Appendix A
Sami's Forcing-Free Proof in Z_2

Abstract In this appendix, I reconstruct in Z_2 Sami's forcing-free proof of "$Det(Turing\text{-}\Sigma^1_1)$ implies HP" in a general setting.

In Sect. 1.2, I give a brief survey of different proofs of "$\mathsf{Z}_2 + Det(\Sigma^1_1)$ implies HP" in the literature. All other proofs of Harrington's Theorem in the literature use forcing. Sami's proof of Harrington's Theorem in [1] is totally forcing-free and only uses effective descriptive set theory. All definitions and results in this section are due to Sami (cf. [1]), only the reconstruction in this form is due to the author.

The structure of this appendix is as follows. I first introduce a general property, called '★', and then show in Theorem A.3 that if there is a Σ^1_1 set of reals with property ★, then $Z_2 + Det(Turing\text{-}\Sigma^1_1)$ implies HP. Then I present Sami's definition of a Σ^1_1 set of reals D and show that D has the ★ property. Hence, by Theorem A.11, we have $Z_2 + Det(Turing\text{-}\Sigma^1_1)$ implies HP.

Definition A.1 A set $Z \subseteq \omega^\omega$ is said to have '*property* ★' if

(1) Z is Turing closed and cofinal in the Turing degree;
(2) If C is a Turing cone with $C \subseteq Z$, $x \in C$, $\beta < \omega_1^x$ and $y \subseteq \beta$, then $y \in L$ implies $y \in L_{\omega_1^x}[x]$.

Now, $A \subseteq \omega^\omega$ is cofinal in the Turing degrees if $\forall x \in \omega^\omega \exists y \in A(x \leq_T y)$. Given a real x, the Turing cone of x is $Cone(x) = \{y \in \omega^\omega \mid x \leq_T y\}$. We say $A \subseteq \omega^\omega$ contains a cone if for some real x, $Cone(x) \subseteq A$.

Lemma A.1 (Martin's Lemma) *For a Turing-closed set $A \subseteq \omega^\omega$, A is determined iff A or its complement includes a cone.*

Proof The converse direction is easy. We only prove the forward direction. Suppose $A \subseteq \omega^\omega$ is Turing-closed and A is determined, there is a real x such that $Cone(x) \subseteq A$ or $Cone(x) \cap A = \emptyset$. We show that if player I has a winning strategy in the game G_A, then A contains a cone. Let σ be a winning strategy for player I. It suffices to show that A contains the cone $\{x \in \omega^\omega : \sigma \leq_T x\}$. Let $x \in \omega^\omega$ be such that $\sigma \leq_T x$.

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Then $a = \sigma * x$ is in A because σ is a winning strategy for player I. Since $x \equiv_T a$ and A is $\equiv_T$-closed, we have $x \in A$. Similarly, we can show that if player II has a winning strategy in the game G_A, then the complement of A contains a cone. $\square$

By Lemma A.1, $Det(Turing\text{-}\Gamma)$ is the statement that if $X \subseteq \omega^\omega$ is in Γ and Turing closed, then:

$$[\exists x \forall y (x \leq_T y \rightarrow y \in X)] \vee [\exists x \forall y (x \leq_T y \rightarrow y \notin X)].$$

Theorem A.2 (G. Sacks, [2]) *If $x \in \omega^\omega$ and α is a countable x-admissible ordinal, then $\alpha = \omega_1^y$ for some $y \in Cone(x)$.*

Theorem A.3 (Z_2) *Suppose there is a Σ^1_1 set of reals with property* $\bigstar$. *Then $Det(Turing\text{-}\Sigma^1_1)$ implies* HP.

Proof Assume that Z is a Σ^1_1 set of reals with property $\bigstar$ and $Det(\text{Turing-}\Sigma^1_1)$ holds. Since Z is Turing closed and cofinal in the Turing degree, by Lemma A.1, Z contains a cone of Turing degrees. Suppose $C = Cone(c) \subseteq Z$. Now we show that for any countable ordinal α, if α is c-admissible, then α is an L-cardinal.

Let α be a countable c-admissible ordinal. By Sacks' Theorem (cf. Theorem A.2), $\alpha = \omega_1^{c \oplus y}$ for some $y \in \omega^\omega$. Thus, $\alpha = \omega_1^y$ for some $y \in C$.

Claim *If $z \in C$, then ω_1^z is an L-cardinal.*

Proof If ω_1^z is not an L-cardinal, then for some $\alpha < \omega_1^z$, there is a well-ordering $R \subseteq \alpha \times \alpha$ such that $R \in L$ and $o.t.(R) = \omega_1^z$. Let $\gamma = \alpha \times \alpha$. Then $\gamma < \omega_1^z$ and $R \subseteq \gamma$. Since Z has property $\bigstar$, we have $R \in L_{\omega_1^z}[z]$. Since $o.t.(R) = \omega_1^z$, by admissibility, we have $\omega_1^z \in L_{\omega_1^z}[z]$, which leads to a contradiction. $\square$

Since $y \in C$, $\alpha = \omega_1^y$ is an L-cardinal. $\square$

In the following, we define such a Σ^1_1 set of reals with property $\bigstar$. First of all, I review some definitions and facts we will use in the following. $\mathscr{R} = \mathscr{P}(\omega \times \omega)$ is the space of relations on ω and S_∞ is the space of permutations of ω, each equipped with its usual recursively presented Polish topology. Let $S \subseteq \mathscr{X} \times \mathscr{Y}$ where $\mathscr{Y}$ is a topological space. The category quantifier $\exists^* y(S(x, y))$ stands for: $\{y \in \mathscr{Y} \mid S(x, y)\}$ is non-meager in $\mathscr{Y}$.

Linear orderings are taken to be reflective, i.e. non-strict. $\mathsf{LO} = \{r \subseteq \omega \times \omega \mid r$ is a linear ordering on its field$\}$. For $r \in \mathsf{LO}$, $\leq_r$ is just binary relation r and $<_r$ has its usual meaning. Note that $\mathsf{WO} = \{r \in \mathsf{LO} : <_r$ is well-founded$\}$, and $\mathsf{WO}_\alpha = \{r \in \mathsf{WO} \mid |r| = \alpha\}$ for $\alpha < \omega_1$. For $r \in \mathscr{R}$ and $k \in \omega$, $r \upharpoonright k = \{(m, n) \mid m <_r k \wedge n <_r k \wedge m \leq_r n\}$. Note that $r \upharpoonright k = \emptyset$ if $k \notin \text{Field}(r)$, and the function $(r, k) \mapsto r \upharpoonright k$ is recursive.

Given $f \in S_\infty$ and $r \subseteq \omega \times \omega$, we denote by $f \cdot r$ the isomorphic copy of r by f. Note that $(f, r) \mapsto f \cdot r$ is a recursive function from $S_\infty \times \mathscr{R}$ to $\mathscr{R}$. Suppose $r, s \subseteq \omega \times \omega$ are isomorphic via $g : (\omega, r) \rightarrow (\omega, s)$. For any $Z \subseteq \mathscr{R}$, we have $\{f : f \cdot r \in Z\} = \{f : f \cdot s \in Z\} \circ g$ since $(f \circ g^{-1}) \cdot s \in Z$ if $f \cdot r \in Z$, and $(f \circ g) \cdot$

$r \in Z$ if $f \cdot s \in Z$.[1] Since right multiplication by g is a homeomorphism of S_∞, the topological properties of $\{f : f \cdot r \in Z\}$ and $\{f : f \cdot s \in Z\}$ are identical.

Definition A.4 ([1]) $r \in \mathsf{LO}$ is a pseudo-well-ordering if any non-empty $\Delta^1_1(r)$ subset of Field(r) has a r-least element. Let PWO denote the set of such orderings.

Note that $\mathsf{WO} \subseteq \mathsf{PWO}$ and PWO is Σ^1_1.

Fact A.5 (Harrison, [3]) *If $r \in \mathsf{PWO} \backslash \mathsf{WO}$, then $o.t.(r) = \omega_1^r \times (1 + \eta) + \rho_r$ where η is the order type of the rationals and $\rho_r < \omega_1^r$.*

Fact A.6 (Boundedness theorem for $\boldsymbol{\Sigma}^1_1$ (Σ^1_1) set, [2, 4])

(1) If $A \subseteq \mathsf{WO}$ is $\boldsymbol{\Sigma}^1_1$, then there is an $\alpha < \omega_1$ such that $A \subseteq \mathsf{WO}_{<\alpha}$;
(2) If $A \subseteq \mathsf{WO}$ is Σ^1_1, then there is an $\alpha < \omega_1^{\mathsf{CK}}$ such that $A \subseteq \mathsf{WO}_{<\alpha}$. As a corollary, if $A \subseteq \mathsf{WO}$ and A is Σ^1_1, then $\sup\{rk(x) : x \in A\} < \omega_1^{\mathsf{CK}}$.

Proposition A.1 (Sami, [1]) *If $r \in \mathsf{PWO}$ and $\omega_1^r = \omega_1^{\mathsf{CK}}$, then r has an isomorphic recursive copy.*

Proof If $r \in \mathsf{WO}$, then $rk(r) < \omega_1^{\mathsf{CK}}$. Thus, r has an isomorphic recursive copy. If $r \in \mathsf{PWO}\backslash\mathsf{WO}$, by Fact A.5, we have $o.t.(r) = \omega_1^{\mathsf{CK}} \times (1 + \eta) + \rho_r$ where η is the order type of the rationals and $\rho_r < \omega_1^{\mathsf{CK}}$.

From Fact A.6, $\{r \in \mathsf{WO} : r$ is recursive$\}$ is not Σ^1_1 (if not, then $\sup\{rk(x) : x$ is a recursive well-ordering on $\omega\} < \omega_1^{\mathsf{CK}}$ which leads to a contradiction). But $\{r \in \mathsf{PWO} : r$ is recursive $\}$ is Σ^1_1. Take a recursive $s \in \mathsf{PWO}\backslash\mathsf{WO}$. Since s is recursive, by trimming some excess, we can assume that $o.t.(s) = \omega_1^{\mathsf{CK}} \times (1 + \eta)$ where η is the order type of the rationals. Thus, by stringing together s and a recursive well-ordering with length ρ_r, we can construct a recursive copy of r. □

Fact A.7 (Kechris, [4]) *If $R \subseteq \mathscr{X} \times \mathscr{Y}$ is Σ^0_α with $\alpha < \omega_1^{\mathsf{CK}}$(resp. R is Δ^1_1) where $\mathscr{X}, \mathscr{Y}$ are recursively presented Polish spaces, then the relation $\exists^* y(R(-, y))$ is Σ^0_α (resp. Δ^1_1).*

Fact A.8 ([5, 6]) *For each $n \geq 1$, there exists a universal $\boldsymbol{\Sigma}^1_n(\Sigma^1_n)$ set in $\mathscr{N}^2$; i.e. a set $U \subseteq \mathscr{N}^2$ such that U is $\boldsymbol{\Sigma}^1_n(\Sigma^1_n)$ and that for every $\boldsymbol{\Sigma}^1_n(\Sigma^1_n)$ set $A \subseteq \mathscr{N}$ there exists some $y \in \mathscr{N}$ such that $A = \{x : (x, y) \in U\}$.*

Given Polish space X, $A \subseteq X$ is nowhere dense if the complement of A contains a dense open set. $A \subseteq X$ is meager (or of first category) if A is the union of countably many nowhere dense sets. A non-meager set is called a set of second category.

Fact A.9 (Baire category theorem, [6]) *In a Polish space, every nonempty open set is non-meager (or of second category).*

Fact A.10 (Gandy's basis theorem, [2, 7]) *Let $A \subseteq \omega^\omega$ and $z \in \omega^\omega$.*

[1] $f \circ g$ means the composition of f and g.

(1) If A is non-empty $\Sigma^1_1(z)$ set, then there is $x \in A$ such that $\omega_1^{x \oplus z} = \omega_1^z$.
(2) If A is non-empty Σ^1_1 set, then there exists $a \in A$ such that $\omega_1^a = \omega_1^{\mathsf{CK}}$.

Theorem A.11 (Sami, [1]) *Given $a \in \omega^\omega$ and $1 \leq \alpha < \omega_1^{\mathsf{CK}}$, if for some $\eta < \omega_1$, a is $\Sigma^0_\alpha(x)$ for all $x \in \mathsf{WO}_\eta$, then a is Σ^0_α.*

Proof Let $U \subseteq \omega \times \mathscr{R} \times \omega$ be ω-universal for Σ^0_α subsets of $\mathscr{R} \times \omega$. Fix $r \in \mathsf{WO}_\eta$. By the hypothesis, for all $f \in S_\infty$ there is $e \in \omega$ such that $a = U(e, f \cdot r, -)$ (i.e. $n \in a \Leftrightarrow U(e, f \cdot r, n)$).

Since $S_\infty = \{f \in S_\infty \mid \exists e \in \omega(a = U(e, f \cdot r, -))\} = \bigcup_{n \in \omega}\{f \in S_\infty \mid a = U(n, f \cdot r, -)\}$, by Baire category theorem (cf. Fact A.9), there exists an $e_0 \in \omega$ such that $\{f \in S_\infty \mid a = U(e_0, f \cdot r, -)\}$ is non-meager in S_∞.

Suppose toward a contradiction that a is not Σ^0_α. Let

$$A = \{(x, s) \mid x \in \omega^\omega \text{ is not } \Sigma^0_\alpha, s \in \mathsf{PWO} \text{ and } \exists^* f \in S_\infty(x = U(e_0, f \cdot s, -))\}. \tag{A.1}$$

Note that "x is Σ^0_α" is a Δ^1_1 property of x and "$x = U(e_0, f \cdot s, -)$" is a Δ^1_1 property of (f, x, s). By Fact A.7, we have "$\exists^* f \in S_\infty(x = U(e_0, f \cdot s, -))$" is Δ^1_1 and hence A is Σ^1_1. A is non-empty since $(a, r) \in A$. By Gandy's basis theorem (cf. Fact A.10), pick $(x_0, s_0) \in A$ such that $\omega_1^{(x_0, s_0)} = \omega_1^{\mathsf{CK}}$. Thus, $\omega_1^{s_0} = \omega_1^{\mathsf{CK}}$.

Since $s_0 \in \mathsf{PWO}$, by Proposition A.1, let w_0 be a recursive copy of s_0. Since $\{f \mid x_0 = U(e_0, f \cdot w_0, -)\}$ is a translate in S_∞ of $\{f \mid x_0 = U(e_0, f \cdot s_0, -)\}$, we have $\exists^* f \in S_\infty(x_0 = U(e_0, f \cdot w_0, -))$. Let $V \subseteq S_\infty$ be a non-empty basic open set such that $\{f \mid x_0 = U(e_0, f \cdot w_0, -)\}$ is co-meager in V.

Claim

$$n \in x_0 \Leftrightarrow (\exists^* f \in V)\, U(e_0, f \cdot w_0, n).$$

Proof Define $P = \{f \in V \mid U(e_0, f \cdot w_0, n)\}$, $Q = \{f \mid x_0 = U(e_0, f \cdot w_0, -)\}$ and $T = \{f \in V \mid x_0 = U(e_0, f \cdot w_0, -)\}$.

($\Rightarrow$): Since $V \setminus Q$ is meager, T is non-meager. Since $n \in x_0$, we have $T \subseteq P$. Thus, P is non-meager.

($\Leftarrow$): If $\exists^* f \in V(U(e_0, f \cdot w_0, n))$, then P is non-meager. Suppose $Q \cap P = \emptyset$. Then $P \subseteq V \setminus Q$. Since Q is co-meager in V if and only if $V \setminus Q$ is meager, we have P is meager which leads to a contradiction. Thus $Q \cap P \neq \emptyset$ and $n \in x_0$. □

Since w_0 is recursive, "$U(e_0, f \cdot w_0, n)$" as a relation in (f, n) is Σ^0_α. Thus "$\exists^* f \in V(U(e_0, f \cdot w_0, n))$" is Σ^0_α. Hence x_0 is Σ^0_α. This contradicts the definition of A in (A.1), and we are done. □

Remark A.1 The proof of Theorem A.11 in fact shows that if $\{f \mid a \text{ is } \Sigma^0_\alpha(f \cdot r)\}$ is non-meager in S_∞ for some $r \in \mathsf{PWO}$, then a is Σ^0_α.

Proposition A.2 (Sami, [1]) *Suppose $\alpha < \omega_1$. Then:*

(1) WO_α is $\boldsymbol{\Sigma}^0_{\alpha+2}$;
(2) Given $r \in \mathsf{WO}_\alpha$, the relation "$s \in \mathsf{WO}_{|r \restriction k|}$" in (s, k) is $\Sigma^0_{\alpha+2}(r)$.

Proof Prove (1) by induction on α. If α is a limit ordinal, then

$$r \in \mathsf{WO}_\alpha \Leftrightarrow (\bigwedge_{\beta<\alpha} \bigvee_{n\in\omega} (r \upharpoonright n \in \mathsf{WO}_\beta)) \wedge (\bigwedge_{n\in\omega} \bigvee_{\beta<\alpha} (r \upharpoonright n \in \mathsf{WO}_\beta)).$$

By induction, "$r \upharpoonright n \in \mathsf{WO}_\beta$" is $\boldsymbol{\Sigma}^0_{\beta+2}$. Thus, "$\bigvee_{\beta<\alpha}(r \upharpoonright n \in \mathsf{WO}_\beta)$" is $\boldsymbol{\Sigma}^0_\alpha$. Similarly, "$\bigwedge_{\beta<\alpha} \bigvee_{n\in\omega}(r \upharpoonright n \in \mathsf{WO}_\beta)$" is $\boldsymbol{\Pi}^0_\alpha$. Hence WO_α is $\boldsymbol{\Sigma}^0_{\alpha+2}$.

If $\alpha = \beta + 1$, then

$$r \in \mathsf{WO}_\alpha \Leftrightarrow r \in \mathsf{LO} \wedge \exists n(n \text{ is } \leq_r \text{-maximum } \wedge r \upharpoonright n \in \mathsf{WO}_\beta). \tag{A.2}$$

The R.H.S of (A.2) is $\boldsymbol{\Sigma}^0_{\beta+2}$ and hence WO_α is $\boldsymbol{\Sigma}^0_{\alpha+2}$. Finally, (2) is just the effective version of (1), and we are done. □

Fact A.12 ([1]) *Given $r, s \in \mathsf{LO}$ with the same order type, there is $s' \leq_T s$ such that $(\omega, r) \cong (\omega, s')$.*

Given $\alpha < \omega_1, r \in \mathsf{WO}_\alpha$ and $X \subseteq \alpha$, let $\pi_r : (field(r), r) \to (\alpha, \leq)$ be the canonical isomorphism and set $Code(X, r) = \pi_r^{-1}(X)$. Note that if M is a transitive set and $r \in M$, then $\pi_r \in M$. Thus, for transitive set M with $r \in M$, we have

$$X \in M \Leftrightarrow Code(X, r) \in M.$$

Now we define a Σ^1_1 set of reals and show that it has property ★. For $x, y \in \omega^\omega$, $x \leq_h y$ stands for: x is hyperarithmetic in y. The reals in $L_{\omega_1^x}[x]$ are precisely the reals hyperarithmetic in x.

Definition A.13 (Sami, [1]) For $a, b \in \omega^\omega$, define $a \sqsubseteq b \Leftrightarrow \forall x \leq_h a\, (x \leq_T b) \wedge \omega_1^a = \omega_1^b$. Define $D = \{x \subseteq \omega \mid \exists y\, (y \sqsubseteq x)\}$.

Fact A.14 ([2]) *$\{(x, y) \in \omega^\omega \times \omega^\omega \mid \omega_1^x \leq \omega_1^y\}$ is Σ^1_1 but not Π^1_1.*

By Fact A.14, D is Σ^1_1. Now we show that D has property ★.

Theorem A.15 *D has property ★.*

Proof Note that D is Turing-closed. In the following, we show that D is cofinal in the Turing degrees and satisfies condition (2) in Definition A.1.

Lemma A.2 *D is cofinal in the Turing degrees.*

Proof We show that $\forall a \in \omega^\omega \exists b \in D(a \leq_T b)$. Let

$$A = \{y \in \omega^\omega \mid \forall x \leq_h a(x \leq_T y)\}.$$

Note that A is $\Sigma^1_1(a)$ and $A \neq \emptyset$ (Since $\{x \subseteq \omega \mid x \leq_h a\}$ is countable, let $\{x_n : n \in \omega\}$ be a list of such x and $y = \oplus_{n\in\omega} x_n$, then $y \in A$). By Gandy basis theorem (cf. Fact A.10), there is $b \in A$ such that $a \leq_T b$ and $\omega_1^a = \omega_1^b$. Thus, $a \sqsubseteq b$ and $b \in D$. □

Lemma A.3 (Sami, [1]) *Suppose* $a \in D, \alpha < \omega_1^a$ *and* $r \in \mathsf{WO}_\alpha$. *If* $X \in \mathscr{P}(\alpha) \cap L_{\omega_1^a}$, *then* $Code(X, r)$ *is* $\Sigma^0_{\alpha+2}(a, r)$.

Proof Let $b \sqsubseteq a$. Since $\omega_1^b = \omega_1^a$ and $\alpha < \omega_1^a$, we have $\alpha < \omega_1^b$. By Fact A.12, pick $s \in \mathsf{WO}_\alpha$ such that $s \leq_T b$ and $(\omega, s) \cong (\omega, r)$ (since $\alpha < \omega_1^b$, there is $r' \in \mathsf{WO}_\alpha$ such that $r' \leq_T b$. For such r', there is $s \leq_T r'$ such that $(\omega, r) \cong (\omega, s)$. So such s exists). Let $x = Code(X, r)$ and $y = Code(X, s)$. Since $s \leq_T b$, we have $s \in L_{\omega_1^b}[b]$. Since $X, s \in L_{\omega_1^b}[b]$, we have $y \in L_{\omega_1^b}[b]$. Thus, $y \leq_h b$. Since $b \sqsubseteq a$, we have $y \leq_T a$. For $k \in \omega$, note that

$$k \in x \Leftrightarrow \exists k'(k' \in y \wedge [(k \in field(r) \leftrightarrow k' \in field(s)) \wedge s \upharpoonright k' \in \mathsf{WO}_{|r \upharpoonright k|}]).$$

Since $y \leq_T a$, we have "$k' \in y$" is a $\Sigma^0_1(a)$ property of k'. Since $s \leq_T b$ and $b \sqsubseteq a$, we have $s \leq_T a$. Thus, by Proposition A.2, x is $\Sigma^0_{\alpha+2}(a, r)$. □

Lemma A.4 (Sami, [1]) *Suppose* C *is a Turing cone with* $C \subseteq D$, $c \in C, \xi < \omega_1^c$, $X \subseteq \xi$ *and* $X \in L$. *Then* $X \in L_{\omega_1^c}[c]$.

Proof Since $\xi < \omega_1^c$, take $r \in \mathsf{WO}_\xi$ such that $r \leq_T c$. Suppose $X \subseteq \xi$ and $X \in L$. Let $X \in L_\delta$ for some $\delta < \omega_1$. Pick any $s \in \mathsf{WO}_\delta$. Since $\delta < \omega_1^s \leq \omega_1^{c \oplus s}$, we have $X \in L_{\omega_1^{c\oplus s}}$. Also $c \oplus s \in Cone(c) \subseteq D$. Since $\xi < \omega_1^c \leq \omega_1^{c \oplus s}$, $X \subseteq \xi$ and $X \in L_{\omega_1^{c \oplus s}}$, by Lemma A.3, we have $Code(X, r)$ is $\Sigma^0_{\xi+2}(c \oplus s, r)$.[2] Since $r \leq_T c$, we have $Code(X, r)$ is $\Sigma^0_{\xi+2}(c \oplus s)$. Theorem A.11 relativized to c gives that:

$$\text{if } a \in \omega^\omega, 1 \leq \alpha < \omega_1^c \text{ and for some } \eta < \omega_1, a \text{ is } \Sigma^0_\alpha(x \oplus c) \text{ for all } x \in \mathsf{WO}_\eta, \text{ then } a \text{ is } \Sigma^0_\alpha(c).$$

Since $Code(X, r)$ is $\Sigma^0_{\xi+2}(c \oplus s)$ for any $s \in \mathsf{WO}_\delta$, by Theorem A.11, we have $Code(X, r)$ is $\Sigma^0_{\xi+2}(c)$. Hence $Code(X, r) \in L_{\omega_1^c}[c]$. Since $r \leq_T c$, we have $X \in L_{\omega_1^c}[c]$. □

By Lemmas A.2 and A.4, D has property ★, and we are done. □

We assume $Z_2 + Det(Turing\text{-}\Sigma^1_1)$. Since D is Turing closed and cofinal in the Turing degrees, D contains a cone of Turing degrees $Cone(c)$ for some $c \in \omega^\omega$. Since D has property ★, by Theorem A.3, if α is a countable c-admissible ordinal, then α is an L-cardinal (i.e. HP holds).

[2] In Lemma A.3, let $a = c \oplus s$ and $\alpha = \xi$.

Appendix B
$\boldsymbol{Det(<\omega^2\text{-}\Pi^1_1)}$ Implies that $0^\sharp$ Exists in Z_3

Abstract In this appendix, I reconstruct Martin's proof of "$Det(<\omega^2\text{-}\Pi^1_1)$ implies that $0^\sharp$ exists" in Z_3 *without* the use of Harrington's Principle.

First of all, besides $Det(\Sigma^1_1)$, there are other determinacy hypotheses equivalent to the existence of $0^\sharp$ in ZF. Examples include: Det(Turing-Σ^1_1), $Det(\Sigma^1_1$-Wadge), and $Det(\Sigma^1_1$-Kleene). For the definition of $Det(<\omega^2\text{-}\Pi^1_1)$, see Definition B.1. For the definition of $Det(\Sigma^1_1$-Wadge) and $Det(\Sigma^1_1$-Kleene), I refer to [8, 9].

Now, Harrington first proved in [8] that $Det(\Sigma^1_1$-Wadge) is equivalent to "$0^\sharp$ exists". G.Weitkamp proved in [9] that $Det(\Sigma^1_1$-Kleene) is equivalent to "$0^\sharp$ exists". Martin proved that $Det(<\omega^2\text{-}\Pi^1_1)$ is equivalent to "$0^\sharp$ exists" (see [10]).

In this chapter, I list some known facts about these determinacy hypotheses. Let Γ be one of the following statements: Det(Turing-Σ^1_1), $Det(\Sigma^1_1$-Wadge) and $Det(\Sigma^1_1$-Kleene). It is provable in Z_2 that $0^\sharp$ exists implies Γ. All known proofs of "Γ implies that $0^\sharp$ exists" are done in two steps: first show that Γ implies HP and then show that HP implies that $0^\sharp$ exists. The first step is provable in Z_2. The minimal system in higher-order arithmetic to show that HP implies that $0^\sharp$ exists is Z_4. As a corollary, it is provable in Z_4 that Γ is equivalent to $0^\sharp$ exists.

However, among the determinacy hypotheses which are equivalent to "$0^\sharp$ exists", one exception is $Det(<\omega^2\text{-}\Pi^1_1)$. Martin's proof of "$Det(<\omega^2\text{-}\Pi^1_1)$ implies $0^\sharp$ exists" does not use Harrington's Principle. I have previously observed that Martin's proof can be done in Z_3. In this Appendix, I reconstruct Martin's proof of "$Det(<\omega^2\text{-}\Pi^1_1)$ implies that $0^\sharp$ exists" in Z_3 without the use of Harrington's Principle.

Definition B.1 ([6])

(1) Suppose α is recursive and there is a recursive well-ordering $E\subseteq\omega^\omega\times\omega^\omega$ such that $o.t.(E)=\alpha$. For $n\in\omega$, let $|n|$ denote the order type of the predecessors of n according to E. For $A\subseteq\omega^\omega$, A is α-Π^1_1 if there exists a sequence $\langle A_\xi \mid \xi<\alpha\rangle$ of subsets of ω^ω such that $\{(n,x)\in\omega\times\omega^\omega \mid x\in A_{|n|}\}\in\Pi^1_1$ and $x\in A$ if the

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least ξ such that $\xi = \alpha$ or $x \notin A_\xi$ is odd.[3] In this case, we say that $\langle A_\xi \mid \xi < \alpha\rangle$ witnesses $A \in \alpha\text{-}\Pi^1_1$.

(2) A is $< \alpha\text{-}\Pi^1_1$ if it is $\beta\text{-}\Pi^1_1$ for some $\beta < \alpha$.

Theorem B.2 (Martin, [10], Z_3) *$Det(<\omega^2\text{-}\Pi^1_1)$ implies that $0^\sharp$ exists.*

Proof From Proposition 2.5, $0^\sharp$ exists iff L_{ω_1} has an uncountable set of indiscernibles. Assume $Det(<\omega^2\text{-}\Pi^1_1)$. It suffices to show that L_{ω_1} has an uncountable set of indiscernibles as follows.

Lemma B.1 *For any formula $\varphi(x_0, \ldots, x_{n-1})$ in $\mathfrak{L}_{st}$, there exists a closed and unbounded subset C_φ of ω_1 such that for any two increasing sequences $\langle\rho_i \mid i \leq n\rangle$ and $\langle\theta_i \mid i \leq n\rangle$ from C_φ we have*

$$L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}] \Leftrightarrow L_{\theta_n} \models \varphi[\theta_0, \ldots, \theta_{n-1}].$$

Proof Fix a formula $\varphi(x_0, \ldots, x_{n-1})$. We will design a game G which is $< \omega^2$-Π^1_1 and hence determined. Let $\pi : \omega \to \omega \cdot (n+1)$ be a recursive function such that for any $\beta < \omega \cdot (n+1)$, $\pi^{-1}(\beta)$ is infinite. For each $\beta < \omega \cdot (n+1)$, let $\pi_\beta : \omega \to \pi^{-1}(\beta)$ be bijection defined by $\pi_\beta(j)$ be the least element of $\pi^{-1}(\beta) \setminus \{\pi_\beta(i) : i < j\}$. If x is a play of the game and $\beta < \omega \cdot (n+1)$, let $((x)_I)_\beta$ be the real $x \upharpoonright \{2 \cdot \pi_\beta(k) : k \in \omega\}$; $((x)_{II})_\beta$ be the real $x \upharpoonright \{2 \cdot \pi_\beta(k) + 1 : k \in \omega\}$.

Let $S_\beta(x)$ and $T_\beta(x)$ be the binary relation on ω coded respectively by the real $((x)_I)_\beta$ and $((x)_{II})_\beta$. Now we define the game G. Given x which is a play of the game, player I loses the game if for some $\beta < \omega \cdot (n+1)$, $S_\beta(x)$ is not a well-ordering but $T_\eta(x)$ is a well-ordering for any $\eta < \beta$; player II loses the game if for some $\beta < \omega \cdot (n+1)$, $T_\beta(x)$ is not a well-ordering but $S_\eta(x)$ is a well-ordering for any $\eta < \beta$. Let $\gamma_\beta(x)$ be the order type of $S_\beta(x)$ if $S_\beta(x)$ is a well-ordering on ω; let $\delta_\beta(x)$ be the order type of $T_\beta(x)$ if $T_\beta(x)$ is a well-ordering on ω. If $\gamma_\beta(x)$ and $\delta_\beta(x)$ are defined for all $\beta < \omega \cdot (n+1)$, let

$$\theta_i = \theta_i(x) = \sup(\{\gamma_{\omega \cdot i + k}(x), \delta_{\omega \cdot i + k}(x) \mid k \in \omega\})$$

for all $i \leq n$. If $\gamma_\beta(x)$ and $\delta_\beta(x)$ are defined for all $\beta < \omega \cdot (n+1)$, then player one wins G iff

$$L_{\theta_n} \models \varphi[\theta_0, \ldots, \theta_{n-1}].$$

Thus G is a $\omega \cdot (n+2)\text{-}\Pi^1_1$ game and hence has a winning strategy τ.

Proposition B.1 *There exists a closed and unbounded set C of ω_1 such that for any increasing sequence $\langle\rho_i \mid i \leq n\rangle$ of elements from C, there exists a play x which is consistent with τ such that for any $i \leq n$, $\theta_i(x)$ are defined and $\theta_i(x) = \rho_i$.*

Proof We assume that τ is a winning strategy for player I. The same argument applies to the case τ is a winning strategy for player II.

[3] An odd ordinal is of the form $\gamma + 2n + 1$ for some limit ordinal γ and $n \in \omega$.

If $\eta < \omega_1$ and $\beta < \omega \cdot (n+1)$, let $C^\beta_\eta = \{y : \exists z \in \omega^\omega$ such that $y = \tau * z$ and $\forall \beta' < \beta(\delta_{\beta'}(y)$ is defined and $\delta_{\beta'}(y) < \eta)\}$. Note that C^β_η is Σ^1_1. Since player I wins the game, for any $y \in C^\beta_\eta$, we have $S_\beta(y) \in \mathsf{WO}$ and $\gamma_\beta(y)$ is defined. By $\boldsymbol{\Sigma}^1_1$ boundedness theorem (cf. Fact A.6), we have $\sup(\{\gamma_\beta(y) \mid y \in C^\beta_\eta\}) < \omega_1$. Let $\upsilon(\beta, \eta) = \sup(\{\gamma_\beta(y) \mid y \in C^\beta_\eta\})$ and $\upsilon(\eta) = \sup(\{\upsilon(\beta, \eta) \mid \beta < \omega \cdot (n+1)\})$. Note that $\upsilon(\eta) < \omega_1$ for any $\eta < \omega_1$. Let

$$C = \{\alpha < \omega_1 \mid \alpha \text{ is limit ordinal and for any } \eta < \alpha, \upsilon(\eta) < \alpha\}.$$

It is easy to check that C is club subset of ω_1. Let $\langle \rho_i \mid i \leq n\rangle$ be a strictly increasing sequence from C. For any $i \leq n$, let $\langle \xi_{\omega \cdot i+m} : m \in \omega\rangle$ be an increasing sequence converging to ρ_i. Given $i \leq n, m \in \omega$, let $z_{i,m}$ be the real which codes a well-ordering on ω with order type $\xi_{\omega\cdot i+m}$. Let player II play $y \subseteq \omega$ such that $((\tau * y)_{II})_{\omega\cdot i+m} = z_{i,m}$. Let $x = \tau * y$. We will show that x is the play we want.

Note that for any $i \leq n, m \in \omega$, $T_{\omega\cdot i+m}(x)$ is a well-ordering on ω coded by $z_{i,m}$, $\delta_{\omega\cdot i+m}(x)$ is defined and $\delta_{\omega\cdot i+m}(x) = \xi_{\omega\cdot i+m}$. Thus, $\delta_\beta(x)$ is defined for all $\beta < \omega \cdot (n+1)$. Since player I wins the game, $\gamma_\beta(x)$ is defined for all $\beta < \omega \cdot (n+1)$. Thus, for all $\beta < \omega \cdot (n+1)$, $\delta_\beta(x)$ and $\gamma_\beta(x)$ are defined. Hence for any $i \leq n$, $\theta_i(x)$ is defined.

Now we show that for any $i \leq n$, we have $\theta_i(x) = \rho_i$. Fix $i \leq n$. Since for all $m \in \omega$, $\delta_{\omega\cdot i+m}(x) = \xi_{\omega\cdot i+m}$, by definition, we have $\theta_i(x) \geq \sup(\{\xi_{\omega\cdot i+m} : m \in \omega\}) = \rho_i$. To show that $\theta_i(x) = \rho_i$, it suffices to show that for any $m \in \omega$, we have $\gamma_{\omega\cdot i+m}(x) < \rho_i$. Given $m \in \omega$, since $\rho_i \in C$ and ρ_i is a limit ordinal, there is $\zeta < \rho_i$ such that for all $\beta' < \omega \cdot i + m$, $\delta_{\beta'}(x)$ is defined and $\delta_{\beta'}(x) < \zeta$. Such ζ exists since $\rho_{i-1} < \rho_i$ and both are limit ordinals. Thus, by definition, we have $x \in C^{\omega\cdot i+m}_\zeta$. Hence $\gamma_{\omega\cdot i+m}(x) \leq \upsilon(\zeta)$. By definition of C, since $\rho_i \in C$ and $\zeta < \rho_i$, we have $\upsilon(\zeta) < \rho_i$. Thus we have shown that $\gamma_{\omega\cdot i+m}(x) < \rho_i$ for any $m \in \omega$. Hence $\theta_i(x) = \rho_i$.

Take C be the club subset of ω_1 as in Proposition B.1. We show that for any increasing sequence $\langle \rho_i \mid i \leq n\rangle$ from C, if τ is a winning strategy for player I, then

$$L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}];$$

if τ is a winning strategy for player II, then

$$L_{\rho_n} \not\models \varphi[\rho_0, \ldots, \rho_{n-1}].$$

Let $\langle \rho_i \mid i \leq n\rangle$ be any increasing sequence from C. By Proposition B.1, there exists a play x, consistent with τ, such that for any $i \leq n$, $\theta_i(x)$ are defined and $\theta_i(x) = \rho_i$. If τ is a winning strategy for player I, then x is a win for player I and, since all $\theta_i(x)$ are defined, $L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}]$. Similarly, if τ is a winning strategy for player II, then since all $\theta_i(x)$ are defined, $L_{\rho_n} \not\models \varphi[\rho_0, \ldots, \rho_{n-1}]$.

Let $C_\varphi = C$. Given any two increasing sequences $\langle \rho_i \mid i \leq n\rangle$, $\langle \theta_i \mid i \leq n\rangle$ from C_φ, if τ is a winning strategy in G for player I, we have $L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}]$ and $L_{\theta_n} \models \varphi[\theta_0, \ldots, \theta_{n-1}]$; if τ is a winning strategy in G for player II, we have $L_{\rho_n} \not\models$

$\varphi[\rho_0, \ldots, \rho_{n-1}]$ and $L_{\theta_n} \nvDash \varphi[\theta_0, \ldots, \theta_{n-1}]$. Thus, for any two increasing sequences $\langle \rho_i \mid i \leq n \rangle$ and $\langle \theta_i \mid i \leq n \rangle$ from C_φ, we have

$$L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}] \Leftrightarrow L_{\theta_n} \models \varphi[\theta_0, \ldots, \theta_{n-1}]$$

and we are done. □

For any formula φ in $\mathfrak{L}_{st}$, let C_φ be the club subset of ω_1 as in Lemma B.1. Let C_0 be the intersection of C_φ for any formula φ in $\mathfrak{L}_{st}$. Since there are only countable many formulas, C_0 is a club subset of ω_1. Let

$$C_1 = \{\alpha < \omega_1 : L_\alpha \prec L_{\omega_1}\}.$$

Then C_1 is a club subset of ω_1. Let $C = C_0 \cap C_1$. C is a club subset of ω_1.

Claim *C is a set of indiscernibles for L_{ω_1}.*

Proof For any formula $\varphi(x_0, \ldots, x_{n-1})$ in $\mathfrak{L}_{st}$, we show that for any two increasing sequences $\langle \rho_i \mid i < n \rangle$ and $\langle \theta_i \mid i < n \rangle$ from C we have

$$L_{\omega_1} \models \varphi[\rho_0, \ldots, \rho_{n-1}] \Leftrightarrow L_{\omega_1} \models \varphi[\theta_0, \ldots, \theta_{n-1}].$$

Take ρ_n and θ_n from C such that $\rho_n > \rho_{n-1}$ and $\theta_n > \theta_{n-1}$. Since $\langle \rho_i \mid i \leq n \rangle$ and $\langle \theta_i \mid i \leq n \rangle$ are from C_φ, by Lemma B.1, we have

$$L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}] \Leftrightarrow L_{\theta_n} \models \varphi[\theta_0, \ldots, \theta_{n-1}].$$

Since ρ_n and θ_n are from C_1, we have $L_{\rho_n} \prec L_{\omega_1}$ and $L_{\theta_n} \prec L_{\omega_1}$. So

$$L_{\rho_n} \models \varphi[\rho_0, \ldots, \rho_{n-1}] \Leftrightarrow L_{\omega_1} \models \varphi[\rho_0, \ldots, \rho_{n-1}]$$

and

$$L_{\theta_n} \models \varphi[\theta_0, \ldots, \theta_{n-1}] \Leftrightarrow L_{\omega_1} \models \varphi[\theta_0, \ldots, \theta_{n-1}].$$

Hence we have

$$L_{\omega_1} \models \varphi[\rho_0, \ldots, \rho_{n-1}] \Leftrightarrow L_{\omega_1} \models \varphi[\theta_0, \ldots, \theta_{n-1}].$$

This finishes the proof of the claim. □

This finishes the proof of Theorem B.2. □

Remark B.1 Martin's proof of "$Det(<\omega^2\text{-}\Pi^1_1)$ implies that $0^\sharp$ exists" does not use HP. But all known proofs of "$Det(\Pi^1_1)$ implies that $0^\sharp$ exists" use HP.

Martin first proved in ZF that $0^\sharp$ exists implies $Det(<\omega^2\text{-}\Pi^1_1)$ (cf. [10]). I have checked that it is in fact provable in Z_2. The following theorem is an observation from Martin's original proof of "$0^\sharp$ exists implies $Det(<\omega^2\text{-}\Pi^1_1)$" in ZF. For details about Martin's proof, I refer to [10].

Theorem B.3 (Martin, [10], Z_2) *If $0^\sharp$ exists and $A \subseteq \omega^\omega$ is in $<\omega^2\text{-}\Pi^1_1$, then G_A has a winning strategy in $L[0^\sharp]$. Thus, $\mathsf{Z}_2 + 0^\sharp$ exists implies $Det(<\omega^2\text{-}\Pi^1_1)$.*

From Theorems B.3 and B.2, $Det(<\omega^2\text{-}\Pi^1_1)$ is equivalent to $0^\sharp$ exists in Z_3. As far as I know, the question "whether $\mathsf{Z}_2 + Det(<\omega^2\text{-}\Pi^1_1)$ implies that $0^\sharp$ exists" is open.

Appendix C
Other Notions of Large Cardinals

Abstract In this appendix, I review some notions of large cardinals used in this book which are not covered in Sects. 2.1.2 and 2.1.3.

The large cardinal notions used in this book are mainly large cardinals compatible with L. The following notions of large cardinals are compatible with L: inaccessible cardinal, reflecting cardinal, Mahlo cardinal, weakly compact, indescribable cardinal, unfoldable cardinal, subtle cardinal, ineffable cardinal, remarkable cardinal, α-iterable cardinalα-iterable cardinal for $\alpha < \omega_1^L$ and α-Erdös cardinal for $\alpha < \omega_1$.

The hierarchy of large cardinals compatible with L in terms of consistency strength is as follows: α-Erdös cardinal ($\omega < \alpha < \omega_1$) > ω-Erdös cardinal > n-iterable cardinal ($2 < n \in \omega$) > 2-iterable cardinal > remarkable cardinal > 1-iterable cardinal > totally ineffable cardinal > n-ineffable cardinal > n-subtle cardinal > unfoldable cardinal > totally indescribable cardinal > Π_m^n-indescribable > weakly compact > Mahlo cardinal > reflecting cardinal > inaccessible cardinal.

For the definition of remarkable cardinals, I refer to Sect. 2.1.3. In the following, I give definitions for the other large cardinal notions used in this book.

Definition C.1 ([5]) For an uncountable cardinal κ, κ is weakly inaccessible if it is a regular limit cardinal; κ is strong limit if $2^\lambda < \kappa$ for any $\lambda < \kappa$; κ is inaccessible if it is a regular strong limit cardinal.

Under GCH, a cardinal is inaccessible iff it is weakly inaccessible. Under AC, if κ is regular, then $\forall x(x \in H_\kappa \leftrightarrow (x \subseteq H_\kappa \wedge |x| < \kappa))$. Under AC, if $\kappa > \omega$ is regular, then $H_\kappa \models \mathsf{ZFC}^-$.

Proposition C.1 ([11], AC) *Let κ be an uncountable regular cardinal. Then the following are equivalent: (1) $H_\kappa \models \mathsf{ZFC}$; (2) $H_\kappa = V_\kappa$; (3) κ is inaccessible.*

Now I introduce the notion of reflecting cardinal and Mahlo cardinal.

Definition C.2 ([12]) A regular cardinal κ is reflecting if for every $a \in H_\kappa$ and every first-order formula $\varphi(x)$, if for some regular cardinal λ, $H_\lambda \models \varphi(a)$, then there exists a cardinal $\delta < \kappa$ such that $H_\delta \models \varphi(a)$.

Y. Cheng, *Incompleteness for Higher-Order Arithmetic*, SpringerBriefs in Mathematics, https://doi.org/10.1007/978-981-13-9949-7

Note that if κ is reflecting, then κ is inaccessible.

Definition C.3 ([6]) An uncountable cardinal κ is Mahlo if κ is inaccessible and $\{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$ is stationary in κ.

If κ is Mahlo, then κ is the κ-th inaccessible cardinal.

Next, I introduce the notion of weakly compact cardinal (cf. [5]). We consider infinitary languages $\mathscr{L}_{\kappa,\lambda}$ for infinite cardinal κ, λ, which are generalizations of the ordinary first-order language. An $\mathscr{L}_{\kappa,\lambda}$ language is formulated as follows: as the usual first-order logic, first specify a supply of non-logical symbols: finitary relation, function and constant symbols. These together with an allowed supply of $\max\{\kappa, \lambda\}$ many variables lead to the terms and atomic formulas. Then the usual formula generating rules are expanded to allow conjunction $\bigwedge_{\xi<\alpha}$ and disjunction $\bigvee_{\xi<\alpha}$ of α many formulas for any $\alpha < \kappa$, and universal quantification $\forall_{\xi<\beta}$ and existential quantification $\exists_{\xi<\beta}$ of β many variables for any $\beta < \lambda$. Finally, a formula is an expression so generated with less than λ free variables, this to allow the possibility of quantification closure. Structures for interpreting the language are as for first-order logic, and the satisfaction relation is extended to incorporate the new infinitary connectives and quantifiers in the expected way.

A set of $\mathscr{L}_{\kappa,\lambda}$ sentences Σ is satisfiable if Σ has a model under the expected interpretation of infinitary conjunction, disjunction and quantification; Σ is θ satisfiable if for every $S \subseteq \Sigma$ with $|S| < \theta$ is satisfiable.[4]

A tree is a partially ordered set $(T, \leq_T)$ such that for all $t \in T$, $\{s \in T \mid s \leq_T t\}$ is well-ordered by $\leq_T$ and T has a unique minimal element, the root of the tree. A branch through a tree $(T, \leq_T)$ is a linearly ordered subset and its length is the order type of the linear order.

Definition C.4 ([5, 6]) Let $\kappa > \omega$ and $\lambda \geq \omega$.

(1) κ is *weakly compact* if for any set of $\mathscr{L}_{\kappa,\kappa}$ sentences Σ with $|\Sigma| \leq \kappa$, if Σ is κ-satisfiable, then Σ is satisfiable.[5]
(2) A regular cardinal κ has the *tree property* if every κ-tree has a κ branch.
(3) κ has the *Extension property* if for any $R \subseteq V_\kappa$ there is a transitive set $X \supseteq V_\kappa$ with $\kappa \in X$ and an $S \subseteq X$ such that $\langle V_\kappa, \in, R\rangle \prec \langle X, \in, S\rangle$.
(4) κ has the *linear order property* if whenever $(L, <_L)$ is a linear ordering of cardinality $\geq \kappa$, there is a monotone $<_L$ sequence of order type κ.

I now introduce the notion of indescribable cardinal. We first define Π^n_m formula (Σ^n_m formula) (cf. [5]). Let $n > 0$ be a natural number and let us consider the n-th order predicate calculus. There are variables of orders $1, 2, \ldots, n$, and the quantifiers are applied to variables of all orders. An n-th order formula contains, in addition to

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[4] The language $\mathscr{L}_{\omega,\omega}$ is just the language of the first-order predicate calculus. $\mathscr{L}_{\omega,\omega}$ satisfies the Compactness Theorem: if Σ is a set of sentences such that every finite $S \subseteq \Sigma$ has a model, then Σ has a model.

[5] $|\Sigma| \leq \kappa$ is equivalent to Σ has at most κ many non-logical symbols.

first-order symbols and higher-order quantifiers, predicates $X(z)$ where X and z are variables of order $k+1$ and k respectively (for any $k < n$).

Satisfaction for an n-th order formula in a model $\mathfrak{A} = (A, P, \ldots, f, \ldots, c \cdots)$ is defined as follows: variables of first-order are interpreted as elements of the set A, variables of second-order as elements of $\mathscr{P}(A)$ etc.; variables of order n are interpreted as elements of $\mathscr{P}^{n-1}(A)$. The predicate $X(z)$ is interpreted as $z \in X$. A Π^n_m formula is a formula of order $n+1$ of the form

$$\underbrace{(\forall X)(\exists Y)\cdots}_{m\text{quantifiers}} \psi \tag{C.1}$$

where $X, Y, \ldots$ are $(n+1)$-th order variables and ψ is such that all quantified variables are of order at most n. Similarly, a Σ^n_m formula is as in (C.1), but with $\exists$ and $\forall$ interchanged.

Definition C.5 ([5, 6])

(1) For Q either Π^m_n or Σ^m_n, κ is *Q-indescribable* if for any $R \subseteq V_\kappa$ and Q sentence φ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$.
(2) κ is *totally indescribable* if κ is Π^m_n-indescribable for any $m, n \in \omega$.

Definition C.6 If f is a function on $[X]^n$, then we say that $Y \subseteq X$ is *homogeneous* for f if f is constant on $[Y]^n$. $\kappa \to (\lambda)^n_\xi$ means that for any $f : [\kappa]^n \to \xi$, there is a $X \subseteq \kappa$ such that $|X| = \lambda$ and X is homogeneous for f. $\kappa \to (\alpha)^{<\omega}_\lambda$ means that for any $f : [\kappa]^{<\omega} \to \lambda$, there is $X \subseteq \kappa$ such that X has order type α and for each $n \in \omega$, X is homogeneous for $f \upharpoonright [\kappa]^n$.

As a summary, we have the following equivalent characterizations of weakly compact cardinals.

Theorem C.7 ([5, 6, 19]) *For uncountable cardinal κ, the following are equivalent:*

(1) κ is weakly compact.
(2) κ is Π^1_1-indescribable.
(3) $\kappa \to (\kappa)^n_\lambda$ for every $n \in \omega, \lambda < \kappa$.
(4) κ is inaccessible and has the tree property.
(5) κ has the Extension property.
(6) κ has the linear order property.

Finally, I introduce the notion of unfoldable cardinal. For definitions and facts about unfoldable cardinal, I refer to [13–15].

Definition C.8 Let (M, E) and (N, F) be models of set theory. We say (N, F) *end extends* (M, E) if for every $a \in M$, $\{b \in M \mid bEa\} = \{b \in N \mid bFa\}$. Then $(M, E) \prec_e (N, F)$ means that (N, F) is an elementary end extension of (M, E). The structure $(\mathfrak{F}_M, \prec_e)$ consists of all non-trivial elementary end extension of M, ordered by the relation $\prec_e$.

Definition C.9 ([13–15]) An inaccessible cardinal κ is unfoldable iff for any $S \subseteq \kappa$ and any α, there exist transitive set N and $S' \subseteq N$ such that $(V_\kappa, \in, S) \prec_e (N, \in, S')$ (i.e. $N \in (\mathfrak{F}_{(V_\kappa, \in, S)}, \prec_e))$ and $o(N) \geq \alpha$, where $o(N) = \mathsf{Ord} \cap N$.

The notions subtle and ineffable cardinals are introduced next. For definitions and facts about subtle ineffable cardinals, I refer to [16, 17].

Definition C.10 ([16, 17])

(1) Let A be a set of ordinals and $1 \leq n < \omega$. A sequence $\mathbf{S} = \langle S_{\beta_1,\ldots,\beta_n} \mid \beta_1 < \cdots < \beta_n, \beta_1, \ldots, \beta_n \in A\rangle$ is an (n, A) sequence iff $S_{\beta_1,\ldots,\beta_n} \subseteq \beta_1$ for all $\beta_1, \ldots, \beta_n$ in A. In particular, it makes sense to speak of an (n, κ) sequence.
(2) We say that X is *homogeneous for an* (n, A) *sequence* $\mathbf{S}$ if $X \subseteq A$ and for any two sequences $\beta_1 < \cdots < \beta_n, \beta'_1 < \cdots < \beta'_n$ from X, if $\beta_1 \leq \beta'_1$, then $S_{\beta_1,\ldots,\beta_n} = \beta_1 \cap S_{\beta'_1,\ldots,\beta'_n}$.
(3) A cardinal κ is *n-subtle* if for every (n, κ) sequence $\mathbf{S}$ and every closed unbounded set $C \subseteq \kappa$, there is $x \in [C]^{n+1}$ such that x is homogeneous for $\mathbf{S}$.
(4) κ is a *subtle cardinal* if κ is 1-subtle. i.e. κ is subtle if for any sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$ and any club C on κ, there exists α, β in C such that if $\alpha < \beta$, then $S_\alpha = S_\beta \cap \alpha$.

Definition C.11 ([16, 17])

(1) κ is *n-ineffable* if every (n, κ) sequence has a homogenous set which is stationary in κ.
(2) κ is *ineffable* if κ is 1-ineffable. i.e. for any sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ with $S_\alpha \subseteq \alpha$, there exists $A \subseteq \kappa$ such that A is stationary and for all $\alpha, \beta \in A$, if $\alpha < \beta$, then $S_\alpha = S_\beta \cap \alpha$.

The notion of α-iterable cardinalα-iterable cardinal is introduced next. For definitions and facts about α-iterable cardinalα-iterable cardinal, I refer to [18].

To define α-iterable cardinalsα-iterable cardinal, we will need the key notion of α-good M-ultrafilters.

Definition C.12

(1) A weak κ-model M of set theory is a transitive set of size κ satisfying ZFC^- with $\kappa \in M$.
(2) For weak κ-model M, an M-ultrafilter U on κ, is 0-good if the ultrapower of M by U is well-founded.
(3) For weak κ-model M, an M-ultrafilter U on κ is weakly amenable if for every $A \in M$ of size κ in M, the intersection $U \cap A$ is an element of M.
(4) For weak κ-model M, an M-ultrafilter on κ is 1-good if it is 0-good and weakly amenable.

Fact C.13 ([18]) *Suppose M is a weak κ-model, U is a α-good M-ultrafilter on κ, and $j : M \to N$ is the ultrapower by U. Define $j(U) = \{A \in \mathscr{P}(j(\kappa))^N \mid A = [f]$ and $\{\alpha < \kappa \mid f(a) \in U\} \in U\}$. Then $j(U)$ is a weakly amenable N-ultrafilter on $j(\kappa)$ such that $j``U \subseteq j(\kappa)$.*

Suppose M is a weak κ-model and U_0 is a 1-good M-ultrafilter on κ. From Fact C.13, let $j(U_0) = U_1$ be the weakly amenable ultrafilter for the ultrapower of M by U. If the ultrapower by U_1 happens to be well-founded, we will say that U_0 is 2-good.

In this way, we can continue iterating the ultrapower construction so long as the ultrapowers are well-founded. For $\alpha \leq \omega$, we will say that U is α-good if the first α-many ultrapowers are well-founded. Suppose next that the first ω-many ultrapowers are well-founded. We can form their direct limit and ask if that is well-founded too.

If the direct limit of the first ω-many iterates turns out to be well-founded, we will say that U is $\omega + 1$-good. Continuing the pattern, we make the following definition.

Definition C.14 ([18]) Suppose M is a weak κ-model and α is an ordinal. An M-ultrafilter on κ is *α-good*, if we can iterate the ultrapower construction for α-many steps.

Fact C.15 ([18]) *Suppose M is a weak κ-model. An ω_1-good M-ultrafilter is α-good for every ordinal α.*

From Fact C.15, to iterate the ultrapower construction through all the ordinals, it suffices to know that we can iterate through all the countable ordinals. Thus, the study of α-good ultranlters only makes sense for $\alpha \leq \omega_1$.

Definition C.16 ([18]) For $\alpha \leq \omega_1$, a cardinal κ is α-iterable if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an α-good M-ultrafilter on κ.

Proposition C.2 ([18])

(1) If κ is an α-iterable cardinalα-iterable cardinal, then the cardinal κ is a limit of β-iterable cardinal for $\beta < \alpha$;
(2) For $\alpha < \omega_1^L$, the α-iterable cardinalsα-iterable cardinal are downward absolute to L. This result is optimal since ω_1-iterable cardinals cannot exist in L;
(3) If κ is a remarkable cardinal, then there is a countable transitive model of ZFC *with a proper class of 1-iterable cardinals.*
(4) If κ is 2-iterable, then κ is a limit of remarkable cardinal;
(5) If there is an ω_1-iterable cardinal, then $0^\sharp$ exists;
(6) If $0^\sharp$ exists, then the Silver indiscernibles are α-iterable in L for all $\alpha < \omega_1^L$;

The notion of α-Erdös cardinal is introduced next. For definitions and facts about α-Erdös cardinal, I refer to [5, 6].

Definition C.17 ([5, 6]) For any limit ordinal α, α-Erdös cardinal is the least cardinal κ such that $\kappa \to (\alpha)_2^{<\omega}$.

Fact C.18 ([5, 18])

(1) For $\omega \leq \alpha < \omega_1^L$, α-Erdös cardinal are downward absolute to L.
(2) An ω-Erdos cardinal implies for every $n \in \omega$, the consistency of the existence of a proper class of n-iterable cardinals.
(3) The existence of ω_1-Erdös cardinal implies $V \neq L$.

Let us consider some equivalences involving "$0^\sharp$ exists" and some consequences of "$0^\sharp$ does not exist" in ZFC.

Theorem C.19 ([5, 6, 19]) *The following statements are equivalent:*

(1) $0^\sharp$ *exists.*
(2) There exists an elementary embedding from L to L.
(3) For some α *and* β*, there exists* $j : L_\alpha \prec L_\beta$ *with* $crit(j) < |\alpha|$.
(4) There exists an L-ultrafilter U such that the ultrapower of L by U is well-founded.
(5) There exists an iterable L-ultrafilter.
(6) For some limit ordinal λ, $(L_\lambda, \in)$ *has uncountable set of indiscernibles.*
(7) There exists a class I of ordinals such that I is a closed unbounded class of indiscernibles for L and the Skolem hull of I in L is L (elements of I are called silver indiscernibles).[6]
(8) (A) If κ, λ *are uncountable cardinal and* $\kappa < \lambda$*, then* $(L_\kappa, \in) \prec (L_\lambda, \in)$.
(B) There exists a unique class sized club of ordinals I containing all uncountable cardinals such that for any uncountable cardinal κ*:*
(a) $|I \cap \kappa| = \kappa$*;*
(b) $I \cap \kappa$ *is a set of indiscernibles for* $(L_\kappa, \in)$*;*
(c) Any $a \in L_\kappa$ *is definable from* $I \cap \kappa$ *in* $(L_\kappa, \in)$.
(9) $\aleph_\omega$ *is regular in L.*
(10) Any $X \in \mathscr{P}(\kappa) \cap L$ *either contains or is disjoint from a closed and unbounded subset of* κ *where* κ *is an uncountable regular cardinal.*
(11) For any Σ^1_1 *game, either player I has a winning strategy or player II has a winning strategy which is recursive in* $0^\sharp$.

Theorem C.20 ([5, 6, 19]) *If* $0^\sharp$ *does not exist, then:*

(1) Any singular cardinal is a singular cardinal in L.
(2) For any singular cardinal κ, $(\kappa^+)^L = \kappa^+$.
(3) SCH *holds. i.e. for singular cardinal* κ*, if* $2^{cf\kappa} < \kappa$*, then* $\kappa^{cf\kappa} = \kappa^+$.
(4) Let κ *be a singular cardinal, if* $\forall \alpha < \kappa (\mathscr{P}(\alpha) \subseteq L)$*, then* $\mathscr{P}(\kappa) \subseteq L$.
(5) (Covering theorem) For any uncountable set X of ordinals, there is a $Y \in L$ *such that* $X \subseteq Y$ *and* $|Y| = |X|$.

In this book, κ-model is a model in the form $L[U]$ such that $\langle L[U], \in, U\rangle \models U$ is a normal ultrafilter over κ. For the theory of $0^\dagger$, I refer to [6].

Theorem C.21 (Solovay, [6]) *The following are equivalent:*

(1) $0^\dagger$ *exists.*
(2) There exists a κ*-model* $\mathscr{M}$ *for some ordinal* κ *that has an uncountable set I of indiscernibles such that* $\min(I) > \kappa$.

[6]Moreover, with $\langle \tau_\xi \mid \xi \in \mathsf{Ord}\rangle$ the increasing enumeration of I, then: (1) I contains every uncountable cardinal; (2) $|\tau_\xi| = |\xi| + \aleph_0$; (3) for any limit ordinal $\alpha \geq \omega$, the Skolem hull of $\{\tau_\xi \mid \xi < \alpha\}$ in L_{τ_α} is L_{τ_α}; (4) if $\xi \leq \eta$, then $L_{\tau_\xi} \prec L_{\tau_\eta} \prec L$.

(3) For every uncountable cardinal λ*, there exists a* λ*-model* $\mathcal{M}$ *and a double class* $\langle X, Y \rangle$ *of indiscernibles for* $\mathcal{M}$ *such that:*

(a) $X \subseteq \lambda$ *is closed and unbounded;*
(b) $Y \subseteq \mathsf{Ord} \setminus (\lambda + 1)$ *is a closed unbounded class;*
(c) $X \cup \{\lambda\} \cup Y$ *contains every uncountable cardinal;*
(d) The Skolem hull of $X \cup Y$ *in* $\mathcal{M}$ *is* $\mathcal{M}$.

Definition C.22 A cardinal κ is *Woodin* if for all $A \subseteq \kappa$ there are arbitrarily large $\delta < \kappa$ such that for all $\alpha < \kappa$ there is some elementary embedding $\pi : V \to M$ with M transitive and with critical point δ such that $V_\alpha \subseteq M$ and $A \cap \alpha = \pi(A) \cap \alpha$.

Definition C.23 Define $M_n^\sharp(x) = \langle L_\gamma[E, x], \in, x, E, U \rangle$ where

- U is an $L[E, x]$-ultrafilter;
- E is a sequence of extenders (a system of ultrafilters) witnessing that $L_\gamma[E, x] \models$ there are n Woodin cardinals;
- $M_n^\sharp(x)$ is iterable.[7]

"$M_n^\sharp(x)$ exists" is a large cardinal axiom which, for $n > 0$, is much stronger than "$0^\sharp$ exists". "$M_0^\sharp(\emptyset)$" is just $0^\sharp$.

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[7] For the definition of iterability, I refer to Sect. 2.1.2.

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Index

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