Answers and Hints for Selected Exercises

2.7 Exercises

1. (b) Use (2.6.8) and (2.6.9).
5. If f vanishes in some bounded interval and $f' = 0$, then $f = 0$.
6. Use the parallelogram law.
8. Use Schwarz's inequality to show that $\|x + y\| = \|x\| + \|y\|$ if and only if x and y are linearly independent.
13. Every finite dimensional normed space is complete.
14. No. Use the result that $C([a, b])$ is incomplete with respect to the norm

$$\|f\| = \left(\int_a^b |f|^2 dx\right)^{\frac{1}{2}}.$$

15. No. Compare with Exercise 14.
16. Yes. Use continuity of the inner product.
18. For (c) and (d), use an orthonormal sequence.
20. No.
35. $x = \sum_{n=1}^{\infty} (\alpha_n a_n)$.
47. Use an example of the set $S = \{(x, 0) : x \in \mathbb{R}\}$ and the point $x_0 = (0, 1)$.
48. $y = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$.
50. Consider an example $x_n(t) = n \exp\left[-n^6 (t - t_0)^2\right]$.
52. Define g as the composition of the orthogonal projection onto F with f.

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53. The sequence $(1, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, 0, \ldots), \ldots$ is a complete orthonormal system in the space ℓ^2.
59. (a) $f \in L^2(\mathbb{R})$, (b) $f \notin L^2(\mathbb{R})$,
(c) $f_r \in L^2(\mathbb{R})$ if $r >< 1$ and $\|f_r\|_1 = \left(\frac{2r}{r-1}\right)$.

3.16 Exercises

1. (b) Hint: $f(t) = -\frac{1}{a}\frac{d}{dt}\left[\exp\left(-at^2\right)\right]$.
1. (c) Hint: $e^t = u$, $\hat{f}(\omega) = \Gamma(1-\omega)$, where $\Gamma(x)$ is the Gamma function.
1. (f) $\sqrt{\frac{\pi}{a}}\exp\left(-\frac{\omega^2}{4a} - \frac{ib\omega}{2a} + \frac{b^2}{4a}\right)$.
1. (g) $(i\omega)^n$.
1. (h) $2\Gamma(a)\cos\left(\frac{a\pi}{2}\right)|\omega|^{-a}$.
1. (i) Hint: Use (3.2.11) and then Duality Theorem 3.4.10. Draw a figure for $f(t)$ and

$$\hat{f}(\omega) = \Delta_{2a}(\omega).$$

1. (j) Hint: Use (3.3.5) combined with Example 3.2.3.

$$\hat{f}(\omega) = \frac{\sin(\omega-\omega_0)\tau}{(\omega-\omega_0)} + \frac{\sin(\omega+\omega_0)\tau}{(\omega+\omega_0)}.$$

Draw the graphs of $f(t)$ and $\hat{f}(\omega)$.
1. (k) $(-i)^n\sqrt{2\pi}\,P_n(\omega)\,\chi_1(\omega)$, $P_n(x)$ is the Legendre polynomial of degree n.
1. (l) $2\pi\delta(\omega - a)$.
11. Hint: $F'(t) = f(t)$ for almost all $t \in \mathbb{R}$, and then take the Fourier transform.
15. (a) $\gamma(t) = \left(1 - \frac{3}{2}|t| + \frac{1}{2}|t|^3\right)H\big(1-|t|\big)$. (b) $\gamma(t) = \exp\big(-a|t|\big)$.
17. Hint: $F_\lambda(\omega) = \lambda F(\lambda\omega) = \lambda\hat{\Delta}(\lambda\omega) = \hat{\Delta}_\lambda(\omega)$.
18. (b) Hint: Use the Dirichlet kernel

$$D_k(z) = \sum_{n=-k}^{k} z^k = z^{-k}\sum_{n=0}^{2k} z^n = z^{-k}\left(\frac{z^{2k+1}-1}{z-1}\right). \quad \text{Put } z = e^{it} \neq 1$$

so that

$$D_k\left(e^{it}\right) = z^{-k}\,\frac{\exp\left\{i(2k+1)\dfrac{t}{2}\right\}}{\exp\left(\dfrac{it}{2}\right)}\cdot\frac{\sin\left\{(2k+1)\dfrac{t}{2}\right\}}{\sin\left(\dfrac{t}{2}\right)} = \frac{\sin\left\{(2k+1)\dfrac{t}{2}\right\}}{\sin\left(\dfrac{t}{2}\right)}.$$

Use $F_n(z) = \dfrac{1}{n+1}\sum_{k=0}^{n} D_k(z)$.

$$\begin{aligned}
2F_n(z)\sin^2\frac{t}{2} &= \frac{1}{n+1}\sum_{k=0}^{n} 2\sin\frac{t}{2}\,\sin(2k+1)\frac{t}{2}\\
&= \frac{1}{n+1}\sum_{k=0}^{n}\left[\cos kt - \cos(k+1)t\right]\\
&= \frac{1}{n+1}\left[1-\cos(n+1)t\right] = \left(\frac{2}{n+1}\right)\sin^2\left\{\left(\frac{n+1}{2}\right)t\right\}.
\end{aligned}$$

19. Hint:

$$\hat{f}(\omega)=\sqrt{\frac{\pi}{\alpha}}\exp\left(-\frac{\omega^2}{4\alpha}\right),\ \int_{-\infty}^{\infty} f^2(t)\,dt=\sqrt{\frac{\pi}{2\alpha}},\ \int_{-\infty}^{\infty}\hat{f}^2(\omega)\,d\omega=\pi\sqrt{\frac{2\pi}{\alpha}}.$$

$$\mathcal{F}\left\{f^2(t)\right\} = \mathcal{F}\left\{\exp\left(-2\alpha t^2\right)\right\} = \sqrt{\frac{\pi}{2\alpha}}\exp\left(-\frac{\omega^2}{8\alpha}\right).$$

$$\mathcal{F}\left\{t^2 f^2(t)\right\} = (-i)^2\frac{d^2}{d\omega^2}\hat{f}(\omega) = -\frac{d^2}{d\omega^2}\left[\sqrt{\frac{\pi}{2\alpha}}\exp\left(-\frac{\omega^2}{8\alpha}\right)\right].$$

$$\int_{-\infty}^{\infty} t^2\exp\left(-2\alpha t^2\right)\,dt = \frac{1}{4\alpha^{3/2}}\cdot\sqrt{\frac{\pi}{2}}.$$

$$\int_{-\infty}^{\infty}\omega^2\hat{f}^2(\omega)\,d\omega = \frac{\pi}{\alpha}\int_{-\infty}^{\infty}\omega^2\exp\left(-\frac{\omega^2}{2\alpha}\right)d\omega = \pi\sqrt{2\pi\alpha}.$$

$$\sigma_t^2 = \int_{-\infty}^{\infty} t^2 f^2(t)\,dt \div \int_{-\infty}^{\infty} f^2(t)\,dt = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{4\alpha^{3/2}}\div\sqrt{\frac{\pi}{2\alpha}} = \frac{1}{4\alpha}.$$

$$\sigma_\omega^2 = \frac{\pi\sqrt{2\pi\alpha}}{\pi\sqrt{\dfrac{2\pi}{\alpha}}} = \alpha.\ \text{Hence,}\ \sigma_t^2\,\sigma_\omega^2 = \frac{1}{4} \Rightarrow \sigma_t\,\sigma_\omega = \frac{1}{2}.$$

20. $\phi(t) = \dfrac{A_0}{2\sqrt{\pi a}}\exp\left\{-\dfrac{(t-t_0)^2}{4a}\right\}.$

21. $\hat{\phi}(\omega) = \left[a + b\cos\left(\dfrac{n\pi\omega}{\omega_0}\right)\right]\exp(-i\omega_0 t)\,\hat{\chi}_{\omega_0}(\omega).$

22. Hint: For (a) and (b), use results (3.3.9) and (3.3.11)

(c) $y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{f}(k)\, e^{ikx}\, dk}{\left(\sigma^2 - k^2 + 2iak\right)}.$

(d) $y(x) = e^{-x} \int_{-a}^{x} e^{\xi} f(\xi)\, d\xi + e^{x} \int_{x}^{a} e^{-\xi} f(\xi)\, d\xi.$

26. Hint: Construct a differential equation similar to that in Exercise 23.

29. (b) $\hat{f}(\omega, \sigma) = \left(\frac{4}{\omega\sigma}\right) \sin(a\omega) \sin(a\sigma).$

31. $u(x) = \int_{-\infty}^{\infty} w(\xi)\, G(\xi, x)\, d\xi$, where $w(x) = W(x)/EI$, and

$$G(\xi, x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos k(x - \xi)\, dk}{k^4 + a^4}, \quad a^4 = \kappa/EI.$$

34. $u(x, t) = \int_{-\infty}^{\infty} f(\xi)\, G(x, t; \xi, 0)\, d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} q(\xi, \tau)\, G(x, t; \xi, \tau)\, d\xi,$

where $G(x, t; \xi, \tau)\, d\xi = \frac{1}{\sqrt{4\pi\kappa(t - \tau)}} \exp\left[-\frac{(x - \xi)^2}{4\kappa(t - \tau)}\right].$

39. $u(x, t) = \left(4at + 1\right)^{-\frac{1}{2}} \exp\left(-\frac{at^2}{4at + 1}\right).$

40. $G(x, t) = \frac{1}{(2\pi)^3} \iiint \exp\{i(\mathbf{k} \cdot \mathbf{x})\} \frac{\sin \alpha t}{\alpha} d\mathbf{k}$, where $\alpha = \left(c^2\kappa^2 + d^2\right)^{\frac{1}{2}}.$

42. $\langle v \rangle = m v_m \exp\left(-\frac{1}{4} \pi v_m^2\right) + v_0,$

and

$$B^2 = \frac{\alpha}{8\pi^2} + \frac{1}{2} m^2 v_m^2 \left[\exp\left\{-\frac{1}{\alpha} \cdot 2\pi^2 v_m^2\right\} - 1\right].$$

4.10 Exercises

1. Since $\int_{-\infty}^{\infty} g(\tau - t)\, dt = 1$, the result follows. The result implies that the set of the Gabor transforms of f with the Gaussian window decomposes the Fourier transform of f exactly.
2. (b) Hint. $\tilde{f}_g(t, \omega) = \langle f, \bar{g}_{t,\omega} \rangle$ and use the Parseval identity of the Fourier transform.
3. Derive $\sigma_t^2 = \sqrt{a}$.

Use $\int_{-\infty}^{\infty} \exp\left(-ax^2\right) dx = \sqrt{\frac{\pi}{2}}$ and then differentiate with respect to a to find

$$\int_{-\infty}^{\infty} x^2 \exp\left(-ax^2\right) dx = \frac{1}{a}\sqrt{\frac{\pi}{4a}}.$$

Replace a by $(2a)^{-1}$ in the above result to derive

$$\|g\|_2 = (8\pi a)^{-\frac{1}{4}},$$

$$\sigma_t^2 = (8\pi a)^{\frac{1}{4}} \left\{ \frac{1}{4\pi a} \cdot \frac{\sqrt{\pi}}{2} \cdot (2a)^{3/2} \right\}^{\frac{1}{2}} = \sqrt{a}.$$

4. For a tight frame, $A = B$.

$$\sum_{n=1}^{3} \left|\langle x, e_n\rangle\right|^2 = \left|x_2\right|^2 + \left|\frac{\sqrt{3}}{2}\, x_1 + \frac{1}{2}\, x_2\right|^2 + \left|\frac{\sqrt{3}}{2}\, x_1 - \frac{1}{2}\, x_2\right|^2 = \frac{3}{2}\,\|x\|^2.$$

10. Put $\tau = 0$ in the second result, multiply the resulting expression by $\exp(2\pi i\,\omega t)$, and integrate the identity over $\omega \in [-b, b]$. The right-hand side is equal to $f(t)$ by the Fourier inversion theorem, and the left-hand side follows from the definition of the Zak transform combined with integrating the exponential.
11. Since

$$\omega^2 \left|\hat{\chi}_{[0,1]}(\omega)\right|^2 = 4\sin^2\left(\frac{\omega}{2}\right),$$

the second integral is infinite.
12.

$$\hat{\chi}_{[0,1]}(\omega) = \exp\left(-\frac{i\,\omega}{2}\right) \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}.$$

5.10 Exercises

1. (a) $W_f(t, \omega) = \sqrt{2\pi}\exp\left[-2\left(\frac{t^2}{\sigma^2} + \frac{\sigma^2\omega^2}{4}\right)\right].$

   (c) $W_f(t, \omega) = 2\exp\left[-\left\{\frac{2\pi t^2}{\sigma^2} + \frac{\sigma^2}{2\pi}\,(\omega - \omega_0 t)^2\right\}\right].$

(d) $W_f(t,\omega) = 2\exp\left[-\left\{\frac{2\pi}{\sigma^2}(t-t_0)^2 + \frac{\sigma^2}{2\pi}(\omega-\omega_0)^2\right\}\right]$.

8. $W_f(t,\omega) = 2\big(\omega-\omega_0\big)^{-1}\sin\left\{2\big(\omega-\omega_0\big)\big(T-|t|\big)\right\}H\big(T-|t|\big)$.

9. Hint: Use Example 5.2.6 and result (5.3.3).

$$W_f(t,\omega) = \frac{\pi}{2}|A|^2\Big[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)+2\delta(\omega)\cos\{2(\omega_0 t+\theta)\}\Big].$$

10. Hint: See Auslander and Tolimieri (1985).
13. Hint: Use $f_n g_n - fg = f_n g_n - f_n g + f_n g - fg$ and then apply Schwarz's inequality.
17. $W_f(n,\theta) = |A|^2 \sum_{k=-\infty}^{\infty} \delta(\theta - an - k\pi)$.
18. Hint: $f_m(n) = f(n)\,m_f(n)$, $g_m(n) = g(n)\,m_g(n)$.
22. $\psi_0(x)$ is a Gaussian signal and $\psi_0\left(\frac{x}{\rho}\right)$ is also a Gaussian signal as in Exercise l(d) with $t_0 = 0$ and $\omega_0 = 0$.

6.6 Exercises

3. Physically, the convolution determines the wavelet transforms with dilated bandpass filters.
11. Write $\cos\omega_0 t = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$ and calculate the Fourier transform. $\hat{f}(\omega)$ has a maximum at $\omega = \pm\omega_0$, and then maximum values become more and more pronounced as σ increases.
12. $\hat{f}(\omega)$ has a maximum at the frequency $\omega = \omega_0$. Due to the jump discontinuity of $f(t)$ at time $t = \pm a$, $\left|\hat{f}(\omega)\right|$ decays slowly as $|\omega| \to \infty$. In fact, $\hat{f}(\omega) \notin L^1(\mathbb{R})$.
13. $\big(\mathcal{W}_\psi f\big)(a,b) = \frac{1}{\sqrt{a}}\left[\int_b^{b+\frac{a}{2}} f(t)\,dt - \int_{b+\frac{a}{2}}^{b+a} f(t)\,dt\right]$. Put $t = x + \frac{a}{2}$ in the integral $\int_{b+\frac{a}{2}}^{b+a} f(t)\,dt$ to get the answer.
15. Check only $\|\psi\| = 1$ and that $\psi_{m,n}$ make up a tight frame with frame constant 1 (see Daubechies 1992, p. 117).

7.7 Exercises

1.

$$\hat{\phi}(\omega) = \int_0^1 (1-t)\, e^{-i\omega t}\, dt + \int_{-1}^0 (1+t)\, e^{-i\omega t}\, dt$$

$$= \int_0^1 (1-t)\, e^{-i\omega t}\, dt + \int_0^1 (1-t)\, e^{i\omega t}\, dt = \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^2 .$$

$$\sum_{k=-\infty}^{\infty} \left| \hat{\phi}(\omega + 2\pi k) \right|^2 = 16 \sin^4 \left(\frac{\omega}{2}\right) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^4}$$

$$= \left(1 - \frac{2}{3} \sin^2 \frac{\omega}{2} \right) \text{ by (7.4.38).}$$

2. Hint: (a) follows from $\hat{\psi}(0) = 0$ and (b) follows from $\left(\frac{d\hat{\psi}}{d\omega} \right)_{\omega=0} = 0.$

(c) follows from $\hat{\psi}(\omega) = \exp\left(\frac{i\omega}{2} \right) \hat{f}(\omega)$ and

$$\psi(-t-1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp\left[- i\omega(t+1) \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp\left[-i\omega \left(t + \frac{1}{2} \right) \right] d\omega.$$

Also, (7.4.43) implies that $\hat{f}(\omega)$ is even and hence,

$$\psi(-t-1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp\left(i\omega t + \frac{1}{2} i\omega \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(\omega)\, e^{it\omega}\, d\omega = \psi(t).$$

5. (a) $\phi(x) = 1, \quad 0 \le x < 1.$
$\psi(x)$ is the Haar wavelet.
(b) $\phi(x) = B_2(x).$
$\psi(x) = x, \quad 0 \le x < \frac{1}{2}; \ \psi(x) = 2 - 3x, \quad \frac{1}{2} \le x < 1.$
(c) $\phi(x) = \delta(x), \quad \psi(x) = \delta(x).$

6. (a) It follows from (7.3.5) that

$$\int_{-\infty}^{\infty} \phi(x)\, dx = \sqrt{2} \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \phi(2x-n)\, dx, \quad (2x-n=t).$$

(b) Use $\hat{m}(\pi) = \dfrac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} (-1)^k c_k = 0.$

7. $\hat{\phi}(2\pi) = \hat{m}(\pi)\, \hat{m}\left(\frac{\pi}{2}\right) \hat{m}\left(\frac{\pi}{2^2}\right) \ldots$ and then $\hat{m}(\omega)$ has a zero of order n at $\omega = \pi$ if $\dfrac{d^m \hat{m}(\omega)}{d\omega^m} = 0$ when $\omega = \pi$ for $m = 0, 1, 2, \ldots (n-1)$. This gives the result.

8. (a) Hint: Write

$$\phi(x) = \sqrt{2} \sum_k c_k\, \phi(2x-k), \quad \phi(x) = \sqrt{2} \sum_m c_m\, \phi(2x-m)$$

$$\int_{-\infty}^{\infty} \phi^2(x)\, dx = 2 \sum_k \sum_m c_k c_m \int_{-\infty}^{\infty} \phi(2x-k)\, \phi(2x-m)\, dx$$

which is zero when $k \neq m$. But, when $k = m$
$\int_{-\infty}^{\infty} \phi^2(x)\, dx = \sum_k c_k^2 \int_{-\infty}^{\infty} \phi^2(t)\, dt$ which gives the result.

(b) Corresponding to the scaling function ϕ defined by (7.3.5), the wavelet may be written as

$$\psi(x) = \sum_k (-1)^k c_k\, \phi(2x + k - N + 1).$$

Use the orthogonality condition

$$\langle \psi(x), \psi(x-m) \rangle = \int_{-\infty}^{\infty} \psi(x)\, \psi(x-m)\, dx = 0 \text{ for all } m \text{ except } m = 0.$$

Substitute $\psi(x)$ in this integral so that

$$\int_{-\infty}^{\infty} \psi(x)\, \psi(x-m)\, dx = \sum_k \sum_s (-1)^{k+s} c_k\, c_s \int_{-\infty}^{\infty} \phi(2x + k - N + 1) \times \phi(2x + s - N + 1 - 2m)\, dx$$

The right-hand integral is zero unless $k = s - 2m$ so that

$$\int_{-\infty}^{\infty} \psi(x)\,\psi(x-m)\,dx = \sum_k (-1)^{2(k+m)} c_k\, c_{k+2m} \int_{-\infty}^{\infty} \phi^2(2t)\,dt.$$

This is always zero, except $m = 0$. This gives the result.

(c) When $m = 0$, we have

$$\sum_k (-1)^k c_k = 0$$

so that

$$\sum_{k=\text{even}} c_k = \sum_{k=\text{odd}} c_k = \frac{1}{\sqrt{2}} \quad \text{and} \quad \left(\sum_{k=\text{even}} c_k\right)^2 + \left(\sum_{k=\text{odd}} c_k\right)^2 = 1.$$

Multiplying gives $\displaystyle\sum_{k=0}^{N-1} c_k^2 + 2\sum_{k=0}^{N-1}\sum_{m=1}^{N/2-1} c_k\, c_{k+2m} = 1.$

This gives the result by 8(a).

9.

$$\begin{aligned}
c_0 + c_1 + c_3 + c_4 + c_5 &= \sqrt{2}\\
c_0 - c_1 + c_2 - c_3 + c_4 - c_5 &= 0\\
-c_1 + 2c_2 - 3c_3 + 4c_4 - 5c_5 &= 0\\
-c_1 + 4c_2 - 9c_3 + 16c_4 - 25c_5 &= 0\\
c_0c_2 + c_1c_3 + c_2c_4 + c_3c_5 &= 0\\
c_0c_4 + c_1c_5 &= 0\\
c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 &= 1.
\end{aligned}$$

11. Hint.

(a)

$$\begin{aligned}
\hat{\phi}(\omega) &= \prod_{k=1}^{\infty} m\left(\frac{\omega}{2^k}\right) = \prod_{k=1}^{\infty} \exp\left(-\frac{3i\omega}{2}\cdot\frac{1}{2^k}\right)\cos\left(\frac{3\omega}{2}\cdot\frac{1}{2^k}\right)\\
&= \prod_{k=1}^{\infty} \exp\left(-\frac{ix}{2^k}\right)\cos\left(\frac{x}{2^k}\right), \quad \left(x = \frac{3\omega}{2}\right)
\end{aligned}$$

$$= \prod_{k=1}^{\infty} \exp\left(-\frac{ix}{2^k}\right) \cdot \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right)$$

$$= \exp(ix) \cdot \frac{\sin x}{x} = \exp\left(-\frac{3i\omega}{2}\right) \frac{\sin\left(\frac{3\omega}{2}\right)}{\left(\frac{3\omega}{2}\right)}.$$

(b)

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\omega + 2\pi k)\right|^2 &= \frac{4}{9} \sin^2\left(\frac{3\omega}{2}\right) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2\pi k)^2} \\
&= \frac{4}{9} \sin^2\left(\frac{3\omega}{2}\right) \cdot \frac{1}{4} \operatorname{cosec}^2\left(\frac{\omega}{2}\right) \\
&= \frac{1}{9}\left(\frac{\sin 3t}{\sin t}\right)^2 \quad \left(t = \frac{\omega}{2}\right) \\
&= \frac{1}{9}\left(\frac{\sin(2t + t)}{\sin t}\right)^2 = \frac{1}{9}(1 + 2\cos\omega)^2 \\
&= \frac{1}{9}(3 + 4\cos\omega + 2\cos 2\omega).
\end{aligned}$$

This means that condition (b) in Theorem 7.3.1 is not satisfied. (c) follows from (a).

12. From (a) and (b), $\phi(x) = 0, \quad x < 0$ or $x > 3$.

$$\phi(x) + \phi(x+1) + \phi(x+2) = 1,$$
$$c\,\phi(x) + (c-1)\,\phi(x-1) + (c-2)\,\phi(x-2) = x.$$

Eliminating $\phi(x+2)$, $\phi(x)$, and $\phi(x+1)$ gives (a), (b), and (c) respectively.

9.6 Exercises

2. (a) Replace $f(x)$ and $\phi(x-k)$ by their Fourier inversion formulas in (9.4.7) and then apply

$$\int_{-\infty}^{\infty} \exp\{i(\omega_1 - \omega_2)x\}\, dx = 2\pi\, \delta(\omega_1 - \omega_2).$$

2. (b) Replace $a_{\phi,k}$ by 2(a) and $\phi(x-k)$ by its Fourier inversion formula to obtain

$$\sum_{k=-\infty}^{\infty} a_{\phi,k}\,\phi(x-k) = 2\pi \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega_1)\,d\omega_1 \int_{-\infty}^{\infty} \overline{\hat{\phi}}(\omega_1)\,\hat{\phi}(\omega_2)\exp(i\omega_2 x) \times \exp\left\{i(\omega_1-\omega_2)k\right\}d\omega_2.$$

Use the Poisson summation formula

$$\frac{1}{2\ell}\sum_{k=-\infty}^{\infty} \exp\left(-ixk\pi/\ell\right) = \sum_{m=-\infty}^{\infty} \delta\left(x-2m\ell\right)$$

with $x=\omega_2-\omega_1$ and $\ell=\pi$ so that

$$\sum_{k=-\infty}^{\infty} \exp\left\{i(\omega_1-\omega_2)k\right\} = 2\pi \sum_{m=-\infty}^{\infty} \delta\left(\omega_2-\omega_1-2\pi m\right).$$

The product $\overline{\hat{\phi}}(\omega_1)\,\hat{\phi}(\omega_2)$ is zero unless ω_1 and ω_2 both lie in $[0,2\pi]$, which is the case when $m=0$. Then, $\omega_1=\omega_2$ and $\overline{\hat{\phi}}(\omega_1)\,\hat{\phi}(\omega_2)=(2\pi)^{-2}$ give the last integral formula.

3. Use the inverse Fourier transform

$$\overline{\psi}\left(2^m x-k\right) = 2^{-m}\int_{-\infty}^{\infty} \overline{\psi}\left(\omega 2^{-m}\right)\exp\left(i\omega k 2^{-m}\right)\exp(-i\omega x)\,d\omega$$

in (9.4.2) and then apply

$$\int_{-\infty}^{\infty} \exp\left\{i(\omega_1-\omega_2)x\right\}dx = 2\pi\,\delta(\omega_1-\omega_2)$$

to obtain 3(a).

4. (9.4.1) and (9.4.9) are identical, provided

$$\sum_{k=-\infty}^{\infty} a_{\phi,k}\,\phi(x-k) = \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} a_{m,k}\,\psi(2^m x-k).$$

Use 2(b) and show that

$$\sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} a_{m,k}\,\psi(2^m x-k)$$

$$= \sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} 2\pi 2^{-m}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} \hat{f}(\omega_1)\,\hat{\psi}\left(\omega_1 2^{-m}\right)\hat{\psi}\left(\omega_2 2^{-m}\right) \times \exp(i\omega_2 x)\exp\left\{i(\omega_1-\omega_2)k2^{-m}\right\}d\omega_2.$$

Use Poisson's summation formula

$$\sum_{k=-\infty}^{\infty} \exp\left\{i(\omega_1-\omega_2)k2^{-m}\right\} = 2\pi 2^m \sum_{r=-\infty}^{\infty} \delta\left(\omega_2-\omega_1-2\pi r 2^m\right)$$

and the fact that $\hat{\psi}\left(\omega_1 2^{-m}\right)\hat{\psi}\left(\omega_2 2^{-m}\right)$ is zero unless $m = 0$ in the above sum and it is equal to (2π). Consequently,

$$\sum_{m=-\infty}^{-1}\sum_{k=-\infty}^{\infty} a_{m,k}\,\psi(2^m x-k)$$
$$= \sum_{m=-\infty}^{-1}\int_{2\pi 2^m}^{2\pi 2^{m-1}} \hat{f}(\omega)\,e^{i\omega x}d\omega = \int_0^{2\pi} \hat{f}(\omega)\,e^{i\omega x}d\omega.$$

5. Use results in 3(a) and 3(b) in $\left|a_{m,k}\right|^2 = a_{m,k}\,\bar{a}_{m,k}$ and $\left|\tilde{a}_{m,k}\right|^2 = \tilde{a}_{m,k}\,\bar{\tilde{a}}_{m,k}$. These lead to double integrals. Then, sum over k and m, which involves

$$\sum_{k=-\infty}^{\infty} \exp\left\{i(\omega_2-\omega_1)k2^{-m}\right\}$$

and its complex conjugate. Then, use the Poisson summation formula

$$\frac{1}{2\ell}\sum_{k=-\infty}^{\infty} \exp\left\{-\frac{ixk\pi}{\ell}\right\} = \sum_{r=-\infty}^{\infty} \delta(x-2r\ell)$$

with $x = \omega_2-\omega_1$ and $\ell = \pi 2^m$ to obtain

$$\sum_{k=-\infty}^{\infty} \exp\left\{i(\omega_1-\omega_2)k2^{-m}\right\} = 2\pi 2^m \sum_{r=-\infty}^{\infty} \delta(\omega_2-\omega_1-2\pi r 2^m).$$

Consequently,

$$\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-m}\left|a_{m,k}\right|^2 = (2\pi)^3 \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\omega)\,\bar{\hat{f}}(\omega)\,\bar{\hat{\psi}}\left(\omega 2^{-m}\right)\hat{\psi}\left(\omega 2^{-m}\right)d\omega$$
$$= 2\pi\int_0^{\infty} \hat{f}(\omega)\,\bar{\hat{f}}(\omega)\,d\omega$$

and hence

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-m} \left(\left| a_{m,k} \right|^2 + \left| \tilde{a}_{m,k} \right|^2 \right) = 2\pi \int_0^{\infty} \left\{ \hat{f}(\omega) \, \bar{\hat{f}}(\omega) + \hat{f}(-\omega) \, \bar{\hat{f}}(-\omega) \right\} d\omega$$

$$= 2\pi \int_{-\infty}^{\infty} \hat{f}(\omega) \, \bar{\hat{f}}(\omega) \, d\omega = \int_{-\infty}^{\infty} \left| f(x) \right|^2 dx.$$

Bibliography

Albeverio, S., Evdokimov, S., and Skopina, M. (2010), p-Adic multiresolution analysis and wavelet frames, *J. Fourier Anal. Appl.* **16**, 693–714.

Almeida, L. B. (1994), The fractional Fourier transform and time-frequency representations, *IEEE Trans. Signal Process.* **42**, 3084–3091.

Alpert, B. K. (1993), A class of basis in L^2 for the sparse representation of integral operators, *SIAM J. Math. Anal.* **24**, 246–262.

Anselmet, F., Gagne, Y., Hopfinger, E.J., and Antonia, R.A. (1984), High-order velocity structure functions in turbulent shear flows, *J. Fluid Mech.* **140**, 63–89.

Argoul, F., Arnéodo, A., Elezgaray, J., Grasseau, G., and Murenzi, R. (1988), Wavelet transform of fractal aggregates, *Phys. Lett.* **A135**, 327–333.

Argoul, F., Arnéodo, A., E1ezgaray, J., Grasseau, G., and Murenzi, R. (1990), Wavelet transform analysis of self-similarity of diffusion limited aggregate and electro-deposition clusters, *Phys. Rev.* **A41**, 5537–5560.

Argoul, F., Arnéodo, A., Grasseau, G., Gagne, Y., Hopfinger, E.J., and Frisch, U. (1989), Wavelet analysis of turbulence reveals the multifractal nature of the Richardson cascade, *Nature* **338**, 51–53.

Aslaksen, E.W. and Klauder, J.R. (1968), Unitary representations of the affine group, *J. Math. Phys.* **9**, 206–211.

Aslaksen, E.W. and Klauder, J.R. (1969), Continuous representation theory using the affine group, *J. Math. Phys.* **10**, 2267–2275.

Auslander, L. and Tolimieri, R. (1985), Radar ambiguity functions and group theory, *SIAM J. Math. Anal.* **16**, 577–601.

Bacchelli, S., Cotronei, M., and Sauer, T. (2002), Wavelets for multichannel signals, *Adv. Appl. Math.* **29**(4), 581–598.

Balian, R. (1981), Un principe d'incertitude fort en théoreie da signal ou en méchanique quantique, *C.R. Acad. Sci., Paris*, **292**, 1357–1362.

Batchelor, G.K. (1967), *The Theory of Homogeneous Turbulence*, Cambridge University Press, Cambridge.

Battle, G. (1987), A block spin construction of ondelettes, Part 1: Lemarié functions, *Commun. Math. Phys.* **110**, 601–615.

Behera, B. (2007), Wavelet packets associated with nonuniform multiresolution analyses, *J. Math. Anal. Appl.* **328**, 1237–1246.

Behera, B. and Jahan, Q. (2012a), Wavelet packets and wavelet frame packets on local fields of positive characteristic, *J. Math. Anal. Appl.* **395**, 1–14.

Behera, B. and Jahan, Q. (2012b), Multiresolution analysis on local fields and characterization of scaling functions, *Adv. Pure Appl. Math.* **3**, 181–202.

L. Debnath, F.A. Shah, *Wavelet Transforms and Their Applications*,
DOI 10.1007/978-0-8176-8418-1

Benedetto, J. J. and Benedetto, R. L. (2004), A wavelet theory for local fields and related groups, *J. Geomet. Anal.* **14** , 423–456.

Benedetto, J.J. and Frazier, M.W. (ed.) (1994), *Wavelets: Mathematics and Applications*, CRC Press, Boca Raton, Florida.

Benzi, R., Paladin, G., Parisi, G., and Vulpiani, A. (1984), On the multifractal nature of fully developed turbulence and chaotic systems, *J. Phys.* **A17**, 3521–3531.

Benzi, R. and Vergassola, M. (1991), Optimal wavelet transform and its application to two-dimensional turbulence, *Fluid Dyn. Res.* **8**, 117–126.

Bertrand, J. and Bertrand, P. (1992), A class of affine Wigner functions with extended covariance properties, *J. Math. Phys.* **33**, 2515–2527.

Beylkin, G. (1992), On the representation of operators in bases of compactly supported wavelets, *SIAM J. Numer. Anal.* **29**, 1716–1740.

Beylkin, G., Coifman, R., and Rokhlin, V. (1991), Fast wavelet transforms and numerical algorithms, *Commun. Pure Appl. Math.* **44**, 141–183.

Boashash, B. (1992), *Time-Frequency Signal Analysis, Methods and Applications*, Wiley-Halsted Press, New York.

Burt, P. and Adelson, E. (1983a), The Laplacian pyramid as a compact image code, *IEEE Trans. Commun.* **31**, 482–540.

Burt, P. and Adelson, E. (l983b), A multiplication spline with application to image mosaics, *ACM Trans. Graphics* **2**, 217–236.

Candés, E. and Donoho, D. (1999a), Ridgelets: A key to higher-dimensional intermittency?. *Phil. Trans. R. Soc. London A.* **357**, 2495–2509.

Candés, E. and Donoho, D. (1999b), Curvelets - a surprisingly effective nonadaptive representation for objects with edges, in *Curves and Surface Fitting*, Cohen et al. Eds, Vanderbilt University Press, Nashville, 105–120.

Candés, E. and Donoho, D. (2003a), Continuous curvelet transform-I: Resolution of the wavefront set, *Appl. Comput. Harmon. Anal.* **19**, 162–197.

Candés, E. and Donoho, D. (2003b), Continuous curvelet transform-II: Discretization and frames, *Appl. Comput. Harmon. Anal.* **19**, 198–222.

Carleson, L. (1966), Convergence and growth of partial sums of Fourier series, *Acta Math.* **116**, 135–157.

Chandrasekhar, S. (1949), On Heisenberg's elementary theory of turbulence, *Proc. Roy. Soc. London* **A200**, 20–33.

Chandrasekhar, S. (1956), A theory of turbulence, *Proc. Roy. Soc. London* **A210**, 1–19.

Chen, Q. and Cheng, Z. X. (2007), A study on compactly supported orthogonal vector-valued wavelets and wavelet packets, *Chaos, Solit. Fract.* **34**(4), 1024–1034

Chen, Q., Cao, H., and Shi, Z. (2009), Construction and characterizations of orthogonal vector-valued multivariate wavelet packets, *Chaos, Solit. Fract.* **40**, 1835–1844.

Choi, H.I. and Williams, W.J. (1989), Improved time-frequency representation of multicomponent signals using exponential kernels, *IEEE Trans. Acoust. Speech, Signal Process.* **37**(6), 862–87l.

Chui, C.K. (1992a), On cardinal spline wavelets, in *Wavelets and Their Applications*, (Ed. R.B. Ruskai et al.), Jones and Bartlett, Boston, 419–438.

Chui, C.K. (1992b), *An Introduction to Wavelets*, Academic Press, New York.

Chui, C.K. (1992c), *Wavelets*: A Tutorial in Theory and Applications, Academic Press, New York.

Chui, C. K. and Li, C. (1993a), Non-orthogonal wavelet packets, *SIAM J. Math. Anal.* **24**(3), 712–738.

Chui, C.K. and Shi, X. (1993b), Inequalities of Littlewood-Paley type for frames and wavelets, *SIAM J. Math. Anal.* **24**(1), 263–277.

Chui, C. K. and Li, J. A. (1996), A study of orthonormal multiwavelets, *J. Appl. Numer. Math.* **20**, 273–298.

Chui, C.K. and Wang, J.Z. (1991), A cardinal spline approach to wavelets, *Proc. Amer. Math. Soc.* **113**, 785–793.

Chui, C.K. and Wang, J.Z. (1992), On compactly supported spline wavelets and a duality principle, *Trans. Amer. Math. Soc.* **330**, 903–915.

Claasen, T.A.C.M. and Mecklenbräuker, W.F.G. (1980), The Wigner distribution-A tool for time-frequency signal analysis, Part I: Continuous time signals; Part II: Discrete time signals; Part III: Relations with other time-frequency signal transformations, *Philips J. Res.* **35**, 217–250; 276–300; 372–389.

Cohen, A. (1990a), Ondelettes, analyses multirésolutions et filtres miroir en quadrature, *Ann. Inst. H. Poincaré, Anal. non linéaire* **7**, 439–459.

Cohen, A. (1990b), *Ondelettes, analyses multirésolutions et traitement numérique du signal*, Ph.D. Thesis, Université Paris, Dauphine.

Cohen, A. (1992), *Biorthogonal Wavelets, in Wavelets - A Tutorial in Theory and Applications*, Academic Press, New York, 123–152.

Cohen, A. (1995), *Wavelets and Multiscale Signal Processing*, Chapman and Hall, London.

Cohen, A., Daubechies, I., and Feauveau, J. (1992), Biorthogonal basis of compactly supported wavelets, *Commun. Pure and Appl. Math.* **45**, 485–560.

Cohen, A. and Daubechies, I. (1993), On the instability of arbitrary biorthogonal wavelet packets, *SIAM J. Math. Anal.* **24** (5), 1340–1354.

Cohen, L. (1966), Generalized phase-space distribution functions, *J. Math. Phys.* **7**(5), 781–786.

Cohen, L. (1989), Time-frequency distributions: Review, *Proc. IEEE* **77**(7), 941–981.

Cohen, L. and Posch, T. (1985), Positive-time frequency distribution functions, *IEEE Trans. Acoust. Speech Signal Process.*, **33**, 31–38.

Cohen, L. and Zaparovanny, Y. (1980), Positive quantum joint distributions, *J. Math. Phys.* **21**, 794–796.

Coifman, R.R., Jones, P., and Semees, S. (1989), Two elementary proofs on the L^2 boundedness of Cauchy integrals on Lipschitz curves, *J. Amer. Math. Soc.* **2**, 553–564.

Coifman, R. R., Meyer, Y., Quake, S., and Wickerhauser, M. V. (1990), Signal processing and compression with wavelet packets, Technical Report, Yale University.

Coifman, R.R., Meyer, Y., and Wickerhauser, M.V. (1992a), Wavelet analysis and signal processing, in *Wavelets and Their Applications* (Ed. M.B. Ruskai et al.), Jones and Bartlett, Boston, 153–178.

Coifman, R.R., Meyer, Y., and Wickerhauser, M.V. (1992b), Size properties of wavelet packets, in *Wavelets and Their Applications*, (Ed. M.B. Ruskai et al.), Jones and Bartlett, Boston, 453–470.

Dahlke, S., Kutyniok, G., Maass, P., Sagiv, C., Stark, H. G., and Teschke, G. (2008), The uncertainty principle associated with the continuous shearlet transform, *Int. J. Wavelets Multiresolut. Inf. Process.* **6**(2), 157–181.

Daubechies, I. (1988a), Time-frequency localization operators: A geometric phase space approach, *IEEE Trans. Inform. Theory* **34**, 605–612.

Daubechies, I. (1988b), Orthogonal bases of compactly supported wavelets, *Commun. Pure Appl. Math.* **41**, 909–996.

Daubechies, I. (1990), The wavelet transform, time-frequency localization and signal analysis, *IEEE Trans. Inform. Theory* **36**, 961–1005.

Daubechies, I. (1992), *Ten Lectures on Wavelets*, NSF-CBMS Regional Conference Series in Applied Math. **61**, SIAM Publications, Philadelphia.

Daubechies, I. (1993), Different perspectives on wavelets, *Proceedings of the Symposium on Applied Mathematics* **47**, 15–37, American Mathematical Society, Providence.

Daubechies, I., Grossmann, A., and Meyer, Y. (1986), Painless nonorthogonal expansion, *J. Math. Phys.* **27**, 1271–1283.

Daubechies, I., Han, B., Ron, A., and Shen, Z. (2003), Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.* **14**, 1–46.

Daubechies, I. and Klauder, J. (1985), Quantum mechanical path integrals with Wiener measure for all polynomial Hamiltonians II, *J. Math. Phys.* **26**, 2239–2256.

Daubechies, I. and Lagarias, J. (1991), Two-scale difference equations I. Existence and global regularity of solutions, *SIAM J. Math. Anal.* **22**, 1388–1410.

Daubechies, I. and Lagarias, J. (1992), Two-scale difference equations II. Local regularity, infinite products of matrices and fractals, *SIAM J. Math. Anal.* **23**, 1031–1055.

De Bruijn, N.G. (1967), Uncertainty principles in Fourier analysis, In *Inequalities* (ed. O. Shisha), Academic Press, Boston, 57–71.

De Bruijn, N.G. (1973), A theory of generalized functions with applications to Wigner distribution and Weyl correspondence, *Nieuw. Arch. Voor Wiskunde* **21**, 205–280.

Debnath, L. (1978), *Oceanic Turbulence*, Memoir Series of the Calcutta Mathematical Society, Calcutta.

Debnath, L. (1995), *Integral Transforms and Their Applications*, CRC Press, Boca Raton, Florida.

Debnath, L. (1998a), Wavelet transforms, fractals, and turbulence, in *Nonlinear Instability, Chaos, and Turbulence*, Vol. I, (Ed. L. Debnath and D.N. Riahi), Computational Mechanics Publications, WIT Press, Southampton, England.

Debnath, L. (1998b), Brief introduction to history of wavelets, *Internat. J. Math. Edu. Sci. Tech.* **29**, 677–688.

Debnath, L. (1998c), Wavelet transforms and their applications, *Proc. Indian Natl. Sci. Acad.* **64A**, 685–713.

Debnath, L. (2001), *Wavelet Transforms and Time-Frequency Signal Analysis*, Birkhäuser, Boston.

Debnath, L. and Bhatta, D. (2014), *Integral Transforms and Their Applications*, Third Edition, Chapman and Hall, CRC Press, Boca Raton, Florida.

Debnath, L. and Mikusinski, P. (1999), *Introduction to Hilbert Spaces with Applications*, Second Edition, Academic Press, Boston.

Demanet, L. and Ying, L. (2007a), Curvelets and wave atoms for mirror-extend images, *Proc. SPIE Wavelets XII*, San Diego.

Demanet, L. and Ying, L. (2007b), Wave atoms and sparsity of oscillatory patterns, *Appl. Comput. Harmon. Anal.* **23**(3), 368–387.

Dirac, P.A.M. (1958), *The Principles of Quantum Mechanics* (Fourth Edition), Oxford University Press, Oxford.

Do, M. N. and Vetterli, M. (2003), Contourlets, in: *Beyond Wavelets*, G. V. Welland, Ed., Academic Press, Amsterdam, 83–105.

Do, M. N. and Vetterli, M. (2005), The contourlet transform: an efficient directional multiresolution image representation, *IEEE Trans. Image Proc.* **14**, 2091–2106.

Donoho, D. (1994), Interpolating wavelet transforms, *Appl. Comput. Harmonic Anal.*, **2**, 1–12.

Donoho, D. L. (2000), Wedgelets: Nearly-minimax estimation of edges, *Anna. Statist.* **27**, 859–897.

Duffin, R.I. and Schaeffer, A.C. (1952), A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72**, 341–366.

Duhamel, P. and Vetterli, M. (1990), Fast Fourier transforms: A tutorial review and a state of the art, *Signal Process.* **19** (4), 259–299.

Erlebacher, G., Hussani, M.Y., and Jameson, L.M. (1966),*Wavelets, Theory and Applications*, Oxford University Press, Oxford.

Everson, R., Sirovich, L., and Sreenivasan, K.R. (1990), Wavelet analysis of turbulent jet, *Phys. Lett.* **A145**, 314–319.

Falconer, K.J. (1990), *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley, Chichester.

Farge, M. (1992), Wavelet transforms and their applications to turbulence, *Ann. Rev. Fluid Mech.* **24**, 395–457.

Farge, M., Goirand, E., Meyer, Y., Pascal, F., and Wickerhauser, M.V. (1992), Improved predictability of two-dimensional turbulent flows using wavelet packet compression, *Fluid Dyn. Res.* **10**, 229–242.

Farge, M., Guezenne, J., Ho, C.M., and Meneveau, C. (1990), *Continuous Wavelet Analysis of Coherent Structures*, Proceedings of the Summer Progress Centre for Turbulence Research, Stanford University-NASA Ames, 331–398.

Farge, M. and Holschneider, M. (1989), Analysis of two-dimensional turbulent flow, *Proc. Scaling, Fractals, and Nonlinear Variability in Geophysics II*, Barcelona.

Farge, M. and Holschneider, M. (1990a), Interpolation of two-dimensional turbulence spectrum in terms of singularity in the vortex cores, *Europhys. Lett.* **15**, 737–743.

Farge, M., Holschneider, M., and Colonna, J.F. (1990b), Wavelet analysis of coherent structures in two-dimensional turbulent flows, in *Topological Fluid Mechanics* (Ed. A.K. Moffatt and A. Tsinober), Cambridge University Press, Cambridge, 765–766.

Farge, M., Kevlahan, N., Perrier, V., and Goirand, E. (1996), Wavelets and turbulence, *Proc. IEEE* **84**(4), 639–669.

Farge, M., Kevlahan, N., Perrier, V., and Schneider, K. (1999a), Turbulence analysis, modelling and computing using wavelets, *Wavelets in Physics* (Ed. J.e. van den Berg), Cambridge University Press, Cambridge, 117–200.

Farge, M., Schneider, K., and Kevlahan, N. (1999b), Non-Gaussianity and coherent vortex simulation for two-dimensional turbulence using an adaptive orthonormal wavelet basis, *Phys. Fluids* **11**(8), 2187–2201.

Farge, M. and Rabreau, G. (1988), Transformé en ondelettes pour detecter et analyser les structures cohérentes dans les ecoulements turbulents bidimensionnels, *C.R. Acad. Sci. Paris Ser.* II **307**, 1479–1486.

Farge, M. and Rabreau, G. (1989), Wavelet transform to analyze coherent structures in two-dimensional turbulent flows, *Proceedings on Scaling, Fractals and Nonlinear Variability in Geophysics I*, Paris.

Farkov, Yu. A. (2005a), Orthogonal p-wavelets on $\mathbb{R}^+$, in: *Proc. Int. Conf. Wavelets and Splines*, St. Petersburg State University, 4–26.

Farkov, Yu. A. (2005b), Orthogonal wavelets with compact support on locally compact Abelian groups, *Izv. Math.* **69**(3), 623–650.

Farkov, Yu. A. (2009), On wavelets related to Walsh series, *J. Approx. Theory*. **161**, 259–279.

Farkov, Yu. A., Maksimov, A. Yu., and Stoganov, S. A. (2011), On biorthogonal wavelets related to the Walsh functions, *Int. J. Wavelets Multiresolut. Inf. Process.* **9**(3), 485–499.

Farkov, Yu. A. and Rodionov, E. A. (2012), Nonstationary wavelets related to the Walsh functions, *Amer. J. Comput. Math.* **2**, 82–87.

Feng, Z. and Chen, G. (2001), Non-orthogonal wavelet packets with r-scaling functions, *J. Comput. Anal. Appl.* **3**(4), 317–330.

Fernandes, F. (2002), Directional, shift-insensitive, complex wavelet transforms with controllable redundancy, *Ph.D Thesis*, Rice University.

Flandrin, P. (1992), Wavelet analysis and synthesis of fractional Brownian motion, *IEEE Trans. Inform. Theory* **38**(2), 910–916.

Fourier, J. (1822). *Théorie Analytique de la Chaleur*, English Translation by A. Freeman, Dover Publications, New York, 1955.

Fowler, J. E. and Li, H. (2002), Wavelet transforms for vector fields using uni-directionally balanced multiwavelets, *IEEE Trans. Signal Process.* **50**(12), 3018–3027.

Freeman, W. and Adelson, E. (1991), The design and use of steerable filters, *IEEE Trans. Pattern Anal. Machine Intell.* **13**(9), 891–906.

Frisch, U. and Parisi, G. (1985), On the singularity structure of fully developed turbulence, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics* (Ed. M. Ghil, R. Benzi, and G. Parisi), North Holland, Amsterdam, 84–125.

Frisch, U., Sulem, P.L., and Nelkin, M. (1978), A simple dynamical model of intermittent fully developed turbulence, *J. Fluid Mech.* **87**, 719–736.

Frisch, U. and Vergassola, M. (1991), A prediction of the multifractal model; the intermediate dissipation range, *Europhys. Lett.* **14**, 439–450.

Frisch, U. (1995), *Turbulence. The Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge.

Fröhlich, J. and Schneider, K. (1997), An adaptive wavelet-vaguelette algorithm for the solution of PDEs, *J. Comput. Phys.* **130**, 174–191.

Fuglede, B. (1974), Commuting self-adjoint partial differential operators and a group theoretic problem, *J. Funct. Anal.* **16**, 101–121.

Gabardo, J. P. and Nashed, M. (1998a), Nonuniform multiresolution analyses and spectral pairs, *J. Funct. Anal.* **158**, 209–241.

Gabardo, J. P. and Nashed, M. (1998b), An analogue of Cohen's condition for nonuniform multiresolution analyses, in: *Wavelets, Multiwavelets and Their Applications*, A. Aldroubi, E. Lin (Eds.), Contemp. Math. **216**, Amer. Math. Soc. Providence, RI, 41–61.

Gabardo, J. P. and Yu, X. (2006), Wavelets associated with nonuniform multiresolution analyses and one-dimensional spectral pairs, *J. Math. Anal. Appl.* **323**, 798–817.

Gabardo, J. P. and Yu, X. (2007), Nonuniform wavelets and wavelet sets related to one-dimensional spectral pairs, *J. Approx. Theory.* **145**, 133–139.

Gabor, D. (1946), Theory of communications, *J. Inst. Electr. Eng. London*, **93**, 429–457.

Gagne, Y. and Castaing, B. (1991), A universal non-globally self-similar representation of the energy spectra in fully developed turbulence, *C.R. A cad. Sci.* Paris, **312**, 414–430.

Gao, X., Nguyen, T. and Strang, G. (2002), A study of two-channel complex-valued filter banks and wavelets with orthogonality and symmetry properties, *IEEE Trans. Signal Process.* **50**(4), 824–833.

Gelfand, I.M. (1950), Eigenfunction expansions for an equation with periodic coefficients, *Dokl. Akad. Nauk. SSR* **76**, 1117–1120.

Geronimo, J., Hardin, D., and Massopust, P. (1994), Fractal functions and wavelet expansions bases on several scaling functions, *J. Approx. Theory.* **78**, 373–401.

Glauber, R.J. (1964), *in Quantum Optics and Electronics* (Ed. C. de Witt, A. Blandin, and C. Cohen-Tannoudji), Gordon and Breach, New York.

Golubov, B. I., Efimov, A. V., and Skvortsov, V. A. (1991), *Walsh Series and Transforms: Theory and Applications*, Kluwer, Dordrecht.

Grant, H.L., Stewart, R.W., and Moilliet, A. (1962), Turbulence spectra from a tidal channel, *J. Fluid Mech.* **12**, 241–268.

Grochenig, K. (2001), *Foundations of Time Frequency Analysis*, Birkhäuser, Boston.

Grossmann, A. and Morlet, J. (1984), Decomposition of Hardy functions into square integrable wavelets of constant shape, *SIAM J. Math. Anal.* **15**, 723–736.

Guo, K. and Labate, D. (2007), Optimally sparse multidimensional representations using shearlets, *SIAM J. Math. Anal.* **39**, 298–318.

Guo, K., Lim, W-Q., Labate, D., Weiss, G., and Wilson, E. (2006), Wavelets with composite dilations and their MRA properties, *Appl. Comput. Harmon. Anal.* **20**, 202–236.

Haar, A. (1910), Zur Theorie der orthogonalen funktionen-systeme, *Math. Ann.* **69**, 331–371.

Hall, M.G. (1972), Vortex breakdown, *Ann. Rev. Fluid Mech.* **4**, 195–218.

Halsey, T.C., Jensen, M.H., Kadanoff, L.P., Procaccia, I., and Shraiman, B.l. (1986), Fractal measures and their singularities: The characterization of strange sets, *Phys. Rev.* **A33**, 1141–1151.

Heil, C. and Walnut, D. (1989), Continuous and discrete wavelet transforms, *SIAM Rev.* **31**, 628–666.

Heisenberg, W. (1948a), Zur statistischen theori der turbulenz, *Z. Phys.* **124**, 628–657.

Heisenberg, W. (1948b), On the theory of statistical and isotropic turbulence, *Proc. Roy. Soc. London* **A195**, 402–406.

Hlawatsch, F. and Boudreaux-Bartels, F. (1992), Linear and quadratic time-frequency signal representations, *IEEE Signal. Process. Mag.* **9**(2), 21–67.

Hlawatsch, F. and Flandrin, P. (1992), The interference structure of the Wigner distribution and related time-frequency signal representations, in *The Wigner Distribution-Theory and Applications in Signal Processing*, (Ed. W.F.G. Mecklenbrauker), Elsevier, Amsterdam.

Holschneider, M. (1988), L'analyse d'objets fractals et leu transformaé en ondelettes, These, Université de Provence, Marseille.

Holschneider, M., Kronland-Martinet, R., Morlet, J., and Tchamitchian, P. (1989), A real-time algorithm for signal analysis with the help of the wavelet transform, in *Wavelets, Time-Frequency Methods and Phase Space*, (Ed. J.M. Combes, et al.), Springer-Verlag, Berlin, 289–297.

Holschneider, M. and Tchamitchian, P. (1991), Pointwise regularity of Riemann's "nowhere-differentiable" function, *Invent. Math.* **105**, 157–175.

Huang, Z., Kawall, J.G., and Keffer, J.F. (1997), Development of structure within the turbulent wake of a porous body. Part 2. Evolution of the three dimensional features, *J. Fluid Mech.* **329**, 117–136.

Hubbard, B. (1996), *The World According to Wavelets*, A. K. Peters, Massachusetts.

Hunt, J.C.R. and Vassilicos, J.C. (1991), Kolmogorov's contributions to the physical and geometrical understanding of small-scale turbulence and recent developments, *Proc. Roy. Soc. London* **A434**, 183–210.

Hunt, J.C.R., Kevlahan, N.K.R., Vassilicos, J.C, and Farge, M. (1993), Wavelets, fractals, and Fourier transforms: Detection and analysis of structure, in *Wavelets, Fractals, and Fourier Transforms*, (Ed. M. Farge, J.CR. Hunt, and J.C Vassilicos), Clarendon Press, Oxford.

Hussain, J.J., Schoppa, W., and Kim, J. (1997), Coherent structures near the wall in a turbulent channel flow, *J. Fluid Mech.* **332**, 185–214.

Jaffard, S. and Meyer, Y. (1996), *Wavelet Methods for Pointwise Regularity and Local Oscillations of Functions*, Volume 123, American Mathematical Society, Providence, RI.

Jähne, B. (1995), *Digital Image Processing*, Springer Verlag, New York.

Janssen, A.J.E.M. (1981a), Weighted Wigner distribution vanishing on lattices, *J. Math. Anal. Appl.* **80**, 156–167.

Janssen, A.J.E.M. (1981b), Gabor representations of generalized functions, *J. Math. Anal. Appl.* **83**, 377–394.

Janssen, A.J.E.M. (1982), On the locus and spread of pseudo-density functions in the time-frequency plane, *Philips J. Res.* **37**, 79–110.

Janssen, A.J.E.M. (1984), Gabor representation and Wigner distribution of signals, *Proc. IEEE* 41, B.2.1-41, B.24.

Janssen, A.J.E.M. (1988), The Zak transform: A signal transform for sampled time-continuous signals, *Philips J. Res.* **43**, 23–69.

Janssen, A.J.E.M. (1992), The Smith-Barnwell condition and non-negative scaling functions, *IEEE Trans. Inform. Theory* **38**, 884–885.

Jiang, H. K., Li, D. F., and Jin, N. (2004), Multiresolution analysis on local fields, *J. Math. Anal. Appl.* **294**, 523–532.

Jimenez, J. (1981), *The Role of Coherent Structures in Modeling Turbulence and Mixing*, Springer-Verlag, New York.

Kahane, J.P. and Katznelson, Y. (1966), Sur les ensembles de divergence des séries trigonometriques, *Studia Math.* **26**, 305–306.

Kaiser, G. (1994), *A Friendly Guide to Wavelets*, Birkhäuser, Boston.

Keinert, F. (2004), *Wavelets and Multiwavelets*, Chapman and Hall, CRC Press, Florida.

Kessler, B. (2000), A construction of orthogonal compactly supported multiwavelets on $\mathbb{R}^2$, *Appl. Comput. Harmon. Anal.* **9**, 146–165.

Khrennikov, A. Yu., Shelkovich, V. M., and Skopina, M. (2009), p-Adic refinable functions and MRA-based wavelets, *J. Approx. Theory.* **161**, 226–238.

Kingsbury, N. (1999), Image processing with complex wavelets, *Phil. Trans. R. Soc. London A.* **357**, 2543–2560.

Kingsbury, N. (2001), Complex wavelets for shift invariant analysis and filtering of signals, *Appl. Comput. Harmon. Anal.* **10**(3), 234–253.

Kirkwood, J.G. (1933), Quantum statistics of almost classical ensembles, *Phys. Rev.* **44**, 31–37.

Klauder, J. and Skagerstam, B.S. (1985), *Coherent States*, World Scientific Publications, Singapore.

Kolmogorov, A.N. (1926), Une série de Fourier-Lebesgue divergent partout, *C.R. Acad. Sci. Paris* **183**, 1327–1328.

Kolmogorov, A.N. (1941a), The local structure of turbulence in an incompressible fluid with very large Reynolds numbers, *Dokl. Akad. Nauk. SSSR* **30**, 301–305.

Kolmogorov, A.N. (1941b), Dissipation of energy under locally isotropic turbulence, *Dokl. Akad. Nauk. SSSR* **32**, 16–18.

Kolmogorov, A.N. (1962), A refinement of previous hypotheses concerning the local structure of turbulence of a viscous incompressible fluid at high Reynolds numbers, *J. Fluid Mech.* **13**, 82–85.

Kovacevic, J. and Vetterli, M. (1992), Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbb{R}^n$, *IEEE Trans. Inform. Theory* **38**, 533–555.

Kozyrev, S. V. (2002), Wavelet analysis as a p-adic spectral analysis, *Izv. Akad. Nauk, Ser. Mat.* **66**(2), 149–158.

Kozyrev, S. V. (2004), p-Adic pseudo differential operators and p-adic wavelets, *Theor. Math. Physics.* **138**(3), 1–42.

Kutyniok, G. and Labate, D. (2009), Resolution of the wavefront set using continuous shearlets, *Trans. Amer. Math. Soc.* **361**, 2719–2754.

Landau, L. and Lifschitz, E. (1959), *Fluid Mechanics*, Addison-Wesley, New York.

Lang, W. C. (1996), Orthogonal wavelets on the Cantor dyadic group, *SIAM J. Math. Anal.* **27**, 305–312.

Lawton, W. (1993), Applications of complex valued wavelet transforms to subband decomposition, *IEEE Trans. Sig. Proc.* **41**(12), 3566–3568.

Lemarié, P.G. (1988), Une nouvelle base d'ondelettes de $L^2(\mathbb{R}^n)$, *J. Math. Pure Appl.* **67**, 227–236.

Lemarié, P.G. (1989), Bases d'ondelettes sur les groupes de Lie stratifiés, *Bull. Soc. Math. France* **117**, 211–232.

Lemarié, P.G., ed., (1990), *Les Ondelettes en 1989*, Lecture Notes in Mathematics, Volume 1438, Springer Verlag, Berlin.

Lemarié, P.G. (1991), La propriété de support minimal dans les analyses multirésolution, *C. R. Acad. Sci. Paris* **312**, 773–776.

Lemarié, P.G. and Meyer, Y. (1986), Ondelettes et bases hilbertiennes, *Rev. Mat. Iberoam.* **2**, 1–18.

Leonard, A. (1974), Energy cascade in large eddy simulations of turbulent fluid flows, *Adv. Geophys.* **18A**, 237–249.

Lewis, A.S. and Knowles, G. (1992), Image compression using the $2D$-wavelet transform, *IEEE Trans. Image Process.* **1**, 244–265.

Li, D. F. and Jiang, H. K. (2008), The necessary condition and sufficient conditions for wavelet frame on local fields, *J. Math. Anal. Appl.* **345**, 500–510.

Lian, Q. F. (2004), Wavelets and wavelet packets related to a class of dilation matrices, *Acta Math. Sinica*. **5**, 1–14.

Lin, C.C. (1948), Note on the law of decay of isotropic turbulence,*Proc. Nat. Acad. Sci.* U.S.A., **34**, 230–233.

Long, R. and Chen, W. (1997), Wavelet basis packets and wavelet frame packets, *J. Fourier Anal. Appl.* **3**, 239–256.

Loughlin, P., Pitton, J., and Atlas, L.E. (1994), Construction of positive time-frequency distributions, *IEEE Trans. Signal Process.* **42**, 2697–2836.

Low, F. (1985), *Complete Sets of Wave packets, A Passion for Physics - Essays in Honor of Geoffrey Chew* (Ed. C. De Tar et al.), World Scientific, Singapore, 17–22.

Lu, Y. and Do, M. N. (2007), Multidimensional directional filter banks and surfacelets, *IEEE Trans. Image Process.* **16**(4), 918–931.

Lukacs, M. (1960), *Characteristic Functions*, Griffin Statistical Monographs, New York.

Lukomskii, S. F. (2012), Multiresolution analysis on product of zero-dimensional Abelian groups, *J. Math. Anal. Appl.* **385**, 1162–1178.

Mallat, S. (1988), Multiresolution representation and wavelets, Ph.D. Thesis, University of Pennsylvania.

Mallat, S. (1989a), Multiresolution approximations and wavelet orthonormal basis of $L^2(\mathbb{R})$, *Trans. Amer. Math. Soc.* **315**, 69–88.

Mallat, S. (1989b), A theory for multiresolution signal decomposition: The wavelet representation, *IEEE Trans. Patt. Recog. And Mach. Intelt.* **11**, 678–693.

Mallat, S. (1989c), Multifrequency channel decompositions of images and wavelet models, *IEEE Trans. Acoust. Signal Speech Process.* **37**, 2091–2110.

Mallat, S. and Hwang, W.L. (1991), Singularity detection and processing with wavelets, *IEEE Trans. Inform. Theory* **38**, 617–643.

Malvar, H.S. (1990a), Lapped transforms for efficient transforms/subband coding, *IEEE Trans. Acoust. Speech Signal Process.* **38**, 969–978.

Malvar, H. (1990b), The design of two-dimensional filters by transformations, *Seventh Annual Princeton Conference on ISS*, Princeton University Press, Princeton, New Jersey, **247**, 251–264.

Mandelbrot, B.B. (1974), Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier. *J. Fluid Mech.* **62**, 331–358.

Mandelbrot, B.B. (1975), On the geometry of homogeneous turbulence with stress on the fractal dimension of the iso-surfaces of scalars, *J. Fluid Mech.* **72**, 401–420.

Mandelbrot, B.B. (1982), *The Fractal Geometry of Nature*, W.H. Freeman, New York.

Margenau, H. and Hill, R.N. (1961), Correlation between measurements in quantum theory, *Prog. Theor. Phys.* **26**, 722–738.

Mecklenbräuker, W. and Hlawatsch, F. (1997), *The Wigner Distribution*, Elsevier, Amsterdam.

Mendlovic, D., Zalevsky, Z., Mas, D., García, J., and Ferreira, C. (1997), Fractional wavelet transform, *Appl. Opt.* **36**, 4801–4806.

Meneveau, C. (1991), Analysis of turbulence in the orthonormal wavelet representation, *J. Fluid Mech.* **232**, 469–520.

Meneveau, C. (1993), Wavelet analysis of turbulence: The mixed energy cascade, in *Wavelets, Fractals, and Fourier Transforms* (Ed. M. Farge, J.C.R. Hunt, and J.e. Vassilicos), Oxford University Press, Oxford, 251–264.

Meneveau, C. and Sreenivasan, K.R. (1987a), Simple multifractal cascade model for fully developed turbulence, *Phys. Rev. Lett.* **59**, 1424–1427.

Meneveau, C. and Sreenivasan, K.R. (1987b), In *Physics of Chaos and Systems Far From Equilibrium* (Ed. M.D. Van, and B. Nichols), Nucl. Phys. B, North Holland, Amsterdam, 2–49.

Métais, O. and Lesieur, M. (1992), Spectral large-eddy simulation of isotropic and stably stratified turbulence, *J. Fluid Mech.* **239**, 157–194.

Meyer, Y. (1990), *Ondelettes et opérateurs*, tomes I, II, and III, Herman, Paris.

Meyer, Y. (1993a), *Wavelets, Algorithms and Applications* (translated by R.D. Ryan), SIAM Publications, Philadelphia.

Meyer, Y. (1993b), *Wavelets and Operators*, Cambridge University Press, Cambridge.

Micchelli, C. (1991), Using the refinement equation for the construction of prewavelets, *Numer. Alg.* **1**, 75–116.

Moffatt, H.K. (1984), Simple topological aspects of turbulent vorticity dynamics, in *Turbulence and Chaotic Phenomena in Fluids* (Ed. T. Tatsumi), Elsevier, New York, 223–230.

Monin, A.S. and Yaglom, A.M. (1975), *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2, MIT Press, Cambridge.

Morlet, J., Arens, G., Fourgeau, E., and Giard, D. (1982a), Wave propagation and sampling theory, Part I: Complex signal land scattering in multilayer media, *J. Geophys.* **47**, 203–221.

Morlet, J., Arens, G., Fourgeau, E., and Giard, D. (1982b), Wave propagation and sampling theory, Part II: Sampling theory and complex waves, *J. Geophys.* **47**, 222–236.

Mouri, H. and Kubotani, H. (1995), Real-valued harmonic wavelets, *Phys. Lett.* **A201**, 53–60.

Mouri, H., Kubotani, H., Fugitani, T., Niino, H., and Takaoka, M. (1999), Wavelet analysis of velocities in laboratory isotropic turbulence, *J. Fluid Mech.* **389**, 229–254.

Myint-U., T. and Debnath, L. (1987), *Partial Differential Equations for Scientists and Engineers* (Third Edition), Prentice Hall, New York.

Neumann, J. and Steidl, G. (2005), Dual-tree complex wavelet transform in the frequency domain and an application to signal classification, *Int. J. Wavelet. Multiresol. Inform. Process.* **3**, 43–66.

Newland, D.E. (1993a), Harmonic wavelet analysis, *Proc. Roy. Soc. London* **A443**, 203–225.

Newland, D.E. (1993b), *An Introduction to Random Vibrations, Spectral and Wavelet Analysis*, (Third Edition), Longman Group Limited, London, England.

Newland, D.E. (1994), Harmonic and musical wavelets, *Proc. Roy. Soc. London* **A444**, 605–620.

Novikov, E.A. (1989), Two-particle description of turbulence, Markov property and intermittency, *Phys. Fluids* **AI**, 326–330.

Novikov, E.A. (1990), The effects of intermittency on statistical characteristics of turbulence and scale similarity of breakdown coefficients, *Phys. Fluids* **A2**, 814–820.

Oboukhov, A.M. (1941), On the distribution of energy in the spectrum of turbulent flow, *Dokl. Akad. Nauk. SSSR* **32**, 19–21.

Oboukhov, A.M. (1962), Some specific features of atmospheric turbulence, *J. Fluid Mech.* **13**, 77–81.

Onsager, L. (1945), The distribution of energy in turbulence, *Phys. Rev.* **68**, 286–292.

Onsager, L. (1949), Statistical hydrodynamics, *Nuovo Omenta Suppl.* **6**, 279–287.

Ozaktas, H. M., Zalevsky, Z., and Kutay, M. A. (2000), *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley, New York.

Perrier, V., Philipovitch, T., and Basdevant, C. (1995), Wavelet spectra compared to Fourier spectra, *J. Math. Phys.* **36**, 1506–1519.

Pinsky, M. (2001), *Introduction to Fourier Analysis and Wavelets*, Brooks-Cole, USA.

Prasad, A. and Mahato, A. (2012), The fractional wavelet transform on spaces of type S, *Integ. Transform. Spec. Funct.* **23**, 237–249.

Protasov, V. Yu. and Farkov, Yu. A. (2006), Dyadic wavelets and refinable functions on a half-line, *Sb. Math.* **197**(10), 1529–1558.

Quak, E. and Weyrich, N. (1997a), Algorithms for trigonometric wavelet packets, *Appl. Comput. Harmon. Anal.* **4**, 74–96.

Quak, E. and Weyrich, N. (1997b), Algorithms for spline wavelet packets on an interval, *BIT.* **37**, 76–95.

Ramakrishnan, D. and Valenza, R. J. (1999), *Fourier Analysis on Number Fields*, Graduate Texts in Mathematics 186, Springer-Verlag, New York.

Ramchandran, K. and VetterIi, M. (1993), Best wavelet packet bases in a rate-distortion sense, *IEEE Trans. Image Process.* **2**(2), 160–175.

Richardson, L.F. (1926), Atmospheric diffusion shown on a distance-neighbour graph, *Proc. Roy. Soc. London* **A110**, 709–737.

Rihaczek, A.W. (1968), Signal energy distribution in time and frequency, *IEEE Trans. Inform. Theory* **14**, 369–374.

Ruskai, M.B., Beylkin, G., Coifman, R., Daubechies, 1., Mallat, S., Meyer, Y., and Raphael, L. (ed.) (1992), *Wavelets and Their Applications*, Jones and Bartlett, Boston.

Saffman, P.G. (1990), Vortex dynamics and turbulence, in *Proceedings of the NATO Advanced Workshop on the Global Geometry of Turbulence: Impact of Nonlinear Dynamics*, 1–30.

Saffman, P.G. and Baker, G.R. (1979), Vortex interactions, *Ann. Rev. Fluid Mech.* **11**, 95–122.

Sarker, S.K. (1985), Generalization of singularities in nonlocal dynamics, *Phys. Rev.* **A31**, 3468–3472.

Schempp, W. (1984), Radar ambiguity functions, the Heisenberg group and holomorphic theta series, *Proc. Amer. Math. Soc.* **92**, 103–110.

Schneider, K. and Farge, M. (1997), Wavelet forcing for numerical simulation of two-dimensional turbulence, *C.R. Acad. Sci. Paris Série II* **325**, 263–270.

Schneider, K. and Farge, M. (2000), Computing and analyzing turbulent flows using wavelets, in *Wavelet Transforms and Time-Frequency Signal Analysis* (Ed. L. Debnath), Birkhäuser, Boston, 181–216.

Schneider, K., Kevlahan, N.K.R., and Farge, M. (1997), Comparison of an adaptive wavelet method and nonlinearly filtered pseudo-spectral methods for two-dimensional turbulence, *Theor. Comput. Fluid Dyn.* **9**, 191–206.

Schwarz, K.W. (1990), Evidence for organized small-scale structure in fully developed turbulence, *Phys. Rev. Lett.* **64**, 415–418.

Schipp, F., Wade, W. R., and Simon, P. (1990), *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, New York.

Sejdić, E., Djurović, I., and Stanković, L. (2011), Fractional Fourier transform as a signal processing tool: an overview of recent developments, *Signal Process.* **91**, 1351–1369.

Selesnick, I. W. (2001), Hilbert transform pairs of wavelet bases, *IEEE Sig. Proc. Letters.* **8**(6), 170–173.

Sen, N.R. (1951), On Heisenberg's spectrum of turbulence, *Bull. Cal. Math. Soc.* **43**, 1–7.

Sen, N.R. (1958), On decay of energy spectrum of isotropic turbulence, *Proc. Natl. Inst. Sci. India* **A23**, 530–533.

Shah, F. A. (2009), Construction of wavelet packets on p-adic field, *Int. J. Wavelets Multiresolut. Inf. Process.* **7**(5), 553–565.

Shah, F. A. (2012a), Biorthogonal wavelet packets related to the Walsh polynomials, *J. Classical Anal.* **1**, 135–146.

Shah, F. A. (2012b), Non-orthogonal p-wavelet packets on a half-line. *Anal. Theory Appl.* **28**(4), 385–396.

Shah, F. A. (2012c), On some properties of p-wavelet packets via the Walsh-Fourier transform, *J. Nonlinear Anal. Optimiz.* **3**(2), 185–193.

Shah, F. A. (2012d), Gabor frames on a half-line, *J. Contemp. Math. Anal.* **47**(5), 251–260.

Shah, F. A. (2013), Tight wavelet frames generated by the Walsh polynomials. *Int. J. Wavelets, Multiresolut. Inf. Process.* **11**(6), 1350042 (15 pages).

Shah, F. A. and Debnath, L. (2011a), Dyadic wavelet frames on a half-line using the Walsh-Fourier transform, *Integ. Transf. Special Funct.* **22**(7), 477–486.

Shah, F. A. and Debnath, L. (2011b), p-Wavelet frame packets on a half-line using the Walsh-Fourier transform, *Integ. Transf. Special Funct.* **22**(12), 907–917.

Shah, F. A. and Debnath, L. (2012), Explicit construction of M-band framelet packets, *Analysis.* **32**(4), 281–294.

Shah, F. A. and Debnath, L. (2013), Tight wavelet frames on local fields, *Analysis.* **33**, 293–307.

Shannon, C.E. (1949), Communications in the presence of noise, *Proc. IRE* **37**, 10–21.

She, Z.S., Jackson, E., and Orszag, S.A. (1991), Structure and dynamics of homogeneous turbulence, models and simulations, *Proc. Roy. Soc. London* **A434**, 101–124.

Sheehan, J.P. and Debnath, L. (1972), On the dynamic response of an infinite Bernoulli-Euler beam, *Pure Appl. Geophys.* **97**, 100–110.

Shen, Z. (1995), Non-tensor product wavelet packets in $L^2(\mathbb{R}^s)$, *SIAM J. Math. Anal.* **26**(4), 1061–1074.

Shi, J., Zhang, N., and Liu, X. (2012), A novel fractional wavelet transform and its applications, *Sci. China Inf. Sci.* **55**, 1270–1279.

Slavakis, K. and Yamada, I. (2001), Biorthogonal unconditional bases of compactly supported matrix valued wavelets, *Numer. Funct. Anal. Optim.* **22**, 223–253.

Spaendonck, R., Fernandes, F., Coates, M., and Burrus, C. (2000), Non-redundant, directionally selective, complex wavelets, *IEEE Proc. Int. Conf. Image Proc.* 379–382.

Sreenivasan, K.R. (1991), Fractals and multifractals in fluid turbulence, *Ann. Rev. Fluid Mech.* **23**, 539–600.

Sreenivasan, K.R. and Meneveau, C. (1986), The fractal facets of turbulence, *J. Fluid Mech.* **173**, 357–386.

Stadler, W. and Shreeves, R.W. (1970), The transient and steady state response of the infinite Bernoulli-Euler beam with damping and an elastic foundation, *Quart. J. Mech. and Appl. Math.* **XXII**, 197–208.

Starck, J. L., Candés, E., and Donoho, D. L. (2002), The curvelet transform for image denoising, *IEEE Trans. Image Process.* **11**(6), 670–684.

Strang, G. (1989), Wavelets and dilation equations, *SIAM Review* **31**, 614–627.

Strang, G. (1993), Wavelet transforms versus Fourier transforms, *Bull. Amer. Math. Soc.* **28**, 288–305.

Strang, G. and Nguyen, T. (1996), *Wavelets and Filter Banks*, Wellesley Cambridge Press, Boston.

Stromberg, J.O. (1982), A modified Franklin system and higher order spline systems on $\mathbb{R}^n$ as unconditional bases for Hardy spaces, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund, II* (Ed. W. Beckner et al.), Wadsworth Mathematics Series, Wadsworth, Belmont, California, 475–493.

Sun, Q., Bi, N., and Huang, D. (2001), *A Introduction to Multiband Wavelets*, Zhejiang University Press, China.

Sun, L. and Cheng, Z. (2007), Construction of a class of compactly supported orthogonal vector-valued wavelets, *Chaos, Solit. Fract.* **34**, 253–261.

Taibleson, M. H. (1975), *Fourier Analysis on Local Fields*, Princeton University Press, Princeton.

Tchamitchan, P. (1987), Biorthogonalité et théorie des opérateurs, *Rev. Mat., Iberoam.* **3**, 163–189.

Tham, J. Y., Shen, L., Lee, S. L., and Tan, H. H. (1998), Good multifilter properties: a new tool for understanding multiwavelets, in: *Proc. Int. Conf. Image Science, System and Tech.*, Las Vegas, 52–59.

Thomson, Sir William and Tait, P.G. (1879), Treatise on natural philosophy, Cambridge University Press, New edition: *Principles of Mechanics and Dynamics* (1962), Dover Publications, New York.

Townsend, A.A. (1951), On the fine scale structure of turbulence, *Proc. Roy. Soc. London* **A208**, 534–556.

Vaidyanathan, P.P. (1987), Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having the perfect reconstruction property, *IEEE Trans. Acoust. Speech Signal Process.* **35**, 476–492.

Vaidyanathan, P.P. (1993), *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, New Jersey.

Vaidyanathan, P.P. and Hoang, P.Q. (1988), Lattice structures for optimal design and robust implementation of two-channel perfect-reconstruction QMF banks, *IEEE Trans. Acoust. Speech Signal Process.* **36**, 81–94.

Vassilicos, J.C (1992), The multi-spiral model of turbulence and intermittency, in *Topological Aspects of the Dynamics of Fluids and Plasmas*, Kluwer, Amsterdam, 427–442.

Vassilicos, J.C (1993), Fractals in turbulence, in *Wavelets, Fractals and Fourier Transforms: New Developments and New Applications*, (Ed. M. Farge, J.CR. Hunt, and J.C Vassilicos), Oxford University Press, Oxford, 325–340.

Vassilicos, J.C and Hunt, J.CR. (1991), Fractal dimensions and spectra of interfaces with applications to turbulence, *Proc. Roy. Soc. London* **A435**, 505–534.

Vetterli, M. (1984), Multidimensional subband coding: Some theory and algorithms, *Signal Process.* **6**, 97–112.

Vetterli, M. and Herley, C. (1992), Wavelet and filter banks: theory and design, *IEEE Trans. Signal Process.* **40**, 2207–2232.

Vetterli, M. and Kovacevic, J. (1995), *Wavelets and Subband Coding*, Prentice Hall, Englewood Cliffs, New Jersey.

Ville, J. (1948), Théorie et applications de la notion de signal analytique, *Cables Transm.* **2A**, 61–74.

Vincent A. and Meneguzzi, M. (1991), The spatial structure and statistical properties of homogeneous turbulence, *J. Fluid Mech.* **225**, 1–20.

Vincent, A. and Meneguzzi, M. (1994), The dynamics of vorticity tubes in homogeneous turbulence, *J. Fluid Mech.* **258**, 245–254.

Von Neumann, J. (1945), *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton.

Walsh, J.L. (1923), A closet set of normal orthogonal functions, *Amer. J. Math.* **45**, 5–24.

Walter, G.G. (1994), *Wavelets and Other Orthogonal Systems with Applications*, CRC Press, Boca Raton.

Wickerhauser, M.V. (1994), *Adapted Wavelet Analysis from Theory to Software*, AK Peters, Wellesley, Massachusetts.

Wigner, E.P. (1932), On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* **40**, 749–759.

Wilcox, C. (1960), The synthesis problem for radar ambiguity functions, MRC Technical Report 157, Mathematics Research Center, U.S. Army, University of Wisconsin, Madison.

Wojtaszczyk, P. (1997), *A Mathematical Introduction to Wavelets*, Cambridge University Press, Cambridge.

Woodward, P.M. (1953), *Probability and Information Theory, with Applications to Radar*, Pergamon Press, London.

Wornell, G.W. (1995), *Signal Processing with Fractals: A Wavelet-Based Approach*, Prentice-Hall, Englewood Cliffs, New Jersey.

Xia, X. G. and Suter, B.W (1996), Vector-valued wavelets and vector filter banks, *IEEE Trans. Signal Process.* **44**(3), 508–518.

Xiao-Feng, W., Gao, H., and Feng, J. (2009), The characterization of vector-valued multivariate wavelet packets associated with a dilation matrix, *Chaos, Solit. Fract.* **42**, 1959–1966.

Yamada, M., Kida, S., and Ohkitani, K. (1993), Wavelet analysis of PDEs in turbulence, in *Unstable and Turbulent Motion of Fluid* (Ed. S. Kida), World Scientific Publishing, Singapore, 188–190.

Yamada, M. and Ohkitani, K. (1991), An identification of energy cascade in turbulence by orthonormal wavelet analysis, *Prog. Theor. Phys.* **86**, 799–815.

Zak, J. (1967), Finite translation in solid state physics,*Phys. Rev. Lett.* **19**, 1385–1397.

Zak, J. (1968), Dynamics of electrons in solids in external fields, *Phys. Rev.* **168**, 686–695.

Zayed, A. (1993), *Advances in Shannon's Sampling Theory*, CRC Press, Boca Raton, Florida.

Zhao, Y., Atlas, L.E., and Marks, R. (1990), The use of cone-shaped kernels for generalized time-frequency representations of nonstationary signals, *IEEE Trans. Acoust., Speech Signal Process.* **38**, 1084–1091.

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