Concluding Remarks

The correspondence between electromagnetism and gravitation is very rich and detailed. Some of these correspondences are still uncovered, while some of them are further developed. This correspondence is reflected in Maxwell-like form of the gravitational field tensor (the Weyl tensor), the super energy–momentum tensor (the Bel–Robinson tensor) and the dynamical equations (the Bianchi identities). It is also known that an electromagnetic field can always be generated through a vector potential A_i. In an analogy to electromagnetism, can we generate a gravitational field through a potential (that should be tensorial in nature)? The answer is in affirmative—indeed it is possible to generate the gravitational field (the Weyl tensor) through the process of covariant differentiation of a third rank tensor L_{ijk}. This is what precisely done by Cornelius Lanczos (a Hungarian mathematician and physicist) in 1962. This tensor is now commonly known as Lanczos potential or Lanczos spin tensor. Unfortunately, this potential exists only in four dimensions and there is no such potential for the Riemann tensor when the space is not Ricci-flat, i.e. $R_{ij} \neq 0$. However, there are several reasons that why the study of such potential is important and some of them are

(i) definition of energy and momentum,
(ii) possibility of 'massive gravitons', then the potential becomes dynamic,
(iii) quantization,
(iv) dealing with a simpler object.

The equations that provide the relationship between this tensor and the Weyl tensor are known as Weyl–Lanczos equations. For a given spacetime geometry, the construction of Lanczos potential is equivalent to solving Weyl–Lanczos equations under certain constraint conditions imposed on L_{ijk}. There are several ways of solving Weyl–Lanczos equations, although none of them are as straightforward as one would like them to be. However, the tetrad formalisms offer some simplifications and this is what we have done in this text.

Using GHP formalism, the solutions of Weyl–Lanczos equations, which in turn leads to Lanczos potential, have been obtained for arbitrary Petrov types II and D

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spacetimes. The results obtained are supported by examples, and it is seen that the Lanczos potential for Robinson–Trautman metric of Petrov type II depends upon the radial coordinate r; while for Kerr black hole, the Lanczos potential is related to the mass parameter of the Kerr black hole and the Coulomb component of the gravitational field. While using NP formalism, a general prescription to obtain the Lanczos potentials for radiative spacetimes (Petrov types III and N) have also been given and consequently the Lanczos potential for the well-known radiative solutions of Einstein field equations have been calculated. In this way, we have found a general prescription for obtaining the Lanczos potential for algebraically special spacetimes (Petrov types II, D, III and N).

Using the method of general observers and the spin-coefficient formalism of Newman and Penrose, a yet another method for obtaining the Lanczos potential for perfect fluid spacetimes has been given. The kinematical quantities and the equations satisfied by them have been translated into NP formalism, and in the process, the Lanczos potential for shear-free irrotational perfect fluid spacetimes and Bianchi Type I spacetimes have been obtained. As an example to Bianchi Type I models, the Lanczos potential for Kasner metric has been found and it is seen that the Lanczos scalars depend upon time and the constants appearing in the Kasner metric.

The Lanczos potential for some well-known solutions of Einstein and Einstein–Maxwell equations have also been obtained using the techniques of tetrad formalisms. It has been observed that the Lanczos scalars can be expressed in terms of the spin-coefficients, and our conjecture is that it shall occur in any Petrov type if we select an adequate null tetrad. Moreover, since Lanczos spin tensor is a geometrical object of spacetime, therefore it can be interpreted physically, and an attempt has been made to assign a possible physical meaning to this tensor. Thus, for example, in case of Gödel spacetime, the Lanczos potential depend upon the parameter that is responsible for the rotation of the fluid. While for a rotating black hole, the Lanczos potential depends upon the mass of the black hole and Coulomb component of the gravitational field. For Schwarzschild exterior solution and Vaidya's solution for the external field of a radiating star, the Lanczos scalars are inversely proportional to the radial distance. Also since Petrov type D fields have only Coulomb component Ψ_2 of the gravitational field with l^i and n^i as the propagation vectors therefore Lanczos scalars can act as the potential of the gravitational field, and thus justifying the name—the Lanczos potential. The non-uniqueness character of Lanczos potential has also been established and it is seen that this character has been achieved by the different choices of the tetrad vectors. This non-uniqueness property of the Lanczos potential is in close analogy with the potential of the electromagnetic field.

Apart from tetrad formalism, there are other methods for the study of Lanczos potential and we have discussed these methods to obtain the Lanczos potential of the Gödel cosmological model. It is seen that the Gauss equation employed in the embedding of a four-dimensional Riemannian manifold into a five-dimensional Euclidean space allows the existence of a symmetric tensor which in turn generates the Lanczos potential. Thus, a connection between the embedding of four-dimensional Riemannian manifold and the Lanczos potential has been established. Using the method of the Lovelock's theorem, the Lanczos potential for the Gödel cosmological model is obtained and a possible physical meaning is assigned to the Lanczos potential for this model. In fact, the Lanczos potential of the Gödel cosmological model represents some type of angular momentum. Considering a second rank symmetric tensor satisfying the wave equation for the Gödel spacetime, a solution of the wave equation has been obtained, which in turn generates the Lanczos potential for this cosmological model.

There are still some areas where the Lanczos potential can be studied—for example, if there is a possibility of 'massive gravitons', then what will happen to Lanczos potential? Will it become dynamic (as in the case of electromagnetic field where the vector potential is dynamic)? Whether or not the Lanczos potential can be used for the quantization of the gravitational field. Moreover, there is no general method available to find the Lanczos potential of an arbitrary Petrov type I (algebraically general) gravitational fields.

The interaction of a Petrov type N gravitational field with a null electromagnetic field has been considered and a metric describing such a situation has been obtained using Newman–Penrose formalism. The geometric and physical properties of the solution have also been discussed. As another application of NP formalism, the geometrical symmetries corresponding to the continuous groups of motions generated by a null vector have been considered and for Petrov type N pure radiation fields, these symmetries have been studied in detail.