Thus, the work done by \( \mathcal{N} \dot{\theta} - 2\mathcal{C} \) is given by
\[ \dot{\theta}^* \left[ \mathcal{N} \dot{\theta} - 2\mathcal{C} \right]^{4.33} \dot{\theta}^* (M_D^* - M_D) \dot{\theta} = 0 \]

The last equality follows since \((M_D^* - M_D)\) is skew-symmetric.
3. The work done by the \( \mathcal{C} \) Coriolis generalized forces vector is given by
\[ \dot{\theta}^* \mathcal{C}(\theta, \dot{\theta})^{4.32} \dot{\theta}^* \left[ M_D - \frac{1}{2} M_D \right] \dot{\theta} = \frac{1}{2} \dot{\theta}^* M_D \dot{\theta} \]

Since \( M_D \) is not skew-symmetric, the work done by \( \mathcal{C} \) is non-zero in general.

Solution 4.5 (pp. 68): Rate of change of the kinetic energy

\[ \frac{d\mathcal{K}_e}{dt}^{4.25} = \dot{\theta}^* \mathcal{M} \dot{\theta} + \frac{1}{2} \dot{\theta}^* \mathcal{M} \dot{\theta} \]
\[ = \dot{\theta}^* \left[ \mathcal{J} - (M_D - \frac{1}{2} M_D^*) \dot{\theta} + \frac{1}{2} M_D \dot{\theta} \right]^{4.32} \dot{\theta}^* \left[ \mathcal{J} - \frac{1}{2} (M_D - M_D^*) \dot{\theta} \right] \dot{\theta} = \dot{\theta}^* \mathcal{J} \]

The last step follows since \((M_D - M_D^*)\) is skew-symmetric.

Solution 4.6 (pp. 69): Christoffel symbols of the first kind

1. The identities in (4.36) follow from simply comparing the definitions of \( \mathcal{C}_i(j, k) \), \( \mathcal{C}_i(k, j) \) and \( \mathcal{C}_i(i, k) \).

For (4.37),
\[ \mathcal{C}_j(i,k) + \mathcal{C}_k(j,i) = 4.35 \frac{1}{2} \left[ \frac{\partial M(i,j)}{\partial \theta(k)} + \frac{\partial M(j,k)}{\partial \theta(i)} - \frac{\partial M(i,k)}{\partial \theta(j)} \right] + 4.35 \frac{1}{2} \left[ \frac{\partial M(k,j)}{\partial \theta(i)} + \frac{\partial M(j,k)}{\partial \theta(j)} - \frac{\partial M(k,i)}{\partial \theta(k)} \right] \]

When the mass matrix is symmetric, the above expression simplifies to \( \frac{\partial M(j,i)}{\partial \theta(i)} \).

2. For multibody systems, the mass matrix is symmetric, and also \( \frac{\partial M(i,k)}{\partial \theta(j)} = 0 \) for \( j \geq i,k \), as discussed in Remark 4.1. Using these properties in the definition of the left- and right-hand sides of the identities in (4.38) establishes the identities. Moreover, since \( \mathcal{C}_i(j,k) \) depends only upon \( M(i,j), \ M(i,k) \) and \( M(j,k) \), and these terms depend only upon \( \theta(l) \) for \( l < \max(i,j,k) \), this implies that \( \mathcal{C}_i(j,k) \) depends only upon \( \theta(l) \) for \( l < \max(i,j,k) \).

3. We have

\[ \mathcal{C}(i) = 4.27 \left\{ M(\theta)\hat{\theta} - \frac{1}{2} \frac{\partial}{\partial \theta} \left[ \hat{\theta}^T M(\theta) \hat{\theta} \right] \right\}(i) = \sum_{k=1}^{N} M(i,k) \hat{\theta}(k) - \frac{1}{2} \hat{\theta} \frac{\partial M}{\partial \theta(i)} \hat{\theta} \]

\[ = \sum_{j,k=1}^{N} \frac{\partial M(i,k)}{\partial \theta(j)} \hat{\theta}(j) \hat{\theta}(k) - \frac{1}{2} \sum_{j,k=1}^{N} \frac{\partial M(j,k)}{\partial \theta(i)} \hat{\theta}(j) \hat{\theta}(k) \]

\[ = \sum_{j,k=1}^{N} \left[ \frac{\partial M(i,k)}{\partial \theta(j)} - \frac{1}{2} \frac{\partial M(j,k)}{\partial \theta(i)} \right] \hat{\theta}(j) \hat{\theta}(k) \]

\[ = \sum_{j,k=1}^{N} \frac{1}{2} \left[ \frac{\partial M(i,k)}{\partial \theta(j)} + \frac{\partial M(i,k)}{\partial \theta(j)} - \frac{1}{2} \frac{\partial M(j,k)}{\partial \theta(k)} \right] \hat{\theta}(j) \hat{\theta}(k) \]

\[ = 4.35 \sum_{j,k=1}^{N} \mathcal{C}_i(j,k) \hat{\theta}(j) \hat{\theta}(k) = \hat{\theta}^T \mathcal{C}_i \hat{\theta} \]

4. We have

\[ M_D(i,j) = 4.29 \frac{\partial [M(\hat{\theta})]}{\partial \theta(j)} = \sum_{k=1}^{N} \frac{\partial M(i,k)}{\partial \theta(j)} \hat{\theta}(k) = 4.37 \sum_{k=1}^{N} \left[ \mathcal{C}_i(j,k) + \mathcal{C}_k(j,i) \right] \hat{\theta}(k) \]

**Solution 4.7 (pp. 70): Hamiltonian form of the equations of motion**

We have

\[ \frac{\partial \mathcal{H}}{\partial p} = 4.42 \mathcal{M}^{-1} p = 4.28 \hat{\theta} \]
This establishes the first expression in (4.43). Also,

\[
\frac{\partial \mathcal{H}}{\partial \theta} = \frac{1}{2} \text{col} \left\{ p^* \frac{\partial \mathcal{M}^{-1}}{\partial \mathcal{O}(k)} p \right\}_{k=1}^N \tag{4.42.A.28} = -\frac{1}{2} \text{col} \left\{ p^* \mathcal{M}^{-1} \frac{\partial \mathcal{M}}{\partial \mathcal{O}(k)} \mathcal{M}^{-1} p \right\}_{k=1}^N
\]

\[
= -\frac{1}{2} \text{col} \left\{ \hat{\mathcal{M}} \hat{\mathcal{O}} \right\}_{k=1}^N \tag{4.28} = -\frac{1}{2} \hat{\mathcal{M}} \hat{\mathcal{O}} + \mathcal{C}
\]

This establishes the latter expression in (4.43).

---

**Solution 4.8 (pp. 71): Lagrangian equations of motion using quasi-velocities**

We have

\[
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \overset{\text{A.27}}{=} [\nabla_{\dot{\theta}} \mathcal{L}]^* \overset{\text{A.26}}{=} [\nabla_{\beta} \dot{\mathcal{L}} \cdot \nabla_{\theta} \beta]^* = [\nabla_{\beta} \dot{\mathcal{L}} \cdot A]^* \overset{\text{A.27}}{=} A^* (\theta) \frac{\partial \dot{\mathcal{L}}}{\partial \beta} \tag{C.7}
\]

Therefore, differentiating this with respect to t we have

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \theta} \overset{\text{C.7}}{=} \dot{A}^* \left[ \frac{d}{dt} \frac{\partial \dot{\mathcal{L}}}{\partial \beta} \right] + \ddot{A}^* \frac{\partial \dot{\mathcal{L}}}{\partial \beta} \tag{C.8}
\]

Similarly,

\[
\frac{\partial \mathcal{L}}{\partial \theta} \overset{\text{A.27}}{=} [\nabla_{\theta} \mathcal{L}]^* \overset{\text{A.26}}{=} [\nabla_{\beta} \dot{\mathcal{L}} \cdot \nabla_{\theta} \beta + \nabla_{\theta} \dot{\mathcal{L}}]^* \overset{\text{A.27}}{=} \frac{\partial \beta}{\partial \theta} \frac{\partial \dot{\mathcal{L}}}{\partial \beta} + \frac{\partial \dot{\mathcal{L}}}{\partial \theta} \tag{C.9}
\]

Combining (C.8) and (C.9) it follows that

\[
\tau \overset{\text{A.24}}{=} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \theta} - \frac{\partial \mathcal{L}}{\partial \theta} = A^* \left( \frac{d}{dt} \frac{\partial \dot{\mathcal{L}}}{\partial \beta} \right) - \frac{\partial \dot{\mathcal{L}}}{\partial \theta} + \left[ \ddot{A}^* - \frac{\partial \beta}{\partial \theta} \right] \frac{\partial \dot{\mathcal{L}}}{\partial \beta}
\]

\[
= A^* \left( \frac{d}{dt} \frac{\partial \dot{\mathcal{L}}}{\partial \beta} \right) - \frac{\partial \dot{\mathcal{L}}}{\partial \theta} + \gamma^* \frac{\partial \dot{\mathcal{L}}}{\partial \beta}
\]

and therefore

\[
\dot{A} (\theta) = \sum_{i=1}^{\mathcal{N}} \frac{\partial A}{\partial \mathcal{O}(k)} \dot{\mathcal{O}}(k) \quad \text{and} \quad \frac{\partial \beta}{\partial \theta} = \frac{\partial \dot{A}}{\partial \theta}
\]

establishing (4.44). Now observe that

\[
\dot{\mathcal{A}} (\theta) = \sum_{i=1}^{\mathcal{N}} \frac{\partial \dot{A}}{\partial \mathcal{O}(k)} \dot{\mathcal{O}}(k) \quad \text{and} \quad \frac{\partial \beta}{\partial \theta} = \frac{\partial \dot{A}}{\partial \theta}
\]

(C.10)
Thus,
\[ \gamma(i, j) = \dot{A}(i, j) - \frac{\partial \beta(i)}{\partial \theta(j)} = \sum_{k=1}^{n} \frac{\partial A(i, j)}{\partial \theta(k)} \dot{\theta}(k) - \frac{\partial}{\partial \theta(j)} \sum_{k=1}^{n} A(i, k) \dot{\theta}(k) \]
\[ = \sum_{k=1}^{n} \left( \frac{\partial A(i, j)}{\partial \theta(k)} - \frac{\partial A(i, k)}{\partial \theta(j)} \right) \dot{\theta}(k) \]
This establishes (4.46).

---

**Solution 4.9 (pp. 72): Lagrangian equations of motion under coordinate transformations**

Since \( \eta = h(\theta) \), therefore
\[ \dot{\eta} = A(\theta) \dot{\theta} \]
Thus, \( A(i, j) = \frac{\partial h(i)}{\partial \theta(j)} \) and
\[ \frac{\partial A(i, j)}{\partial \theta(k)} = \frac{\partial A(i, k)}{\partial \theta(j)} = \frac{\partial^2 h(i)}{\partial \theta(j) \partial \theta(k)} \]
Therefore, the (4.46) expression for the \( \gamma(i, j) \) elements are zero. Hence, \( \gamma = 0 \).

---

**Solution 4.10 (pp. 72): Non-working \( \hat{\beta} - 2 \hat{C} \) generalized force**

We have
\[ \beta^* A^{-1} \frac{\partial \mathcal{L}}{\partial \theta} = \dot{\theta}^* \frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} \dot{\theta}^* [\nabla_\theta \beta^* \mathcal{M}(\theta) \beta] = \frac{1}{2} [\nabla_\theta \beta^* \mathcal{M}(\theta) \beta] \dot{\theta} \]
\[ = \frac{1}{2} \beta^* \mathcal{M}(\theta) \beta \]
Also,
\[ \gamma A^{-1} \beta = \gamma \dot{\theta}^* 4.48 = 0 \]
Therefore
\[ \beta^* (\hat{\beta} - 2 \hat{C}) = 4.50 \beta^* \left( -\hat{\mathcal{M}} \beta + 2A^{-1} \frac{\partial \mathcal{L}}{\partial \theta} - 2A^{-1} \gamma^* \frac{\partial \mathcal{L}}{\partial \beta} \right) \]
\[ \overset{C.11,C.12}{=} -\beta^* \hat{\mathcal{M}} \beta + \beta^* \hat{\mathcal{M}} \beta - 0 = 0 \]
**Solution 4.11 (pp. 73): Nonlinear diagonalizing coordinate transformations**

1. (4.49) on page 72 describes the effect of a coordinate transformation upon the Lagrangian equations of motion which in the \((\eta, \dot{\eta})\) coordinates are given by

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \frac{\partial \mathcal{L}}{\partial \eta} = A^{-*}J
\]  

(C.13)

Since \(\mathcal{M}(\theta) = A^*(\theta)A(\theta)\), this means that

\[
\mathcal{R}_e = \mathcal{L}(\theta, \dot{\theta}) = \dot{\mathcal{L}}(\eta, \dot{\eta}) = \frac{1}{2} \eta^* \dot{\eta} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \eta} = 0
\]  

(C.14)

Using these in (C.13) leads to

\[\dot{\eta} = A^{-*}J = \zeta\]

establishing (4.51).

2. This time

\[
\mathcal{R}_e = \dot{\mathcal{L}}(\theta, \beta) = \frac{1}{2} \beta^* \beta \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \theta} = 0
\]  

(C.15)

Since \(\beta\) are now quasi-velocities, the appropriate Lagrangian equations of motion are those in (4.44) and using (C.15) take the form:

\[\dot{\beta} = A^{-*}\gamma^* \beta = \zeta\]

and the result follows.

3. From Exercise 4.10 we have

\[0 = \beta^* (\dot{\mathcal{M}} \beta - 2 \mathcal{C}) = -2 \beta^* \mathcal{C}\]

The above uses the fact that \(\mathcal{M}\) is the constant identity matrix. This implies that the \(\mathcal{C}(\theta, \beta)\) Coriolis forces vector do no work.

---

**Solutions for Chapter 5**

**Solution 5.1 (pp. 79): \(a(k)\) for helical & cylindrical hinges**

The hinge map matrix for a helical hinge with axis \(h_\omega(k)\) and pitch \(p\) has the form

\[
H^*(k) = \begin{bmatrix} h_\omega(k) \\ ph_\omega(k) \end{bmatrix} \implies \Delta_V(k) = \begin{bmatrix} \Delta_\omega(k) \\ p\Delta_\omega(k) \end{bmatrix}
\]
Thus,
\[ a(k) \stackrel{5.12}{=} - \begin{bmatrix} \tilde{\Delta}_\omega(k)\omega(k) \\ \Delta_\omega(k)[v(k) + p\omega(k)] \end{bmatrix} \]

For a cylindrical hinge with axis \( h_\omega(k) \), the axis of rotation and translation are the same. For this case
\[ H^*(k) = \begin{pmatrix} h_\omega(k) & 0 \\ 0 & h_\omega(k) \end{pmatrix} \implies \Delta_V(k) = \begin{bmatrix} \Delta_\omega(k) \\ \Delta_V(k) \end{bmatrix} = \begin{bmatrix} h_\omega(k)\dot{\theta}_1(k) \\ h_\omega(k)\dot{\theta}_2(k) \end{bmatrix} \]

Thus,
\[ a(k) \stackrel{5.15}{=} - \begin{bmatrix} \tilde{\Delta}_\omega(k)\omega(k) \\ \Delta_\omega(k)v(k) + \Delta_V(k)\omega(k) \end{bmatrix} \]

**Solution 5.2 (pp. 80):** \( a_B(k) \) with body frame derivatives but \( \mathbb{B}_k \neq \mathbb{O}_k \)

Since the reference for the \( k \)th body does not coincide with \( \mathbb{O}_k \), we have from (3.20) that
\[ V(k) \triangleq V(\mathbb{B}_k) = \phi^*(\mathbb{O}_k, \mathbb{B}_k)V(\mathbb{O}_k) \quad (C.16) \]
and the spatial acceleration \( \alpha_B(k) \) is defined as
\[ \alpha_B(k) \triangleq \frac{d_k V(\mathbb{O}_k)}{dt} \]

Differentiating (C.16) with respect to the \( k \)th body frame and noting that \( \phi^*(\mathbb{O}_k, \mathbb{B}_k) \) is constant in the \( k \)th body frame, we have
\[ \alpha_B(k) \stackrel{C.16}{=} \phi^*(\mathbb{O}_k, \mathbb{B}_k)\frac{d_k V(\mathbb{O}_k)}{dt} = \phi^*(\mathbb{O}_k, \mathbb{B}_k)\alpha(k) \]
\[ \stackrel{5.8}{=} \phi^*(\mathbb{O}_k, \mathbb{B}_k)\left\{ \phi^*(\mathbb{O}_{k+1}, \mathbb{O}_k)\phi^*(\mathbb{B}_{k+1}, \mathbb{O}_{k+1})\alpha_B(k+1) \\ + H^*(k)\ddot{\theta}(k) + a(k) \right\} \]
\[ = \phi^*(\mathbb{B}_{k+1}, \mathbb{B}_k)\alpha_B(k+1) + \phi^*(\mathbb{O}_k, \mathbb{B}_k)H^*(k)\ddot{\theta}(k) + \phi^*(\mathbb{O}_k, \mathbb{B}_k)a(k) \quad (3.22) \]

This establishes (5.18). Note further that
\[ \phi^*(\mathbb{O}_k, \mathbb{B}_k)\ddot{V}(\mathbb{O}_k)\Delta_V(k) \stackrel{1.35,C.16}{=} \ddot{V}(k)\phi^*(\mathbb{O}_k, \mathbb{B}_k)\Delta_V(k) \quad (C.17) \]
and that
\[ \phi^*(\mathbb{O}_k, k)\overline{\Delta}_\omega(k)\Delta_V(k) = \phi^*(\mathbb{O}_k, k)\begin{bmatrix} 0 \\ \tilde{\Delta}_\omega(k)\Delta_V(k) \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{\Delta}_\omega(k)\Delta_V(k) \end{bmatrix} \]
\[ = \begin{bmatrix} 0 \\ \tilde{\Delta}_\omega(k)\Delta_V(k) \end{bmatrix} = \overline{\Delta}_\omega(k)\Delta_V(k) \quad (C.18) \]
Thus (5.18) can be re-expressed as:

\[ a_B(k) \overset{C.17,C.18,3.22}{=} \tilde{\omega}(k) \Delta_B(k) - \Delta_V(k) \Delta_V k + \phi^*(\Omega, k) \frac{d_{k+1} H^*(k)}{dt} \dot{\theta}(k) \]

This establishes (5.18).

---

**Solution 5.3 (pp. 81): a_J(k) with inertial frame derivatives**

Differentiating (5.6) with respect to the inertial frame \( I \), we obtain

\[ a_J(k) \triangleq \frac{d_I \mathcal{V}(k)}{dt} \overset{1.28}{=} \alpha(k) + \tilde{\omega}(k) \mathcal{V}(k) \overset{1.36}{=} \alpha(k) + \mathcal{V}(k) \mathcal{V}(k) \tag{C.19} \]

Using (C.19) in (5.8) we have

\[ a_J(k) = \phi^*(k+1, k) a_J(k+1) + H^*(k) \dot{\theta}(k) + a_J(k) \tag{C.20} \]

where the velocity dependent Coriolis acceleration term, \( a_J(k) \), is given by

\[ a_J(k) \triangleq a(k) + \mathcal{V}(k) \mathcal{V}(k) - \mathcal{V}(k+1) \mathcal{V}(k+1) \]

This establishes the first equality in (5.19).

For the latter equality in (5.19), we have

\[ a(k) + \mathcal{V}(k) \mathcal{V}(k) - \mathcal{V}(k+1) \mathcal{V}(k+1) \]

\[ \overset{5.11}{=} \tilde{\omega}(k) \Delta_V(k) - \Delta_V(k) \Delta_V(k) + \frac{d_{k+1} H^*(k)}{dt} \dot{\theta}(k) \]

\[ + \mathcal{V}(k) \mathcal{V}(k) - \mathcal{V}(k+1) \mathcal{V}(k+1) \tag{C.21} \]

The individual terms in (C.21) can be re-expressed as follows:

\[ \tilde{\omega}(k) \Delta_V(k) = \left[ \tilde{\omega}(k) + \tilde{\omega}(k) \right] \Delta_V(k) \overset{1.36}{=} \tilde{\omega}(k) \Delta_V(k) + \tilde{\omega}(k) \Delta_V(k) \]

\[ \overset{1.36}{=} \left[ \tilde{\omega}(k) + \tilde{\omega}(k) \right] \Delta_V(k) + \tilde{\omega}(k) \Delta_V(k) \]

\[ = \tilde{\omega}(k+1) \Delta_V(k) + \tilde{\omega}(k) \Delta_V(k) - \tilde{\omega}(k) \Delta_V(k) \]

\[ = \tilde{\omega}(k+1) \Delta_V(k) + \tilde{\omega}(k) \Delta_V(k) \]

\[ = \tilde{\omega}(k+1) \Delta_V(k) \]

\[ \mathcal{V}(k) \mathcal{V}(k) = \tilde{\omega}(k+1) \mathcal{V}(k) = \left[ \tilde{\omega}(k+1) + \tilde{\omega}(k) \right] \mathcal{V}(k) \]

\[ \mathcal{V}(k+1) \mathcal{V}(k+1) \overset{1.36}{=} \tilde{\omega}(k+1) \mathcal{V}(k+1) \]

Substituting the above expressions into (C.21) and simplifying we obtain the latter equality in (5.19).

Equation (5.20a) is a component-level expansion of the first equality in (5.19).
The top angular half of (5.20b) is a simple carryover from \(a(k)\). For the lower linear half, observe from (5.12) and (5.20a) that

\[-\Delta_\nu (k) \nu(k+1) - \Delta_\omega (k) \nu(k) + \nu(k) - \nu(k+1) \omega(k+1) = \nu(k) - \nu(k+1) + \Delta_\nu (k)\]

---

**Solution 5.4 (pp. 88): Inclusion of gravitational forces**

1. The application of a pseudo-acceleration \(g\) at the base-body results in the following altered version of the expression for \(\alpha\) in (5.22):

\[\bar{\alpha} = \mathcal{E}_\phi \bar{\alpha} + H^* \ddot{\theta} + \alpha + E^* g\]

Using (3.36), we then obtain

\[\bar{\alpha} = \phi^* [H^* \ddot{\theta} + \alpha + E^* g]\]

Thus, the effect of the pseudo-acceleration can be handled by replacing \(a\) by \(a + E^* g\). This leads to the Coriolis forces vector expression in (5.39).

2. Adding the additional gravitational force leads to the following body level equations of motion:

\[f(k) = \phi(k, k-1) f(k-1) + M(k) \alpha(k) + b(k) + M(k) g\]

This in turn leads to the following altered expression for \(f\) in (5.22):

\[f = \mathcal{E}_\phi f + M[\alpha + E^* g] + b \Rightarrow f = \phi [M[\alpha + E^* g] + b]\]

Using this expression for \(f\) with the rest of the expressions in (5.23) leads to (5.40).

3. We have that

\[\phi^* E^* g = \text{col} \left\{ \phi^*(n, k) g \right\}_{k=1}^n\]

However, since \(g\) is a purely linear acceleration, \(\phi^*(n, k) g = g\). Hence, the above equation simplifies to \(E^* g\) and establishes (5.41).
Solution 5.5 (pp. 92): Inverse dynamics using composite body inertias

1. \[ f = \phi \left[ M \phi^*(H^* \dot{\theta} + a) + b \right] \]
\[ = \phi \left[ H^* \dot{\theta} + a \right] \]

establishing (5.44).

2. Define \[ y^+ \triangleq y + R(H^* \ddot{\theta} + a) \]
\[ = \phi b + \phi R(H^* \ddot{\theta} + a) + R(H^* \ddot{\theta} + a) = \phi[b + R(H^* \ddot{\theta} + a)] \]

Thus \[ E \phi y^+ \overset{\text{C.22}}{=} \phi[b + R(H^* \ddot{\theta} + a)] \]
\[ = y - b \Rightarrow y = E \phi y^+ + b \]

The first expression in the (5.45) recursion is a component level expression of the above. The second expression follows from the \[ y^+ = y + R(H^* \ddot{\theta} + a) \] definition of \[ y^+ \].

---

Solution 5.6 (pp. 94): Expression for \( H^*_i(k) \)

\[ \frac{d_i H^*_i(k)}{dt} = \frac{d_i \phi^*(\phi^* + \phi^* \phi)}{dt} H^*(k) + \phi^*(\phi^* + \phi^* \phi) \]
\[ = \phi^*(\phi^* + \phi^* \phi) \left( \tilde{\gamma}^v(k) H^*(k) + \tilde{\gamma}^\omega(k) H^*(k) + \frac{d_{k+1} H^*(k)}{dt} \right) \]
\[ = \phi^*(\phi^* + \phi^* \phi) \left( \tilde{\gamma}_i(k) H^*(k) + \frac{d_{k+1} H^*(k)}{dt} \right) \]

---
Solutions for Chapter 6

Solution 6.1 (pp. 102): Properties of the $\tau(k)$ and $\tau^*(k)$ projection matrices

1. We have

\[ \tau(k) \cdot \tau(k) \overset{6.16}{=} \mathcal{G}(k)H(k)\mathcal{G}(k) \overset{6.14}{=} \mathcal{G}(k)H(k) = \tau(k) \]

This implies that $\tau(k)$ is a projection operator. Since $\tau(k)$ is a projection operator, it follows directly that so are $\tau^*(k) = I - \tau(k)$ and their transposes.

2. These identities follow by using (6.14) and the definitions in (6.16).

Solution 6.2 (pp. 104): The $\mathcal{G}(k)$ Kalman gain and the link spatial acceleration

We have

\[ \mathcal{G}^*(k)\alpha(k) \overset{6.15}{=} \mathcal{G}^*\tau^*(k)\alpha(k) + \overset{6.18}{=} 0. \]

Solution 6.3 (pp. 105): Properties of $\mathcal{P}^+(k)$

1. We have

\[ \mathcal{P}^+(k) \overset{6.24}{=} \mathcal{P}(k)\tau^*(k) \overset{6.16}{=} \mathcal{P}(k)\left[I - H^*(k)\mathcal{G}^*(k)\right] \]

\[ \overset{6.13}{=} \mathcal{P}(k) - \mathcal{P}(k)H^*(k)D^{-1}(k)H(k)\mathcal{P}(k) \]

\[ = \left[I - \mathcal{P}(k)H^*(k)D^{-1}(k)H(k)\right]\mathcal{P}(k) \overset{6.13}{=} \mathcal{P}(k)\mathcal{P}(k) \]

That is,

\[ \mathcal{P}^+(k) = \mathcal{P}(k)\tau^*(k) = \tau(k)\mathcal{P}(k) \]

(C.23)

Thus

\[ \tau(k)\mathcal{P}(k)\tau^*(k) \overset{C.23}{=} \mathcal{P}(k)[\tau^*(k)]^2 = \mathcal{P}(k)\tau^*(k) \]

2. We have

\[ H(k)\mathcal{P}^+(k) \overset{6.25}{=} H(k)\tau(k)\mathcal{P}(k) \overset{6.18}{=} 0. \]

The latter equality is obtained by transposing this equality and using the symmetry of $\mathcal{P}^+(k)$. 
Solution 6.4 (pp. 107): Ordering of $\mathcal{R}(k)$, $\mathcal{P}(k)$ and $\mathcal{M}(k)$

We have

$$\mathcal{P}^+(k) \overset{6.28}{=} \mathcal{P}(k) - \tau(k)\mathcal{P}(k)\tau^*(k)$$

Since $\tau(k)\mathcal{P}(k)\tau^*(k) \geq 0$ this implies that $\mathcal{P}(k) \geq \mathcal{P}^+(k)$.

$\mathcal{P}(k) \geq \mathcal{M}(k)$ follows from (6.28).

For link 1, $\mathcal{R}(1) = \mathcal{P}(1) = \mathcal{M}(1)$, and hence $\mathcal{R}(1) \geq \mathcal{P}(1)$. Now we prove that if $\mathcal{R}(k) \geq \mathcal{P}(k)$ then necessarily $\mathcal{R}(k+1) \geq \mathcal{P}(k+1)$. For this, observe that $\mathcal{R}(k) \geq \mathcal{P}(k)$ implies that $\mathcal{R}(k) \geq \mathcal{P}(k+1)$. Then comparing (4.8) on page 60 with (6.28) implies that $\mathcal{R}(k+1) \geq \mathcal{P}(k+1)$.

Solution 6.5 (pp. 111): Relationship of $\nu(k)$ to $\alpha(k)$

$$\mathcal{S}^*(k)\alpha(k) \overset{6.43}{=} \mathcal{S}^*(k)[\psi^*(k+1,k)\alpha(k+1) + H^*(k)\nu(k)]$$

$$= \mathcal{S}^*(k)\overline{\pi}^*(k)\phi^*(k+1,k)\alpha(k+1) + \mathcal{S}^*(k)H^*(k)\nu(k) \overset{6.18,6.14}{=} \nu(k)$$

Solutions for Chapter 7

Solution 7.1 (pp. 121): Decomposition of $\phi\mathcal{M}\phi^*$ using $\mathcal{P}$

1. Equation (7.7) on page 117 states that $\mathcal{M} = \mathcal{P} - \mathcal{E}_\phi\mathcal{P}\mathcal{E}^*_\phi$. Pre and post multiplying this equation by $\phi$ and $\psi^*$, respectively leads to

$$\phi\mathcal{M}\psi^* \overset{3.41,7.8}{=} \phi\mathcal{P}\psi^* - \tilde{\phi}\tilde{\mathcal{P}}\psi^* = (\tilde{\phi} + I)\mathcal{P}(\psi^* + I) - \tilde{\phi}\tilde{\mathcal{P}}\psi^*$$

and the result follows.

2. We have

$$\phi\mathcal{M}\phi^* = (\phi\psi^{-1})\psi\mathcal{M}\phi^* \overset{7.15}{=} (\phi\psi^{-1})[\psi\mathcal{P} + \tilde{\mathcal{P}}\phi^*] \overset{7.11}{=} \phi\mathcal{P} + [I + \phi\mathcal{K}\mathcal{H}]\tilde{\mathcal{P}}\phi^* = \tilde{\mathcal{P}} + \tilde{\phi}\mathcal{P} + \tilde{\mathcal{P}}\phi^* + \phi\mathcal{K}\mathcal{H}\mathcal{E}^*_\phi\phi^*$$

$$\overset{6.13}{=} \mathcal{P} + \tilde{\phi}\mathcal{P} + \tilde{\mathcal{P}}\phi^* + \phi\mathcal{K}\mathcal{D}\mathcal{G}^*_\phi\phi^* \overset{6.36}{=} \mathcal{P} + \tilde{\mathcal{P}}\phi^* + \phi\mathcal{K}\mathcal{D}\mathcal{G}^*_\phi\phi^*$$

3. From (4.10) on page 61 it follows that $\mathcal{R}$ forms the diagonal part of $\phi\mathcal{M}\phi^*$, while (7.16) shows that its diagonal part is given by $\mathcal{P}$ plus the positive semi-definite diagonal part of $\phi\mathcal{K}\mathcal{D}\mathcal{G}^*_\phi\phi^*$. This establishes that $\mathcal{R} \geq \mathcal{P}$. 

Solution 7.2 (pp. 125): Expression relating \( z \) and \( z^+ \)

1. We have

\[
\epsilon \overset{7.25b}{=} [I - H\psi \mathcal{K}]T - H\psi[\mathcal{P}a + b]
\]

Therefore

\[
T \overset{C.24}{=} [I - H\psi \mathcal{K}]^{-1}\{\epsilon + H\psi[\mathcal{P}a + b]\}
\]

\[
\overset{7.17,7.12}{=} [I + H\phi \mathcal{K}]\epsilon + H\phi[\mathcal{P}a + b]
\]

Therefore

\[
z \overset{7.25a}{=} \psi[\mathcal{K}T + \mathcal{P}a + b] \overset{C.25}{=} \psi\mathcal{K}[I + H\phi \mathcal{K}]\epsilon + \{\psi\mathcal{K}H\phi + \psi\}[\mathcal{P}a + b]
\]

\[
\overset{7.10,7.12}{=} \phi\mathcal{K}\epsilon + \{\phi - \psi + \psi\}[\mathcal{P}a + b]
\]

from which (7.26) follows.

2. Pre-multiplying (7.26) by \( \psi^{-1} = (I - E\phi) \) and rearranging terms, we have

\[
\bar{z} = \mathcal{E}\phi \bar{z} + [\mathcal{K}\epsilon + \mathcal{P}a + b] \overset{7.3}{=} \mathcal{E}\phi (\bar{z} + \mathcal{G}\epsilon) + \mathcal{P}a + b \overset{7.27}{=} \mathcal{E}\phi \bar{z}^+ + \mathcal{P}a + b
\]

3. Now

\[
\bar{z}^+ \overset{7.27}{=} \bar{z} + \mathcal{G}\epsilon \overset{7.26}{=} \phi[\mathcal{K}\epsilon + \mathcal{P}a + b] + \mathcal{G}\epsilon \overset{7.3}{=} \phi[\mathcal{G}\epsilon + \mathcal{P}a + b]
\]

Solution 7.3 (pp. 125): Expression for \( \alpha \) in terms of \( \nu \)

From (7.25d) we have \( \bar{\theta} = [I - H\psi \mathcal{K}]^*\nu - \mathcal{K}^*\psi^*a \). Thus, it follows that

\[
\alpha \overset{5.23}{=} \phi^*[H^*\bar{\theta} + a] \overset{7.25d}{=} \phi^*H^*[\{I - H\psi \mathcal{K}]^*\nu - \mathcal{K}^*\psi^*a\} + \phi^*a
\]

\[
\overset{7.12}{=} \psi^*H^*\nu - \phi^*H^*\mathcal{K}^*\psi^*a + \phi^*a \overset{7.10}{=} \psi^*H^*\nu + \psi^*a = \psi^*[H^*\nu + a]
\]

This establishes (7.30).

The first equality in (7.31) results from combining (7.25d) and (7.30) as follows:

\[
\bar{\theta} \overset{7.25d}{=} [I - H\psi \mathcal{K}]^*\nu - \mathcal{K}^*\psi^*a = \nu - \mathcal{K}^*\psi^*H^*\nu - \mathcal{K}^*\psi^*a \overset{7.30}{=} \nu - \mathcal{K}^*\alpha
\]

The second equality follows from the additional use of (7.3) in the first equality.
Solution 7.4 (pp. 127): Relationship between $\nu$, $\alpha$ and $a$

The first identity in (7.33) follows from
\[
\alpha \overset{7.30}{=} \psi^*[H^*\nu + a] \overset{7.8}{=} \tilde{\psi}^*[H^*\nu + a] + [H^*\nu + a] \\
\overset{7.8}{=} \mathcal{E}_\psi \psi^*[H^*\nu + a] + [H^*\nu + a] \overset{7.254}{=} \mathcal{E}_\psi \alpha + H^*\nu + a \\
\overset{7.32e}{=} \tau^* \alpha^+ + H^*\nu + a
\]

The second identity in (7.33) follows by multiplying both sides of the first identity by $\mathcal{G}^*$ and noting that $\mathcal{G}^* \tau^* = 0$ and $\mathcal{G}^* H^* = I$.

Solution 7.5 (pp. 128): Computing inter-link spatial force $f(k)$

We have
\[
f \overset{5.23}{=} \phi[\mathbf{M}\alpha + b] \overset{7.30}{=} \phi[\mathbf{M}\psi^* (H^*\nu + a) + b] \\
\overset{7.15}{=} [\phi^\mathcal{P} + \mathcal{P}\phi](H^*\nu + a) + \phi b \overset{7.30}{=} \phi^\mathcal{P} H^*\nu + \phi^\mathcal{P}a + \mathcal{P}\alpha + \phi b \\
\overset{7.3}{=} \phi \mathcal{K} e + \phi^\mathcal{P}a - a + \mathcal{P}\alpha + \phi b \overset{7.26}{=} \mathcal{P}[\alpha - a] + \tilde{\mathcal{J}}
\]

This establishes the first half of (7.34).

From (7.33) we have that $\alpha = \tau^* \alpha^+ + H^*\nu + a$. It thus, follows that
\[
f \overset{7.34}{=} \mathcal{P}(\alpha - a) + \tilde{\mathcal{J}} \overset{7.33}{=} \mathcal{P}[\tau^* \alpha^+ + H^*\nu + a] - \mathcal{P}a + \tilde{\mathcal{J}} \\
\overset{6.25, 7.32b}{=} \mathcal{P}^* \alpha^+ + \mathcal{P}H^*\nu + \tilde{\mathcal{J}}^+ - \mathcal{G}_e \overset{7.25e}{=} \mathcal{P}^* \alpha^+ + \mathcal{J}^+
\]

Solution 7.6 (pp. 129): Including gravitational accelerations

1. The proof here is an extension of the proof of Lemma 7.6 on page 123 to include in the additional gravitational term. The Coriolis term expression with gravity included is defined by (5.39), and consists of replacing $a$ with $a'$ where $a' \overset{\Delta}{=} a + E^* g$. The new version of (7.23) with the gravitational term requires a similar change to obtain
\[
[I - H\psi \mathcal{K}]H\phi(\mathbf{M}\phi^* a' + b)
\]
\[
= (H\psi \mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)a' + H\psi b \\
= (H\psi \mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)/1a + (H\mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)E^* g + H\psi b \\
\overset{7.3}{=} (H\psi \mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)a + \mathcal{D}(\mathcal{G}^* + \mathcal{K}^* \phi^*)E^* g + H\psi b \\
\overset{7.3, 3.41}{=} (H\psi \mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)a + \mathcal{D}\mathcal{G}^* (I + \tilde{\mathcal{G}}^*)E^* g + H\psi b \\
\overset{3.40}{=} (H\psi \mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)a + \mathcal{D}\mathcal{G}^* \mathcal{E}^* g + H\psi b \\
\overset{5.41}{=} (H\psi \mathcal{P} + \mathcal{D}\mathcal{K}^* \phi^*)a + \mathcal{D}\mathcal{G}^* \mathcal{E}^* g + H\psi b
\]

(C.26)
When compared with (7.23), (C.26) contains the additional $\mathcal{D}\mathcal{G}^*\tilde{E}^*\mathcal{g}$ gravitational term. Updating (7.24) to include this additional term results in the following altered expression:

\[
[I - H\psi\mathcal{K}]^*\mathcal{D}^{-1} [I - H\psi\mathcal{K}] H\psi(M\phi^* a + b)
= [I - H\psi\mathcal{K}]^*\mathcal{D}^{-1} [H\psi(Pa + b)] + \mathcal{K}^*\psi^* a + [I - H\psi\mathcal{K}]^* \mathcal{G}^*\tilde{E}^*\mathcal{g}
\]

The additional $[I - H\psi\mathcal{K}]^* \mathcal{G}^*\tilde{E}^*\mathcal{g}$ term also needs to be included in (7.21) resulting in the desired (7.36) expression.

2. We repeat the steps in the proof of Exercise 7.3, but this time using the expression for $\tilde{\mathcal{D}}$ from (7.36) instead of from (7.21). We obtain:

\[
\alpha \overset{5.23}{=} \phi^* [H^*\tilde{\mathcal{D}} + a] \overset{7.30,7.36}{=} \psi^* [H^*\nu + a] - \phi^* H^* [I - H\psi\mathcal{K}]^* \mathcal{G}^*\tilde{E}^*\mathcal{g} \overset{7.12}{=} \psi^* [H^*\nu + a] - \psi^* H^* \mathcal{G}^*\tilde{E}^*\mathcal{g} = \psi^* \left[ H^* \left( \nu - \mathcal{G}^*\tilde{E}^*\mathcal{g} \right) + a \right] \overset{7.37}{=} \psi^* \left[ H^*\tilde{\nu} + a \right]
\]

establishing (7.38a).

Equation (7.38b) is obtained by simply using (7.25c) in (7.36). For (7.38c) repeat the steps in the proof of (7.31).

3. The proof of (7.39) exactly parallels the proof of Exercise 7.4 on page 127 except for using the expression for $\alpha$ in (7.38a) as a starting point.

4. The proof of (7.40) is similar to that of Exercise 7.5. We have

\[
f \overset{5.23}{=} \phi \left[ M(\alpha + \tilde{E}^*\mathcal{g}) + b \right] \overset{7.38a}{=} \phi \left[ M \left[ \psi^* (H^*\tilde{\nu} + a) + \tilde{E}^*\mathcal{g} \right] + b \right] \overset{7.15}{=} \left[ \phi P + P\phi \right] (H^*\tilde{\nu} + a) + \phi b + \phi M\tilde{E}^*\mathcal{g} \overset{7.38a,7.37}{=} \phi P H^*\nu + \phi P a + P\alpha + \phi b + \left[ \phi M - \phi P H^* \mathcal{G}^* \right] \tilde{E}^*\mathcal{g} \overset{5.41,7.25c}{=} \phi \mathcal{K} c + \phi P a - P\alpha + P\phi + \left[ \phi M \phi^* - \phi P H^* \mathcal{G}^* \phi^* \right] E^*\mathcal{g} \overset{7.26,7.16}{=} P[\alpha - a] + \tilde{\phi} P + P\tilde{\phi} - \tilde{\phi} P H^* \mathcal{G}^* E^*\mathcal{g} \overset{5.41,7.3}{=} P \left[ \alpha + \tilde{E}^*\mathcal{g} - a \right] + \tilde{\phi} P + P\tilde{\phi} \mathcal{G}^* E^*\mathcal{g} = P \left[ \alpha + \tilde{E}^*\mathcal{g} - a \right] + \tilde{\phi} P \mathcal{G}^* E^*\mathcal{g}
\]

The last step follows from the $\tilde{\phi} P \mathcal{G}^* E^*\mathcal{g} = 0$ identity which is a consequence of the strictly lower-triangular nature of $\tilde{\phi}$. This establishes the first half of (7.40).

Now using (7.39) in the above we have

\[
f \overset{7.40}{=} P \left[ \alpha + \tilde{E}^*\mathcal{g} - a \right] + \tilde{\phi} P \mathcal{G}^* E^*\mathcal{g} \overset{7.39}{=} P \left[ \mathcal{T}^*\alpha^+ + H^* \left( \nu - \mathcal{G}^*\tilde{E}^*\mathcal{g} \right) + a + \tilde{E}^*\mathcal{g} - a \right] + \tilde{\phi} P \mathcal{G}^* E^*\mathcal{g} \overset{6.25,7.32b}{=} P^+ \left( \alpha^+ + \tilde{E}^*\mathcal{g} \right) + P H^* \nu + \tilde{\phi} P \mathcal{G}^* E^*\mathcal{g} \overset{7.25c}{=} P^+ \left( \alpha^+ + \tilde{E}^*\mathcal{g} \right) + \tilde{\phi} P \mathcal{G}^* E^*\mathcal{g}
\]

5. (7.41) follows directly from the operator expression in (7.38c) for $\tilde{\phi}$. 


Solutions for Chapter 8

Solution 8.1 (pp. 147): Root/tip nodes and BW A matrices

1. The first part of (8.20) is a generalized restatement of (8.4) for BW A matrices. The latter half follows from the following:

\[ A^{-1} \cdot e_r \overset{8.15}{=} (I - E_A)e_r \overset{8.20}{=} e_r \]

2. The proof here is completely analogous to that of the first part with the use of (8.5).

Solutions for Chapter 9

Solution 9.1 (pp. 165): Recursive evaluation of \( \bar{A}x \) and \( \bar{A}^*x \)

We have that \( y \triangleq \bar{A}x = \bar{A}\bar{y} - x \) where \( \bar{y} = \bar{A}x \). This implies that

\[ y(k) = \bar{y}(k) - x(k) \overset{9.5}{=} \sum_{i \in \mathcal{G}(k)} A(k,i)\bar{y}(i) = \sum_{i \in \mathcal{G}(k)} A(k,i)[y(i) + x(i)] \]

Thus, the computational algorithm for \( y \) is the one in (9.5) except that the computational step in the loop is now given by the above expression with initial condition \( y(0) = 0 \).

Similarly \( y \triangleq \bar{A}^*x = \bar{y} - x \) where \( \bar{y} = \bar{A}^*x \). It thus follows that the computational algorithm for \( \bar{A}^*x \) is the same as in (9.8) except that the computational step in the loop is now given by

\[ y(k) = \bar{A}^*(\varphi(k),k)[y(\varphi(k)) + x(\varphi(k))] \]

with initial condition \( y(n + 1) = 0 \).

Solution 9.2 (pp. 180): Determinant of the mass matrix

1. We have

\[ I - HAJK \overset{9.37}{=} I - HA_E \bar{A}G \overset{8.19}{=} I - H\bar{A}G \]

For a canonical tree, \( \bar{A} \) is strictly lower triangular, while \( H \) and \( G \) are block-diagonal. Hence, \( [I + HAJK] \) is lower triangular with identity blocks along the diagonal. Moreover, from (9.50) we know that \( [I - H\psi K] \) is its inverse. From
matrix theory, we know that the inverse of a lower-triangular matrix is also lower-triangular, and that the diagonal elements are inverses of each other. It thus, follows that for a canonical tree, \([I - H\psi \mathcal{K}]\) is also lower-triangular with identity blocks along its diagonal.

2. For canonical trees, the above part established that \([I + HA\mathcal{K}]\) is a lower-triangular matrix with identity matrices along the diagonal. Since, the determinant of a lower-triangular matrix is the product of the determinants of the block elements along its diagonal, it follows that (9.52) holds for canonical trees. Since all tree can be converted into canonical trees by a simple renumbering of the bodies, there exists a permutation matrix which transforms the Newton–Euler factors for a tree into the corresponding factors for a canonical version of the tree. Since permutation matrices are orthogonal, their determinants are 1, and hence, the determinant of the canonical and non-canonical versions of the Newton–Euler factors are equal to each other and are both 1. This establishes (9.52).

3. For (9.53), we have

\[
\det\{\mathcal{M}\} \overset{9.49}{=} \det\{[I + HA\mathcal{K}] \mathcal{D} [I + HA\mathcal{K}]^*\} \\
= \det([I + HA\mathcal{K}]) \det\{\mathcal{D}\} \det\{[I + HA\mathcal{K}]^*\} \\
\overset{9.52}{=} \det\{\mathcal{D}\} = \prod_{k=1}^{n} \det\{\mathcal{D}(k)\}
\]

---

**Solutions for Chapter 10**

**Solution 10.1 (pp. 197): \(\Upsilon(k)\) for a micro/macro manipulator system**

From (10.16) we have the following general expression for \(\Upsilon(k)\):

\[
\Upsilon(k) = \sum_{\forall i: i \geq k} \psi^*(i,k)H^*(i)\mathcal{D}^{-1}(i)H(i)\psi(i,k)
\]

Thus,

\[
\Upsilon(1) = \sum_{\forall i: i \geq 1} \psi^*(i,1)H^*(i)\mathcal{D}^{-1}(i)H(i)\psi(i,1) \\
= \sum_{\forall i: \rho(\mathcal{S}) \succ i \geq 1} \psi^*(i,1)H^*(i)\mathcal{D}^{-1}(i)H(i)\psi(i,1) \\
+ \sum_{\forall i: i \geq \rho(\mathcal{S})} \psi^*(i,1)H^*(i)\mathcal{D}^{-1}(i)H(i)\psi(i,1)
\]


\[ C.27 \Rightarrow \mathcal{Y}(1) + \psi^*(\varphi(\mathcal{G}),1) \left\{ \sum_{\forall i: i \succ \varphi(\mathcal{G})} \psi^*(i,\varphi(\mathcal{G}))H^*(i)^* \right\} D^{-1}(i)H(i)\psi(i,\varphi(\mathcal{G})) \right\} \psi^*(\varphi(\mathcal{G}),1) \]

The above steps have used the fact that \( D(i) \) does not depend on the generalized coordinates and other properties of the bodies inboard of the \( i \)th body in including \( \mathcal{Y}(1) \) in the above expressions. This establishes (10.23). (10.24) is a direct consequence of (10.23) and the positive semi-definite nature of \( \psi^*(\mathcal{P}(\varphi(\mathcal{G}),1)\mathcal{Y}(\varphi(\mathcal{G}))\psi(\mathcal{P}(\varphi(\mathcal{G}),1). \)

---

**Solution 10.2 (pp. 199): Computation of the mass matrix inverse**

1. Expanding out the factorized form of the mass matrix inverse, it follows that

\[ M^{-1} \stackrel{9.51}{=} D^{-1} - \mathcal{K}^*\psi^*H^*D^{-1} - D^{-1}H\psi\mathcal{K} + \mathcal{K}^*\Omega\mathcal{K} \]

\[ = D^{-1} - \mathcal{K}^*\psi^*H^*D^{-1} - D^{-1}H\psi\mathcal{K} + \mathcal{K}^*\left[ \mathcal{Y} + \bar{\psi}^*\mathcal{Y} + \mathcal{Y}\bar{\psi} + \mathcal{R} \right] \mathcal{K} \]

Therefore,

\[ M^{-1} \stackrel{9.51}{=} D^{-1} - \mathcal{K}^*\psi^*H^*D^{-1} - D^{-1}H\psi\mathcal{K} + \mathcal{K}^*\left[ \mathcal{Y} + \bar{\psi}^*\mathcal{Y} + \mathcal{Y}\bar{\psi} + \mathcal{R} \right] \mathcal{K} \]

\[ = (D^{-1} + \mathcal{G}^*\mathcal{Y}\mathcal{G}) - \mathcal{K}^*\psi^*(H^*D^{-1} - \bar{\tau}\,\bar{\mathcal{G}}) \]

\[ - (D^{-1}H - \mathcal{G}^*\mathcal{Y}\tau)\psi\mathcal{K} + \mathcal{K}^*\mathcal{R}\mathcal{K} \]

\[ \stackrel{10.30}{=} L - \mathcal{K}^*\psi^*U - U\psi\mathcal{K} + \mathcal{K}^*\mathcal{R}\mathcal{K} \]

The last equality used the following

\[ D^{-1}H - \mathcal{G}^*\mathcal{Y}\tau = D^{-1}H - \mathcal{G}^*\mathcal{Y} + \mathcal{G}^*\mathcal{Y}\tau = D^{-1}H - \mathcal{G}^*\mathcal{Y} + \mathcal{G}^*\mathcal{Y}\mathcal{H} \]

\[ = LH - \mathcal{G}^*\mathcal{Y} \stackrel{10.30}{=} U \]

This establishes the decomposition of \( M^{-1} \).

2. For a serial-chain system, \( \mathcal{R} = 0 \), and hence (10.29) reduces to (10.31). Also, for a serial-chain system \( \mathcal{E}_\varphi^*\mathcal{Y}\mathcal{E}_\varphi \) is block-diagonal, and hence, \( \mathcal{Y} = \mathcal{Y}^+ \) follows from (10.20). Since the component terms of \( L \) in (10.30) are block-diagonal, so is \( L \). Similarly, \( U \) is block-diagonal as well. The product \( \psi\mathcal{K} \) is strictly lower-triangular, establishing that (10.31) is a decomposition into block-diagonal, strictly upper-triangular and strictly lower-triangular terms.
Algorithm 10.4 is a recursive implementation of the above expressions for the elements of $M^{-1}$.

**Solution 10.3 (pp. 207):** Ground connected equations of motion

In Lemma 10.6, assume that system B is the inertial frame. Since the inertial frame is immovable, we have

$$\Delta^B = 0 \quad \text{and} \quad \bar{c}^B_{os} = 0$$

Substituting these into (10.48) and keeping just the upper part results in (10.53).

---

**Solutions for Chapter 11**

**Solution 11.1 (pp. 214):** Mapping between $T$ and $\bar{\theta}$ for closed-chain systems

This result is obtained by setting combining together (11.9a)-11.9c and (11.8).

**Solution 11.2 (pp. 215):** Torque minimization using squeeze forces

The squeeze force has the parametric form $T_{sq} = G_c^* \lambda$ for some $\lambda$. The norm of the overall generalized force is given by

$$\|T\|^2_W = \|T_{mv} + T_{sq}\|^2_W = \|T_{mv} + G_c^* \lambda\|^2_W = (T_{mv} + G_c^* \lambda)^T W (T_{mv} + G_c^* \lambda)$$

The gradient of the above with respect to $\lambda$ must be zero for the minimum norm solution. Taking the gradient and setting it to zero yields,

$$0 = G_c W (T_{mv} + G_c^* \lambda) \quad \Rightarrow \quad \lambda = -(G_c W G_c^*)^{-1} G_c W T_{mv} \quad (C.28)$$

Thus $X = G_c W$ minimizes the norm. Since the squeeze force is $T_{sq} = G_c^* \lambda$, the minimum norm $T$ value is

$$T_{mv} + G_c^* \lambda \quad \Rightarrow \quad T_{mv} = P_{ms} T_{mv} \quad (11.15) \quad (I - P_{ms}) (I - P_{ms}) T = (I - P_{ms}) T$$

The last step used the projection matrix property of $(I - P_{ms})$.

**Solution 11.3 (pp. 218):** Expression for $\frac{Y}{C}$ with loop constraints

This result is obtained by combining the expressions in and (11.20) and (11.21).

**Solution 11.4 (pp. 223):** Explicit expression for $P$

The result follows by directly using the expression for $P$ in (11.37).
Solution 11.5 (pp. 224): Transformed and partitioned augmented dynamics
Substituting $\ddot{\theta} = P\ddot{\theta}$ into (11.31) transforms it as follows:

$$
\begin{pmatrix}
M & G^*_c \\
G_c & 0
\end{pmatrix}
\begin{pmatrix}
P \ddot{\theta} \\
-\lambda
\end{pmatrix}
= 
\begin{pmatrix}
\mathcal{T} - \mathcal{C} \\
\dot{\mathcal{U}}
\end{pmatrix}
$$

(C.29)

Pre-multiplying the top half of both sides of (C.29) with $P^*$ leads to

$$
\begin{pmatrix}
M \ 
\end{pmatrix}
\begin{pmatrix}
G^*_c \\
0
\end{pmatrix}
\begin{pmatrix}
\ddot{\theta} \\
-\lambda
\end{pmatrix}
= 
\begin{pmatrix}
P^*(\mathcal{T} - \mathcal{C}) \\
\dot{\mathcal{U}}
\end{pmatrix}
$$

(C.30)

Applying the partitioned structure in $G = [G_r, 0]$ from (11.37) to (C.30) leads to the partitioned equations in (11.42).

Solution 11.6 (pp. 224): Reduction of augmented dynamics
We begin by rearranging the blocks in the transformed equations of motion in (11.42) to obtain:

$$
\begin{pmatrix}
M_{11} & G^*_r \\
G_r & 0 \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
\ddot{\theta}_1 \\
-\lambda \\
\ddot{\theta}_2
\end{pmatrix}
= 
\begin{pmatrix}
\mathcal{T}_1 \\
\dot{\mathcal{U}}
\end{pmatrix}
$$

(C.31)

Using (A.10) to solve the above equation, we obtain

$$
\ddot{\theta}_2 = \left\{M_{22} - [M_{21}, 0] \left( \begin{pmatrix} M_{11} & G^*_r \\ G_r & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} M_{12} \\ 0 \end{pmatrix} \right\}^{-1}
$$

(C.32)

Additionally, from (A.11) we have

$$
\begin{pmatrix}
M_{11} & G^*_r \\
G_r & 0
\end{pmatrix}^{-1} =
\begin{pmatrix}
0 & G_r^{-1} \\
G_r^{-*} & -G_r^{-*} M_{11}^{-1} G_r^{-1}
\end{pmatrix}
$$

(C.33)

Substituting this into (C.32) and expanding out the matrix products we see that the equation simplifies to

$$
M_{22}^{-1} (\mathcal{T}_2 - M_{21} G_r^{-1} \dot{\mathcal{U}}) \quad \text{or} \quad M_{22} \ddot{\theta}_2 = \mathcal{T}_2 - M_{21} G_r^{-1} \dot{\mathcal{U}}
$$

(C.34)
Solution 11.7 (pp. 224): Transformed projected dynamics
Using (11.39) in (11.36) we see that
\[ M_{r11.39,11.36} = X_\star c P^* M X_c \]
\[ M_{r11.38} = X_\star c M X_c \]
Thus, the projected dynamics equations of motion in (11.36) becomes
\[ M_{22} \theta \dddot{r} = X_\star c (T - e - M \dddot{\theta}_p) \]
\[ = (T - X_\star c M P^{-1} \dddot{\theta}_p) \]
Note however, that \( \dddot{\theta}_p \) defined by
\[ \dddot{\theta}_p \triangleq G_r^{-1} \hat{U} \]
satisfies the particular solution condition, (11.35), since
\[ G_c \dddot{\theta}_p \triangleq G_c P \begin{bmatrix} G_r^{-1} \hat{U} \\ 0 \end{bmatrix} \in \mathcal{R}_N \]
Substituting (C.37) in (C.36) leads to
\[ M_{22} \dddot{\theta}_r = \begin{bmatrix} T - X_\star c M \end{bmatrix} \begin{bmatrix} G_r^{-1} \hat{U} \\ 0 \end{bmatrix} \]
\[ = (T - M_{21} G_r^{-1} \hat{U}) \]
With \( \dddot{\theta}_r \equiv \dddot{\theta}_2 \), this last expression agrees with (11.43) and establishes the result.

Solutions for Chapter 12

Solution 12.1 (pp. 238): Simplification of \( R_G(k) \)
The expression for \( R_G(k) \) in (12.18) and the recursion in (12.19) for \( R_{G1}(k) \) follows from substituting the expression for \( \Phi_G(k+1, k) \) from (12.4) into (12.14) and expanding out the partitioned products.
A similar process leads the expressions for \( X_{1}(k+1), X_{1}(j+1) \) and \( M_G(k, k) \).

Solution 12.2 (pp. 242): Structure of \( P_G(k) \) and related quantities
The partitioned structure and related recursive algorithms follow by starting with the expressions in (12.21) and (12.25) and using the partitioned structure of \( \Phi_G(k+1, k) \) to explicitly compute the products.
Solutions for Chapter 13

Solution 13.1 (pp. 269):  Additional forward dynamics simplifications
From (13.69), \( \mathcal{D}_m(k) = H_{Mfl}(k) \left[ A_{fl}(k) \Gamma_{fl}(k), A_{fl}^*(k) + M_{fl}(k) \right] H_{Mfl}^*(k). \) Applying the (A.19) matrix identity,
\[
[A + BCB^*]^{-1} = A^{-1} - A^{-1}B[C^{-1} + B^*A^{-1}B]^{-1}B^*A^{-1}
\]
to the expression for \( \mathcal{D}_m(k) \) with
\[
A = H_{Mfl}(k) M_{fl}(k) H_{Mfl}^*(k), \quad B = H_{Mfl}(k) A_{fl}(k), \quad C = \Gamma_{fl}(k)
\]
establishes (13.70).

Solutions for Chapter 14

Solution 14.1 (pp. 274):  Non-tree path-induced sub-graphs
1. Since a directed cycle containing an edge, represents a directed path connecting the node pair for the edge, the path (and the cycle) must belong to \( \mathcal{S} \) since it is path-induced.
2. Similarly, all paths connecting a pair of nodes in \( \mathcal{S} \) must be in the sub-graph since it is path-induced, and the result follows.

Solution 14.2 (pp. 278):  Mass matrix invariance of the outer sub-system
The expressions in (14.6) and (14.7) are obtained by directly evaluating \( M = H \mathcal{A} M \mathcal{A}^* H^* \) using the following component partitioned expressions from (14.4) and (14.5):
\[
H = \begin{pmatrix} H_{\mathcal{S}} & 0 \\ 0 & H_{\mathcal{P}} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{A}_{\mathcal{S}} & 0 \\ \mathcal{A}_{\mathcal{P}} E_{\mathcal{S}} \mathcal{A}_{\mathcal{S}} & \mathcal{A}_{\mathcal{P}} \end{pmatrix}, \quad M = \begin{pmatrix} M_{\mathcal{S}} & 0 \\ 0 & M_{\mathcal{P}} \end{pmatrix}
\]

Solution 14.3 (pp. 292):  Mass matrix invariance with aggregation
First, we have
\[
H_{\mathcal{A}}^{14.18,14.20} = (H_{\alpha} \mathcal{J}_{\alpha}^{-1}) (\mathcal{J}_{\alpha} \mathcal{A}_{\alpha}) = H_{\alpha} \mathcal{A}_{\alpha}
\]
Using this directly establishes the \( M \) equalities in (14.24).
Moreover
\[ H_A(\mathbf{M}^*a + b) \overset{C.39}{=} H_A(\mathbf{M}^*a + b) \overset{14.18}{=} H_A(\mathbf{M}^*a\mathbf{J}^*a + b) \]
\[ \overset{14.21}{=} H_A(\mathbf{M}^*a + b) \]
This establishes the \( C \) equalities in (14.24).

---

**Solutions for Chapter 15**

**Solution 15.1 (pp. 310): Expression for \( \dot{X}_\Theta \)**

With \( Z \overset{\triangle}{=} Y_1^{-1}Y_2 \),
\[
\frac{dZ}{dt} = \frac{dY_1^{-1}}{dt}Y_2 + Y_1^{-1}\frac{dY_2}{dt} \overset{A.28}{=} -Y_1^{-1}\frac{dY_1}{dt}Y_1^{-1}Y_2 + Y_1^{-1}\frac{dY_2}{dt}
= Y_1^{-1}\left[ \frac{dY_2}{dt} - \frac{dY_1}{dt}Z \right]
\]
Thus,
\[
\dot{X}_\Theta = \begin{bmatrix} \dot{Y}_1 \end{bmatrix} \begin{bmatrix} Y_1^{-1} \left[ \dot{Y}_1 Z - \dot{Y}_2 \right] \end{bmatrix} = \begin{bmatrix} Y_1^{-1} \dot{Y}X_\Theta \end{bmatrix}
\]
This establishes the first half of (15.27).
Using this, we have
\[
\dot{X}_\Theta \dot{\theta}_R = \begin{bmatrix} Y_1^{-1} \dot{Y}X_\Theta \dot{\theta}_R \end{bmatrix} = \begin{bmatrix} Y_1^{-1} \dot{Y} \dot{\theta}_\Theta \end{bmatrix}
\]
This establishes the second half of (15.27).

---

**Solutions for Chapter 16**

**Solution 16.1 (pp. 320): Alternate expression for \( S_{\alpha\alpha} \)**

We have
\[
\phi\mathbf{M}_\perp^*\mathbf{M}_\perp\phi^* \overset{10.36}{=} \phi\mathbf{M}_\perp^*\phi^* \overset{10.39a}{=} \phi\mathbf{M}(\psi^* - \Omega\mathcal{P}) = \phi\mathbf{M}\psi^* - \phi\mathbf{M}^*\mathcal{P}
\]
\[ \overset{9.41,10.38a}{=} \phi\mathcal{P} + \mathcal{P}\phi^* - (\phi - \psi + \mathcal{P}\Omega)\mathcal{P} = (\psi - \mathcal{P}\Omega)\mathcal{P} + \mathcal{P}\phi^* \]
Pre- and post-multiplying the above with \( H_\alpha \) and \( H_\alpha^* \) and comparing with the expression for \( S_{\alpha\alpha} \) in (16.11c) establishes the lemma.
Solution 16.2 (pp. 325): Expression for the generalized Jacobian

Ignoring all the velocity dependent terms, we have

\[ \alpha_{nd}^{16.18} = J_{\dot{\theta}} = J_a \ddot{\theta}_a + J_p \ddot{\theta}_p \]

\[ \alpha_{nd}^{16.20} = J_{\dot{\theta}} = J_a \ddot{\theta}_a + J_p J_D \ddot{\theta}_a = (J_a + J_p J_D) \ddot{\theta}_a \]

Solution 16.3 (pp. 330): The disturbance Jacobian for a free-flying system

Equation (4.18) on page 63 showed that the center of mass spatial velocity, \( V_C \) is given by the following relationship when the base-body generalized forces are zero:

\[ V_C = V(n) + R^{-1}(n) \sum_{k=1}^{n-1} \phi(n, k) R(k) H^*(k) \dot{\theta}(k) \]  \( \text{(C.40)} \)

For the case when the system has zero spatial momentum, \( V_C = 0 \) and so we can re-express (C.40) as:

\[ V(n) = -R^{-1}(n) \sum_{k=1}^{n-1} \phi(n, k) R(k) H^*(k) \dot{\theta}(k) \]

The expression for the \( J_D \) disturbance Jacobian is obtained by converting (16.28) into a matrix form.

Solutions for Chapter 17

Solution 17.1 (pp. 333): Weight matrices for the dual model

In the dual model, the \( k \)th link is the parent of the \( (k+1) \)th link. Thus the velocity recursion in (3.19b) can be re-expressed as:

\[ V(k) = \phi^*(k-1, k) V(k-1) - \phi^*(k-1, k) H^*(k-1) \dot{\theta}(k-1) \]

Thus \( \phi^*(k-1, k) \) matrices are the weight matrices in the dual model.

Solution 17.2 (pp. 335): Dual articulated body inertia properties

1. Using (17.4) we have

\[ S(k) H^*(k) = S^+(k) [I + H^*(k) S^*_{dl}(k)] H^*(k) \]

\[ = S^+(k) [I - H^*(k) D_{dl}^{-1}(k) H(k) S^+(k)] H^*(k) = 0 \]

This establishes (17.5).
2. Again using (17.4) we have

$$\tau_{d1}(k)\overline{\sigma}(k) = \tau_{d1}(k) + \mathcal{G}_{d1}(k)\mathcal{H}(k)\mathcal{G}(k) = \tau_{d1}(k) + \mathcal{G}_{d1}(k)\mathcal{H}(k) = 0$$

This establishes the first expression in (17.6).
The latter expression has an analogous proof.

---

**Solution 17.3 (pp. 347): Relationship between \(\phi^*_G\) and \(\phi^*_R\)**

$$\tilde{\phi}^*_R(\phi^*_G)^{-1} = \tilde{\phi}^*_R(I - \mathcal{E}^*_{\phi_G}) = (\phi^*_R - \mathcal{I}) - \phi^*_R\mathcal{E}^*_{\phi_R}\mathcal{E}^*_{\phi_G}$$

Equations 17.22, 17.28a

Post-multiplying the above with \(\phi^*_G\) establishes (17.36).

---

**Solution 17.4 (pp. 347): The inverse transformation \(\hat{J}_{GR}\)**

Observe that

$$\hat{J}_{GR}\hat{\theta}_R = (\mathcal{S}_{RG}^* + \mathcal{E}^*_{\phi_R}\mathcal{H}_{RG})\hat{\theta}_R = \sum_{j=k}^{n-1} e_{Gj}\hat{\theta}_R(j+1) + e_{Gn}e_n\mathcal{V}_G$$

Equations 17.29, 17.30, 17.18

This establishes (17.38).

Now,

$$\hat{J}_{GR} \ast \hat{J}_{RG} = (\mathcal{S}_{RG}^* + \mathcal{E}^*_{\phi_R}\mathcal{H}_{RG})\mathcal{S}_{RG}^* + \mathcal{E}^*_{\phi_R}\mathcal{H}_{RG}$$

Equation 17.38

(C.41)

However,

$$\phi^*_R \mathcal{H}_{RG} = \phi^*_R(-\mathcal{E}^*_{\phi_R}\mathcal{S}_{RG}^* + e_k^*\mathcal{H}_{RG})$$

Equation 17.37a

(C.42)

Also,

$$\phi^*_R \mathcal{H}_{RG} e_k^* = \phi^*_R(-\mathcal{E}^*_{\phi_R}\mathcal{S}_{RG}^* + e_k^*\mathcal{H}_{RG}) e_k^*$$

Equation 17.37a

(C.43)
Therefore
\[ J_{GR}^* J_{RG} = S_{RG}^* S_{RG} + e_{Gn}^* e_n^* (\Phi_R^* H^* + \Phi_k^* e_k^* \Phi_S^* H_S^*) \]
\[ = S_{RG}^* S_{RG} + e_{Gn}^* (\Phi_R^* + \Phi_S^*) H_S^* \]
\[ = S_{RG}^* S_{RG} + e_{Gn}^* \Phi_S^* H_S^* \]
\[ = S_{RG}^* S_{RG} + e_{Gn}^* H_S^* \]
\[ = I \]

This establishes (17.39).

Solution 17.5 (pp. 348): Transformed mass matrix

1. We have
\[ \Phi_R^* H_R^* J_{RG} = \Phi_R^* (e_{C_k}^* e_k^* \Phi_S^*) H_S^* \]
\[ \Rightarrow (\Phi_R^* + \Phi_S^*) H_S^* = \Phi_S^* H_S^* \]
establishing (17.43).

2. The \( \hat{\theta} = J_{RG} \hat{\theta} \) directly from the use of (17.28c) and the definition of \( \hat{\theta} \) in (17.40).
For the latter equality, we have
\[ \Phi_H^* J_{RG} \]
\[ \Rightarrow \begin{pmatrix} \Phi_C^* H_C^* & \Phi_C^* B_C^* \Phi_R^* H_R^* J_{RG} \\ 0 & \Phi_R^* H_R^* J_{RG} \end{pmatrix} \]
\[ \Rightarrow \begin{pmatrix} \Phi_C^* H_C^* & \Phi_C^* B_C^* \Phi_S^* H_S^* \\ 0 & \Phi_S^* H_S^* \end{pmatrix} \]
\[ \Rightarrow \Phi_H^* \]

This establishes (17.44).

3. We have
\[ M = H \Phi M \Phi^* H^* = J_{RG}^* \Phi M \Phi^* H^* J_{RG} = J_{RG}^* M J_{RG} \]

This establishes (17.45).

4. Now,
\[ \delta e = \frac{1}{2} \delta^* M \delta \]
\[ = \frac{1}{2} \delta^* J_{RG}^* M J_{RG} \delta \]
\[ = \frac{1}{2} \delta^* M \delta \]
establishing (17.46).

5. \( \phi \) is an SPO operator and \( H \) and \( M \) are both block-diagonal. Also, \( H \) is full rank because \( H \) and \( H_R \) are both full rank. Thus, the SKO model requirements are satisfied.
Solution 18.1 (pp. 356): Identities for $\tilde{V}$
At the component level,

$$\mathcal{V}^+(k) \overset{3.15}{=} \phi^*(\varphi(k), k) \mathcal{V}(\varphi(k)) \Rightarrow \tilde{\mathcal{V}}^+(k) \phi^*(\varphi(k), k) \overset{1.35}{=} \phi^*(\varphi(k), \mathcal{V}(\varphi(k)))$$

The above relationship establishes the component level equivalence of the elements of the operator expressions on the left and right of (18.9a).

Equation (18.9b) then follows from using the fact that $\mathcal{V} = \mathcal{V}^+ + \Delta \mathcal{V}$ in (18.9a).

Equation (18.9c) follows from pre- and post-multiplying (18.9b) by $\phi^*$ and simplifying.

Equation (18.9d) is simply a transposed version of (18.9c).

Solution 18.2 (pp. 358): Time derivative of $H\phi$

$$\frac{dH\phi}{dt} = H\phi + H\dot{\phi}^ {18.12,18.5} = -H\tilde{V}_S^\omega \phi + H \left[ \phi \Delta \mathcal{V} \phi + \tilde{\mathcal{V}}^\omega \phi - \phi \tilde{\mathcal{V}}^\omega \right]$$

$$\overset{18.5}{=} H\tilde{\Delta}^\omega \phi + H\tilde{\phi} \Delta \mathcal{V} \phi - H\phi \tilde{\mathcal{V}}^\omega$$

$$= H\tilde{\Delta}^\omega \phi + H\tilde{\phi} \left[ \Delta \mathcal{V}^\omega + \Delta \mathcal{V}^\nu \right] \phi - H\phi \tilde{\mathcal{V}}^\omega$$

$$= H\phi \tilde{\Delta}^\omega \phi + H\tilde{\phi} \Delta \mathcal{V}^\nu \phi - H\phi \tilde{\mathcal{V}}^\omega = H\phi \left[ \tilde{\Delta}^\omega \phi + E^\nu \Delta \mathcal{V}^\nu \phi - \tilde{\mathcal{V}}^\omega \right]$$

This establishes the result.

Solution 18.3 (pp. 358): Operator expression for the $a_j$ Coriolis acceleration

1. $\phi^* a_j ^ {8.43} \frac{d\phi^* H^*}{dt} \overset{18.13}{=} H \left[ \tilde{\mathcal{V}}^\nu \phi - \phi^* \tilde{\Delta}^\nu \phi - \phi^* \tilde{\Delta}^\nu E^*_\phi \right] \phi^* H^* \dot{\theta} ^ {C.44}$

Hence,

$$a_j ^ {C.44} = \phi^* \left[ \tilde{\mathcal{V}}^\nu - \phi^* \tilde{\Delta}^\nu \phi - \phi^* \tilde{\Delta}^\nu E^*_\phi \right] \mathcal{V}$$

$$= \phi^* \left[ \tilde{\mathcal{V}}^\nu - \tilde{\Delta}^\nu \phi - \tilde{\Delta}^\nu E^*_\phi \right] \mathcal{V}$$

$$= -\tilde{\Delta}^\nu \mathcal{V} + \left[ I - E^*_\phi \right] \tilde{\mathcal{V}}^\nu \mathcal{V} - \tilde{\Delta}^\nu \mathcal{V}$$. 

(C.45)

This establishes (18.14).
2. The kth element of $a_j$ from (C.45) is given by:

$$a_j(k) = \left[ \begin{array}{c} \tilde{\omega}(g(k)) \omega(k) \\ \tilde{\omega}(g(k)) \nu(k) \end{array} \right] - \phi^*(g(k), k) \left[ \begin{array}{c} 0 \\ \tilde{\omega}(g(k)) \nu(g(k)) \end{array} \right] - \left[ \begin{array}{c} 0 \\ \Delta_v(k) \omega(g(k)) \end{array} \right]$$

$$= \left[ \begin{array}{c} -\tilde{\omega}(k) \omega(g(k)) \\ \tilde{\omega}(g(k)) \left[ \nu(k) - \nu(g(k)) + \Delta_v(k) \right] \end{array} \right]$$

$$= \left[ \begin{array}{c} -\tilde{\omega}(k) \left[ \omega(k) - \Delta_\omega(k) \right] \\ \tilde{\omega}(g(k)) \left[ \nu(k) - \nu(g(k)) + \Delta_v(k) \right] \end{array} \right]$$

The further use of $g(k) = k + 1$ establishes the result

---

**Solution 18.4 (pp. 359): Time derivative of $\phi(k+1,k)$ with $\mathcal{O}_k \neq \mathbb{B}_k$**

1. Equation (18.17) follows from (1.37) when applied to the $\mathbb{B}_k$ and $\mathcal{O}_k$ frame pair on the kth rigid link.
2. Equation (18.18a) follows directly from Exercise 1.8 on page 13.
   Equation (18.18b) follows from differentiating (18.15).
   Equation (18.18c) and (18.18d) follow from further manipulation and rearrangement of the earlier expressions.

---

**Solution 18.5 (pp. 360): Operator time derivatives with $\mathcal{O}_k \neq \mathbb{B}_k$**

1. (18.21a) follows from the use of (18.17) in its operator definition in (18.20).
   For (18.21b), we have

$$\frac{d\Delta_{\mathbb{B}/\mathcal{O}}^{-1}}{dt} \stackrel{18.21a,A.28}{=} -\Delta_{\mathbb{B}/\mathcal{O}}^{-1} \frac{d\Delta_{\mathbb{B}/\mathcal{O}}^{-1}}{dt} \Delta_{\mathbb{B}/\mathcal{O}}^{-1} \stackrel{18.21a}{=} -\Delta_{\mathbb{B}/\mathcal{O}}^{-1} \mathcal{V}_{\mathbb{B}/\mathcal{O}} \Delta_{\mathbb{B}/\mathcal{O}}^{-1} \stackrel{1.36}{=} -\mathcal{V}_{\mathbb{B}/\mathcal{O}}$$

In the last equality we have used the fact that $\mathcal{V}_{\mathbb{B}/\mathcal{O}}$ only contains non-zero linear velocity values together with (1.36) in:

$$\mathcal{V}_{\mathbb{B}/\mathcal{O}} = \mathcal{V}_{\mathbb{B}/\mathcal{O}} \Delta_{\mathbb{B}/\mathcal{O}}^{-1} = \Delta_{\mathbb{B}/\mathcal{O}} \mathcal{V}_{\mathbb{B}/\mathcal{O}} \quad (C.46)$$
2. We have

\[
\frac{d\mathcal{E}_{\phi}}{dt} = \frac{d\Delta_{B/O} - \mathcal{E}_{\phi} \Delta_{B/O}}{dt} = \Delta_{B/O} \frac{d\mathcal{E}_{\phi}}{dt} \Delta_{B/O}^{-1} + \nabla_{B/O} \mathcal{E}_{\phi} \Delta_{B/O}^{-1} - \Delta_{B/O} \mathcal{E}_{\phi} \nabla_{B/O}
\]

This establishes (18.22a).

For (18.22b),

\[
\frac{d\phi_{\phi}}{dt} = \phi_{\phi} \frac{d(I - \epsilon_{\phi})}{dt} \phi_{\phi} = \phi_{\phi} \left( \Delta_{B/O} \frac{d\mathcal{E}_{\phi}}{dt} \Delta_{B/O}^{-1} + \nabla_{B/O} \mathcal{E}_{\phi} \Delta_{B/O}^{-1} - \Delta_{B/O} \mathcal{E}_{\phi} \nabla_{B/O} \right) \phi_{\phi}
\]

This establishes (18.22a).

For (18.22c), we have

\[
\frac{dH_{\phi}}{dt} = \frac{d\Delta_{B/O}}{dt} = \frac{d\Delta_{B/O}^{-1}}{dt} = \dot{H} \Delta_{B/O}^{-1} - H \nabla_{B/O}
\]

Solution 18.6 (pp. 361): Alternative derivation of \( \dot{M} \) expression

1. We have

\[
\frac{d\phi M \phi^*}{dt} = \dot{\phi} M \phi^* + \phi \dot{M} \phi^* + \phi M \phi^* = 18.12 \left[ \phi \Delta_{V} \phi + \tilde{\omega} \phi - \phi \tilde{\omega} \right] M \phi^* + \phi \left[ \tilde{\omega} M - M \tilde{\omega} \phi \right] \phi^*
\]

This establishes (18.24).
2. For \( \dot{M} \), we have

\[
\dot{M} = H\phi M\phi^* H^* + H \frac{d\phi M\phi^*}{dt} H^* + H\phi M\phi^* \dot{H}^*
\]

\[\overset{18.12,18.24}{=} -H \tilde{\nu}_v^\omega \phi M\phi^* H^* + H\phi M\phi^* \tilde{\nu}_v^\omega H^*
\]

\[
+ H \left\{ \left[ \tilde{\phi} \Delta v + \tilde{\nu}_v^\omega \phi M\phi^* - \phi M\phi^* \left[ \Delta v \phi^* + \tilde{\nu}_v^\omega \right] \right] H^* \right\}
\]

\[\overset{18.5}{=} H \left\{ \left[ \tilde{\phi} \Delta v + \tilde{\Delta}^v_\omega \right] \phi M\phi^* - \phi M\phi^* \left[ \Delta v \phi^* + \tilde{\Delta}^v_\omega \right] \right\} H^*
\]

\[
= H\phi \left\{ \left[ \epsilon_{\phi} \Delta v + (I - \epsilon_{\phi}) \tilde{\Delta}^v_\omega \right] \phi M
\]

\[
- \left[ \Delta v \epsilon_{\phi}^* \phi M\phi^* - \phi M\phi^* \left[ \Delta v \epsilon_{\phi}^* + \tilde{\Delta}^v_\omega \right] \right] \right\} \phi^* H^*
\]

This establishes (18.23).

---

**Solution 18.7 (pp. 363): Sensitivities of \( \phi(\varphi(k), k) \), \( H(k) \) and \( M(k) \)**

The expressions in (18.29) follow directly from applying (18.26) and (18.28) to the time derivative expressions in (18.10), together with the expressions in (18.28).

---

**Solution 18.8 (pp. 364): Sensitivity of \( H\phi \)**

Equation (18.34) follows from applying (18.26) and (18.31) to the time derivative expressions in (18.13).

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**Solution 18.9 (pp. 368): Equivalence of Lagrangian and Newton–Euler equations of motion**

1. From (2.2) we see that the \( b_j \) stacked vector can be expressed as

\[
b_j = [ \tilde{\nu} M - M \tilde{\nu} ] \nu
\]

Also, we have seen in (C.44) that

\[
\phi^* a_j = [ \tilde{\nu}_v^\omega - \phi^* \tilde{\Delta}_v^\omega - \phi^* \tilde{\Delta}_v^\epsilon_{\phi}^* ] \nu
\]

Using these in (18.39) leads to

\[
\epsilon(\theta, \dot{\theta}) = H\phi \left[ \tilde{\nu} M - M \tilde{\nu} + M \left( \tilde{\nu}_v^\omega - \phi^* \tilde{\Delta}_v^\omega - \phi^* \tilde{\Delta}_v^\epsilon_{\phi}^* \right) \right] \nu
\]

\[\overset{1.22}{=} H\phi \left[ \tilde{\nu} M - M \tilde{\nu} - M \phi^* \left( \Delta_v^\omega + \Delta_v^\epsilon_{\phi}^* \right) \right] \nu
\]
Noting that $\nabla^V V = 0$ helps simplify the above expression and leads to the expression in (18.41). This establishes the first part of this exercise.

2. We have

$$\dot{\mathcal{M}}(\theta)\dot{\theta} = \frac{1}{2} \frac{\partial [\dot{\theta}^* \mathcal{M}(\theta) \dot{\theta}]}{\partial \theta}$$

$$= \frac{\partial [\dot{\theta}^* \mathcal{M}(\theta) \dot{\theta}]}{\partial \theta}$$

$$= H\phi \left[ \nabla - \left( \varepsilon_\phi \tilde{\Delta}_V^\nu + \tilde{\Delta}_V^\omega \right) \phi \right] \mathcal{M} V$$

$$+ H\phi \left[ \left( \tilde{\Delta}_V^\nu + \varepsilon_\phi \tilde{\Delta}_V^\nu \right) \phi \mathcal{M} - \mathcal{M} \phi^* \left( \tilde{\Delta}_V^\nu \varepsilon_\phi^* + \tilde{\Delta}_V^\omega \right) \right] V$$

$$= H\phi \left[ \nabla \mathcal{M} - \mathcal{M} \phi^* \left( \tilde{\Delta}_V^\nu \varepsilon_\phi^* + \tilde{\Delta}_V^\omega \right) \right] V$$

This is indeed the expression in (18.41) and establishes the second part of this exercise.

---

**Solution 18.10 (pp. 373): Physical interpretation of $\tilde{\lambda}$**

In general

$$\dot{\mathcal{P}}(k) = \frac{d_t \mathcal{P}(k)}{dt} = \frac{d_{\mathcal{O}_k^+} \mathcal{P}(k)}{dt} + \tilde{\nu}(\varphi(k)) \mathcal{P}(k) - \mathcal{P}(k) \tilde{\nu}(\varphi(k))$$

Comparing this expression with that of $\dot{\mathcal{P}}(k)$ in (18.50) establishes the equality for $\tilde{\lambda}(k)$ in (18.51). Also,

$$\tilde{\lambda}(k) = \frac{d_{\mathcal{O}_k^+} \mathcal{P}(k)}{dt} = \frac{d_{\mathcal{O}_k} \mathcal{P}(k)}{dt} + \tilde{\Delta}_V^\nu(k) \mathcal{P}(k) - \mathcal{P}(k) \tilde{\Delta}_V^\nu(k)$$

Comparing this expression with that for $\tilde{\lambda}(k)$ in (18.48) establishes the equality for $\dot{\mathcal{P}}(k)$ in (18.51). On a similar note,

$$\dot{\mathcal{P}}^+(k) = \frac{d_t \mathcal{P}^+(k)}{dt} = \frac{d_{\mathcal{O}_k^+} \mathcal{P}^+(k)}{dt} + \tilde{\nu}(\varphi(k)) \mathcal{P}^+(k) - \mathcal{P}^+(k) \tilde{\nu}(\varphi(k))$$

Comparing this expression with that for $\dot{\mathcal{P}}^+(k)$ in (18.50) establishes the equality for $\dot{\mathcal{P}}^+(k)$ in (18.51).

---

**Solution 18.11 (pp. 375): Time derivative of $(I + H\phi K)$**

We have

$$[I + H\phi K] = [I + H\phi \mathcal{G}] = [I - \mathcal{G} + H\phi \mathcal{G}] = H\phi \mathcal{G} \quad \text{(C.47)}$$
Thus,
\[
\frac{d[I + H\Phi K]}{dt} = \frac{dH\phi}{dt} [I + H\Phi \dot{K}] = \frac{dH\phi}{dt} [I + H\Phi \dot{K}] = H\phi \left[ (\tilde{\Delta}_\omega \phi - \tilde{\nabla}_\omega + \varepsilon_\phi \tilde{A}_V \phi) \frac{\partial}{\partial \theta} + \tau \lambda H^* D^{-1} + \tilde{\nabla}_\omega \phi \right]
\]

\textbf{Solution 18.12 (pp. 376): Time derivative of } [I + H\Phi K] \textbf{D}

We have
\[
\frac{d[I + H\Phi K]}{dt} = \frac{d[I + H\Phi K]}{dt} [I + H\Phi K] \frac{dD}{dt}
\]

\textbf{Solution 18.13 (pp. 376): Time derivative of } [I - H\Psi K]

Since [I - H\Psi K]^{-1} = [I + H\Phi K] it follows from (A.28) on page 402 that
\[
\frac{d[I - H\Psi K]}{dt} = -[I + H\Phi K] \frac{d[I - H\Psi K]}{dt} [I - H\Psi K]
\]

\textbf{Solution 18.14 (pp. 376): Sensitivities of } [I + H\Phi K] \textbf{ and } [I - H\Psi K]

The expressions are obtained by starting with the time derivative expressions in (18.61) and (18.63) and using the standard process for converting them into sensitivity expressions.
**Solution 18.15 (pp. 377): Sensitivity of \( \log \{ \det \{ \mathcal{M} \} \} \)**

1. From (9.53) on page 180, we have

\[
\det \{ \mathcal{M} \} = \prod_{k=1}^{n} \det \{ \mathcal{D}(k) \}
\]

Hence,

\[
\log \{ \det \{ \mathcal{M} \} \} = \sum_{k=1}^{n} \log \{ \det \{ \mathcal{D}(k) \} \}
\]  \hspace{1cm} \text{(C.48)}

Differentiating with respect to \( t \) we have

\[
\frac{d \log \{ \det \{ \mathcal{M} \} \}}{dt} \overset{\text{C.48}}{=} \frac{d}{dt} \sum_{k=1}^{n} \log \{ \det \{ \mathcal{D}(k) \} \} = \sum_{k=1}^{n} \frac{d \log \{ \det \{ \mathcal{D}(k) \} \}}{dt} = \sum_{k=1}^{n} \frac{d}{dt} \det \{ \mathcal{D}^{-1}(k) \} \dot{\mathcal{D}}(k)
\]

\[
= \text{Trace} \left\{ \mathcal{D}^{-1} \dot{\mathcal{D}} \right\} \overset{\text{18.43a}}{=} \text{Trace} \left\{ \mathcal{D}^{-1} \mathcal{H} \dot{\mathcal{H}}^* \right\}
\]

To establish the first expression in (18.66), differentiate both sides by \( \dot{\theta}_i \) and use (18.53). Use the component level expressions in (18.55) to establish the latter half of (18.66).

2. For (18.68), we have

\[
2 \text{Trace} \{ \mathcal{P} \Omega \mathcal{H}_i^\omega \} \overset{\text{A.22}}{=} 2 \text{Trace} \{ \mathcal{H}_i^\omega \mathcal{P} \Omega \}
\]

\[
\overset{\text{A.22}}{=} \text{Trace} \{ \mathcal{H}_i^\omega \mathcal{P} \Omega \} + \text{Trace} \{ [\mathcal{H}_i^\omega]^* \mathcal{P} \Omega \}
\]

\[
\overset{\text{A.22,A.21}}{=} \text{Trace} \{ \mathcal{H}_i^\omega \mathcal{P} \Omega \} - \text{Trace} \{ \mathcal{P} \mathcal{H}_i^\omega \}
\]

\[
= \text{Trace} \{ \mathcal{H}_i^\omega \mathcal{P} \Omega \} - \text{Trace} \{ \mathcal{P} \mathcal{H}_i^\omega \}
\]

\[
= \text{Trace} \{ [\mathcal{H}_i^\omega] [\mathcal{P} - \mathcal{P} \mathcal{H}_i^\omega] \Omega \}
\]

\[
= \text{Trace} \{ [\mathcal{H}_i^\omega] [\mathcal{P} - \mathcal{P} \mathcal{H}_i^\omega] \Omega \}
\]

\[
\overset{\text{18.54g}}{=} \text{Trace} \{ [\tilde{\lambda}_{\theta_i} - \mathcal{E}_\psi \tilde{\lambda}_{\theta_i} \mathcal{E}_\psi^* \mathcal{P} \Omega] \}
\]

\[
\overset{\text{10.12}}{=} \text{Trace} \{ [\tilde{\lambda}_{\theta_i} - \mathcal{E}_\psi \tilde{\lambda}_{\theta_i} \mathcal{E}_\psi^* \mathcal{P} \Omega] \mathcal{H}^* \mathcal{D}^{-1} \mathcal{H} \}
\]

\[
= \text{Trace} \{ [\mathcal{D}^{-1} \mathcal{H} \mathcal{E}_\psi \mathcal{D}^{-1} \mathcal{H} \mathcal{P} \Omega] \mathcal{H}^* \mathcal{D}^{-1} \mathcal{H} \}
\]

\[
= \text{Trace} \{ [\mathcal{D}^{-1} \mathcal{H} \mathcal{E}_\psi \mathcal{D}^{-1} \mathcal{H} \mathcal{P} \Omega] \mathcal{H}^* \mathcal{D}^{-1} \mathcal{H} \}
\]

The last equality above uses the zero block-diagonal elements property of \( \tilde{\psi} \). The last expression is the same as (18.66), and establishes the equivalency of (18.66) and (18.67).
3. For (18.68), we have

\[
\frac{d \log \{ \det \{ M \} \}}{dt} \overset{A.29}{=} \text{Trace} \{ M^{-1} \dot{M} \}
\]
\[
\overset{9.51,18.23}{=} \text{Trace} \left\{ [I - H\psi K]^{*} D^{-1} [I - H\psi K] H\phi \right\} \left[ (\tilde{\Delta}_V^{\omega} + \mathcal{E}_\phi \tilde{\Delta}_V^{v}) \phi M - M \phi^* (\tilde{\Delta}_V^{\omega} \mathcal{E}_\phi^* + \tilde{\Delta}_V^{\omega}) \right] \phi^* H^*
\]
\[
\overset{9.45}{=} \text{Trace} \left\{ D^{-1} H\psi \left[ (\tilde{\Delta}_V^{\omega} + \mathcal{E}_\phi \tilde{\Delta}_V^{v}) \phi M - M \phi^* (\tilde{\Delta}_V^{\omega} \mathcal{E}_\phi^* + \tilde{\Delta}_V^{\omega}) \right] \psi^* H^* \right\}
\]
\[
\overset{10.12}{=} 2 \text{Trace} \left\{ \left[ (\tilde{\Delta}_V^{\omega} + \mathcal{E}_\phi \tilde{\Delta}_V^{v}) \phi M - M \phi^* (\tilde{\Delta}_V^{\omega} \mathcal{E}_\phi^* + \tilde{\Delta}_V^{\omega}) \right] \Omega \right\}
\]
\[
\overset{10.38a}{=} 2 \text{Trace} \left\{ \tilde{\Delta}_V^{\omega} \phi M \Omega \right\} + 2 \text{Trace} \left\{ \mathcal{E}_\phi \tilde{\Delta}_V^{v} \phi M \Omega \right\}
\]
\[
\overset{10.38a}{=} 2 \text{Trace} \left\{ \tilde{\Delta}_V^{\omega} \mathcal{P} \Omega \right\} + 2 \text{Trace} \left\{ \mathcal{E}_\phi \tilde{\Delta}_V^{v} \phi M \Omega \right\}
\]

The last equality uses the zero diagonal elements property of \( \phi - \psi \).

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**Solutions for Chapter 19**

**Solution 19.1 (pp. 387):** Time derivative of \( D^{\frac{1}{2}} \)

We have

\[
D = \frac{d D^{\frac{1}{2}} \cdot D^{\frac{1}{2}}}{dt} = \frac{d D^{\frac{1}{2}}}{dt} D^{\frac{1}{2}} + D^{\frac{1}{2}} \frac{d D^{\frac{1}{2}}}{dt}
\]

For a system with 1 degree of freedom joints, the diagonal block elements of \( D^{\frac{1}{2}} \) are in fact scalar values, so that \( D^{1/2} \) and \( \frac{d D^{1/2}}{dt} \) commute. Thus, the above can be transformed into

\[
\frac{d D^{\frac{1}{2}}}{dt} = \frac{1}{2} D^{-\frac{1}{2}} D^{\frac{1}{2}} \overset{18.43a}{=} \frac{1}{2} D^{-\frac{1}{2}} H\lambda H^*
\]

---

**Solution 19.2 (pp. 391):** Non-working \( C(\theta, \eta) \) Coriolis Vector

\[
\eta^* C(\theta, \eta) \overset{19.29}{=} \eta^* D^{-\frac{1}{2}} H\psi \left[ \tilde{\Delta}_V^{\omega} H - \frac{1}{2} \left( \tilde{\Delta}_V^{\omega} \mathcal{P} + \mathcal{P} \tilde{\Delta}_V^{\omega} \right) \right] \mathcal{E}_\phi^* + \mathcal{E}_\phi \tilde{\lambda} \tilde{\lambda} \mathcal{E}_\psi^* + \mathcal{E}_\psi \tilde{\lambda} \mathcal{E}_\phi^* \right] \n\]
Observe that
\[
\eta^* \mathcal{D}^{-\frac{1}{2}} \mathbf{H} \psi 19.23 = \hat{\theta}^* \mathbf{H} [I + \mathbf{H} \phi] \mathbf{H} \psi 9.45 = \hat{\theta}^* \mathbf{H} \phi
\]

Thus,
\[
\eta^* \mathcal{C}(\theta, \eta) = \hat{\theta}^* \mathbf{H} \phi \left[ \nabla \mathbf{M} - \frac{1}{2} \left( \tilde{\Delta}_V^\omega \mathcal{P} + \mathcal{P} \tilde{\Delta}_V^\omega \right) + \varepsilon_\phi \left\{ \tilde{\Delta}_V^\omega \mathcal{P} + \mathcal{P} \tilde{\Delta}_V^\omega \right\} \varepsilon_\phi + \varepsilon_\psi \hat{\lambda} - \hat{\lambda} \varepsilon_\phi^* \right] \psi
\]
\[
\mathcal{C}(\theta, \eta) = \hat{\theta}^* \mathbf{H} \phi \left[ \nabla \mathbf{M} - \frac{1}{2} \left( \tilde{\Delta}_V^\omega \mathcal{P} + \mathcal{P} \tilde{\Delta}_V^\omega \right) + \varepsilon_\phi \left\{ \tilde{\Delta}_V^\omega \mathcal{P} + \mathcal{P} \tilde{\Delta}_V^\omega \right\} \varepsilon_\phi + \varepsilon_\psi \hat{\lambda} - \hat{\lambda} \varepsilon_\phi^* \right] \psi
\]

Since the matrix expression in the middle is skew-symmetric, the expression evaluates to zero.

**Solution 19.3 (pp. 392): Expression for \( \ell \)**

It follows from \( \ell^{-1} = \mathbf{m} \) and (A.28) on page 402 that
\[
\ell A.28 = -\ell \mathbf{m} \ell
\]
\[
= -\mathcal{D}^{-\frac{1}{2}} [I - \mathbf{H} \phi] \mathbf{H} \phi \left[ \tilde{\Delta}_V^\omega \phi \mathbf{K} + \left( I + \bar{\tau} \right) \hat{\lambda} \mathcal{D}^{-1} \right] [I - \mathbf{H} \phi] \mathbf{K}
\]

**Solution 19.4 (pp. 392): Sensitivity of Innovations factors**

The expressions are obtained by starting with the time derivative expressions in (19.25) and (19.33) and using the standard process for converting them into sensitivity expressions.
Solutions for Appendix A

Solution A.1 (pp. 397): Norm of $\tilde{s}$

We have

$$\|\tilde{s}x\|^2 = -x^*\tilde{s}sx = -x^*[ss^* - \|s\|^2I]x = \|s\|^2\|x\|^2 - (x^*s)^2$$

Clearly the above achieves a maximum of $\|s\|^2\|x\|^2$ when $x^*s = 0$. Hence, using (A.2) it follows that $\|\tilde{s}\| = \|s\|$.

---

Solution A.2 (pp. 401): Product gradient and chain rules

1. The $(k, j)$ element of $\nabla_x [f(x)g(x)]$ is

$$\nabla_x [f(x)g(x)](k, j) = \frac{\partial f_k \cdot g}{\partial x_j} = \frac{\partial f_k}{\partial x_j} \cdot g + f_k \cdot \frac{\partial g}{\partial x_j} \quad (C.49)$$

On the other hand, the $(k, j)$ element of $\nabla_x f(x) \cdot g(x) + f(x) \cdot \nabla_x g(x)$, the right-hand side of (A.25), is given by:

$$\left[\nabla_x f(x) \cdot g(x) + f(x) \cdot \nabla_x g(x)\right](k, j) = \frac{\partial f_k}{\partial x_j} \cdot g + f_k \cdot \frac{\partial g}{\partial x_j}$$

This is identical to the expression in (C.49). Thus, the component-level values on both sides of (A.25) agree and establish (A.25).

2. The $(k, j)$ element of $\nabla_x f(x)$ is

$$\left[\nabla_x f(x)\right](k, j) = \frac{\partial f_k}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial f_k}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_j} \quad (C.50)$$

On the other hand, the $(k, j)$ element of $\nabla_y f(y) \cdot \nabla_x y(x)$, the right-hand side of (A.26), is given by:

$$\left[\nabla_y f(y) \cdot \nabla_x y(x)\right](k, j) = \sum_{i=1}^{n} \left[\nabla_y f(y)\right](k, i) \cdot \left[\nabla_x y(x)\right](i, j) = \sum_{i=1}^{n} \frac{\partial f_k}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_j}$$

This is identical to the expression in (C.50). Thus, the component-level values on both sides of (A.26) agree and establish (A.26).
Solutions for Appendix B

Solution B.1 (pp. 403): Time derivative of a rotation matrix

Since rotation matrices are orthogonal, \( \dot{\mathbf{R}} \mathbf{R}^* = I \). Differentiating this with respect to time yields

\[
\dot{\mathbf{R}} \mathbf{R}^* + \mathbf{R} \dot{\mathbf{R}}^* = 0 \quad \Rightarrow \quad \dot{\mathbf{R}} \mathbf{R}^* = -[\dot{\mathbf{R}} \mathbf{R}^*]^*
\]

This implies that \( \dot{\mathbf{R}} \mathbf{R}^* \) is skew-symmetric, i.e., there exists a 3-vector \( \mathbf{w} \), such that,

\[
\dot{\mathbf{R}} \mathbf{R}^* = -\dot{\mathbf{w}}
\]

This establishes the result.

Solution B.2 (pp. 405): Derivation of the Euler–Rodrigues formula

1. While this can be established by direct verification using an arbitrary vector \( \mathbf{s} \) we use alternative derivation. Thus

\[
\tilde{\mathbf{s}}^3 \overset{\text{A.1}}{=} \tilde{\mathbf{s}} \left( s s^* - \|s\|^2 I \right) = -\sigma^2 \tilde{\mathbf{s}} \tag{C.51}
\]

We know that the characteristic polynomial of \( \tilde{\mathbf{s}} \) is a polynomial of order 3 and, that it is the unique polynomial of order 3 that \( \tilde{\mathbf{s}} \) satisfies as well. It follows therefore, from (C.51) that the characteristic polynomial of \( \tilde{\mathbf{s}} \) is \( \lambda^3 + \sigma^2 \lambda \).

2. We have that

\[
\mathbf{R}_B(\mathbf{n}, \theta) \overset{\text{B.4}}{=} \exp[\tilde{\mathbf{n}} \theta] = I + \tilde{\mathbf{n}} \theta + \frac{\tilde{\mathbf{n}}^2 \theta^2}{2!} + \frac{\tilde{\mathbf{n}}^3 \theta^3}{3!} + \ldots 
\]

\[
\overset{\text{C.51}}{=} I + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \tilde{\mathbf{n}} + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+2}}{(2k+2)!} \tilde{\mathbf{n}}^2 
\]

\[
= I + \sin(\theta) \tilde{\mathbf{n}} + \left[ 1 - \cos(\theta) \right] \tilde{\mathbf{n}}^2 
\]

\[
\overset{\text{A.1}}{=} \cos(\theta) I_3 + \left[ 1 - \cos(\theta) \right] \mathbf{n}^* + \sin(\theta) \tilde{\mathbf{n}}
\]

Solution B.3 (pp. 405): Trace and characteristic polynomial of a rotation matrix

1. The result follows by applying the trace operation to (B.6) on page 405 and noting that \( \text{Trace}\{1\} = 3, \text{Trace}\{\mathbf{n}^*\} = 1, \text{and Trace}\{\tilde{\mathbf{n}}\} = 0 \).

2. Since \( \mathbf{R} \) is a \( 3 \times 3 \) matrix, its characteristic polynomial is of the form \( \lambda^3 + a\lambda^2 + b\lambda + c \) for some constants \( a, b \) and \( c \). From matrix theory we know that \( -a \) is equal to the trace \( \gamma \), and \( -c \) is the determinant \( 1 \) of the matrix. That is the polynomial is of the form \( \lambda^3 - \gamma \lambda^2 + b\lambda - 1 \). Furthermore, we know that \( 1 \) is an eigen-value of \( \mathbf{R} \) and hence, must satisfy the characteristic polynomial. Using this fact implies that \( b = \gamma \), establishing the result.
3. Use the second expression from (B.6) to verify that

\[ \mathcal{R} n = n \]

This establishes that \( n \) is an eigen-vector with eigen-value 1.

4. Since \( \tilde{n}_\tilde{n} \tilde{n} \) is a symmetric matrix, from (B.6) we obtain

\[ \mathcal{R} - \mathcal{R}^* = 2 \sin(\theta) \tilde{n} \]

from which (B.9) follows.

---

**Solution B.4 (pp. 406): Angular velocity from the angle/axis rates**

Differentiating \( n^* n = 1 \), we obtain \( n^* \dot{n} = 0 \). Thus

\[ \tilde{n} \tilde{n} \tilde{n} = \tilde{n} [n n^* - n^* n I] = 0 \quad (C.52) \]

Also, it is easy to verify that

\[ \tilde{n} \tilde{n} \tilde{n} = -\tilde{n} \quad (C.53) \]

Using these cross product identities, we have,

\[ \tilde{\omega} = \mathcal{B} \mathcal{R}_I \mathcal{R}_{I_b} = [I_3 + (1 - \cos(\theta)) \tilde{n} \tilde{n} - \sin(\theta) \tilde{n}] \times \]

\[ \left[ (\sin(\theta) \tilde{n} \tilde{n} + \cos(\theta) \tilde{n}) \dot{\theta} + (\sin(\theta) + (1 - \cos(\theta)) \tilde{n}) \tilde{n} \right. \]

\[ + (1 - \cos(\theta)) \tilde{n} \tilde{n} \]

\[ = \left[ \sin(\theta) \tilde{n} \tilde{n} + \cos(\theta) \tilde{n} - \sin^2(\theta) \tilde{n} \tilde{n} \tilde{n} - \sin(\theta) \cos(\theta) \tilde{n} \tilde{n} \right. \]

\[ + \sin(\theta)(1 - \cos(\theta)) \tilde{n} \tilde{n} \tilde{n} + \cos(\theta)(1 - \cos(\theta)) \tilde{n} \tilde{n} \tilde{n} \]

\[ + \sin(\theta)(1 - \cos(\theta)) \tilde{n} \tilde{n} \tilde{n} + (1 - \cos(\theta))^2 \tilde{n} \tilde{n} \tilde{n} \tilde{n} + (1 - \cos(\theta))^2 \tilde{n} \tilde{n} \tilde{n} \]

\[ = \tilde{n} \dot{\theta} + \sin(\theta) \tilde{n} + (1 - \cos(\theta)) \tilde{n} \tilde{n} - \sin^2(\theta) \tilde{n} \tilde{n} \]

\[ - (1 - \cos(\theta))^2 \tilde{n} \tilde{n} + (1 - \cos(\theta))^2 \tilde{n} \tilde{n} \]

\[ = \tilde{n} \dot{\theta} + \sin(\theta) \tilde{n} - (1 - \cos(\theta)) \tilde{n} \tilde{n} \]

\[ = \tilde{n} \dot{\theta} + \sin(\theta) \tilde{n} - (1 - \cos(\theta)) \tilde{n} \tilde{n} \quad (C.54) \]

Hence

\[ \tilde{\omega} = n \dot{\theta} + \sin(\theta) \tilde{n} - (1 - \cos(\theta)) \tilde{n} \tilde{n} \]

This establishes (B.10).
Solution B.5 (pp. 406): Angle/axis rates from the angular velocity
We have

\[
\sin(\theta) - (1 - \cos \theta) \tilde{n}, \quad n \left[ \frac{1}{2} [ \tilde{n} - \cot(\theta/2) \tilde{n} \tilde{n} ] \right] n^* \]

\[
= \frac{1}{2} \left[ - (1 - \cos \theta) \tilde{n} \tilde{n} + (1 - \cos \theta) \cot(\theta/2) \tilde{n} \tilde{n} + \sin(\theta) \tilde{n} - \sin(\theta) \cot(\theta/2) \tilde{n} \tilde{n} \right] + nn^*
\]

(C.55)

However,

\[
\tilde{n} \tilde{n} = nn^* - I, \quad \tilde{n} \tilde{n} \tilde{n} = -\tilde{n}
\]

\[
\sin(\theta) \cot(\theta/2) = 2 \cos^2(\theta/2), \quad (1 - \cos \theta) \cot(\theta/2) = \sin(\theta/2)
\]

Using these in (C.55) leads to

\[
\sin(\theta) - (1 - \cos \theta) \tilde{n}, \quad n \left[ \frac{1}{2} [ \tilde{n} - \cot(\theta/2) \tilde{n} \tilde{n} ] \right] n^* \]

\[
= \frac{1}{2} \left[ - (1 - \cos \theta) \tilde{n} \tilde{n} - \sin \theta) \tilde{n} + \sin(\theta) \tilde{n} - (1 + \cos(\theta/2)) \tilde{n} \tilde{n} \right] + nn^*
\]

\[
= -\tilde{n} \tilde{n} + nn^* = I
\]

This implies that expressions in (B.11) and (B.10) are indeed inverses of each other, establishing the result.

Solution B.6 (pp. 407): Quaternion expression for a rotation matrix
Using \( \cos(\theta) = 2 \cos^2(\theta/2) - 1 = 1 - 2 \sin^2(\theta/2) \), \( \sin(\theta) = 2 \sin(\theta/2) \cos(\theta/2) \), and (B.12) in (B.6), we obtain

\[
\mathcal{I}_B(n, \theta) = \cos(\theta) I_3 + [1 - \cos(\theta)] nn^* + \sin(\theta) \tilde{n} \]

\[
\overset{\text{B.12}}{=} (2q_0^2 - 1)I_3 - qq^* + 2q_0 \tilde{q} \quad \overset{\text{B.12}}{=} (q_0^2 - q^* q)I_3 - qq^* + 2q_0 \tilde{q}
\]

This establishes the first equality in (B.14). The remaining equalities follow from using, and manipulating, 3-vector cross-product identities.
Solution B.7 (pp. 407): Basic properties of unit quaternions

1. We have

\[ \mathcal{R}(\tilde{q})\mathcal{R}^*(\tilde{q}) = \begin{bmatrix} I + 2\tilde{q}^2 & 0 \\ 0 & I \end{bmatrix} - 4\tilde{q}_0 \tilde{q}^2 \]

= \[ I + 4\tilde{q}^2 + 4\tilde{q}^4 - 4\tilde{q}_0^2 \tilde{q}^2 \]

= \[ I + 4\tilde{q}^2(I + \tilde{q}^2 - \tilde{q}_0^2 I) \]

\[ \stackrel{A.1}{=} I + 4\tilde{q}^2(q^* q I + q q^* - q^* q I) = I \]

2. It is easy to verify using (B.14) that \( \mathcal{R}(\tilde{q}) = \mathcal{R}(-\tilde{q}) \).

3. Once again it is easy to check from (B.14) that \( \mathcal{R}(q_0, -\tilde{q}) = \mathcal{R}^*(q_0, q) \). Since the inverse of a rotation matrix is its transpose, the result follows.

4. We have

\[ \text{Trace}\{\mathcal{R}(\tilde{q})\} = 1 + 2\cos(\theta) = 1 + 4\cos^2(\theta/2) - 2 \stackrel{B.12}{=} 4\tilde{q}_0^2 - 1 \]

establishing (B.16).

5. Exercise B.3 established that \( \mathbf{n} \) is an eigen-vector of \( \mathcal{R} \) with eigen-value 1. Since \( q = \sin(\theta/2)\mathbf{n}, q \) is also an eigen-vector of \( \mathcal{R} \) with eigen-value 1.

6. From (B.9) we obtain

\[ \tilde{q} \stackrel{B.12}{=} \sin(\theta/2)(\mathcal{R} - \mathcal{R}^*)/(2\sin(\theta)) = (\mathcal{R} - \mathcal{R}^*)/(4\cos(\theta/2)) \]

\[ \stackrel{B.12}{=} (\mathcal{R} - \mathcal{R}^*)/(4\tilde{q}_0) \]

This establishes (B.18).

Solution B.8 (pp. 409): Properties of \( \mathbf{E}_- \) and \( \mathbf{E}_+ \) matrices

1. The identities in (B.20) are established by evaluating and simplifying the various products.

2. Similarly, the identities in (B.21) are established by evaluating and simplifying the various products.

3. These identities are easily established by examining (B.19).

4. We have

\[ \mathbf{E}_+(p)q = \begin{bmatrix} p_0 q + pq_0 + \tilde{p}q \\ p_0 q_0 - p^* q \end{bmatrix} = \begin{bmatrix} q_0 I - \tilde{q} \\ -q^* \end{bmatrix} \begin{bmatrix} p \\ p_0 \end{bmatrix} = \mathbf{E}_-(q)p \]

establishing (B.23).
5. We have

\[ E_-(q)E_+(p) = \begin{pmatrix} q_0I_3 - \tilde{q} & q \\ -q^* & q_0 \end{pmatrix} \begin{pmatrix} p_0I_3 + \tilde{p} \\ -p^* \\ p_0 \end{pmatrix} \]

\[ = \begin{pmatrix} (q_0I_3 - \tilde{q})(p_0I_3 + \tilde{p}) - qp^* \\ -q^*(p_0I_3 + \tilde{p}) + q_0p^* \\ q_0p_0I_3 - \tilde{q}p_0 + q_0\tilde{p} - \tilde{q}p - qp^* \\ -q^*p_0 - q^*\tilde{p} - q_0p^* \end{pmatrix} \begin{pmatrix} (q_0I_3 - \tilde{q})p + q_0p_0 \\ -q^*p + q_0p_0 \end{pmatrix} \]

Similarly,

\[ E_+(p)E_-(q) = \begin{pmatrix} p_0I_3 + \tilde{p} \\ -p^* \\ p_0 \end{pmatrix} \begin{pmatrix} q_0I_3 - \tilde{q} \\ -q^* \\ q_0 \end{pmatrix} \]

\[ = \begin{pmatrix} q_0p_0I_3 - \tilde{q}p_0 + q_0\tilde{p} - \tilde{q}p - qp^* \\ -q^*p_0 + p^*\tilde{q} - q_0p^* \\ -q^*p + q_0p_0 \end{pmatrix} \begin{pmatrix} q_0p - \tilde{q}p + q_0p_0 \end{pmatrix} \]

Comparing the matrix terms in this expression with those in (C.56) shows that they are the same. This establishes (B.24a).

A similar explicit evaluation process establishes (B.24b). (B.24c) is simply the transpose of (B.24b), while (B.24d) is the transpose of (B.24a).

6. Since \( q \) is an eigen-vector of \( \mathfrak{R}(q, q_0) \) from (B.17), (B.25) follows directly from the expression for \( \mathbb{T}(q) \) in (B.20).

---

Solution B.9 (pp. 409): Composition of unit quaternions

1. The (B.27) follows from direct evaluation of the product in (B.26).

The unit norm property of \( r = p \otimes q \) follows from

\[ \|r\|^2 = r^*r = q^*E^*_+(p)E_+(p)q = B.20 = q^*q = 1 \]

2. Define \( r \triangleq p \otimes q = (r, r_0) \). The (B.28) identity can be established by the brute force use of (B.14) on both sides of the equation and a series of algebraic manipulations. We however, use a simpler approach, and show that \( r \) defined by (B.27) is an eigen-vector of \( \mathbb{T}(p)\mathbb{T}(q) \) with eigenvalue 1 as required by (B.25).
\[ T(p)T(q) = \begin{pmatrix} R(p)R(q) & 0 \\ 0 & 1 \end{pmatrix} B.21 = E_+(p)E_-(p)E_+(q)E_-(q) \]

Thus,

\[ T(p)T(q)r B.26 = E + (p)E(r)q - (p)E + (q)E(r)q - (q)E(r)q \]

This establishes that \( r \) is an eigen-vector corresponding to the eigen-value of 1, and hence it is the quaternion corresponding to the \( T(p)T(q) \) rotational transformation and verifies \( (B.28) \).

3. Now

\[ p \otimes (q \otimes r) B.26 = E_+(p)E_-(r)q B.24a = E_-(r)E_+(p)q \]

\[ = E_-(r)(p \otimes q) B.26 = (p \otimes q) \otimes r \]

4. We have

\[ p^{-1} \otimes q B.26 = E_+(p^{-1})q B.22 = E^+(p)q \]

establishing the first half of \( (B.30) \). For the latter half,

\[ p \otimes q^{-1} B.26 = E_-(q^{-1})p B.22 = E^-(q)p \]

Solution B.10 (pp. 409): The quaternion identity element

It is easy to verify that \( E_+(e_q) = E_-(e_q) = I_4 \). Thus

\[ q \otimes e_q B.26 = E_-(e_q)q = q \]

The latter identity in \( (B.31) \) can be established by explicitly evaluating \( E_+(q)q^{-1} \) and verifying that it is simply \( e_q \).

Solution B.11 (pp. 410): Unit quaternion products and inverses

1. Now

\[ (p \otimes q) \otimes (q^{-1} \otimes p^{-1}) B.29 = p \otimes (q \otimes q^{-1}) \otimes p^{-1} B.31 = p \otimes p^{-1} = e_q \]

This establishes that \( (q^{-1} \otimes p^{-1}) \) is the inverse of \( p \otimes q \).
2. Now \( \mathbf{r} \overset{\text{B.26}}{=} E_+ (\mathbf{p}) \mathbf{q} \). Thus

\[
\mathbf{q} \overset{\text{B.20}}{=} E_+^* (\mathbf{p}) \mathbf{r} \overset{\text{B.22}}{=} E_+ (\mathbf{p}^{-1}) \mathbf{r} \overset{\text{B.26}}{=} \mathbf{p}^{-1} \otimes \mathbf{r}
\]

A similar derivation leads to \( \mathbf{p} = \mathbf{r} \otimes \mathbf{q}^{-1} \).

Solution B.12 (pp. 410): Transforming vectors with unit quaternions
We have

\[
\begin{bmatrix}
^I \mathbf{x} \\
0
\end{bmatrix} = \begin{bmatrix}
^I \mathbf{R}_B (\mathbf{q}) & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
^B \mathbf{x} \\
0
\end{bmatrix} \overset{\text{B.21}}{=} E_+ (\mathbf{q}) E_+^* (\mathbf{q}) \begin{bmatrix}
^B \mathbf{x} \\
0
\end{bmatrix} \overset{\text{B.22}}{=} E_+ (\mathbf{q}) E_- (\mathbf{q}^{-1}) \begin{bmatrix}
^B \mathbf{x} \\
0
\end{bmatrix} \overset{\text{B.26}}{=} \mathbf{q} \otimes \begin{bmatrix}
^B \mathbf{x} \\
0
\end{bmatrix} \otimes \mathbf{q}^{-1}
\]

Solution B.13 (pp. 410): Quaternion rates from the angular velocity
With \( q_0 = \cos (\theta / 2) \),

\[
\dot{q}_0 = -\frac{1}{2} \sin (\theta / 2) \dot{\theta} \overset{\text{B.11}}{=} -\frac{1}{2} \sin (\theta / 2) \mathbf{n}^* \mathbf{B} \omega = -\frac{1}{2} q^* \mathbf{B} \omega
\]

This establishes the lower half of the first equality in (B.34).

Also, with \( q = \sin (\theta / 2) \mathbf{n} \),

\[
\dot{q} = -\frac{1}{2} \cos (\theta / 2) \dot{\theta} \mathbf{n} + \sin (\theta / 2) \dot{\mathbf{n}}
\]

\[
\overset{\text{B.11}}{=} \frac{1}{2} q_0 \mathbf{n} \mathbf{n}^* \mathbf{B} \omega + \frac{\sin (\theta / 2)}{2} \left[ \mathbf{n} - \cot (\theta / 2) \mathbf{n} \mathbf{n}^* \mathbf{B} \mathbf{B} \right] \mathbf{B} \omega
\]

\[
= \frac{1}{2} [q_0 \mathbf{n} \mathbf{n}^* + \mathbf{n} \mathbf{n} - \cos (\theta / 2) \mathbf{n} \mathbf{n}^* \mathbf{B} \mathbf{B} \mathbf{B} \mathbf{B} \omega
\]

\[
= \frac{1}{2} [q_0 \mathbf{n} \mathbf{n}^* + \mathbf{n} \mathbf{n} - q_0 \mathbf{n} \mathbf{n}^* \mathbf{B} \mathbf{B} \mathbf{B} \mathbf{B} \omega
\]

\[
= \frac{1}{2} [q_0 (\mathbf{n} \mathbf{n}^* - \mathbf{n} \mathbf{n}^*) + \mathbf{n} \mathbf{n}^* \mathbf{B} \mathbf{B} \mathbf{B} \mathbf{B} \omega
\]

This establishes the upper half of the first equality in (B.34).

The second equality in (B.34) is a rearrangement of the first one.

The third equality follows by using the definition of \( E_+ (\mathbf{q}) \) from (B.19).

The last equality follows from (B.26).
Solution B.14 (pp. 410): Constant unit quaternion norms

1. The rate of change of the norm of $q(t)$ is given by

$$\frac{d}{dt} \|q(t)\|^2 = \frac{1}{2} \frac{dq^*q}{dt} = \frac{1}{2} q^* \dot{q} \overset{B.34}{=} \frac{1}{2} q^* E_+ (q) \begin{bmatrix} B \omega \\ 0 \end{bmatrix} \overset{B.22,B.26}{=} \frac{1}{2} e^*_q \begin{bmatrix} B \omega \\ 0 \end{bmatrix} = 0$$

2. From the above we have that

$$0 = q^* \dot{q} = q^* \ddot{q} + q_0 \dot{q}_0$$

from which (B.35) follows.

Solution B.15 (pp. 411): Unit quaternion rate to angular velocity

1. (B.36) follows by using (B.32) in (B.34). From this it follows that

$$\overset{B}{\omega} \overset{B.19}{=} 2 [q_0 I_3 - \tilde{q}] \dot{q} - q \dot{q}_0 \overset{B.35}{=} 2 [q_0 I_3 - \tilde{q}] \dot{q} + q (q^* \ddot{q} / q_0)$$

from which (B.37) follows.

2. We have

$$\begin{bmatrix} \overset{\mathbb{I}}{\omega} \\ 0 \end{bmatrix} \overset{B.33}{=} q \otimes \begin{bmatrix} \overset{B}{\omega} \\ 0 \end{bmatrix} \otimes q^{-1} \overset{B.36}{=} 2 \dot{q} \otimes q^{-1} \overset{B.30}{=} 2 E^* \dot{q}$$

establishing (B.39). From this it follows that

$$\overset{\mathbb{I}}{\omega} \overset{B.19}{=} 2 [q_0 I_3 + \tilde{q}] \dot{q} - q \dot{q}_0 \overset{B.35}{=} 2 [q_0 I_3 + \tilde{q}] \dot{q} + q (q^* \ddot{q} / q_0)$$

establishing (B.41).

Solution B.16 (pp. 411): Quaternion double time derivatives

Differentiating (B.34) we have

$$\ddot{q} = \frac{1}{2} q \otimes \begin{bmatrix} \overset{B}{\omega} \\ 0 \end{bmatrix} + \frac{1}{2} \dot{q} \otimes \begin{bmatrix} \overset{B}{\omega} \\ 0 \end{bmatrix} \overset{B.34}{=} \frac{1}{2} q \otimes \begin{bmatrix} \overset{B}{\dot{\omega}} \\ 0 \end{bmatrix} + \frac{1}{2} \left\{ \frac{1}{2} q \otimes \begin{bmatrix} \overset{B}{\omega} \\ 0 \end{bmatrix} \right\} \otimes \begin{bmatrix} \overset{B}{\omega} \\ 0 \end{bmatrix}$$

$$\overset{B.27}{=} \frac{1}{2} q \otimes \begin{bmatrix} \overset{B}{\omega} \\ 0 \end{bmatrix} + \frac{1}{4} q \otimes (\| \overset{B}{\omega} \|^2 e_q)$$

establishing (B.42).

(B.43) follows by composing both sides of (B.42) with $q^{-1}$ and rearranging terms.
**Solution B.17 (pp. 412): Gibbs vector attitude representation**

1. From Exercise (B.2) on page 405 it follows that

\[
\tilde{s}^3 = -\sigma^2 \tilde{s}, \quad \tilde{s}^5 = \sigma^4 \tilde{s}, \quad \ldots \quad \tilde{s}^{2k+1} = (-1)^k \sigma^{2k} \tilde{s}
\]

Thus,

\[
\tilde{s}^4 = -\sigma^2 \tilde{s}^2, \quad \tilde{s}^6 = \sigma^4 \tilde{s}^2, \quad \ldots \quad \tilde{s}^{2k+2} = (-1)^k \sigma^{2k} \tilde{s}^2
\]

Using these expressions for the powers of \(\tilde{s}\), we have

\[
[I - \tilde{s}]^{-1} = I + \tilde{s} + \tilde{s}^2 + \ldots = I + \sum_{k=0}^{\infty} (-1)^k \sigma^{2k} (\tilde{s}^2 + \tilde{s})^k
\]

\[
= I + (\tilde{s} + \tilde{s}^2) \sum_{k=0}^{\infty} (-1)^k \sigma^{2k} = I + (\tilde{s}^2 + \tilde{s})/(1 + \sigma^2)
\]

2.

\[
\mathcal{R}(s) \overset{B.44}{=} (I + \tilde{s}) + \tilde{s}(I + \tilde{s})^2/(1 + \sigma^2) = I + (\tilde{s}^3 + 2\tilde{s}^2 + (2 + \sigma^2)\tilde{s})/(1 + \sigma^2)
\]

\[
\overset{B.5}{=} I + 2(\tilde{s}^2 + \tilde{s})/(1 + \sigma^2)
\]

To establish that \(\mathcal{R}(s)\) is a rotation matrix it suffices to verify that \(\mathcal{R}(s)\mathcal{R}^*(s) = I\) and that \(\det\mathcal{R}(s) = 1\). Since

\[
I - \tilde{s}^2 = [I - \tilde{s}][I + \tilde{s}] = [I + \tilde{s}][I - \tilde{s}]
\]

it follows that

\[
\mathcal{R}(s) \overset{B.45}{=} [I - \tilde{s}]^{-1}[I + \tilde{s}] = [I + \tilde{s}][I - \tilde{s}]^{-1}
\]

and that

\[
\mathcal{R}^*(s) \overset{B.45}{=} [I + \tilde{s}]^{-1}[I - \tilde{s}]
\]

Thus,

\[
\mathcal{R}(s)\mathcal{R}^*(s) = [I - \tilde{s}]^{-1}[I + \tilde{s}][I + \tilde{s}]^{-1}[I - \tilde{s}] = I
\]

establishing the orthogonality of \(\mathcal{R}(s)\). Furthermore,

\[
d \triangleq \det[I - \tilde{s}] = \det[I - \tilde{s}]^* = \det[I + \tilde{s}]
\]

Hence, \(\det[I - \tilde{s}]^{-1} = 1/d\). Putting these together we have

\[
\det\mathcal{R}(s) = \det[I - \tilde{s}]^{-1} \det[I + \tilde{s}] = (1/d) d = 1
\]

This establishes that \(\mathcal{R}(s)\) is a rotation matrix since it is orthogonal and has determinant 1.
3. From (B.45) it follows that $\mathbf{R} - \mathbf{R} \tilde{s} = I + \tilde{s}$ from which the first half of (B.46) follows. Now let us assume that for some constants $a$, $b$ and $c$ to be determined, we have

$$-[I - \mathbf{R}][I + \mathbf{R}]^{-1} = a\mathbf{R}^2 + b\mathbf{R} + c\mathbf{I}$$

Then it follows that

$$-\mathbf{I} + \mathbf{R} = [I + \mathbf{R}][a\mathbf{R}^2 + b\mathbf{R} + c\mathbf{I}]$$

$$= a\mathbf{R}^3 + (a + b)\mathbf{R}^2 + (b + c)\mathbf{R} + c\mathbf{I}$$

$$\overset{B.8}{=} ((1 + \gamma)a + b)\mathbf{R}^2 + (-a\gamma + b + c)\mathbf{R} + (a + c)\mathbf{I}$$

For this to hold, the coefficients must satisfy the following linear equations:

$$(1 + \gamma)a + b = 0, \quad -a\gamma + b + c = 1, \quad a + c = -1$$

The solution to these equations is

$$a = -1/(1 + \gamma), \quad b = 1, \quad c = -\gamma/(1 + \gamma)$$

which establishes the result.

4. For $\mathbf{s}$ to the axis of rotation, we need to show that it an eigen-vector of $\mathbf{R}(\mathbf{s})$ with eigen-value 1. We have

$$\mathbf{R}(\mathbf{s})\mathbf{s} \overset{B.45}{=} \left[I + 2(\tilde{s}^2 + \tilde{s})/(1 + \sigma^2)\right] = \mathbf{s}$$

This establishes the result.

5. With $\mathbf{s}$ defined by (B.47), we have $\sigma = \tan(\theta/2)$. Thus,

$$\mathbf{R}(\mathbf{s}) \overset{B.45}{=} \mathbf{I} + 2(\sigma^2 \tilde{n}^2 + \sigma \tilde{n})/(1 + \sigma^2)$$

$$= \mathbf{I} + 2\cos^2(\theta/2)(\tan^2(\theta/2) \tilde{n}^2 + \tan(\theta/2) \tilde{n})$$

$$= \mathbf{I} + (1 - \cos(\theta))\tilde{n}^2 + \sin(\theta)\tilde{n}$$

The last expression agrees with the expression for a rotation matrix in (B.6) establishing the result.

---

**Solution B.18 (pp. 413): Rotation of vectors**

1. Since $\mathbf{R}(\mathbf{s})\mathbf{a} = \mathbf{b}$,

$$\mathbf{b} \overset{B.45}{=} \left[I - \tilde{s}\right]^{-1}[I + \tilde{s}]\mathbf{a} \implies [I - \tilde{s}]\mathbf{b} = [I + \tilde{s}]\mathbf{a}$$

from which (B.48) follows.

2. Since any vector $\mathbf{s}$ satisfying (B.48) will generate a rotation matrix $\mathbf{R}(\mathbf{s})$ with the desired properties, we look for the general solution to the linear matrix equation
in (B.48). We now verify that $s_p \triangleq \lambda (\overline{a-b})(a+b)$ is a particular solution to the linear equations. Since $a$ and $b$ have the same norm, the $(a+b)$ and $(a-b)$ vectors are mutually orthogonal because
\[
(a+b)^*(a-b) = a^*a - a^*b + b^*a - b^*b = 0
\]

Now
\[
(\overline{a+b})s_p = -\lambda (\overline{a+b})(\overline{a+b})(a-b)
\]
\[
\overset{A.1}{=} -\lambda [(a+b)(a+b)^* - \|a+b\|^2 \mathbf{I}] (a-b) = (a-b)
\]

Thus, $s_p$ is indeed a particular solution for (B.48). To determine all the remaining solutions observe that the class of homogeneous solutions for (B.48) is simply $s = \alpha (a+b)$ for an arbitrary scalar $\alpha$. Combining the particular and homogeneous solution leads to the general solution in (B.49).
References


References


## List of Notation

### Symbols

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<td>(X^\omega)</td>
<td>the spatial vector with just the angular component of the spatial vector</td>
<td>9</td>
</tr>
<tr>
<td>(X^v)</td>
<td>the spatial vector with just the linear component of the spatial vector</td>
<td>9</td>
</tr>
<tr>
<td>(X_S)</td>
<td>(= \text{col}\left{X(\varphi(k))\right}_{k=1}^n) – a shifted version of a (X) stacked vector with the (k)th slot being occupied by the (X(\varphi(k))) element</td>
<td>353</td>
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<tr>
<td>([ij,k])</td>
<td>Christoffel symbols of the first kind</td>
<td>381</td>
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<tr>
<td>({k}_{ij})</td>
<td>Christoffel symbols of the second kind</td>
<td>381</td>
</tr>
<tr>
<td>(\text{col}\left{x(i)\right}_{i=m})</td>
<td>the stacked vector consisting of the (x(i)) elements for bodies (m) through (p) in the multibody system</td>
<td>48</td>
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<tr>
<td>(F_X)</td>
<td>representation of vector (x) in frame (F)</td>
<td>4</td>
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<tr>
<td>(I)</td>
<td>the identity matrix of appropriate dimension</td>
<td>12</td>
</tr>
<tr>
<td>(\mathbb{1}_{\text{set}})</td>
<td>indicator function that returns a 1 if the element belongs to the set, and 0 otherwise</td>
<td>138</td>
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<tr>
<td>(\nabla_{\theta}f(\theta))</td>
<td>the gradient of a the vector-valued function (f) with respect to the (\theta) vector</td>
<td>401</td>
</tr>
<tr>
<td>(\hat{z})</td>
<td>an operator related to the cross-product operator for spatial vectors</td>
<td>10</td>
</tr>
<tr>
<td>(\hat{l})</td>
<td>cross product operator (l \otimes [\cdot])</td>
<td>6</td>
</tr>
<tr>
<td>(\tilde{X})</td>
<td>(= \text{diag}\left{X(k)\right}_{k=1}^n) – a block-diagonal matrix with (X(k)) diagonal elements from the (X) stacked vector</td>
<td>353</td>
</tr>
<tr>
<td>({l}^{-})</td>
<td>same as (\hat{l})</td>
<td>6</td>
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<tr>
<td>(\overline{X})</td>
<td>(= \text{diag}\left{\overline{X}(k)\right}_{k=1}^n) – block-diagonal matrix with (\overline{X}(k)) diagonal elements from the (X) stacked vector</td>
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</tr>
<tr>
<td>(\overline{x})</td>
<td>(= -{\overline{x}}^\ast) – cross-product related operator for spatial vectors</td>
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0 the zero matrix of appropriate dimension 9
i \not\in j node \( j \) is not the ancestor of node \( i \) in a digraph 136
i \prec j node \( j \) is the ancestor of node \( i \) in a digraph 136
j \not\succ i node \( j \) is not the ancestor of node \( i \) in a digraph 136
j \succ i node \( j \) is the ancestor of node \( i \) in a digraph 136

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<td>the rigid body generalized velocity coordinates defined as the inertial frame representation of the body’s spatial velocity</td>
</tr>
<tr>
<td>$\beta^I_V$</td>
<td>the rigid body generalized velocity coordinates defined as the inertially referenced spatial velocity of the body</td>
</tr>
<tr>
<td>$b_I$</td>
<td>the gyroscopic spatial force associated with the equations of motion about an inertially fixed velocity reference point</td>
</tr>
<tr>
<td>$\mathbb{B}$</td>
<td>a body-fixed coordinate frame</td>
</tr>
<tr>
<td>$\mathbb{B}_k$</td>
<td>the reference frame for the $k$th link</td>
</tr>
<tr>
<td>$\mathcal{B} \in \mathbb{R}^{6n \times 6n}$</td>
<td>the pick-off operator for body nodes</td>
</tr>
<tr>
<td>$\mathcal{B}_C$</td>
<td>the connector block whose non-zero elements define the parent/child connectivity between links in the $\mathcal{S}$ sub-graph and their children in the $\mathcal{C}$ child sub-graph</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>the set of immediate child nodes of the $k$th node in a digraph</td>
</tr>
<tr>
<td>$\mathcal{C}(k)$</td>
<td>the induced sub-graph consisting of the descendant nodes of the $\mathcal{S}$ sub-graph</td>
</tr>
<tr>
<td>$\mathcal{C}(\theta, \dot{\theta})$</td>
<td>the system-level Coriolis and gyroscopic forces vector</td>
</tr>
<tr>
<td>$\mathcal{C}_i(j, k)$</td>
<td>Christoffel symbol of the first kind</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>the location of the center of mass for a rigid body</td>
</tr>
<tr>
<td>$D$</td>
<td>derivative of vector $x(s)$ with respect to $s$ in frame $F$</td>
</tr>
<tr>
<td>$\mathcal{D}(k)$</td>
<td>$H(k)P(k)H^*(k)$ – the articulated body hinge inertia for the $k$th body</td>
</tr>
<tr>
<td>$\Delta_{\mathbb{B}/\mathcal{O}} = \text{diag} \left{ \phi(\mathbb{B}_k, \mathcal{O}_k) \right}$</td>
<td>the $\mathbb{B}_k$ body frame to the $\mathcal{O}_k$ transformation operator</td>
</tr>
<tr>
<td>$\mathcal{V}_{\mathbb{B}/\mathcal{O}}(k)$</td>
<td>the relative spatial velocity of the $\mathcal{O}_k$ frame with respect to the $\mathbb{B}_k$ frame on the $k$th body</td>
</tr>
<tr>
<td>$\Delta_{\mathcal{V}}^\omega = \text{col} \left{ \Delta_{\mathcal{V}}^\omega(k) \right}_{k=1}^{n}$</td>
<td>the stacked vector of angular relative hinge spatial velocities for all the links</td>
</tr>
<tr>
<td>$\Delta_{\mathcal{V}}^\nu = \text{col} \left{ \Delta_{\mathcal{V}}^\nu(k) \right}_{k=1}^{n}$</td>
<td>the stacked vector of linear relative hinge spatial velocities for all the links</td>
</tr>
<tr>
<td>$\Delta_{\mathcal{V}}(k)$</td>
<td>the relative linear velocity across the $k$th hinge</td>
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<tr>
<td>$\Delta_{\mathcal{V}}(k)$</td>
<td>relative spatial velocity across the $k$th hinge</td>
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<tr>
<td>$\Delta_{\mathcal{V}}(k)$</td>
<td>the relative angular velocity across the $k$th hinge</td>
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<tr>
<td>$\Delta_{\mathcal{V}}^\omega(k)$</td>
<td>the hinge relative spatial velocity $\Delta_{\mathcal{V}}(k)$ referenced to the $k$th link body frame $\mathbb{B}_k$</td>
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<tr>
<td>$\Delta^i_{\mathcal{V}}(k)$</td>
<td>the spatial velocity across the $k$th hinge in the inertially referenced formulation of the equations of motion</td>
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<td><strong>E</strong></td>
<td>the articulated body inertia innovations generalized force</td>
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<tr>
<td>$\epsilon(k)$</td>
<td>$\in \mathbb{R}^{N \times m_k}$ — a $n$ elements block-vector with all zero elements except for the $k$th element which is the identity matrix</td>
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<tr>
<td>$\mathcal{E}_\phi$</td>
<td>the SKO operator for rigid-link multibody systems</td>
</tr>
<tr>
<td>$\mathcal{E}_{\phi_B}$</td>
<td>$= \Delta_{\mathbb{B}/\mathbb{O}} \mathcal{E}<em>{\phi} \Delta</em>{\mathbb{B}/\mathbb{O}}^{-1}$ — the SKO operator for rigid-link multibody systems for the case when $\mathbb{B}_k \neq \mathbb{O}_k$</td>
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<tr>
<td>$\mathcal{E}_{\Lambda_c}$</td>
<td>the SKO operator for the $c$ child sub-graph</td>
</tr>
<tr>
<td>$\mathcal{E}_{\Lambda_p}$</td>
<td>the SKO operator for the $p$ parent sub-graph</td>
</tr>
<tr>
<td>$\mathcal{E}_{\Lambda_S}$</td>
<td>the SKO operator for the $S$ sub-graph</td>
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<td>$\mathcal{E}_{\Lambda_A}$</td>
<td>the SKO operator for the aggregated tree</td>
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<tr>
<td>$\mathcal{E}_{\Lambda}$</td>
<td>a generic SKO operator</td>
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<tr>
<td>$\mathcal{E}_E$</td>
<td>a transformed version of the $E$ base pick-off operator</td>
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<tr>
<td>$\mathcal{E}_A$</td>
<td>the base pick-off operator</td>
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<tr>
<td>$\mathcal{E}_S$</td>
<td>the connector block whose non-zero elements define the parent/child connectivity between links in $\mathcal{S}$ and their parents in the $P$ sub-graph</td>
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<tr>
<td>$\mathcal{E}_\psi$</td>
<td>$= \mathcal{E}_\phi \overline{\mathcal{E}}$ — the articulated body SKO operator</td>
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<th>Symbol</th>
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<td>$f(\mathbb{F})$</td>
<td>spatial force at the $\mathbb{F}$ frame</td>
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<tr>
<td>$f(k)$</td>
<td>the spatial force of interaction between the $(k+1)$th and the $k$th links</td>
</tr>
<tr>
<td>$f_c$</td>
<td>the constraint spatial forces being applied at nodes on the multibody system</td>
</tr>
<tr>
<td>$f_{i_c}(k)$</td>
<td>the constraint spatial force at the $i$th node on the $k$th body</td>
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<tr>
<td>$f_{ext}$</td>
<td>the stacked vector of external spatial forces on the system</td>
</tr>
<tr>
<td>$f_{ext}(k)$</td>
<td>the external spatial forces on the $i$th node on the $k$th body</td>
</tr>
<tr>
<td>$f_I$</td>
<td>the equivalent spatial force on a rigid body associated with an inertially fixed velocity reference point</td>
</tr>
<tr>
<td>$f_{II}(k)$</td>
<td>$\in \mathbb{R}^6$ — the spatial force of interaction between the $(k+1)$th and the $k$th link referred to frame $\mathbb{I}$</td>
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<th>Symbol</th>
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<tr>
<td>$g$</td>
<td>the gravity spatial acceleration vector</td>
</tr>
<tr>
<td>$g_l$</td>
<td>the gravity linear acceleration vector</td>
</tr>
<tr>
<td>$g(k) \mathbb{P}(k) \mathbb{H}^+(k) \mathbb{D}^{-1}(k)$</td>
<td>$= \mathcal{G}(k)$ — the articulated body Kalman gain operator for the $k$th body</td>
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<td>$h_{\omega}(k)$</td>
<td>the angular sub-block of the joint map matrix $\mathbb{H}^+(k)$ for the $k$th body</td>
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<tr>
<td>$h_v(k)$</td>
<td>the linear sub-block of the joint map matrix $H^*(k)$ for the $k$th body</td>
</tr>
<tr>
<td>$H^*(k)$</td>
<td>the hinge map matrix for the $k$th hinge</td>
</tr>
<tr>
<td>$H$</td>
<td>the block-diagonal spatial operator of hinge map matrices</td>
</tr>
<tr>
<td>$H_B = H\Delta_B^{-1}$</td>
<td>the block-diagonal spatial operator of hinge map matrices when $B_k \neq 0_k$</td>
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<tr>
<td>$H_I$</td>
<td>the block-diagonal spatial operator of hinge map matrices for inertially referenced formulation of the equations of motion</td>
</tr>
<tr>
<td>$H_C$</td>
<td>the $H$ operator for the $C$ child sub-graph</td>
</tr>
<tr>
<td>$H_P$</td>
<td>the $H$ operator for the $P$ parent sub-graph</td>
</tr>
<tr>
<td>$H_{\mathcal{G}}$</td>
<td>the $H$ operator for the $\mathcal{G}$ sub-graph</td>
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<td>$H_a$</td>
<td>the $H$ operator for the aggregated tree</td>
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<td>$\mathcal{H}_{i=i}$</td>
<td>$\text{col}\left{H^*(i) \cdot \mathbb{I}<em>{[k=i]}\right}</em>{k=1}^n$ - the derivative of $\Delta_V$ with respect to $\dot{\theta}_i$</td>
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<tr>
<td>$\mathcal{H}_{\omega=i}$</td>
<td>$\text{col}\left{H^\omega_{\omega}(i) \cdot \mathbb{I}<em>{[k=i]}\right}</em>{k=1}^n$ - the derivative of $\Delta_\omega^V$ with respect to $\dot{\theta}_i$</td>
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<tr>
<td>$\mathcal{H}_{\omega&lt;\omega(i)}$</td>
<td>$\text{col}\left{H^\omega_{\omega}(i) \cdot \mathbb{I}<em>{[k&lt;i]}\right}</em>{k=1}^n$ - the derivative of $\mathcal{V}_\omega^\mathcal{G}$ with respect to $\dot{\theta}_i$</td>
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<tr>
<td>$\mathcal{H}_{\omega\omega=i}$</td>
<td>$\text{col}\left{H^\omega_{\omega}(i) \cdot \mathbb{I}<em>{[k=i]}\right}</em>{k=1}^n$ - the derivative of $\mathcal{V}_\omega^V$ with respect to $\dot{\theta}_i$</td>
</tr>
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<td>$h(z)$</td>
<td>spatial momentum of a rigid body</td>
</tr>
<tr>
<td>$I$</td>
<td>an inertially-fixed coordinate frame</td>
</tr>
<tr>
<td>$J$</td>
<td>$\in \mathbb{R}^{6n_d \times N}$ - the Jacobian matrix</td>
</tr>
<tr>
<td>$J(k)$</td>
<td>rotational inertia of a rigid body</td>
</tr>
<tr>
<td>$K$</td>
<td>$\mathcal{K} = \mathcal{E}_\phi \mathcal{J}$ - the spatial operator formed from the shifted Kalman gain elements</td>
</tr>
<tr>
<td>$\mathcal{K}(k+1,k)$</td>
<td>$= \Phi(k+1,k)\mathcal{J}(k)$ - the shifted Kalman gain operator</td>
</tr>
<tr>
<td>$\mathcal{K}_{\text{sys}}$</td>
<td>the system kinetic energy</td>
</tr>
<tr>
<td>$L$</td>
<td>$\dot{\mathcal{P}}^+ = \dot{\mathcal{P}}^+ + \Delta_\mathcal{V}^V \mathcal{P}^+ - \mathcal{P}^+ \Delta_\mathcal{V}^V$ - an operator associated with the time derivative of the articulated body inertia, $\dot{\mathcal{P}}$</td>
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<td>Notation</td>
<td>Description</td>
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<tr>
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</tr>
<tr>
<td>$\dot{\lambda}$</td>
<td>the forward Lyapunov equation solution associated with the time derivative of the $P$ articulated body inertia.</td>
</tr>
<tr>
<td>$\lambda_{0i}$</td>
<td>an intermediate spatial operator used for the computation of the sensitivity of the $P$ articulated body inertia spatial operator</td>
</tr>
<tr>
<td>$l(F,G)$</td>
<td>vector from frame $F$ to frame $G$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>the operational space inertia matrix</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>the operational space compliance matrix</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>the Lagrangian function</td>
</tr>
<tr>
<td>$M$</td>
<td>mass of a rigid body</td>
</tr>
<tr>
<td>$M_D(\theta,\dot{\theta})$</td>
<td>the gradient of the generalized momentum</td>
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<td>$M$</td>
<td>the mass matrix of a multibody system</td>
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<tr>
<td>$M_B = \Delta_B/\mathcal{O} M \Delta_B^*/\mathcal{O}$</td>
<td>the block-diagonal spatial operator of body spatial inertias about the body frame when $B_k \neq \mathcal{O}_k$</td>
</tr>
<tr>
<td>$M(k)$</td>
<td>the spatial inertia of the $k$th link</td>
</tr>
<tr>
<td>$M(x)$</td>
<td>spatial inertia of a rigid body referenced to point $x$</td>
</tr>
<tr>
<td>$M_I \triangleq \phi(I,C)M\phi^*(I,C) \in \mathbb{R}^{6 \times 6}$, the inertially referenced spatial inertia of a body</td>
<td></td>
</tr>
<tr>
<td>$M_S$</td>
<td>rigid body spatial inertia matrix for the $\mathcal{S}$ sub-graph</td>
</tr>
<tr>
<td>$N$</td>
<td>the total number of velocity degrees of freedom for the system</td>
</tr>
<tr>
<td>$n$</td>
<td>the number of bodies in the multibody system</td>
</tr>
<tr>
<td>$n_G$</td>
<td>the number of bodies in the $\mathcal{S}$ sub-graph</td>
</tr>
<tr>
<td>$n_{nd}(k)$</td>
<td>number of nodes on the $k$th body</td>
</tr>
<tr>
<td>$n_{nd}$</td>
<td>the number of nodes on the system</td>
</tr>
<tr>
<td>$\nu(k)$</td>
<td>the articulated body inertia innovations generalized acceleration</td>
</tr>
<tr>
<td>$\tilde{\nu} = \nu - \mathcal{G}^* \mathcal{E}^* g$</td>
<td>the articulated body innovations acceleration with gravity contribution included</td>
</tr>
<tr>
<td>$N$</td>
<td>outboard hinge reference frame for the $k$th link</td>
</tr>
<tr>
<td>$\mathcal{O}_{k}^+$</td>
<td>inboard hinge reference frame for the $k$th link</td>
</tr>
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<td>Notation</td>
<td>Description</td>
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<tr>
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</tr>
<tr>
<td>P(k)</td>
<td>the set of parent nodes of the kth node in a digraph</td>
</tr>
<tr>
<td>P</td>
<td>the induced sub-graph for the nodes not in, or descendant of, the ( \mathcal{S} ) sub-graph</td>
</tr>
<tr>
<td>p(k)</td>
<td>the 3-vector from the point k to the center of mass of a rigid body</td>
</tr>
<tr>
<td>( \phi )</td>
<td>((I - \mathcal{E}_\phi)^{-1}) – the SPO operator for rigid-link multibody systems</td>
</tr>
<tr>
<td>( \phi(x,y) )</td>
<td>rigid body transformation matrix for the x and y frames</td>
</tr>
<tr>
<td>( \phi(\mathcal{B}) )</td>
<td>( \Delta_{\mathcal{B}/\mathcal{O}}\phi\Delta_{\mathcal{B}/\mathcal{O}}^{-1} ) – the SPO operator for the case when ( \mathcal{B}_k \neq \mathcal{O}_k )</td>
</tr>
<tr>
<td>( \phi(k,k-1) )</td>
<td>the rigid body transformation matrix from frame ( \mathcal{B}<em>k ) to frame ( \mathcal{B}</em>{k-1} )</td>
</tr>
<tr>
<td>( \tilde{\phi} )</td>
<td>( \phi - I ) – the strictly lower-triangular spatial operator derived from the ( \phi ) SPO operator</td>
</tr>
<tr>
<td>( \psi )</td>
<td>((I - \mathcal{E}_\psi)^{-1}) – the articulated body SPO operator</td>
</tr>
<tr>
<td>( \psi(k+1,k) )</td>
<td>( \phi(k+1,k)\tilde{\tau}(k) ) – the articulated body transformation matrix for the kth body</td>
</tr>
<tr>
<td>( \tilde{\psi} )</td>
<td>( \psi - I ) – the articulated body spatial operator derived from the ( \psi ) SPO operator</td>
</tr>
<tr>
<td>( \mathcal{P}(k) )</td>
<td>the articulated body inertia of the kth link</td>
</tr>
<tr>
<td>( \tilde{\mathcal{P}} )</td>
<td>( \mathcal{E}<em>\phi\tilde{\mathcal{P}}^*\mathcal{E}</em>\phi^* ) – an operator associated with the time derivative of the articulated body inertia, ( \mathcal{P} )</td>
</tr>
<tr>
<td>( \mathcal{P}^+(k) )</td>
<td>( \Delta \mathcal{P}(k)\tilde{\tau}^*(k) ) the articulated body inertia ( \mathcal{P}(k) ) transformed across the joint from ( \mathcal{O}<em>k ) to ( \mathcal{O}</em>{k+1} ) for the kth body</td>
</tr>
<tr>
<td>( \tilde{\mathcal{P}}^+ )</td>
<td>( \tilde{\tau}\lambda\tilde{\tau}^* ) – an operator associated with the time derivative of the articulated body inertia, ( \mathcal{P} )</td>
</tr>
<tr>
<td>R</td>
<td>number of hinge generalized coordinates for the kth hinge</td>
</tr>
<tr>
<td>( r_p(k) )</td>
<td>number of hinge generalized coordinates for the kth hinge</td>
</tr>
<tr>
<td>( r_v(k) )</td>
<td>number of generalized velocity coordinates for the kth hinge</td>
</tr>
<tr>
<td>( R(k) )</td>
<td>the composite rigid body inertia associated with the kth hinge</td>
</tr>
<tr>
<td>( R )</td>
<td>the block-diagonal spatial operator consisting of composite body inertia of the links</td>
</tr>
<tr>
<td>( F_RG )</td>
<td>( \in \mathcal{R}^{3\times3} ) – a rotation matrix that transforms vector representations from the ( G ) frame to the ( F ) frame</td>
</tr>
<tr>
<td>S</td>
<td>the adjacency matrix for digraphs</td>
</tr>
<tr>
<td>( S )</td>
<td>a BWA matrix for a tree digraph</td>
</tr>
<tr>
<td>Notation</td>
<td>Description</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
</tr>
<tr>
<td>T</td>
<td>the vector of generalized coordinates for the system</td>
</tr>
<tr>
<td>θ</td>
<td>the articulated body projection operator for the kth hinge</td>
</tr>
<tr>
<td>τ(k)</td>
<td>the complement of projection operator τ(k) for the kth hinge</td>
</tr>
<tr>
<td>Π(k)</td>
<td>the generalized force vector for the kth hinge</td>
</tr>
<tr>
<td>FTG</td>
<td>homogeneous transform</td>
</tr>
<tr>
<td>Ξ</td>
<td>a rooted directed tree of nodes and edges</td>
</tr>
<tr>
<td>U</td>
<td>the projection operator for operational space dynamics</td>
</tr>
<tr>
<td>U⊥</td>
<td>the complement of the U projection operators for operational space dynamics</td>
</tr>
<tr>
<td>Υ+</td>
<td>matrix associated with Υ</td>
</tr>
<tr>
<td>Υ</td>
<td>kernel matrix associated with operational space inertias</td>
</tr>
<tr>
<td>V</td>
<td>the linear velocity of the G frame with respect to the F frame</td>
</tr>
<tr>
<td>v(x)</td>
<td>linear velocity vector for point x</td>
</tr>
<tr>
<td>vI</td>
<td>the linear velocity of the inertially fixed velocity reference frame I for a rigid body</td>
</tr>
<tr>
<td>vI(k)</td>
<td>the linear velocity of the inertially fixed velocity reference frame I for the kth body</td>
</tr>
<tr>
<td>Υω</td>
<td>the stacked vector of angular spatial velocities for all the links</td>
</tr>
<tr>
<td>Υω(F)</td>
<td>the angular spatial velocity component of the V(F) spatial velocity of F frame</td>
</tr>
<tr>
<td>Vωn</td>
<td>the stacked vector of spatial velocities of the task space nodes (also used as generalized velocities for Operational Space dynamics)</td>
</tr>
<tr>
<td>Vω(F,G)</td>
<td>the spatial velocity of the G frame with respect to the F frame</td>
</tr>
<tr>
<td>Vν</td>
<td>the stacked vector of linear spatial velocities for all the links</td>
</tr>
<tr>
<td>Vν(F)</td>
<td>the linear spatial velocity component of the V(F) spatial velocity of F frame</td>
</tr>
<tr>
<td>Vν(Oi)k</td>
<td>the spatial velocity of the ith node on the kth links</td>
</tr>
<tr>
<td>Vν(F,G)</td>
<td>the stacked vector of link spatial velocities</td>
</tr>
<tr>
<td>V(F)</td>
<td>spatial velocity of the F frame</td>
</tr>
<tr>
<td>V(k)</td>
<td>spatial velocity of the Bk frame</td>
</tr>
<tr>
<td>V+(k)</td>
<td>spatial velocity of O+(k) frame</td>
</tr>
</tbody>
</table>
\[ \hat{V}^\omega (z) \quad a \in \mathbb{R}^{6 \times 6} \text{ cross-product matrix associated with the } V^\omega (z) \]

\[ \tilde{V}_S^\omega = \text{diag} \left\{ \hat{V}^\omega (\varphi(k)) \right\}_{k=1}^n - \text{the block-diagonal matrix with } \hat{V}^\omega (\varphi(k)) \text{ elements} \]

\[ \tilde{V}_S^v = \text{diag} \left\{ \tilde{V}^v (\varphi(k)) \right\}_{k=1}^n - \text{the block-diagonal matrix with } \tilde{V}^v (\varphi(k)) \text{ elements} \]

\[ V = \text{diag} \left\{ V(k) \right\} - \text{the block-diagonal spatial operator with } V(k) \text{ diagonal elements} \]

\[ \bar{V}_S = \text{diag} \left\{ \bar{V}(\varphi(k)) \right\} - \text{the block-diagonal, shifted up version of the } V \text{ spatial operator} \]

\[ V_S \text{ the stacked sub-vector corresponding to the } S \text{ sub-graph bodies} \]

\[ V_I \text{ the inertially referenced spatial velocity of a point on a body, with } \omega \text{ and } v_I \text{ denoting the angular and linear velocity components, respectively} \]

\[ V_{II}(k) \text{ the spatial velocity of the } k\text{th link referred to frame } II, \text{ with } \omega(k) \text{ and } u(k) \text{ denoting the angular and linear velocity components, respectively} \]

\[ Z \]

\[ z(k) \text{ the residual spatial force for the } k\text{th link} \]

\[ z^+(k) \text{ the } z(k) \text{ articulated body inertia residual force propagated across the } k\text{th hinge} \]

\[ z_\delta \text{ the correction residual spatial force due to a non-zero tip force} \]
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