Appendices

Appendix A

Multifractal Detrended Fluctuation Analysis (MF-DFA)

Multifractal detrended fluctuation analysis was proposed by Kantelhardt et al. (2002) for the study of nonstationary time series which are affected by trends or cannot be normalized. This method, which aims to identify the scaling behavior of the fluctuations of the time series for different qth order moments, is based on the detrended fluctuation analysis (Peng et al. 1994). The method is detailed below:

First let us consider $x(i)$ for $i = 1, \ldots\ldots\ldots\ldots, N$, to be a nonstationary time series of length N. The mean of the above series is given by

$$x_{\mathrm{ave}} = \frac{1}{N}\sum_{i=1}^{N} x\,(i) \tag{1}$$

Assuming $x(i)$ to be the increments of a random walk process around the average, the trajectory can be obtained by integration of the signal.

$$Y(i) = \sum_{k=1}^{i} [x(k) - x_{\mathrm{ave}}] \text{for } i = 1 \ldots\ldots N \tag{2}$$

The level of measurement noise present in experimental records and the finite data are also reduced by the integration thereby dividing the integrated time series into N_s nonoverlapping bins, where $N_s =$ int (N/s) where s is the length of the bin. As N is not a multiple of s, a small portion of the series is left at the end. Again, to include that left part, the entire process is repeated in a similar way starting from the opposite end, leaving a small portion at the beginning. Hence, $2N_s$ bins are obtained

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altogether, and for each bin least sqaure fit of the series is done followed by determination of the variance.

$$F^2\ (s,\nu\ ) = \frac{1}{s}\ \sum_{i=1}^{s} \{Y[(\nu-1)s+i] - y_\nu\ (i)\}^2 \tag{3}$$

For each bin ν, $\nu = 1$ .... N_s and

$$F^2\ (s,\nu\ ) = \frac{1}{s}\sum_{i=1}^{s} \{Y\ [\ N-(\nu-N_s)s+i\ \ ] - y_\nu\ (i)\}^2 \tag{4}$$

For $\nu = N_s + 1$........, $2N_s$, where $y_\nu\ (i)$ is the least sqaure fitted value in the bin ν. In our research work, we have performed a least sqaure linear fit (MF-DFA -1). The study can also be extended to higher orders by fitting quadratic, cubic, or higher-order polynomials.

The qth order fluctuation function $F_q(s)$ is obtained after averaging over $2N_s$ bins:

$$F_q(s) = \left\{1/2N_s \sum_{\nu=1}^{2N_s} \left[F^2\left(s,\nu\right)^{\frac{q}{2}}\right)\right]\right\}^{1/q} \tag{5}$$

where q is an index which can take all possible values except zero, as the factor $1/q$ becomes infinite with zero value. The procedure can be repeated by varying the value of s. With the increase in the value of s, $F_q(s)$ increases, and for the long-range power correlated series, $F_q(s)$ shows power-law behavior:

$$F_q(s) \propto s^{h(q)}$$

If such a scaling exists, $\ln F_q$ will depend linearly on s with slope $h(q)$. In general, the exponent $h(q)$ depends on q. For a stationary time series, $h(2)$ is identical with the Hurst exponent H. $h(q)$ is said to be the generalized exponent. The value of h (0) cannot be obtained directly, because F_q blows up at $q = 0$. F_q cannot be obtained by normal averaging procedure; instead a logarithmic averaging procedure is applied.

$$F_0(s) \equiv \exp\left\{1/4N_s \sum_{\nu=1}^{2N_s} \ln\left[F^2(s,\nu\ )\right]\right\}\ \sim\ \ s^{h(0)} \tag{6}$$

A monofractal time series is characterized by unique $h(q)$ for all values of q. If small and large fluctuations scale differently, then $h(q)$ will depend on q, or in other words the time series is multifractal. Kantelhardt et al. (2003) have explained that the

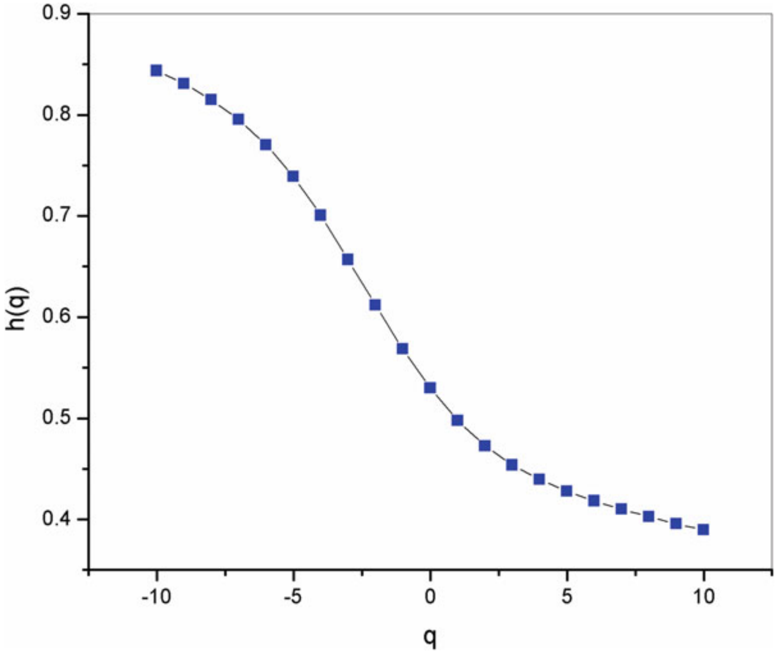

Fig. 1 Plot of $h(q)$ vs. q

values of $h(q)$ for $q < 0$ will be larger than that for $q > 0$. A typical plot of $h(q)$ vs. q is shown in Fig. 1.

The generalized Hurst exponent $h(q)$ of MF-DFA is related to the classical scaling exponent $\tau(q)$ by the relation:

$$\tau(q) = qh(q) - 1 \tag{7}$$

A typical plot of $\tau(q)$ vs. q is shown in Fig. 2.

A monofractal series with long-range correlation is characterized by linearly dependent q- order exponent $\tau(q)$ with a single Hurst exponent H. Multifractal signals have multiple Hurst exponent, and $\tau(q)$ depends non-linearly on q (Ashkenazy et al. 2003a). The singularity spectrum $f(\alpha)$ is related to $\tau(q)$ by Legendre transform (Parisi and Frisch 1985).

$$\propto = \frac{d\tau}{dq} \qquad\qquad f(\propto) = q \propto -\tau(q)$$

where α is the singularity strength or Holder exponent and $f(\alpha)$ specifies the dimension of the subset series that is characterized by α. Using Eq. (7) we can write α and $f(\alpha)$ in terms of $h(q)$

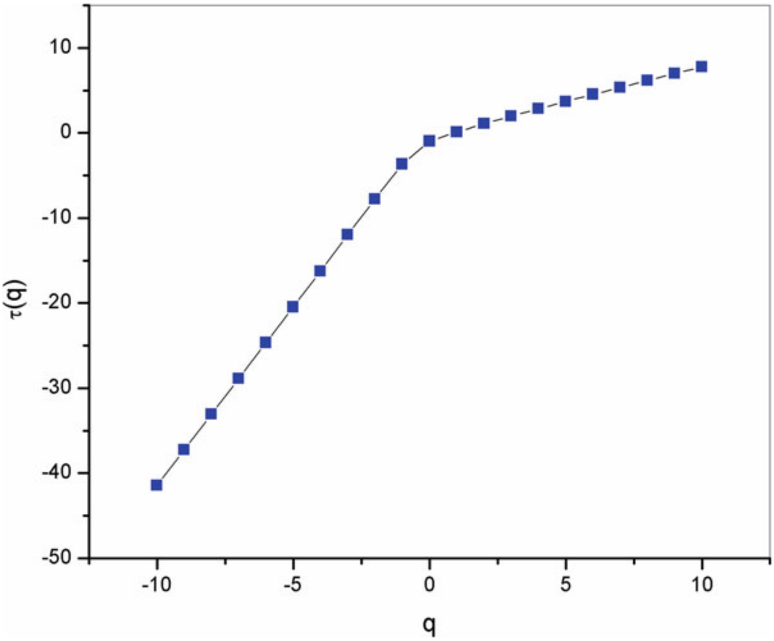

Fig. 2 Plot of $\tau(q)$ vs. q

$$\propto = h(q) + qh'(q) \tag{8}$$

$$f(\propto) = q[\propto - h(q)] + 1 \tag{9}$$

In general, the singularity spectrum quantifies the long-range correlations property of the time series (Ashkenazy et al. 2002). The multifractal spectrum is capable of providing information about the relative importance of various fractal exponents in the time series, e.g., the width of the spectrum denotes range of exponents. A quantitative characterization of the spectra can be done by least square fitting it to quadratic function (Shimizu et al. 2002) around the position of maximum α_0:

$$f(\propto) = A\,(\propto - \propto_0)^2 + B(\propto - \propto_0) + C \tag{10}$$

where C is an additive constant, $C = f(\alpha_0) = 1$, B indicates the asymmetry of the spectrum, and is zero for a symmetric spectrum. The width of the spectrum can be obtained by extrapolating the fitted curve to zero. Width W is defined as $W = \alpha_1 - \alpha_2$ with $f(\alpha_1) = f(\alpha_2) = 0$. It has been proposed by some workers (Ashkenazy et al. 2003b) that the width of the multifractal spectrum is a measure of the degree of multifractality. Singularity strength or Holder exponent α and the dimension of subset series $f(\alpha)$ can be obtained from relations 8 and 9. For a monofractal series, $h(q)$ is independent of q. Hence from relations 8 and 9, it is evident that there will be a unique value of α and $f(\alpha)$, the value of α being the generalized Hurst exponent H

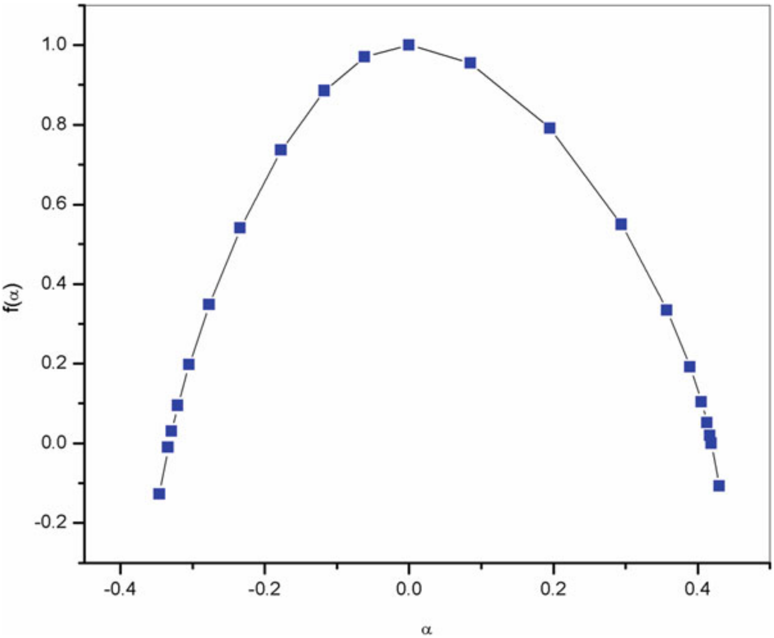

Fig. 3 Plot of $f(\alpha)$ vs. α

and the value of $f(\alpha)$ being 1. Hence the width of the spectrum will be zero for a monofractal series. The more the width, the more multifractal is the spectrum. A sample plot of multifractal spectrum is depicted in Fig. 3.

The auto-correlation exponent γ can be estimated from the relation given below (Kantelhardt et al. 2001; Movahed and Hermanis 2008):

$$\gamma = 2 - 2(h)(q = 2) \tag{11}$$

For uncorrelated or short-range correlated data, $h(2)$ is expected to have a value of 0.5, while a value greater than 0.5 is expected for long-range correlations. Therefore for uncorrelated data, γ has a value of 1, and the lower the value, the more correlated is the data.

Multifractality may be of two types: (i) "due to broad probability density function for the values of time series and (ii) due to different long-range correlation for small and large fluctuation." To ascertain the origin of multifractality, the time series is randomly shuffled and then analyzed. While shuffling the values are arranged randomly so that all correlations are destroyed. The shuffled series will exhibit non-multifractal scaling if multifractality is due to long-range correlation, and if it is due to broad probability density, then, the original $h(q)$ dependence is not changed, $h(q) = h_{shuf}(q)$. "But if both kinds of multifractality are present in a given series, then the shuffled series will show weaker multifractality than the original one" (Kantelhardt et al. 2002).

Appendix B

Multifractal Detrended Cross-Correlation Analysis (MF-DXA)

In 2008, Zhou (2008), extended the detrended cross-correlation analysis (DXA) method to multifractal detrended cross-correlation analysis (MF-DXA), an advanced version of the DXA method to investigate multifractal behavior between two time series in one or higher dimensions that are recorded simultaneously. The MF-DXA method is a combination of multifractal analysis and detrended cross-correlation analysis and is based on the order detrended covariance (Campillo and Paul 2003; Cottet et al. 2004; Podobnik et al. 2009). Just same as the MF-DFA method, MF-DXA consists of the four steps.

Let us suppose two nonstationary time series $x(i)$ and $y(i)$ for $i = 1 \ldots\ldots N$ of length N. The means of the above series' are given by

$$x_{\text{avg}} = \frac{1}{N}\sum_{i=1}^{N} x(i) \quad \& \quad y_{\text{avg}} = \frac{1}{N}\sum_{i=1}^{N} y(i) \tag{12}$$

The profiles of the underlying data series $x(i)$ and $y(i)$ are computed as

$$X(i) \equiv \sum_{k=1}^{i}\left[x(k) - x_{\text{avg}}\right] \text{for } i = 1 \ldots\ldots N.$$

$$Y(i) \equiv \sum_{k=1}^{i}\left[y(k) - y_{\text{avg}}\right] \text{for } i = 1 \ldots\ldots N \tag{13}$$

The integration also reduces the level of measurement noise present in experimental records and finite data. Each of the integrated time series is divided to N_s nonoverlapping bins where $N_s = \text{int}(N/s)$ where s is the length of the bin. Now since N is not a multiple of s, a short part of the series is left at the end. So in order to include this part of the series, the entire process is repeated starting from the opposite end thus leaving a short part at the beginning thus obtaining $2N_s$ bins. For each bin, least sqaure linear fit is performed, and the fluctuation function is given by

$$F(s,\nu) = \frac{1}{s}\sum_{i=1}^{s}\{Y[(\nu-1)s+i] - y_\nu(i)\} \times \{X[(\nu-1)s+i] - x_\nu(i)\}$$

for each bin ν,$\nu = 1, \ldots\ldots N_s$ and

$$F(s,\nu) = \frac{1}{s}\sum_{i=1}^{s}\{Y[N-(\nu-N_s)s+i]-y_\nu(i)\}\times\{X[N-(\nu-N_s)s+i]-x_\nu(i)\}$$

for $\nu = N_{\mathrm{s}}+1,\ldots\ldots.0.2N_{\mathrm{s}}$, where $x_\nu(i)$ and $y_\nu(i)$ are the least sqaure fitted values in the bin ν.

The qth order detrended covariance $F_q(\mathrm{s})$ is obtained after averaging over $2N_{\mathrm{s}}$ bins.

$$F_q(s) = \left\{\frac{1}{2N_s}\sum_{\nu=1}^{2N_s}[F(s,\nu)]^{q/2}\right\}^{1/q} \tag{14}$$

where q is an index which can take all possible values except zero because in that case the factor $1/q$ blows up. The procedure can be repeated by varying the value of s. $F_q(s)$ increases with increase in value of s. If the series is long-range power correlated, then $F_q(s)$ will show power-law behavior:

$$F_q(s) \propto s^{\lambda(q)}$$

If such a scaling exists, ln F_q will depend linearly on ln s, with $\lambda(q)$ as the slope. Scaling exponent $\lambda(q)$ represents the degree of the cross-correlation between the two time series. In general the exponent $\lambda(q)$ depends on q. We cannot obtain the value of $\lambda(0)$ directly because F_q blows up at $q = 0$. F_q cannot be obtained by the normal averaging procedure; instead a logarithmic averaging procedure is applied:

$$F_0(s) \equiv \exp\left\{\frac{1}{4N_{\mathrm{s}}}\sum_{\nu=1}^{2N_{\mathrm{s}}}\ln[F(s,\nu)]\right\}\sim s^{\lambda(0)} \tag{15}$$

For $q = 2$ the method reduces to standard DXA.

$F(s,\nu)$ may obtain negative values in general. To eliminate the problem in evaluation of fluctuation functions which may be complex valued for different values of q, we have taken the modulus of $F(s,\nu)$ to eliminate the negative values. However, Oswiecimka et al. (2014) proposed an alternative more rigorous method multifractal cross-correlation analysis (MFCCA) to take care of the negative values in cross covariances. The authors suggest that the proposed method is a more natural generalization of DCCA compared to MF-DXA. It prohibits losing information that is stored in the negative cross-covariance. The method is yet to be tested in various systems.

If scaling exponent $\lambda(q)$ is independent of q, the cross-correlations between two time series are monofractal; on the other hand if $\lambda(q)$ depends on q, the cross-correlations between two time series are multifractal. Furthermore, for positive q, $\lambda(q)$ describes the scaling behavior of the segments with large fluctuations, and for negative q, $\lambda(q)$ describes the scaling behavior of the segments with small fluctuations. Scaling exponent $\lambda(q)$ represents the degree of the cross-correlation between the two time series $x(i)$ and $y(i)$. The value $\lambda(q) = 0.5$ denotes the absence of cross-correlation. $\lambda(q) > 0.5$ indicates persistent long-range cross-correlations where a large value in one variable is likely to be followed by a large value in another variable, while the value $\lambda(q) < 0.5$ indicates anti-persistent cross-correlations where a large value in one variable is likely to be followed by a small value in another variable and vice versa (Movahed and Hermanis 2008; Shadkhoo and Jafari 2009).

Zhou (2008) found that for two time series constructed by binomial measure from p model, there exists the following relationship between scaling exponent and Hurst exponent:

$$\lambda(q = 2) \approx \left[h_{\mathrm{x}}(q = 2) + h_{\mathrm{y}}(q = 2)\right]/2 \tag{16}$$

Podobnik and Stanley have studied the above relation for monofractal autoregressive fractional integral moving average (ARFIMA) signals and EEG time series (Podobnik and Stanley 2008; Shadkhoo and Jafari 2009). Zhou has shown that the above relation holds for any q for multifractal random walks (MRW) and binomial measures generated from the p model (Mars and Lopes da Silva 1983; Hajian and Movahed 2010). However, there are also examples in which the above relation does not exist for all values of q, such as daily price changes for DJIA and NASDAQ indices (Podobnik and Stanley 2008; Shadkhoo and Jafari 2009), but for $q = 2$ it is still correct (Podobnik and Stanley 2008; Shadkhoo and Jafari 2009). The other example is the case of two time series generated by using two uncoupled ARFIMA processes, each of both is auto-correlated, but there is no power-law cross-correlation with a specific exponent (Podobnik and Stanley 2008; Shadkhoo and Jafari 2009).

According to auto-correlation function given by

$$C\ (\tau) = \langle [x(i+\tau) - \langle x\rangle][x(i) - \langle x\rangle]\rangle \tilde{}\ \tau^{-\gamma} \tag{17}$$

Hajian and Movahed (2010) introduced the cross-correlation function as

$$C_{\mathrm{x}}(\tau) = \langle [x(i+\tau) - \langle x\rangle][y(i) - \langle y\rangle]\rangle \tilde{}\ \tau^{-\gamma_{\mathrm{x}}} \tag{18}$$

where γ and γ_{x} are the auto-correlation and cross-correlation exponents, respectively. Due to the non-stationarities and trends superimposed on the collected data, direct calculation of these exponents is usually not recommended; rather the reliable method to calculate auto-correlation exponent is the DFA method, namely, $\gamma = 2 - 2h$ $(q = 2)$ (Kantelhardt et al. 2001; Movahed and Hermanis 2008). Podobnik and Stanley (2008) have demonstrated the relation between cross-correlation exponent, γ_{x}, and scaling exponent $\lambda(q)$ derived by Eq. (15) according to $\gamma_{\mathrm{x}} = 2 - 2\lambda(q = 2)$. For uncorrelated data, γ_{x} has a value of 1, and the lower the value of γ and γ_{x}, the more correlated is the data.

In general, $\lambda(q)$ depends on q, indicating the presence of multifractality. In other words, we want to point out how two series are cross-correlated in various time scales. To clarify this correlation, we generalize the singularity spectrum, $f(\alpha)$, concept to two cross-correlated series. This generalized concept gives useful information about the distribution of the degree of cross-correlation in different time scales. The way to characterize multifractality of cross-correlation between two series is to relate via a $\lambda(q)$ Legendre transform, as in the case of one series (Peitgen et al. 1992; Wang et al. 2012).

$$\alpha = \lambda(q) + q\lambda'(q) \tag{19}$$

$$f(\alpha) = q[\alpha - \lambda(q)] + 1 \tag{20}$$

Here, α is the singularity strength or Hölder exponent, while $f(\alpha)$ denotes the dimension of the subset of the series that is characterized by α. Unique Hölder exponent denotes monofractality, while in the multifractal case, the different parts of the structure are characterized by different values of α, leading to the existence of the spectrum $f(\alpha)$. The width of the spectrum can be obtained by extrapolating the fitted curve to zero. Width W is defined as

$$W = \alpha_1 - \alpha_2 \tag{21}$$

with $f(\alpha_1) = f(\alpha_2) = 0$. The growth of the width of $f(\alpha)$ shows the increase in the degree of multifractality of two coupled signals.

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