Notes

1 INTRODUCTION TO DERIVATIVES

1. As described in Hull (2006, pp. 35–36), in the case of an exchange-traded derivatives contract with physical delivery, a party with a short position in the derivative delivers the physical underlying to a party designated by the exchange who has a long position in the same derivative. In addition, with certain exchange-traded contracts, the party with the short position may have a choice of what underlying to deliver. For example, a seller of a Chicago Board of Trade (CBOT) US Treasury future can deliver one of a set of delivery-grade bonds. For more details, see the Rules and Regulations on the CBOT website, http://www.cbot.com.


3. Another widely traded derivative that pays out linearly in the value of the underlying is a “swap,” which is a generalized version of a forward contract. With a forward contract, there is one exchange of cash on a date in the future, while a swap contract has exchanges of cash on multiple dates in the future. See Chapter 7 of Hull (2006) for more details.

4. In addition to European-style exercise, options are often “American-style” exercise. In this case, the option holder can exercise at many different times up to the expiration of the option. For other exotic-option styles, see Chapter 22 of Hull (2006).

5. We avoid using the term “at-the-money” (which means that the value of the underlying is equal to the strike upon expiration) because using this term with digital options can cause confusion. For example, as we see in moment, a digital call pays out if the underlying is equal to the strike upon expiration while a digital put does not pay out if the underlying is equal to the strike upon expiration. See Hull (2006,
p. 188) for alternative definitions of in-the-money, at-the-money, and out-of-the-money.

6. Digital options are also called “binary” options or “all-or-nothing” options.

7. Section 5.1 will discuss these derivatives in more detail.

8. While derivatives-market participants consider digital options to be exotic, academic researchers typically consider digital options to be the fundamental building blocks of finance. See the upcoming discussions in Sections 2.1 and 5.3.

9. For additional details on customers and market makers, see Harris (2003), particularly Sections 3.1 and 19.3.

10. See Chapter 15 of Hull (2006) for further discussion on delta hedging.

11. See Weber (1999) for details on electronic trading of derivatives on an exchange, and see Chapter 6 of Liebenberg (2002) for details on electronic trading of derivatives OTC.

12. See Harris (2003, pp. 90–91) for a brief introduction to call auctions. Garbade and Silber (1979) show that auctions in general can reduce overall transaction costs. Several papers show that call auctions in particular can be useful for aggregating liquidity. See, among many, Economides and Schwartz (1995), Theissen (2000), and Kalay, Wei, and Wohl (2002).


15. See Hull (2006, pp. 36–37) for a brief discussion of these and other order types. For further details, see Chapter 4 of Harris (2003).


17. See Barbour (1963, pp. 74–84), and De La Vega (1996) for details on trading in Amsterdam during the 1600s.


19. See Gates (1973) for a history of early options trading in the US, and see Lurie (1979) for a history of the early years of the CBOT.


22. In fact, the BIS reports daily numbers for April 2004. We convert these to annual numbers by assuming 250 trading days in a year. See Bank for International Settlements (2004, p. 14) for more details.


26. See Anderson (1984) on the development of the CFTC.

27. See Miller (1986) and Telser (1986) for a discussion of the legal aspects involving the CFTC and cash settlement.

28. Of the top ten contracts listed in Burghardt (2005), the following six are cash-settled: KOSPI 200 options, 3-month Eurodollar futures, 28-day Mexican Peso deposit futures, E-Mini S&P 500 index futures, 3-month Euribor futures, and 3-month Eurodollar options.

29. See Cravath, Swaine, and Moore (2001) and White (2001) for more details on the CFMA.

2 INTRODUCTION TO PARIMUTUEL MATCHING

1. Most financial underlyings are likely to require many more than four states. Parimutuel matching can handle such cases, but we chose a small state-space example here to most simply illustrate the properties of parimutuel matching.

2. State claims are also often called “state-contingent” claims or “Arrow–Debreu” securities.

3. In practice, most parimutually traded derivatives expire on the same date as the auction so premium and payouts are often exchanged on the same date, satisfying this assumption. See Sections 3.2 and 3.3 for details.

4. In fact, a customer can approximate selling a state claim as follows. For illustration, consider a customer who wants to sell the first state claim. This customer can submit orders to buy every other state claim. In this case, the customer loses money if the first state occurs and may profit if any of the other states occur, as a seller of the first state claim would. However, this customer pays premium up front (as opposed to receiving premium up front as a seller typically would), and this customer receives different payouts depending on what state occurs (since there is no way to guarantee that the customer receives the same payout for every state purchased, since the state prices are unknown at the time the orders are submitted).

5. In fact, it may be the case that a specific state is sufficiently unlikely that no customer invests premium in that state. However, Equation (2.1) is a convenient assumption that simplifies exposition and avoids any dividing-by-zero issues, which would appear for example in Equation (2.8).

6. Equation (2.4) holds because the premium amounts are non-zero, as specified in Equation (2.1).
7. Chapter 6 will add a third no-arbitrage restriction to handle the pricing of derivatives that pay out in multiple states. See Equation (6.4).
8. Equation (2.8) resembles the result from the classical Arrow-Debreu equilibrium, which shows that the ratio of state prices equals the ratio of marginal utilities in those two states. See Arrow (1964).
9. It is not hard to check that the self-hedging conditions of Equation (2.6) and the relative-demand pricing of Equation (2.8) are mathematically equivalent when state prices are positive and sum to one. Appendix 6D will prove this result.
10. It is worth pointing out that parimutuel matching does not always fill all orders when customers can submit limit orders. See the example in Section 5.2.
11. Although this is somewhat non-standard, in this example we fill customers for a fractional number of contracts to make sure that the self-hedging property of Equation (2.6) is satisfied exactly.
12. Transaction costs may include fees for trading on an exchange, clearing fees, and costs of capital.
13. This example illustrates a very simple no-arbitrage relationship. For a detailed discussion of other no-arbitrage relationships, see Chapter 4 of Cox and Rubinstein (1985).
15. In addition to the research described here, parimutuel matching is closely related to the academic field of “combinatorial auctions.” Section 3.5 will describe combinatorial auctions and compare them to parimutuel matching.
16. The main difference between parimutuel wagering on horses to win and the framework in Section 2.1 is that the horse racing association typically takes 20 cents of every one dollar bet as a fee, whereas the framework in Section 2.1 assumed that fees equal zero.
17. In parimutuel wagering, arbitrage opportunities are generally not possible within one wagering type.
20. The strongest evidence that at least some bettors have consistently made money wagering on horse races is with strategies based on computer handicapping systems for Hong Kong races. Benter (1994) and Chapman (1994) describe some of these econometric models for handicapping Hong Kong horse races based on information from past races.
Kaplan (2002) describes Benter’s consistent success and the success of others in Hong Kong.

21. In somewhat related work, Chapter 8 will prove that state prices are unique in a more general parimutuel setting (which includes orders for vanilla options and limit orders).

22. Also, Norvig (1967) proves that the Eisenberg and Gale (1959) result holds in a convergence sense, that is, bettors reach the same equilibrium iteratively by being able to respond to changing parimutuel prices by altering their wagers through time.

23. In related work, Levin (1994) provides a comprehensive solution methodology for computing optimal parimutuel wagers under a variety of different assumptions.

24. In addition, Hanson (2003) discusses adapting scoring rules for use as a combinatorial information market mechanism. See Section 3.5.

25. Hanson, Oprea, and Porter (2006) provide additional experimental evidence that it is difficult to distort prices in a related setting.

3 PARIMUTUEL APPLICATIONS

1. See Buck (1977, pp. 3–14) for a history of parimutuel wagering on horse races, and see Epstein (1977, pp. 287–288) for a short history of horse racing. Many lotteries have features in common with parimutuel wagering on horse races and date back to biblical times. See Brenner and Brenner (1990, pp. 1–18) for a detailed history of lotteries. Of further note, insurance or risk-sharing pools have many features in common with parimutuel wagering and existed in ancient Greece. See Bernstein (1996, p. 92).

2. For country by country parimutuel wagering statistics on thoroughbred horse racing, see the International Federation of Horseracing Authorities website: http://www.horseracingintfed.com.


4. See Freeman, Freeman, and McKinley (1982) for more information on these sports.

5. The International Federation of Horseracing Authorities reports that Hong Kong wagered 65 million Hong Kong dollars parimutuelly in 2003. Based on the exchange rate of 7.8 Hong Kong dollars per 1 US dollar (the exchange rate on 31 December 2003), this converts to 8.3 billion US dollars.

6. For Hong Kong population information, see Chapter 20 of the Hong Kong 2003 Yearbook published on the Hong Kong Special Administrative Region Government website: http://www.info.gov.hk/eindex.htm.
7. Although customers cannot submit limit orders at the racetrack, customers can sometimes cancel wagers after they are made. See Camerer (1998) for further discussion.

8. See the discussion regarding the win, place, and show pools in Section 2.3.3.

9. A financial-method patent is considered to be a type of “business-method” patent. In a recent decision, the USPTO’s Board of Patent Appeals overturned a patent examiner’s rejection of a business-method patent as “outside the technological arts,” finding that there was no basis in law for such a rejection. This reversal is widely viewed as favorable to business-method patents (and by implication for financial-method patents). See Ex Parte Carl A. Lundgren, Appeal No. 2003-2088, heard 20 April 2004. Also, see Falloon (1999), Heaton (2000), and Lerner (2002) for further discussion of the patentability of financial methods.

10. See also Shiller (2003) for some discussion of Longitude.

11. Such short-dated options have very high “gamma” and “theta” exposures. See Hull (2006, Chapter 15) for the definition of these “Greeks.”

12. The CME began allowing their customers to trade economic derivatives via their Globex platform in March 2006.


14. As discussed by Horrigan (1987), CPI futures were first traded on the Coffee, Sugar, and Cocoa Exchange in the 1980s without generating significant volume.

15. For more details on this contract, see the CME website http://www.cme.com/trading.

16. It is worth noting that Miller (1986) does not discuss the possibility of a parimutuel auction for trading futures and options on CPI.

17. In a similar vein to TIPS, several academics and practitioners have proposed trading GDP-indexed bonds. See Shiller (1993, 2003) and Borensztein and Mauro (2004).

18. Wolfers and Zitzewitz (2004) note that such markets are also often called prediction markets, information markets, or event futures.


20. Bonds and credit-default swaps have an event-market flavor. For example, a corporate bond makes scheduled payments so long as the company does not default. A catastrophic risk or “CAT” bond makes scheduled payments as long as the underlying event (i.e., the hurricane or earthquake) does not occur.

21. It is worth mentioning two additional call-auction frameworks that were used for trading US equities, although neither is in operation today. Woodward (2001) and Domowitz and Madhavan (2001) describe the Arizona Stock Exchange, which used a standard call auction in


23. As is frequently noted in this literature, a combinatorial auction is an instance of the well-known “set-packing problem,” which itself is of the same class of problem as the “knapsack problem.” In the standard version of the knapsack problem, a decision maker can choose from among different types and shapes of objects. Each object type has a size or volume and a value to the decision maker. The goal of the decision maker is to choose a set of objects from among the different types in order to maximize the total value. The decision maker is constrained in that the total volume of objects selected must be below some determined constant or the size of the knapsack.

24. See de Vries and Vohra (2003) for citations on these applications.

25. Recall from endnote 23 that a combinatorial auction is closely related to the knapsack problem. A parimutuel derivatives auction relates to the knapsack problem as follows. We can think of the objects as the derivative strategies, and we can think of their shapes as the payout functions on these orders. The parimutuel knapsack must have a flat edge on top representing the criteria that irrespective of which state occurs, the same level of payout has been provided to the group of auction participants.

26. Sandholm, Suri, Gilpin, and Levine (2001) use the term “combinatorial auction” to strictly mean a multiple-item auction with a single seller where participants bid for bundles of items, while they use the term “combinatorial exchange” to refer to a multiple-item auction with multiple buyers and multiple sellers where participants bid and offer for bundles of items. We use the term “combinatorial auction” more generally to mean any auction where bids and/or offers may be made for bundles of items.

27. As will be shown in Equation (4.4), a parimutuel derivatives auction requires that a customer’s fill can equal any value between zero and the customer’s requested amount.

28. As discussed by Sandholm, Suri, Gilpin, and Levine (2001), de Vries and Vohra (2003), and Sandholm and Suri (2006), the solution to the standard combinatorial auction problem is “NP-hard,” meaning that no “polynomial-time” solution is known to exist and worst-case computation times are high. Although Chapter 2 shows that solving the parimutuel wagering problem is trivial, solving for prices and fills in a parimutuel derivatives auction requires an iterative approach (see Chapter 7). Even in this case, though, computation times for typical auctions are under half a second. Peters, So, and Ye (2006) prove an important and related theoretical result. They show that parimutuel derivatives auctions can be solved in polynomial time.

30. See Chapter 7 for more details.

31. While advances in computing have helped bring parimutuel matching to the derivatives markets, it is interesting to note that in the past, parimutuel wagering has spurred on innovation in computing. The Julius Totalisator, used for calculating odds on horse races in the early 1900s, was one of the leading computers of its time. See Swade (1987).

32. Without mentioning parimutuel call auctions specifically, several authors have pointed out the usefulness of electronic call auctions. See, for example, Domowitz and Madhavan (2001), the collection of papers in Schwartz (2001), and the collection of papers in Schwartz, Byrne, and Colaninno (2003). Schwartz (2003, p. xvi) argues that “computer technology is essential for unleashing the power of a modern call.” Cohen and Schwartz (2001) emphasize that electronic call auctions can be run with low operating costs. Economides and Schwartz (1995) stress the importance of electronic trading for call auctions: “with computerization, participants can see the order flow and interact with the system on a real-time basis, entering their orders while the computer broadcasts the orders and indicated clearing prices.”

4 A CASE STUDY USING NONFARM PAYROLLS

1. Even though NFP is the main variable of interest in the employment report, Fair (2003, p. 311) notes that the unemployment rate and the average-hourly earnings also get some attention.

2. For more details on how the BLS calculates this statistic, see Chapter 2 of the BLS Handbook of Methods, found on http://www.bls.gov, and see Krueger and Fortson (2003).

3. Krueger and Fortson (2003, pp. 947–948) show that fixed-income markets respond in a statistically insignificant way to revisions on NFP. Most market participants consider the first released value of NFP to be the “headline” number.

4. For more recent work in this area, see the papers by Tashjian (1995), Tashjian and Weissman (1995), and Corkish, Holland, and Vila (1997).

5. See the following six articles in the Wall Street Journal in their C Section on 4 June 2004:
   1. E. S. Browning – “Stocks Fall on Interest-Rate Fears Ahead of Jobs Report;”
   2. Jesse Eisinger – “A Yawner;”
5. Karen Talley – “Intel, Agere Fall Amid Broad Selling;” and

These forecasts can be accessed on a Bloomberg Terminal™ by typing “ECO” and then hitting the “GO” button.

7. Note that if indicator 2 holds for an underlying, then it is very likely to be the case that indicator 1 holds as well. In other words, given that the release of an underlying has a large impact on financial markets (indicator 2), then that release is likely to be widely followed and forecasted by financial-market participants (indicator 1).


9. Academic researchers disagree on whether a close relationship between a new futures contract and existing futures contracts is likely to be associated with the success or failure of the new futures contract. For evidence that a close relationship decreases the chance of success, see Duffie and Jackson (1989) and Cuny (1993), who develop theoretical models showing that new futures markets for which there are no close substitutes may have the greatest chance for success. Further, using statistical tests, Black (1986) and Corkish, Holland, and Vila (1997) find empirical evidence that introducing a contract that is highly correlated with other contracts is a negative factor for a contract’s success. Our view (which is expressed above) is that the fact that several financial markets have a predictable reaction to NFP increases demand for NFP derivatives. Providing evidence consistent with that view, Merton (1992b) postulates that highly correlated contracts may, in certain cases, complement one another as opposed to competing with each other, creating a virtuous circle of liquidity. Further, Tashjian (1995) and Tashjian and Weissman (1995) point out that exchanges often introduce futures contracts with a high correlation to other existing contracts, providing indirect evidence that there must be some advantage to doing so.

10. Section 4.4.4 discusses the range forward in more detail. In addition, Section 5.1.5 will present the payout on the range forward, and Section 6.2.1 will present the formula for the price of the range forward.

11. Gürkaynak and Wolfers (2006) show empirically that the forecast from a parimutuel derivatives auction is more accurate than the median from a survey of economists (which is often used to measure the market consensus). More generally, there is a significant body of academic research that shows that forecasts based on real money at stake are more accurate than forecasts based on surveys. See Section 2.3.2 and the box at the end of Section 4.4 for more details.
12. If the surprise is defined as the difference between NFP and the median forecast of the Bloomberg survey, then the standard deviation of surprises is 101 thousand jobs over the sample period. This value is similar to the standard deviation of surprises based on the auction’s range forward price.

13. In studying other financial markets, Andersen, Bollerslev, Diebold, and Vega (2005) show that NFP has a predictable impact in the five minutes after its release on foreign equity indexes, currency markets, and foreign bond markets.

14. When the underlying is the traded price of a commodity, similar principles apply. Gray (1978), Jones (1982), Garbade and Silber (1983), and Paul (1985) argue that the underlying used for cash settlement must be a reliable and accurate measure of the commercial value of that commodity. They suggest that the underlying must be immune to manipulation and pricing distortions such as short squeezes. Relatedly, Duffie and Rahi (1995) note that for trading to be successful, there must be confidence that no person trading has material private information on the value of the underlying.


16. See, for example, Enders (2004) for a discussion of time-series techniques.

17. In fact, the standard deviation of surprises is statistically indistinguishable from the one-month conditional standard deviation of NFP, as we now show. Define the statistic $A$ as the standard deviation of NFP surprises divided by the one-month conditional standard deviation of NFP.

$$A = \frac{SD[\psi_t]}{SD_{t-1}[u_t]}$$

Consider the null hypothesis $H_0$ that the population value of $A$ is equal to one, versus the one-sided alternative hypothesis $H_1$ that the population value of $A$ is less than one. If the random variables in the numerator and the denominator of $A$ are independent and normally distributed, then under the null hypothesis, $A^2$ is $F$ distributed and the lower $A^2$ is the greater the evidence against the null hypothesis. In this case, $A^2$ equals 0.61. Using 34 degrees of freedom for the numerator and the denominator of the $F$ statistic, this value of $A^2$ equates to a $p$-value of 8%. Thus, $A$ is statistically indistinguishable from one. See Snedecor and Cochran (1989, pp. 98–99) or another standard statistics text for more details on this type of “equality of variance” test.
18. The BLS publishes the dates and times for the release of economic statistics on their website http://www.bls.gov typically months ahead of each release.

19. In related work, Brenner, Eldor, and Hauser (2001) show that traders are willing to pay a premium for options that are liquid.

20. The diversity of forecasts shows that any measurement of the “market’s forecast” for NFP is really an average of the forecasts of market participants, as opposed to there being any consensus among market participants on the upcoming value of NFP.

21. Since it represents the monthly change in jobs in the nonfarm sector, NFP can be positive or negative. Although the lowest strike in the auctions on August 2005 NFP was 0 jobs, there is nothing that precludes NFP options that have negative strikes. In fact, in several past NFP auctions, customers have traded options with negative strikes.

22. Section 5.1 describes vanilla capped calls, vanilla floored puts, and the range forward in more detail.

23. To keep the notation simple, this section uses the variables $a, r, x, w$, and $\pi$ without any subscripts to denote a particular customer order. By necessity, however, Chapters 5 and higher use subscripts to indicate that variables are related to specific customer orders. Hence, Chapters 5 and higher use the variables $a_j, r_j, x_j, w_j$, and $\pi_j$ to denote that these variables are related to the $j$th customer order.

24. Replicating derivatives for these NFP auctions using 377 state claims is computationally very expensive, that is, computing prices and fills takes a long time. Fortunately, derivatives can be replicated using a much smaller set of fundamental building blocks. Longitude’s parimutuel matching engine uses such an approach, and that approach is described in Lange, Baron, Walden, and Harte (2003, Chapter 11). This replication approach is related to the “supershare approach,” which was introduced by Hakansson (1976, 1978) and discussed in Cox and Rubinstein (1985).

25. To guarantee that each state has a positive price, a small amount of premium is invested in each state at the start of every auction. See the discussion in Section 6.1.

26. Recall the assumptions from Section 2.1 that (1) all auction participants meet their financial obligations, and so there is no credit risk in a parimutuel auction; and (2) there is no discounting required between the date that premium is paid and the date that option payouts are made. These two assumptions are crucial for the second no-arbitrage condition.

27. These three no-arbitrage conditions imply that the prices in a parimutuel derivatives auction satisfy “put-call parity,” an important condition in option-pricing theory as described in Hull (2006, pp.212–215). In this context, put-call parity relates the price of the range forward to the price of vanilla capped calls and vanilla floored puts. See Appendix 6B for further discussion.
28. Section 6.2.4 will discuss this in further detail. In particular, that section will rigorously define auction volume as being equal to a quantity known as the total “market exposure.”

29. In fact, Appendix 6D will prove that, under no-arbitrage conditions, the parimutuel principle of self-hedging and relative-demand pricing are equivalent.

30. The fact that prices are available on all tradable derivatives is a unique feature of parimutuel derivatives auctions. Section 6.1 will discuss this further.

31. Parimutuel wagering works in a similar fashion to parimutuel derivatives auctions, as bettors can view indicative odds (calculated based on all wagers submitted up to that time) throughout the wagering period.

32. Note that there is no price for the vanilla floored put struck at 0 jobs (the floor) or for the vanilla capped call struck at 375 thousand jobs (the cap). Both of these derivatives pay out zero regardless of the value of the underlying, and consequently, neither derivative is tradable in this auction.

33. The fee for an option is typically 1% of the option’s maximum payout. Since a digital option pays out $1 if it expires in-the-money, the fee is $0.01. For example, the price to buy the digital call struck at 275 thousand jobs is 0.111 (0.111 = 0.101 + 0.01) and the price to sell the digital call struck at 275 thousand jobs is 0.091 (0.091 = 0.101 – 0.01).

34. When the adjacent auction strikes are one NFP outcome or one “tick” apart, the state claims and the digital options in the implied distribution are identical. The strikes in the NFP auctions are 25 ticks apart, and so these derivatives differ.

35. For example, an auction participant might take the view that the implied distribution should be relatively smooth, that is, the price of a digital range should be closely related to the prices of the adjacent digital ranges. Based on this view, an auction participant might buy (resp., sell) a digital range whose price is low (resp., high) relative to the prices of the adjacent digital ranges.

36. Many authors have pointed out the close relationship between the implied probability and the price of a digital option. See, for example, Ingersoll (2000, p. 70) and Hull (2006, p. 535).

37. More generally, each tradable option in a parimutuel derivatives auction can be represented as a portfolio of the state claims.

38. Prices in a parimutuel derivatives auction satisfy the principle of “first degree stochastic dominance,” which is described in, for example, Chapter 2 of Huang and Litzenberger (1988). Using first degree stochastic dominance, one can explicitly bound the price of the range forward based on the prices in the implied distribution. Based on this, we can derive that the price of the range forward must lie between 177.5 and 201.1 in the 2 September 2005 auction on August 2005 NFP. Of course, the price of the range forward can be determined exactly by using the prices of the state claims. See Equation (6.4).
39. In fact, Gürkaynak and Wolfers (2006) use the “implied market forecast” instead of the range forward price in their analysis. The implied market forecast is based on the range forward price and is an estimate of the price of an uncapped forward. The implied market forecast and the price of the range forward are very highly correlated.

40. Fair and Shiller’s (1990) methodology relies on a multiple regression that uses both the price of the range forward and the median economist forecast as independent variables. Ironically, most econometricians would refer to this testing methodology as “a horse race” methodology.

41. For related studies based on somewhat smaller data sets, see McCabe (2004) and McKelvey (2004).

5 DERIVATIVE STRATEGIES AND CUSTOMER ORDERS

1. For ease of exposition in Chapters 5 and 6, the no-arbitrage and self-hedging principles, which were the first and second mathematical principles in the wagering framework, are the third and fourth mathematical principles in the derivatives framework.

2. Digital options are also called “binary” options or “all-or-nothing” options.

3. For further details on this material, see Lange and Baron (2002), Baron and Lange (2003), Lange, Baron, Walden, and Harte (2003), and Lange and Economides (2005).

4. Another way to exposit the material in Chapters 5 and 6 would be to present the complete mathematical specification of the PEP before presenting an example with equilibrium values included. Because of the large number of equilibrium equations and variables for this problem, we do not take this approach. Instead, to make the exposition as readable as possible, we present the values of variables in equilibrium before the complete equilibrium specification has been presented.

5. Although this assumption is not required, it is used because it simplifies the replication formulas for vanilla options in Section 5.3.

6. Most financial exchanges list and trade many more than three strikes on a single underlying. Parimutuel derivatives auctions can handle such cases, but we chose a small number of strikes to most simply illustrate the properties of parimutuel derivatives auctions.

7. The variable $d$ typically depends on one or more parameters, such as the option strike. To keep the notation simple, we will suppress that dependence in the notation.

8. As shown in a moment, the function $d$ takes on a finite set of values. Therefore, the minimum and maximum of $d$ are well defined.
9. It is possible to create derivatives with unbounded payouts by including a condition that such derivatives are in zero net supply in a parimutuel derivatives auction.

10. Readers with a derivatives background will recognize that a vanilla capped call struck at $k_e$ is simply a “vanilla call spread” with a lower strike equal to $k_e$ and a higher strike equal to $k_E$. For a vanilla capped call, $k_e$ must be strictly less than $k_E$ since Equation (5.10) implies that a vanilla capped call struck at $k_E$ pays out zero everywhere.

11. A vanilla floored put struck at $k_e$ is simply a “vanilla put spread” with the higher strike equal to $k_e$ and the lower strike equal to $k_1$. For a vanilla floored put, $k_e$ must be strictly greater than $k_1$ since Equation (5.11) implies that a vanilla floored put struck at $k_1$ pays out zero everywhere.

12. A parimutuel derivatives auction assumes that all derivative payouts are settled based on the value of the underlying at a specific point in time (as opposed to a range of dates). Thus, this chapter studies forwards, instead of futures, because forwards typically have a single date on which the forward contract is settled, whereas futures often have a range of dates for settlement.

13. Of all the derivatives described in this chapter, the forward and the range forward (described in a moment) are the only derivative strategies with negative payouts. This property implies that an owner of a forward or a range forward may have to make a payout upon expiration.


15. Chapter 6 describes in detail how the price of the range forward is determined.


17. Appendix 6A will treat the case where the premium settlement date is on or before the payout settlement date and the discount factor for that time period is greater than zero and less than or equal to one.

18. If $\rho \neq 1$, then it is worth noting that one point in-the-money is not the same as one tick in-the-money. For example, a vanilla capped call struck at $k_e$ is one point in-the-money if $U = k_e + 1$, whereas a vanilla capped call is one tick in-the-money if $U = k_e + \rho$.

19. Recall from Sections 5.1.4 and 5.1.5 that payouts on vanilla options and the range forward are capped below $k_1$ and above $k_E$. See Equations (5.10), (5.11), and (5.13).

20. Section 1.2 introduced market orders and limit orders.

21. Chapters 6 and 7 will describe in more detail how $x_j$ and $\pi_j$ are determined.

22. Uniform-price auctions are used, for instance, by the US Treasury to auction off new Treasury securities. See, for example, the discussion in Garbade and Ingber (2005).

23. To see why such an order is always fully filled requires material from Chapter 6. Chapter 6 restricts the prices of every state claim to be positive (Equation (6.2)). This implies that the price of an option is strictly greater than the option’s minimum payout and strictly less
than the option's maximum payout. Similarly, the price of the range forward is strictly greater than $k_1$ and strictly less than $k_E$. Therefore, Equation (5.17) implies that such a market order is fully filled and at a price better than the limit price.

24. The fact that the number of states is finite makes the mathematics in this and in upcoming chapters tractable.

25. Since the strikes are multiples of $\rho$, Equation (5.18) implies that $S$, the number of states, is at least one greater than $E$, the number of option strikes, that is, $S \geq E + 1$.

26. Depending on how far away the strikes are set in an auction, the strikes $k_1 + \rho, k_1 + 2\rho, \ldots, k_E - \rho$ may or may not be tradable by customers. See the discussion in endnote 28.

27. We represent the payout function for the $s$th state claim with the symbol $\tilde{d}_s$ to evoke that it is similar to the payout function $d$ on a derivative strategy. Thus, the tilde symbol is not used to denote that $d_s$ is a random variable, as is sometimes done in the statistics literature.

28. The first state claim is the digital put struck at $k_1$ and the $S$th state claim is the digital call struck at $k_E$, both of which are tradable by customers in a parimutuel derivatives auction. If adjacent auction strikes are one tick apart, then the 2nd, 3rd, $\ldots$, $S - 1$st state claims are digital ranges that customers can trade in a parimutuel derivatives auction. However, if all adjacent strikes are more than one tick apart, then customers cannot trade the 2nd, 3rd, $\ldots$, $S - 1$st state claims in the auction. In fact, when strikes are more than one tick apart, derivatives can be replicated using a much smaller set of fundamental building blocks. Longitude's parimutuel matching engine uses such an approach, and that approach is described in Lange, Baron, Walden, and Harte (2003, Chapter 11). This replication approach is related to the “supershare approach,” which was introduced by Hakansson (1976, 1978) and discussed in Cox and Rubinstein (1985).

29. These four state claims were also described in Chapter 2.

30. Each state claim is a digital strategy, so one contract of a state claim pays out one dollar if the state claim expires in-the-money.

31. It is important to note that the replication weights for customer order $j$ neither depend on $b_j$, the side of the customer order, nor on $r_j$, the number of contracts requested. Thus, the replication weights are for a unit case of $b_j = +1$ and $r_j = +1$.

32. Section 2.3.2 and 4.3.4 also discussed this property.

6 THE PARIMUTUEL EQUILIBRIUM

1. The principles of no-arbitrage pricing and self-hedging were used in Section 2.1 to develop the parimutuel wagering framework discussed there.
2. For further information on this material, see Lange and Baron (2002), Baron and Lange (2003), Lange, Baron, Walden, and Harte (2003), and Lange and Economides (2005).

3. For the parimutuel derivatives applications described in Section 3.3, Goldman Sachs is the initial-liquidity provider.

4. At the start of the auction (before customers have submitted any orders), the opening orders imply that the initial state prices are

\[ p_s = \frac{\theta_s}{\sum_{s=1}^{S} \theta_s} \quad s = 1, 2, \ldots, S \]

Based on these initial state prices, Equation (6.4) can be used to calculate prices on all derivatives in a parimutuel derivatives auction.

5. Endnote 21 provides intuition as to how positive premium amounts for all opening orders guarantee unique prices in a parimutuel derivatives auction. Chapter 8 provides a proof of this result.

6. Equation (6.1) plays a similar role to Equation (2.1) in Chapter 2.

7. Column six of Table 6.2 shows the equilibrium state prices, which will be discussed in more detail in the next section.

8. Chapter 7 will show how the prices and the customer fills are determined in a parimutuel derivatives auction.

9. Appendix 6A will consider the case where the premium settlement date is before the payout settlement date, which implies that the sum of the state prices equals the risk-free discount factor between those two dates.

10. See, for example, Theorem 3.4.1 in LeRoy and Werner (2001, p. 26).

11. See Section 5.3.1 for the definition of \( \tilde{d}_s \) and recall that \( \tilde{d}_s \) is a function, not a random variable.

12. Note that the left-hand side includes any payments that customers who have sold options must make. These payments are included as negative quantities on the left-hand side.

13. Appendix 8A will show that Equation (6.16) can be represented as an eigensystem.

14. Chapter 9 will discuss this case in detail.

15. See also Tashjian and Weissman (1995), who generalize Duffie and Jackson’s (1989) model. For a different approach, see Cuny (1993), who assumes that the exchange maximizes revenue earned from charging its members for seats. See also Duffie and Rahi (1995) and Tashjian (1995) for further discussion on this topic.

16. To derive this result, we can first use Equations (5.6), (5.5), and (5.11) to determine the payout functions on orders A, B, and C, respectively. Next, we can show that: the price of the digital put struck at 0.3 is \( p_1 + p_2 \) by Equation (6.6); the price of the digital call struck at 0.3 is \( p_3 + p_4 \) using Equation (6.5), which equals \( 1 - p_1 - p_2 \) by Equation (6.3); and the price of the vanilla floored put is \( (p_1 + p_2)/10 \) by Equation (6.9).
These results imply that all three orders have P&L profiles as defined in Equation (6.22).

17. The fact that a vanilla order (the buy of ten contracts of the vanilla floored put) has the same P&L profile as the digital orders (the buy of one contract of the digital put and the sell of one contract of the digital call) is somewhat unusual and comes about because the adjacent strikes in this auction are one tick apart.

18. There is no need to take the absolute value of the opening order amounts (since they are positive by Equation (6.1)) or the customer fills (since they are non-negative by Equation (5.17)).

19. The fact that the objective function is a linear function of the customer fills allows us to employ a linear program in part two of our solution algorithm. See Chapter 7 for additional discussion.

20. Other objective functions satisfy our two criteria. For example, the maximum possible gain satisfies both criteria, and the standard deviation of the P&L profile (where the probabilities used in the standard deviation are the state prices) satisfy both criteria.

21. To see how positive premium amounts for all opening orders guarantee unique prices in a parimutuel derivatives auction, let us examine the following simple example. Consider an auction with one strike $k_1$ and $S = 2$ state claims. Let $\theta_1$ be the premium amount of the opening order for the first state claim, the digital put struck at $k_1$, and let $\theta_2$ be the premium amount of the opening order for the second state claim, the digital call struck at $k_1$. Assume there are $J = 2$ customer orders. Let the first customer order be an order to buy 100 contracts of the digital put struck at $k_1$ with a limit price of 0.6, and let the second customer order be an order to buy 100 contracts of the digital call struck at $k_1$, also with a limit price of 0.6. Let $p_1$ denote the price of the digital put struck at $k_1$, and let $p_2$ denote the price of the digital call struck at $k_1$. Note that $p_1$ and $p_2$ must sum to one by Equation (6.3). If $\theta_1 = \theta_2 = 0$, then $x_1 = x_2 = 100, y_1 = y_2 = 100, M = 100, 0.4 \leq p_1 \leq 0.6$, and $p_2 = 1 - p_1$ satisfy the parimutuel equilibrium conditions (except for the conditions that $\theta_1$ and $\theta_2$ must be positive). Thus, with opening-order premium amounts equal to zero, the prices $p_1$ and $p_2$ are not unique. In contrast, when both opening-order premium amounts are positive, that is, $\theta_1 > 0$ and $\theta_2 > 0$, it is not hard to check in this simple example that there exist unique prices. For example, if $\theta_1 = \theta_2 = 1$, then $x_1 = x_2 = 100, y_1 = y_2 = 100, M = 102$, and $p_1 = p_2 = 0.5$ are the only values that satisfy the parimutuel equilibrium conditions. Chapter 8 provides a rigorous proof that the state prices are unique in the most general case.

22. Chapter 7 will describe how we solve for the unknown variables.

23. Some of this material was first introduced in the parimutuel wagering framework of Chapter 2.

24. Recall that a range forward trade has zero premium exchanged at the time of the trade. Even so, the range forward does contribute to the
individual state premiums, as the range forward has a payout in every state.

25. For this result, see Equation (6D.8) in Appendix 6D.

26. Note that since the $p_s$'s are all positive, Equation (6.32) and $M \neq 0$ implies that $M, m_1, m_2, \ldots, m_S$ are either all positive or all negative. If $M = 0$, then Equation (6.16) implies that $\theta_s + p_s y_s = 0$, and so Equation (6.29) implies that $m_s = 0$ for $s = 1, 2, \ldots, S$. Consequently, in all cases, $M, m_1, m_2, \ldots, m_S$ have the same sign.

27. See Aït-Sahalia and Lo (1998) and the citations therein.

28. See Section 4.4.4 for additional discussion on this topic.

29. As argued in Section 2.2.4, a parimutuel derivatives auction may allow for more active trading of low-delta options since no one party has to be short an option strategy that may result in the seller having to make a large payout.

30. This property will be discussed in more detail in Section 7.3. A portfolio of one contract of each of the derivatives requested in the first three customer orders is risk-free, and so we could also illustrate how parimutuel derivatives auctions aggregate liquidity with these three orders.

31. See also Section 2.3.3 for discussion of the impact of no-arbitrage restrictions on parimutuel prices in the wagering framework.

32. The initial-liquidity provider's worst loss is generally smaller the closer the state prices are to the opening prices. For instance, the initial-liquidity provider's worst loss is zero if the state prices exactly equal the opening prices.

33. See Section 2.3.5, Section 3.2, and Section 4.3.3 for further discussion on this point.

34. To those with a derivatives background, it may not be surprising that prices in a parimutuel derivatives auction satisfy put-call parity, since a parimutuel derivatives auction enforces no-arbitrage restrictions and put-call parity is one such no-arbitrage restriction.

35. Appendix 6A presents put-call parity incorporating a discount factor.

7 THE SOLUTION ALGORITHM FOR THE PARIMUTUEL EQUILIBRIUM PROBLEM

1. For details on such numerical solution techniques in general, see Gill, Murray, and Wright (1981) or Luenberger (2005). The PEP is an example of what is referred to in the academic literature as a “mathematical program with equilibrium constraints.” As described in Luo, Pang, and Ralph (1996), such mathematical programs are commonly found in engineering and economics. They are typically two-part or bilevel problems in the sense that there is a variational inequality problem (VIP) nested within an optimization problem. Similarly, part one of our algorithm solves the VIP for the unique set of state and strategy prices,
and then part two of our algorithm maximizes the objective function. See Chapter 8 for further discussion on the relationship between the PEP and a VIP.

2. The no-arbitrage restriction of Equation (6.4) is shown in Equation (7.9).

3. Theorem 8.1 will show that Equation (7.4) can be represented as an eigensystem. Although we do not use this representation in the numerical algorithm presented in this chapter, the eigensystem representation will be used in Chapter 8.

4. Theorem 8.4 will show that these equilibrium state and strategy prices are unique.

5. As shown in the steps below, every unknown variable in the PEP can be determined from the customer fills. Put another way, if the customer fills are known, then all other PEP unknown variables (including prices) can be determined. Thus, constraints (7.1), (7.2), (7.3), and (7.4) can be expressed in terms of the customer fills, though our notation does not represent that explicitly.

6. Although the prices from the final iteration are the equilibrium prices, the customer fills from the final iteration of part one may not maximize the market exposure. As described in Sections 7.2 and 7.3, part two adjusts, if possible, those customer fills to maximize the market exposure.

7. The results from Appendix 7A show that \( p_1, p_2, \ldots, p_S \) satisfy constraints (7.2) and (7.3) based on the following. Note that \( M \) greater than \( M_0 \) (see Equation (7A.8)) implies that \( M - y_s \) is positive for \( s = 1, 2, \ldots, S \). Since \( \theta_s \) is positive, we conclude by Equation (7.14) that \( p_s \) is positive for \( s = 1, 2, \ldots, S \). Further, \( f(M) = 0 \) (see Equation (7A.12)) implies that the sum of the state prices equals one. Consequently, \( p_1, p_2, \ldots, p_S \) satisfy constraints (7.2) and (7.3).

8. In practice, we stop stepping either when \( V \) equals zero or when \( V \) is close to zero, that is, when \( V \) is less than one dollar.

9. As a rigorous matter, the results from Appendix 8C (see Theorem 8C.2 in particular) will imply that the Jacobian matrix of changes in derivative strategy prices with respect to changes in customer fills adjusted for the order direction is positive semi-definite.

10. This exercise illustrates the market-driven nature of prices in a parimutuel derivatives auction – the greater the demand for a particular derivative strategy, the greater the price of that strategy.

11. Allowing customer order \( j \) to be a sell order and possibly for a vanilla option or range forward does not change the conclusion that modifying \( x_j \) based on Equation (7.12) will reduce \( v_j \).

12. Appendix 8C will show analytically that the changes in derivative strategy prices with respect to changes in customer fills are smaller in magnitude when the opening orders are large.

13. Due to space considerations, Table 7.3 does not include the values of the variables between the second iteration and the final iteration.

14. A note on fill precision. Although customers typically receive a whole number of contracts when trading in a parimutuel derivatives auction,
Table 7.3 shows the customer fills $x_1, x_2, \ldots, x_6$ to hundredths of a contract (two decimal places of precision). Doing this gives a reader who is so interested the ability to verify the values of the other variables in Table 7.3, as these variables are computed based on the customer fills before they are rounded to the nearest whole number.

15. A note on price precision. Exchanges allow customers to trade derivatives that have market prices and limit prices with typically four digits of precision or less. Exchanges restrict price precision for two reasons: first, as the number of decimal places of price precision becomes large, additional price precision becomes more of a nuisance than a help to customers; and second, exchange clearing systems have a fixed number of decimal places that they can handle to process trades. We determine equilibrium prices in the PEP to a large number of decimal places to satisfy constraints more closely (7.1), (7.2), (7.3), and (7.4). To follow standard exchange-traded derivatives conventions, parimutuel derivatives auctions display a pre-determined number of those decimal places of prices to customers. For instance, in this CPI example, we round and display $p_1, p_2, p_3,$ and $p_4$ and $\pi_1, \pi_2, \ldots, \pi_6$ to four decimal places. In parimutuel derivatives auctions, premium is determined based on the rounded values of $\pi_1, \pi_2, \ldots, \pi_J$.

16. Table 7.3 contains two rows with dashes, which are used to separate variables with different numbers of decimal places.

17. For ease of exposition, we use a step size of one in the first and second iteration to keep the numbers simple.

18. This point was also made in Section 6.3.4.

19. It is not hard to check that a portfolio containing one contract of the derivative requested in the first customer order (the digital call struck at 0.3), one contract of the derivative requested in the second customer order (the digital put struck at 0.2), and one contract of the derivative requested in the third customer order (the digital range with strikes of 0.2 and 0.3) pays out one dollar in all four states (see the replication weights in Table 7.2). These three orders are complementary orders, as discussed in more detail in Section 7.3.

20. The algorithm in part one is also an example of an “active set” method, which is a type of primal method. See Luenberger (2005, p. 326) and Gill, Murray, and Wright (1981, p. 168).

21. The gradient projection method is based on the method of “steepest descent,” which is used for unconstrained problems. See the discussion in Chapter 11 of Luenberger (2005).

22. An algorithm that steps the customer fills large distances without sufficient intelligence is not likely to have good convergence properties. For example, if $\delta_j$ is too large, then $x_j$ might oscillate between 0 and $r_j$ on successive iterations.

23. See, for example, Chapter 6 of Luenberger (2005) for a discussion of convex programs. In a convex program, any local minimum is in fact a global minimum.
24. For details on how to solve LPs, see Chapter 5 of Gill, Murray, and Wright (1981) or Part One of Luenberger (2005).
25. The state prices from part one satisfy Equations (7.2) and (7.3), as shown in endnote 7. Since the state prices are the same in part two as in part one, we do not have to re-check that Equations (7.2) and (7.3) are satisfied in part two.
26. Section 6.2.4 argued that the objective function should be an increasing function of the customer fills. Equation (7.22) verifies that this criteria is met here.
27. Section 9.3 will describe “competing orders,” which have some similar properties to complementary orders.
28. With some additional notation, we can write Equation (7.26) in matrix form. Let \( \tilde{A} \) denote the \( J \) by \( S \) matrix whose element in the \( j \)th row and \( s \)th column is \( a_{js}b_j \). Let \( T \) denote the transpose operator. Let \( c \) denote the column vector of length \( J \) whose element in the \( j \)th row is \( c_j \) for \( j = 1, 2, \ldots, J \). Let \( \mathbf{1} \) denote the \( S \) by 1 vector of all 1’s. Then,

\[
\tilde{A}^T c = \kappa \mathbf{1}
\]

is the matrix version of Equation (7.26). If \( \kappa = 0 \), then this equation is called a “homogenous system.” See Chapter 2 of Strang (1988) for further discussion.
29. In the special case that all the customer orders in \( C \) are buys of option strategies, then \( \kappa \) is strictly positive.
30. See Pindyck and Rubinfeld (2004) for a discussion of “complementary goods,” a somewhat related concept from the field of economics.
31. In addition to these three complementary orders in the CPI auction, one can check that the fourth and fifth customer orders complement one another, and one can check that the fifth and sixth customer orders complement one another.
32. Conditions 1 and 2 imply significant restrictions on the limit prices of complementary orders. See Appendix 7C.
33. Because the customer requested amounts can be different orders of magnitude, one might consider scaling the variables \( x_1, x_2, \ldots, x_J \) before calling the LP. See, for example, the scaling discussions in Luenberger (2005).
34. Theorem 8.1 will show that Equation (7A.4) can be represented as an eigensystem. Because of this, a number of results in this appendix can be derived using eigensystem machinery. For example, Equation (7A.8) can be derived using bounds on the “Perron root,” as described in Kolotilina (2004, p. 2482). In addition, the uniqueness of \( M \) follows from the “Perron Theorem,” which will be described in Chapter 8. We avoid introducing that machinery here and instead derive results from first principles.
36. See Chapter 4 of Gill, Murray, and Wright (1981) or Chapter 7 of Luenberger (2005) for more details on Newton’s method.

37. It is straightforward to check that $f$ has a continuous second derivative over the range $G \in [M, \infty)$. Luenberger (2005, p. 202) shows that Newton’s method converges at a quadratic rate under this condition provided that the initial value $M$ is sufficiently close to $M$. Regarding the closeness of $M$ to $M$, it is not hard to check that

$$0 < M - M < \sum_{s=1}^{S} \theta_s$$

by using the upper bound in Equation (1.3) of Kolotilina (2004, p. 2482).

38. As will be discussed further in Chapter 8, $M$ can be solved for as part of an eigensystem problem. Numerical solution techniques are well known for eigensystems, and Chapter 7 of Strang (1988) provides an overview of some of the standard methods. Newton’s method has the advantage that it is computationally inexpensive.

39. To emphasize that these iterations take place inside a particular iteration of the part one algorithm, Table 7A.1 shows different iterations in different rows, whereas Table 7.3 shows different iterations in different columns.

40. The variables $p_1, p_2, \ldots, p_S$ do not require superscripts since these variables do not change between part one and part two.

8 MATHEMATICAL PROPERTIES OF PARIMUTUEL EQUILIBRIUM PRICES

1. Eisenberg and Gale (1959), Norvig (1967), and Owen (1997) study the mathematical properties of parimutuel prices in the wagering framework described in Chapter 2. See Section 2.3.4 for more details.

2. Section 5.2.2 showed that a market order can also be represented as a limit order with an aggressive enough limit price.

3. See, for example, Chapter 5 of Strang (1988) for more details on eigensystems.

4. We thank Professor Michael Overton of the Courant Institute of New York University for first pointing out that Equation (8.5) might be represented as an eigensystem.

5. The self-hedging restrictions are equivalent to the eigensystem representation in Equation (8.7) if customers only submit market orders, or if customers submit both market orders and limit orders.

6. This eigensystem representation for the PEP was first presented in Lange and Economides (2005, pp. 37–38).
7. We can rearrange the eigensystem of Equation (8.7), as is commonly done, into the following form

\[(H - MI)p = 0\]

where \(I\) is the \(S\) by \(S\) identity matrix and \(0\) is the column vector of length \(S\) of all zeros. Based on this equation, we say that the vector \(p\) lies in the “null space” of the matrix \(H - MI\), and the equilibrium state prices are the prices that drive \(H - MI\) to zero.

8. Theorem 8.3 can be proven from first principles (and without the eigensystem machinery) based on the approach used in Appendix 7A. Without limit orders, \(y_1, y_2, \ldots, y_S\) are known and given by Equation (8.4). Appendix 7A showed that there exists a unique amount of net auction premium \(M\) based on a fixed set of net customer pay-outs \(y_1, y_2, \ldots, y_S\). Based on these quantities, Appendix 7A proved that there exists a unique \(M\) and a unique set of state prices \(p_1, p_2, \ldots, p_S\).

9. As described in Section 5.2.2, one can also think of these four market orders as limit orders with limit prices of one.

10. Note that \(p_3\) and \(p_4\) are close to zero. When customers only submit market orders, it is not unusual for certain state prices to be close to zero.

11. See, for example, Magill and Quinzii (1996) and Starr (1997), who discuss these issues in the context of “incomplete markets” and “general equilibrium theory,” respectively.

12. Although the SPEP has an objective function, Section 8.1.3 showed that its objective function is not necessary, since there is only one solution to the problem.

13. Recall that Theorem 8.5 implies Theorem 8.4.

14. The pseudo-orders are not used in the PEP solution algorithm, which was described in Chapter 7.

15. The requested amount for a pseudo-order is arbitrary as long as it is positive.

16. The pseudo-orders resemble the opening orders in that both the pseudo-orders and the opening orders are orders to buy the \(S\) state claims. However, the resemblance ends there. The opening orders are always filled, while the pseudo-orders are never filled. Further, the opening orders are submitted in premium terms, while the pseudo-orders request specific numbers of contracts.

17. These four relationships represent step 2 through step 5 of part one of the PEP solution algorithm from Section 7.1.2.


19. In fact, Theorem 8.2 is often stated in terms of \(H\) being a non-negative and “irreducible matrix,” of which a positive matrix is a special case. See Horn and Johnson (1985, p. 361) for the definition of an irreducible matrix.
20. Recall that the GPEP in Equation (8.14) does not include an objective function, and consequently, it is a more general problem than the PEP. Thus, proving Theorem 8.5 implies that Theorem 8.4 holds.


22. A function $G$ is monotone if and only if $(G(w) - G(v))^T(w - v) \geq 0 \ \forall \ v, w \in X$.

23. Equations (8C.4) and (8C.5) do not handle the pricing of the range forward. However, because put–call parity holds in a parimutuel derivatives auction, we can represent a buy (resp., sell) of a range forward with limit price $w$ as a buy (resp., sell) of a vanilla capped call struck at $k$ with a limit price of $w - k$. See Appendix 6B for more discussion on put–call parity.

24. Laplacian matrices are used for the study of “networks,” and they encode how different “nodes” in a network “communicate.” In the parimutuel derivatives auction framework, the nodes are the states, and the matrix of first derivatives tells us that all the states communicate when the price of one state is changed.

25. Note that $\tilde{D}$ is also “diagonally dominant,” which is defined and discussed in Horn and Johnson (1985, p. 349).

9 MATHEMATICAL PROPERTIES OF CUSTOMER FILLS IN THE PARIMUTUEL EQUILIBRIUM

1. We use the term “vector” in this chapter to describe the $J$ customer fills, as we define below a $J$ by 1 column vector $x$ with customer fill $x_j$ in the $j$th row for $j = 1, 2, \ldots, J$.

2. When there are multiple vectors of customer fills that satisfy the PEP for a given set of auction inputs, an objective approach is needed to choose a particular vector of customer fills. As discussed in Harris (2003, pp. 112–120), there are a variety of standard approaches, including “time priority” and “pro-rata allocation.” Parimutuel derivatives auctions currently allocate fills using a pro-rata based approach.

3. This representation is the shorthand representation of the PEP. For all the constraints and definitions, see Table 6.7.

4. Section 9.4 will explain why this CPI example has exactly one vector of customer fills.

5. The opening orders for this auction are the same as the opening orders for the auction example from Chapters 5, 6, and 7, and for the auction example in Section 8.1.4.

6. Theorem 9.3 from Section 9.2 will prove that there are not just three, but in fact an infinite number of vectors of customer fills for this example.
7. If an auction has multiple vectors of customer fills and all filled customer orders are orders to buy options, then the net customer payouts are equal across all vectors of customer fills.

8. Theorems 9.1, 9.2, and 9.3 will relate to the first, second, and third observations, respectively.

9. It is worth noting that although the customer orders that are worse than the market are the same for every vector of fills, the customer orders that receive zero fill can differ for different vectors of customer fills. In Table 9.3, note that: in the first vector, the second and third customer orders receive no fill; in the second vector, the fourth customer order receives no fill; and in the third vector, all customer orders receive some fill. In a similar vein, one can show that although the customer orders that are better than the market are the same for every vector of fills, the customer orders that are fully filled can differ for different vectors of customer fills.

10. In fact, we verified in Section 9.1.2 that this vector of customer fills satisfies the PEP conditions.

11. Since $C$ is non-empty, at least one value of $c_1, c_2, \ldots, c_J$ is strictly non-zero.

12. We can write Equation (9.24) in matrix form. See endnote 28 in Chapter 7, and see Section 9.5.

13. The concept of competing orders is closely related to the concept of “redundant securities.” See LeRoy and Werner (2001, pp. 5, 10, 36–37), and see Ingersoll (1987, pp. 49–50) for a discussion of redundant securities. LeRoy and Werner (2001, p. 10) point out that the existence of redundant securities leads to multiple portfolio allocations associated with a “market-clearing consumption allocation.” Theorem 9.4 will show a somewhat similar result – that the existence of competing orders is required for there to be multiple equilibrium customer fill vectors.

14. Note that the two examples of competing orders considered here both have $\kappa = 0$ in Equation (9.24). In Section 9.5, Theorem 9.6 shows that $\kappa$ equals zero if all the customer orders in $C$ are buys of options.

15. Domowitz and Madhavan (2001, p. 379) and Harris (2003, p. 134) point out that excess demand can lead to multiple vectors of fills in traditional auctions.

16. See Theorem 9.8 for what these conditions imply about the limit prices of the customer orders in $C$.

17. Here, $C = \{4, 6\}, c_1 = c_2 = c_3 = c_5 = 0, c_4 = 1, c_6 = -1,$ and $\kappa = \pi_6$.

18. Note that $C$ is non-empty, which implies that $c$ contains at least one non-zero element. Thus, if Equation (9.34) holds, then it holds for a $c \neq 0$.

19. Based on Equation (9.34), it is easy to see that competing orders are closely related to “redundant securities.” See endnote 13.

20. The converse of Theorem 9.8 is worth stating: if all the customer orders in an auction are buy orders, and there does not exist a non-zero
vector $c$ such that both $A^Tc = 0$ and $w^Tc = 0$, then there exists a unique vector of customer fills. Thus, even if $A$ is not of full rank, there may exist a unique vector of customer fills in the auction, depending on the relationship between the customers’ limit prices. This result is (obviously) stronger than Theorem 9.7.

21. The state prices $p_1, p_2, \ldots, p_S$ do not depend on the customer fills over this narrow range of customer fill vectors. However, in the more general case, $p_1, p_2, \ldots, p_S$ do depend on the customer fills.

22. Although the results in Appendix 7C are for complementary orders, those results are based on Equation (7.26), which is identical to Equation (9.24). Thus, those results can be applied here.

23. If the customer orders are all buys of options, and $A$ has full rank, then we can prove that the PEP has a unique set of prices and fills by using the fact that the PEP can be represented as a BVIP (see Chapter 8) which has a positive definite Jacobian (see, e.g., Luo, Pang, and Ralph (1996, pp. 54–55)). Such a proof would parallel the proof of Theorem 8B.2, which relied on a positive semi-definite Jacobian.


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