

Answers and Hints for Selected Exercises

2.7 Exercises

1. (b) Use (2.6.8) and (2.6.9).
5. If f vanishes in some bounded interval and $f' = 0$, then $f = 0$.
6. Use the parallelogram law.
8. Use Schwarz's inequality to show that $\|x + y\| = \|x\| + \|y\|$ if and only if x and y are linearly independent.
13. Every finite dimensional normed space is complete.
14. No. Use the result that $C([a, b])$ is incomplete with respect to the norm

$$\|f\| = \left(\int_a^b |f|^2 dx \right)^{\frac{1}{2}}.$$

15. No. Compare with Exercise 14.
16. Yes. Use continuity of the inner product.
18. For (c) and (d), use an orthonormal sequence.
20. No.
35. $x = \sum_{n=1}^{\infty} (\alpha_n a_n)$.
47. Use an example of the set $S = \{(x, 0) : x \in \mathbb{R}\}$ and the point $x_0 = (0, 1)$.
48. $y = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$.
50. Consider an example $x_n(t) = n \exp[-n^6(t - t_0)^2]$.
52. Define g as the composition of the orthogonal projection onto F with f .

53. The sequence $(1, 0, 0, \dots)$, $(0, 1, 0, 0, \dots)$, $(0, 0, 1, 0, 0, \dots)$, \dots is a complete orthonormal system in the space ℓ^2 .
59. (a) $f \in L^2(\mathbb{R})$, (b) $f \notin L^2(\mathbb{R})$,
 (c) $f_r \in L^2(\mathbb{R})$ if $r > 1$ and $\|f_r\|_1 = \left(\frac{2r}{r-1}\right)$.

3.16 Exercises

1. (b) Hint: $f(t) = -\frac{1}{a} \frac{d}{dt} [\exp(-at^2)]$.
1. (c) Hint: $e^t = u$, $\hat{f}(\omega) = \Gamma(1 - \omega)$, where $\Gamma(x)$ is the Gamma function.
1. (f) $\sqrt{\frac{\pi}{a}} \exp\left(-\frac{\omega^2}{4a} - \frac{ib\omega}{2a} + \frac{b^2}{4a}\right)$.
1. (g) $(i\omega)^n$.
1. (h) $2\Gamma(a) \cos\left(\frac{a\pi}{2}\right) |\omega|^{-a}$.
1. (i) Hint: Use (3.2.11) and then Duality Theorem 3.4.10. Draw a figure for $f(t)$ and

$$\hat{f}(\omega) = \Delta_{2a}(\omega).$$

1. (j) Hint: Use (3.3.5) combined with Example 3.2.3.

$$\hat{f}(\omega) = \frac{\sin(\omega - \omega_0)\tau}{(\omega - \omega_0)} + \frac{\sin(\omega + \omega_0)\tau}{(\omega + \omega_0)}.$$

Draw the graphs of $f(t)$ and $\hat{f}(\omega)$.

1. (k) $(-i)^n \sqrt{2\pi} P_n(\omega) \chi_1(\omega)$, $P_n(x)$ is the Legendre polynomial of degree n .
1. (l) $2\pi\delta(\omega - a)$.
11. Hint: $F'(t) = f(t)$ for almost all $t \in \mathbb{R}$, and then take the Fourier transform.
15. (a) $\gamma(t) = \left(1 - \frac{3}{2}|t| + \frac{1}{2}|t|^3\right) H(1 - |t|)$. (b) $\gamma(t) = \exp(-a|t|)$.
17. Hint: $F_\lambda(\omega) = \lambda F(\lambda\omega) = \lambda \hat{\Delta}(\lambda\omega) = \hat{\Delta}_\lambda(\omega)$.
18. (b) Hint: Use the Dirichlet kernel

$$D_k(z) = \sum_{n=-k}^k z^n = z^{-k} \sum_{n=0}^{2k} z^n = z^{-k} \left(\frac{z^{2k+1} - 1}{z - 1}\right). \quad \text{Put } z = e^{it} \neq 1$$

so that

$$D_k(e^{it}) = z^{-k} \frac{\exp\left\{i(2k+1)\frac{t}{2}\right\}}{\exp\left(\frac{it}{2}\right)} \cdot \frac{\sin\left\{(2k+1)\frac{t}{2}\right\}}{\sin\left(\frac{t}{2}\right)} = \frac{\sin\left\{(2k+1)\frac{t}{2}\right\}}{\sin\left(\frac{t}{2}\right)}.$$

Use $F_n(z) = \frac{1}{n+1} \sum_{k=0}^n D_k(z)$.

$$\begin{aligned} 2F_n(z) \sin^2 \frac{t}{2} &= \frac{1}{n+1} \sum_{k=0}^n 2 \sin \frac{t}{2} \sin(2k+1) \frac{t}{2} \\ &= \frac{1}{n+1} \sum_{k=0}^n [\cos kt - \cos(k+1)t] \\ &= \frac{1}{n+1} [1 - \cos(n+1)t] = \left(\frac{2}{n+1}\right) \sin^2 \left\{ \left(\frac{n+1}{2}\right) t \right\}. \end{aligned}$$

19. Hint:

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\omega^2}{4\alpha}\right), \int_{-\infty}^{\infty} f^2(t) dt = \sqrt{\frac{\pi}{2\alpha}}, \int_{-\infty}^{\infty} \hat{f}^2(\omega) d\omega = \pi \sqrt{\frac{2\pi}{\alpha}}.$$

$$\mathcal{F}\{f^2(t)\} = \mathcal{F}\{\exp(-2\alpha t^2)\} = \sqrt{\frac{\pi}{2\alpha}} \exp\left(-\frac{\omega^2}{8\alpha}\right).$$

$$\mathcal{F}\{t^2 f^2(t)\} = (-i)^2 \frac{d^2}{d\omega^2} \hat{f}(\omega) = -\frac{d^2}{d\omega^2} \left[\sqrt{\frac{\pi}{2\alpha}} \exp\left(-\frac{\omega^2}{8\alpha}\right) \right].$$

$$\int_{-\infty}^{\infty} t^2 \exp(-2\alpha t^2) dt = \frac{1}{4\alpha^{3/2}} \cdot \sqrt{\frac{\pi}{2}}.$$

$$\int_{-\infty}^{\infty} \omega^2 \hat{f}^2(\omega) d\omega = \frac{\pi}{\alpha} \int_{-\infty}^{\infty} \omega^2 \exp\left(-\frac{\omega^2}{2\alpha}\right) d\omega = \pi \sqrt{2\pi\alpha}.$$

$$\sigma_t^2 = \int_{-\infty}^{\infty} t^2 f^2(t) dt \div \int_{-\infty}^{\infty} f^2(t) dt = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{4\alpha^{3/2}} \div \sqrt{\frac{\pi}{2\alpha}} = \frac{1}{4\alpha}.$$

$$\sigma_\omega^2 = \frac{\pi \sqrt{2\pi\alpha}}{\pi \sqrt{\frac{2\pi}{\alpha}}} = \alpha. \text{ Hence, } \sigma_t^2 \sigma_\omega^2 = \frac{1}{4} \Rightarrow \sigma_t \sigma_\omega = \frac{1}{2}.$$

20. $\phi(t) = \frac{A_0}{2\sqrt{\pi a}} \exp\left\{-\frac{(t-t_0)^2}{4a}\right\}.$

21. $\hat{\phi}(\omega) = \left[a + b \cos\left(\frac{n\pi\omega}{\omega_0}\right) \right] \exp(-i\omega_0 t) \hat{\chi}_{\omega_0}(\omega).$

22. Hint: For (a) and (b), use results (3.3.9) and (3.3.11)

$$(c) \quad y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{f}(k) e^{ikx} dk}{(\sigma^2 - k^2 + 2iak)}.$$

$$(d) \quad y(x) = e^{-x} \int_{-a}^x e^{\xi} f(\xi) d\xi + e^x \int_x^a e^{-\xi} f(\xi) d\xi.$$

26. Hint: Construct a differential equation similar to that in Exercise 23.

$$29. (b) \quad \hat{f}(\omega, \sigma) = \left(\frac{4}{\omega\sigma} \right) \sin(a\omega) \sin(a\sigma).$$

$$31. \quad u(x) = \int_{-\infty}^{\infty} w(\xi) G(\xi, x) d\xi, \text{ where } w(x) = W(x)/EI, \text{ and}$$

$$G(\xi, x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos k(x - \xi) dk}{k^4 + a^4}, \quad a^4 = \kappa/EI.$$

$$34. \quad u(x, t) = \int_{-\infty}^{\infty} f(\xi) G(x, t; \xi, 0) d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} q(\xi, \tau) G(x, t; \xi, \tau) d\xi,$$

$$\text{where } G(x, t; \xi, \tau) d\xi = \frac{1}{\sqrt{4\pi\kappa(t - \tau)}} \exp\left[-\frac{(x - \xi)^2}{4\kappa(t - \tau)}\right].$$

$$39. \quad u(x, t) = (4at + 1)^{-\frac{1}{2}} \exp\left(-\frac{at^2}{4at + 1}\right).$$

$$40. \quad G(x, t) = \frac{1}{(2\pi)^3} \iiint \exp\{i(\mathbf{k} \cdot \mathbf{x})\} \frac{\sin \alpha t}{\alpha} d\mathbf{k}, \text{ where } \alpha = (c^2 k^2 + d^2)^{\frac{1}{2}}.$$

$$42. \quad \langle v \rangle = mv_m \exp\left(-\frac{1}{4} \pi v_m^2\right) + v_0,$$

and

$$B^2 = \frac{\alpha}{8\pi^2} + \frac{1}{2} m^2 v_m^2 \left[\exp\left\{-\frac{1}{\alpha} \cdot 2\pi^2 v_m^2\right\} - 1 \right].$$

4.10 Exercises

1. Since $\int_{-\infty}^{\infty} g(\tau - t) dt = 1$, the result follows. The result implies that the set of the Gabor transforms of f with the Gaussian window decomposes the Fourier transform of f exactly.
2. (b) Hint. $\hat{f}_g(t, \omega) = \langle f, \bar{g}_{t, \omega} \rangle$ and use the Parseval identity of the Fourier transform.
3. Derive $\sigma_f^2 = \sqrt{a}$.

Use $\int_{-\infty}^{\infty} \exp(-ax^2) dx = \sqrt{\frac{\pi}{a}}$ and then differentiate with respect to a to find

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2) dx = \frac{1}{a} \sqrt{\frac{\pi}{4a}}.$$

Replace a by $(2a)^{-1}$ in the above result to derive

$$\begin{aligned} \|g\|_2 &= (8\pi a)^{-\frac{1}{4}}, \\ \sigma_i^2 &= (8\pi a)^{\frac{1}{4}} \left\{ \frac{1}{4\pi a} \cdot \frac{\sqrt{\pi}}{2} \cdot (2a)^{3/2} \right\}^{\frac{1}{2}} = \sqrt{a}. \end{aligned}$$

4. For a tight frame, $A = B$.

$$\sum_{n=1}^3 |\langle x, e_n \rangle|^2 = |x_2|^2 + \left| \frac{\sqrt{3}}{2} x_1 + \frac{1}{2} x_2 \right|^2 + \left| \frac{\sqrt{3}}{2} x_1 - \frac{1}{2} x_2 \right|^2 = \frac{3}{2} \|x\|^2.$$

10. Put $\tau = 0$ in the second result, multiply the resulting expression by $\exp(2\pi i \omega t)$, and integrate the identity over $\omega \in [-b, b]$. The right-hand side is equal to $f(t)$ by the Fourier inversion theorem, and the left-hand side follows from the definition of the Zak transform combined with integrating the exponential.

11. Since

$$\omega^2 \left| \hat{\chi}_{[0,1]}(\omega) \right|^2 = 4 \sin^2 \left(\frac{\omega}{2} \right),$$

the second integral is infinite.

12.

$$\hat{\chi}_{[0,1]}(\omega) = \exp\left(-\frac{i\omega}{2}\right) \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}.$$

5.10 Exercises

1. (a) $W_f(t, \omega) = \sqrt{2\pi} \exp\left[-2\left(\frac{t^2}{\sigma^2} + \frac{\sigma^2 \omega^2}{4}\right)\right].$
- (c) $W_f(t, \omega) = 2 \exp\left[-\left\{\frac{2\pi t^2}{\sigma^2} + \frac{\sigma^2}{2\pi} (\omega - \omega_0 t)^2\right\}\right].$

$$(d) \quad W_f(t, \omega) = 2 \exp \left[- \left\{ \frac{2\pi}{\sigma^2} (t - t_0)^2 + \frac{\sigma^2}{2\pi} (\omega - \omega_0)^2 \right\} \right].$$

$$8. \quad W_f(t, \omega) = 2(\omega - \omega_0)^{-1} \sin \left\{ 2(\omega - \omega_0)(T - |t|) \right\} H(T - |t|).$$

9. Hint: Use Example 5.2.6 and result (5.3.3).

$$W_f(t, \omega) = \frac{\pi}{2} |A|^2 \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) + 2\delta(\omega) \cos \{ 2(\omega_0 t + \theta) \} \right].$$

10. Hint: See Auslander and Tolimieri (1985).

13. Hint: Use $f_n g_n - fg = f_n g_n - f_n g + f_n g - fg$ and then apply Schwarz's inequality.

$$17. \quad W_f(n, \theta) = |A|^2 \sum_{k=-\infty}^{\infty} \delta(\theta - an - k\pi).$$

18. Hint: $f_m(n) = f(n) m_f(n)$, $g_m(n) = g(n) m_g(n)$.

22. $\psi_0(x)$ is a Gaussian signal and $\psi_0\left(\frac{x}{\rho}\right)$ is also a Gaussian signal as in Exercise 1(d) with $t_0 = 0$ and $\omega_0 = 0$.

6.6 Exercises

3. Physically, the convolution determines the wavelet transforms with dilated bandpass filters.

11. Write $\cos \omega_0 t = \frac{1}{2} (e^{i\omega_0 t} + e^{-i\omega_0 t})$ and calculate the Fourier transform. $\hat{f}(\omega)$ has a maximum at $\omega = \pm\omega_0$, and then maximum values become more and more pronounced as σ increases.

12. $\hat{f}(\omega)$ has a maximum at the frequency $\omega = \omega_0$. Due to the jump discontinuity of $f(t)$ at time $t = \pm a$, $|\hat{f}(\omega)|$ decays slowly as $|\omega| \rightarrow \infty$. In fact, $\hat{f}(\omega) \notin L^1(\mathbb{R})$.

13. $(W_\psi f)(a, b) = \frac{1}{\sqrt{a}} \left[\int_b^{b+\frac{a}{2}} f(t) dt - \int_{b+\frac{a}{2}}^{b+a} f(t) dt \right]$. Put $t = x + \frac{a}{2}$ in the integral $\int_{b+\frac{a}{2}}^{b+a} f(t) dt$ to get the answer.

15. Check only $\|\psi\| = 1$ and that $\psi_{m,n}$ make up a tight frame with frame constant 1 (see Daubechies 1992, p. 117).

7.7 Exercises

1.

$$\begin{aligned} \hat{\phi}(\omega) &= \int_0^1 (1-t) e^{-i\omega t} dt + \int_{-1}^0 (1+t) e^{-i\omega t} dt \\ &= \int_0^1 (1-t) e^{-i\omega t} dt + \int_0^1 (1-t) e^{i\omega t} dt = \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^2. \end{aligned}$$

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2 &= 16 \sin^4 \left(\frac{\omega}{2} \right) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^4} \\ &= \left(1 - \frac{2}{3} \sin^2 \frac{\omega}{2} \right) \text{ by (7.4.38).} \end{aligned}$$

2. Hint: (a) follows from $\hat{\psi}(0) = 0$ and (b) follows from $\left(\frac{d\hat{\psi}}{d\omega} \right)_{\omega=0} = 0$.

(c) follows from $\hat{\psi}(\omega) = \exp\left(\frac{i\omega}{2}\right) \hat{f}(\omega)$ and

$$\begin{aligned} \psi(-t-1) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp[-i\omega(t+1)] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp\left[-i\omega\left(t + \frac{1}{2}\right)\right] d\omega. \end{aligned}$$

Also, (7.4.43) implies that $\hat{f}(\omega)$ is even and hence,

$$\begin{aligned} \psi(-t-1) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp\left(i\omega t + \frac{1}{2}i\omega\right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(\omega) e^{i\omega t} d\omega = \psi(t). \end{aligned}$$

5. (a) $\phi(x) = 1, \quad 0 \leq x < 1$.

$\psi(x)$ is the Haar wavelet.

(b) $\phi(x) = B_2(x)$.

$\psi(x) = x, \quad 0 \leq x < \frac{1}{2}; \quad \psi(x) = 2 - 3x, \quad \frac{1}{2} \leq x < 1$.

(c) $\phi(x) = \delta(x), \quad \psi(x) = \delta(x)$.

6. (a) It follows from (7.3.5) that

$$\int_{-\infty}^{\infty} \phi(x) dx = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \phi(2x - n) dx, \quad (2x - n = t).$$

(b) Use $\hat{m}(\pi) = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} (-1)^k c_k = 0$.

7. $\hat{\phi}(2\pi) = \hat{m}(\pi) \hat{m}\left(\frac{\pi}{2}\right) \hat{m}\left(\frac{\pi}{2^2}\right) \dots$ and then $\hat{m}(\omega)$ has a zero of order n at $\omega = \pi$ if $\frac{d^m \hat{m}(\omega)}{d\omega^m} = 0$ when $\omega = \pi$ for $m = 0, 1, 2, \dots, (n-1)$. This gives the result.

8. (a) Hint: Write

$$\phi(x) = \sqrt{2} \sum_k c_k \phi(2x - k), \quad \phi(x) = \sqrt{2} \sum_m c_m \phi(2x - m)$$

$$\int_{-\infty}^{\infty} \phi^2(x) dx = 2 \sum_k \sum_m c_k c_m \int_{-\infty}^{\infty} \phi(2x - k) \phi(2x - m) dx$$

which is zero when $k \neq m$. But, when $k = m$

$$\int_{-\infty}^{\infty} \phi^2(x) dx = \sum_k c_k^2 \int_{-\infty}^{\infty} \phi^2(t) dt \text{ which gives the result.}$$

(b) Corresponding to the scaling function ϕ defined by (7.3.5), the wavelet may be written as

$$\psi(x) = \sum_k (-1)^k c_k \phi(2x + k - N + 1).$$

Use the orthogonality condition

$$\langle \psi(x), \psi(x - m) \rangle = \int_{-\infty}^{\infty} \psi(x) \psi(x - m) dx = 0 \text{ for all } m \text{ except } m = 0.$$

Substitute $\psi(x)$ in this integral so that

$$\begin{aligned} \int_{-\infty}^{\infty} \psi(x) \psi(x - m) dx &= \sum_k \sum_s (-1)^{k+s} c_k c_s \int_{-\infty}^{\infty} \phi(2x + k - N + 1) \\ &\quad \times \phi(2x + s - N + 1 - 2m) dx \end{aligned}$$

The right-hand integral is zero unless $k = s - 2m$ so that

$$\int_{-\infty}^{\infty} \psi(x) \psi(x - m) dx = \sum_k (-1)^{2(k+m)} c_k c_{k+2m} \int_{-\infty}^{\infty} \phi^2(2t) dt.$$

This is always zero, except $m = 0$. This gives the result.

(c) When $m = 0$, we have

$$\sum_k (-1)^k c_k = 0$$

so that

$$\sum_{k=\text{even}} c_k = \sum_{k=\text{odd}} c_k = \frac{1}{\sqrt{2}} \quad \text{and} \quad \left(\sum_{k=\text{even}} c_k \right)^2 + \left(\sum_{k=\text{odd}} c_k \right)^2 = 1.$$

Multiplying gives $\sum_{k=0}^{N-1} c_k^2 + 2 \sum_{k=0}^{N-1} \sum_{m=1}^{N/2-1} c_k c_{k+2m} = 1.$

This gives the result by 8(a).

9.

$$\begin{aligned} c_0 + c_1 + c_3 + c_4 + c_5 &= \sqrt{2} \\ c_0 - c_1 + c_2 - c_3 + c_4 - c_5 &= 0 \\ -c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 &= 0 \\ -c_1 + 4c_2 - 9c_3 + 16c_4 - 25c_5 &= 0 \\ c_0c_2 + c_1c_3 + c_2c_4 + c_3c_5 &= 0 \\ c_0c_4 + c_1c_5 &= 0 \\ c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 &= 1. \end{aligned}$$

11. Hint.

(a)

$$\begin{aligned} \hat{\phi}(\omega) &= \prod_{k=1}^{\infty} m\left(\frac{\omega}{2^k}\right) = \prod_{k=1}^{\infty} \exp\left(-\frac{3i\omega}{2} \cdot \frac{1}{2^k}\right) \cos\left(\frac{3\omega}{2} \cdot \frac{1}{2^k}\right) \\ &= \prod_{k=1}^{\infty} \exp\left(-\frac{ix}{2^k}\right) \cos\left(\frac{x}{2^k}\right), \quad \left(x = \frac{3\omega}{2}\right) \end{aligned}$$

$$\begin{aligned}
 &= \prod_{k=1}^{\infty} \exp\left(-\frac{ix}{2^k}\right) \cdot \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) \\
 &= \exp(ix) \cdot \frac{\sin x}{x} = \exp\left(-\frac{3i\omega}{2}\right) \frac{\sin\left(\frac{3\omega}{2}\right)}{\left(\frac{3\omega}{2}\right)}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 \sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2 &= \frac{4}{9} \sin^2\left(\frac{3\omega}{2}\right) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^2} \\
 &= \frac{4}{9} \sin^2\left(\frac{3\omega}{2}\right) \cdot \frac{1}{4} \operatorname{cosec}^2\left(\frac{\omega}{2}\right) \\
 &= \frac{1}{9} \left(\frac{\sin 3t}{\sin t}\right)^2 \quad \left(t = \frac{\omega}{2}\right) \\
 &= \frac{1}{9} \left(\frac{\sin(2t + t)}{\sin t}\right)^2 = \frac{1}{9} (1 + 2\cos \omega)^2 \\
 &= \frac{1}{9} (3 + 4\cos \omega + 2\cos 2\omega).
 \end{aligned}$$

This means that condition (b) in Theorem 7.3.1 is not satisfied. (c) follows from (a).

12. From (a) and (b), $\phi(x) = 0$, $x < 0$ or $x > 3$.

$$\begin{aligned}
 \phi(x) + \phi(x+1) + \phi(x+2) &= 1, \\
 c\phi(x) + (c-1)\phi(x-1) + (c-2)\phi(x-2) &= x.
 \end{aligned}$$

Eliminating $\phi(x+2)$, $\phi(x)$, and $\phi(x+1)$ gives (a), (b), and (c) respectively.

9.6 Exercises

2. (a) Replace $f(x)$ and $\phi(x-k)$ by their Fourier inversion formulas in (9.4.7) and then apply

$$\int_{-\infty}^{\infty} \exp\{i(\omega_1 - \omega_2)x\} dx = 2\pi \delta(\omega_1 - \omega_2).$$

2. (b) Replace $a_{\phi,k}$ by 2(a) and $\phi(x - k)$ by its Fourier inversion formula to obtain

$$\sum_{k=-\infty}^{\infty} a_{\phi,k} \phi(x - k) = 2\pi \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega_1) d\omega_1 \int_{-\infty}^{\infty} \bar{\hat{\phi}}(\omega_1) \hat{\phi}(\omega_2) \exp(i\omega_2 x) \times \exp\{i(\omega_1 - \omega_2)k\} d\omega_2.$$

Use the Poisson summation formula

$$\frac{1}{2\ell} \sum_{k=-\infty}^{\infty} \exp(-ik\pi/\ell) = \sum_{m=-\infty}^{\infty} \delta(x - 2m\ell)$$

with $x = \omega_2 - \omega_1$ and $\ell = \pi$ so that

$$\sum_{k=-\infty}^{\infty} \exp\{i(\omega_1 - \omega_2)k\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega_2 - \omega_1 - 2\pi m).$$

The product $\bar{\hat{\phi}}(\omega_1) \hat{\phi}(\omega_2)$ is zero unless ω_1 and ω_2 both lie in $[0, 2\pi]$, which is the case when $m = 0$. Then, $\omega_1 = \omega_2$ and $\bar{\hat{\phi}}(\omega_1) \hat{\phi}(\omega_2) = (2\pi)^{-2}$ give the last integral formula.

3. Use the inverse Fourier transform

$$\bar{\psi}(2^m x - k) = 2^{-m} \int_{-\infty}^{\infty} \bar{\psi}(\omega 2^{-m}) \exp(i\omega k 2^{-m}) \exp(-i\omega x) d\omega$$

in (9.4.2) and then apply

$$\int_{-\infty}^{\infty} \exp\{i(\omega_1 - \omega_2)x\} dx = 2\pi \delta(\omega_1 - \omega_2)$$

to obtain 3(a).

4. (9.4.1) and (9.4.9) are identical, provided

$$\sum_{k=-\infty}^{\infty} a_{\phi,k} \phi(x - k) = \sum_{m=-\infty}^{-1} \sum_{k=-\infty}^{\infty} a_{m,k} \psi(2^m x - k).$$

Use 2(b) and show that

$$\begin{aligned} & \sum_{m=-\infty}^{-1} \sum_{k=-\infty}^{\infty} a_{m,k} \psi(2^m x - k) \\ &= \sum_{m=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2\pi 2^{-m} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}(\omega_1) \hat{\psi}(\omega_1 2^{-m}) \hat{\psi}(\omega_2 2^{-m}) \\ & \quad \times \exp(i\omega_2 x) \exp\{i(\omega_1 - \omega_2)k 2^{-m}\} d\omega_2. \end{aligned}$$

Use Poisson's summation formula

$$\sum_{k=-\infty}^{\infty} \exp \{i(\omega_1 - \omega_2)k2^{-m}\} = 2\pi 2^m \sum_{r=-\infty}^{\infty} \delta(\omega_2 - \omega_1 - 2\pi r 2^m)$$

and the fact that $\hat{\psi}(\omega_1 2^{-m}) \hat{\psi}(\omega_2 2^{-m})$ is zero unless $m = 0$ in the above sum and it is equal to (2π) . Consequently,

$$\begin{aligned} & \sum_{m=-\infty}^{-1} \sum_{k=-\infty}^{\infty} a_{m,k} \psi(2^m x - k) \\ &= \sum_{m=-\infty}^{-1} \int_{2\pi 2^m}^{2\pi 2^{m+1}} \hat{f}(\omega) e^{i\omega x} d\omega = \int_0^{2\pi} \hat{f}(\omega) e^{i\omega x} d\omega. \end{aligned}$$

5. Use results in 3(a) and 3(b) in $|a_{m,k}|^2 = a_{m,k} \bar{a}_{m,k}$ and $|\bar{a}_{m,k}|^2 = \bar{a}_{m,k} a_{m,k}$. These lead to double integrals. Then, sum over k and m , which involves

$$\sum_{k=-\infty}^{\infty} \exp \{i(\omega_2 - \omega_1)k2^{-m}\}$$

and its complex conjugate. Then, use the Poisson summation formula

$$\frac{1}{2\ell} \sum_{k=-\infty}^{\infty} \exp \left\{ -\frac{ik\pi}{\ell} \right\} = \sum_{r=-\infty}^{\infty} \delta(x - 2r\ell)$$

with $x = \omega_2 - \omega_1$ and $\ell = \pi 2^m$ to obtain

$$\sum_{k=-\infty}^{\infty} \exp \{i(\omega_1 - \omega_2)k2^{-m}\} = 2\pi 2^m \sum_{r=-\infty}^{\infty} \delta(\omega_2 - \omega_1 - 2\pi r 2^m).$$

Consequently,

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-m} |a_{m,k}|^2 &= (2\pi)^3 \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega) \bar{\hat{f}}(\omega) \bar{\hat{\psi}}(\omega 2^{-m}) \hat{\psi}(\omega 2^{-m}) d\omega \\ &= 2\pi \int_0^{\infty} \hat{f}(\omega) \bar{\hat{f}}(\omega) d\omega \end{aligned}$$

and hence

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-m} (|a_{m,k}|^2 + |\tilde{a}_{m,k}|^2) &= 2\pi \int_0^{\infty} \{ \hat{f}(\omega) \tilde{\hat{f}}(\omega) + \hat{f}(-\omega) \tilde{\hat{f}}(-\omega) \} d\omega \\ &= 2\pi \int_{-\infty}^{\infty} \hat{f}(\omega) \tilde{\hat{f}}(\omega) d\omega = \int_{-\infty}^{\infty} |f(x)|^2 dx. \end{aligned}$$

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