# Appendix A
# General Dissipativity Constraint

The asymptotically positive realness constraint (APRC) and quadratic dissipativity constraint (QDC) introduced in Chap. 2 are the special cases of the general dissipativity constraint (GDC) to be presented in this Appendix. The absolute function is employed for the supply rate, and the $\mathscr{K}\mathscr{L}-$bounded functions are used for representing the GDC stability. In Chap. 2, the asymptotic attractivity conditions with the quadratic constraints have been stated and applied to the decentralised MPC problem. However, the stability of a controlled system in general has not been fully analysed therein. The stability obtained from the stabilisation with the GDC is included here in the context of Lyapunov stability, Lagrange stability and asymptotic stability. With the advantage of having the storage function behave as a relaxed non-monotonic Lyapunov function, the GDC method is less conservative than the original Lyapunov's method with monotonic Lyapunov functions. The Lyapunov stability is not, nevertheless, assuredly obtained in the GDC method. The GDC provides a type of stability that is similar to the Lyapunov stability starting from a future time instant $k^* > 0$ plus the convergence property. A controlled system with the GDC is said to be *'stable in the GDC sense'* or simply *GDC stable*. The GDC also provides a boundedness property that is similar to the Lagrange uniform boundedness with an extra feasible condition. The *'input-to-power-and-state stability'* (IpSS) is introduced as an extension of the GDC stability for systems having internal and external perturbations, similarly to the input-to-state stability as an extension to Lyapunov stability. Also, a GDC stability condition for constrained controlled systems with the model predictive control is included at the end of this Appendix.

## A.1 General Dissipativity Constraint

Consider a discrete-time system $\mathscr{S}$ of the form:

$$\mathscr{S} : x(k+1) = f(x(k)) + B(x(k))u(k) + Ld(k), \tag{A.1}$$

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where $x \in \mathbb{X} \subset \mathbb{R}^n$, $u \in \mathbb{U} \subset \mathbb{R}^m$ are the state and control vectors, respectively; $f(x)$ is a vector field, $f : \mathbb{R}^n \to \mathbb{R}^n$; $B(x)$ is a matrix field, $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$; the elements of $f$ and $B$, denoted as $f_{[i]}(x)$; and $B_{[i,j]}(x)$, are not necessarily continuous functions of $x$, $f_{[i]} : \mathbb{R}^n \to \mathbb{R}$ and $B_{[i,j]} : \mathbb{R}^n \to \mathbb{R}$; $d(k)$ represents an unknown disturbance, $d(k) \in \mathbb{R}^q$, but bounded: $\|d(k)\|^2 \leqslant \theta < +\infty$. Without loss of generality, we assume that $\mathbb{X}$ is compact, $0 \in \mathbb{X}$, $0 \in \mathbb{U}$, and $f(0) = 0$.

The general dissipativity constraint (GDC) to be defined in this section is applicable to the discrete-time systems of the form $\mathscr{S} : x(k+1) = f(x(k), u(k)) + Ld(k)$, in general. However, the compound output vector introduced next will be more suitable for the input-affine system (A.1).

The following supply-rate function can be implemented in various engineering problems:

$$\xi_{\triangle}(x, w) = [x^T\ f(x)^T]Q\begin{bmatrix} x \\ f(x) \end{bmatrix} + 2[x^T\ f(x)^T]Sw + w^T Rw,$$

where $Q,\ S,\ R$ are coefficient matrices with appropriate dimensions, $Q = Q^T$ and $R = R^T$. The quadratic form of the supply-rate function is well perceived as the general quadratic supply-rate function for a dissipative system in the control literature, see, e.g. [17, 172]. Such a dissipative system can also be called $(Q, S, R)$-dissipative system [59]. In some developments, $Q = \alpha I,\ R = \gamma I$, and $S = 0$ may be employed to derive the input-output gains for linear systems.

Next, define the controlled supply rate $\xi(k, x(k), u(k))$, which is also a real-valued piecewise-continuous function in $x$ and $u$, $\xi : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. The initial $\xi_{(0)} := \xi\big(0, x(0), u(0)\big)$ is finite. For any $x(k) \in \mathbb{X}$, the control sequence $\{u(k) \in \mathbb{U}\}$ is such that the supply rate satisfies the following bounded condition:

$$\exists \theta \in \mathbb{R}^+ : \sum_{k=0}^{\kappa} \left|\xi_{(k)}\right| \leqslant \theta \quad \text{for all } \kappa > 0, \tag{A.2}$$

where $\xi_{(k)} := \xi\ (k, x(k), u(k))$ .

**Definition A.1** The controlled motion $\big(x(k), u(k)\big)$ of $\mathscr{S}$ is said to satisfy the general dissipativity constraint (GDC), if there exists a supply rate $\xi\big(k, x(k), u(k)\big)$ and there is a function $\alpha(.)$ of class $\mathscr{K}\mathscr{L}$, such that

$$\left|\xi\big(k, x(k), u(k)\big)\right| \leqslant \alpha\big(|\xi_{(0)}|, k\big) \quad \forall k \in \mathbb{Z}^+. \tag{A.3}$$

**Definition A.2** The controlled motion $\big(x(k), u(k)\big)$ of $\mathscr{S}$ is said to satisfy the GDC, practically, if there exists a supply rate $\xi\big(k, x(k), u(k)\big)$ and there is a function $\alpha(.)$ of class $\mathscr{K}\mathscr{L}$, such that

$$\left|\xi\big(k, x(k), u(k)\big)\right| \leqslant \alpha\big(|\xi_{(0)}|, k\big) + \varphi(k) \quad \forall k \in \mathbb{Z}^+,\ \varphi(k) > 0. \tag{A.4}$$

**Definition A.3** A function $\zeta : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is called $\mathcal{KL}$ bounded, if there exists a class $\mathcal{KL}$ function $\alpha(.,.)$, such that for all $\zeta(k) \in \mathbb{R}^p$, $k \in \mathbb{R}_0^+$, we have the inequality $\|\zeta(k)\| \leqslant \alpha(\|\zeta(0)\|, k)$.

The above GDC can then be simply stated as follows: The controlled motion $\big(x(k), u(k)\big)$ of $\mathcal{S}$ is said to satisfy the GDC, if there exists a supply rate $\xi\big(k, x(k), u(k)\big)$ satisfying the bounded condition (A.2), that is also $\mathcal{KL}$-bounded.

**Lemma A.1** *Consider $\mathcal{S}$ and the supply rate $\xi\big(k, x(k), u(k)\big)$ – a real-valued piecewise-continuous function of $x$ and $u$, $\xi : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$.*

1. *If the supply rate $\xi\big(k, x(k), u(k)\big)$ satisfies the bounded condition* (A.2)*, then $\big|\xi\big(k, x(k), u(k)\big)\big| \rightarrow 0$ as $k \rightarrow +\infty$.*

2. *If the supply rate $\xi\big(k, x(k), u(k)\big)$ is $\mathcal{KL}$-bounded, then the boundedness of* (A.2) *holds.*

*Proof* (1) The boundedness of (A.2) $\Rightarrow \lim_{k\to\infty} \big|\xi\big(k, x(k), u(k)\big)\big| = 0$:

Assume, on the contrary, that $\lim_{k\to\infty} \big|\xi\big(k, x(k), u(k)\big)\big| = \delta > 0$, i.e. for any $k > k_0$ there is always a small $\nu(k) > 0$ such that $\big|\xi\big(k, x(k), u(k)\big)\big| \geqslant \delta + \nu(k)$. Therefore, $\lim_{\kappa\to+\infty} \sum_{k=k_0+1}^{\kappa} \big|\xi\big(k, x(k), u(k)\big)\big| = +\infty$. This means the bounded condition (A.2) is not true.

(2) GDC $\Rightarrow$ the boundedness (A.2): In what follows, we denote $\xi_{(k)} := \xi\big(k, x(k), u(k)\big)$, and thus, $\xi_{(k_0)} := \xi\big(k_0, x(k_0), u(k_0)\big)$ for conciseness.

The $\mathcal{KL}$ function $\alpha\big(\big|\xi_{(k_0)}\big|, k-k_0\big)$ is expressed as a product of a class $K$ function and a small real number in this proof. Consider an indexed set of real numbers for each $\alpha(.,.)$ and $k_0$

$$\mathcal{E} = \{\varepsilon_k \in \mathbb{R}^+ \big| k \in \mathbb{N},\ k > k_0 :\ \varepsilon_k \leqslant 1\ \wedge\ \lim_{k\to+\infty} \prod_{i=1+k_0}^{k} \varepsilon_i = 0\}.$$

The value of $\alpha$ at the time step $k$ can be related to its value at the time step $k-1$, as follows:

$$\alpha\big(\big|\xi_{(k_0)}\big|, k-k_0\big) \leqslant \varepsilon_k \times \alpha\big(\big|\xi_{(k_0)}\big|, k-1-k_0\big),$$

for all $k > k_0$, and to its value at the initial time step $k_0$:

$$\alpha\big(\big|\xi_{(k_0)}\big|, k-k_0\big) \leqslant \alpha\big(\big|\xi_{(k_0)}\big|, 0\big) \times \prod_{i=1+k_0}^{k} \varepsilon_i. \tag{A.5}$$

It is noted here that, $\varepsilon_k \leqslant 1$ is due to the decreasing (not strictly) property of the $\mathscr{KL}$ function $\alpha(s, k)$ for a fixed $s$. And $\prod_{i=1+k_0}^{k} \varepsilon_i \to 0$ as $k \to +\infty$ due to $\alpha(., k)$ also goes to zero as $k \to +\infty$. Then, it follows from (A.4) that

$$\sum_{k=k_0+1}^{\kappa} \left|\xi\big(k, x(k), u(k)\big)\right| \leqslant \alpha\big(\left|\xi_{(k_0)}\right|, 0\big) \sum_{k=k_0+1}^{\kappa} \Big( \prod_{j=1+k_0}^{k} \varepsilon_j\Big). \tag{A.6}$$

For $\varepsilon_m > \max_{k=k_0+1}^{\kappa} \varepsilon_k > 1$, denote $\varepsilon_{(k)} := \dfrac{\varepsilon_k}{\varepsilon_m}$. It is obviously that $0 < \varepsilon_{(k)} < 1$. (A.6) is then equivalent to

$$\sum_{k=k_0+1}^{\kappa} \left|\xi\big(k, x(k), u(k)\big)\right| \leqslant \varepsilon_m\, \alpha\big(\left|\xi_{(k_0)}\right|, 0\big) \sum_{k=k_0+1}^{\kappa} \Big( \prod_{j=1+k_0}^{k} \varepsilon_{(j)}\Big). \tag{A.7}$$

Applying the sum of consecutive powers, we get

$$\sum_{k=k_0+1}^{\kappa} \left|\xi\big(k, x(k), u(k)\big)\right| \leqslant \varepsilon_m\, \alpha\big(\left|\xi_{(k_0)}\right|, 0\big) \frac{1-\varepsilon_{(m)}^{\kappa+1}}{1-\varepsilon_{(m)}}, \tag{A.8}$$

where $\varepsilon_{(m)} := \max_{k=k_0+1}^{\kappa} \varepsilon_{(k)}$. The boundedness of (A.2) is then obtained with $\theta = \varepsilon_m\, \alpha\big(\left|\xi_{(k_0)}\right|, 0\big) \dfrac{1}{1-\varepsilon_{(m)}}$. The proof is complete ■

In other contexts, the GDC can be defined in association with the dissipation inequality for a more general nonlinear system of the form

$$\mathscr{S} : x(k+1) = f(x(k), u(k)), \tag{A.9}$$

where $x \in \mathbb{X} \subset \mathbb{R}^n$, $u \in \mathbb{U} \subset \mathbb{R}^m$ are the state and control vectors, respectively; $\mathbb{X}$ is compact, $0 \in \mathbb{X}$, $0 \in \mathbb{U}$, and $f(0, 0) = 0$; $f$ is not necessarily continuous. The controlled system with MPC is usually discontinuous in $x$, even when the open-loop system is continuous. The function $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_0^+$ in the GDC method is generally piecewise continuous, whereas smooth or locally Lipschitz continuous Lyapunov functions are usually considered for discontinuous dynamical systems in previous works. The assumption on the locally Lipschitz continuity is not made, but the bounds of the form $\underline{\alpha}\big(\|x(k)\|\big) \leqslant V\big(k, x(k)\big) \leqslant \overline{\alpha}\big(\|x(k)\|\big)$ are employed in the stability conditions instead.

**Definition A.4** The controlled system $\mathscr{S}$ (A.9) is said to be *GDC stable* around the zero equilibrium with respect to the supply rate $\xi\big(k, x(k), u(k)\big)$ and the real-valued, non-negative and radially unbounded storage function $V(k, x(k))$ if the following conditions hold for all $k > 0$ with some control sequences $u(k) \in \mathbb{R}^m$, irrespective of the initial state $x(0)$:

$$\underline{\alpha}\big(\|x(k)\|\big) \leqslant V\big(k, x(k)\big) \leqslant \overline{\alpha}\big(\|x(k)\|\big),$$

$$V\,(k, x(k)) - \boldsymbol{\tau}\, V\,(k-1, x(k-1)) \leqslant |\xi(k, x(k), u(k))|\,, \quad 0 < \boldsymbol{\tau} < 1, \text{ and}$$

$$|\xi(k, x(k), u(k))| \leqslant \alpha\left(\big|\xi(0, x(0), u(0))\big|, k\right).$$

for some $\mathscr{K}_\infty$ functions $\underline{\alpha}(.)$ and $\overline{\alpha}(.)$, and some $\mathscr{K}\mathscr{L}$ function $\underline{\alpha}(.)$.

Alternatively, the GDC stability can be defined with a separated dissipation inequality, as follows:

**Definition A.5** The controlled system $\mathscr{S}$ (A.9) is said to be *GDC stable* around the zero equilibrium with respect to the supply rate $\xi\big(k, x(k), u(k)\big)$ and the real-valued, non-negative, and radially unbounded storage function $V(k, x(k))$ if the following conditions hold for all $u(k) \in \mathbb{R}^m$ and all $k > 0$, irrespective of the initial state $x(0)$:

$$\underline{\alpha}\big(\|x(k)\|\big) \leqslant V\big(k, x(k)\big) \leqslant \overline{\alpha}\big(\|x(k)\|\big),$$

$$V\,(k, x(k)) - \boldsymbol{\tau}\, V\,(k-1, x(k-1)) \leqslant |\xi(k, x(k), u(k))|\,, \quad 0 < \boldsymbol{\tau} < 1$$

for some $\mathscr{K}_\infty$ functions $\underline{\alpha}(.)$ and $\overline{\alpha}(.)$, and there exist some control sequences $u(k) \in \mathbb{U} \subset \mathbb{R}^m$, such that the following inequality is fulfilled for all $k > 0$:

$$|\xi(k, x(k), u(k))| \leqslant \alpha\left(\big|\xi(0, x(0), u(0))\big|, k\right),$$

for some $\mathscr{K}\mathscr{L}$ function $\underline{\alpha}(.)$.

The GDC will be applied to the problem of designing the closed-form control law or in the optimisation-based control algorithms such as the model predictive control for $\mathscr{S}$. In the next section, the input-to-power-and-state stabilisation with the GDC is introduced. The GDC method is different to the previous constructive method by considering the controlled system as a single system instead of as an interconnection of two open-loop systems. As a result, controlled system $\mathscr{S}$ (A.9) can be simply said to satisfy some GDCs in Definition A.4; i.e., the dissipation inequality is assumed an integrated part of the GDC.

## A.2 Input-to-Power-and-State Stabilisation

In the GDC method, an input-to-power-and-state stable (IpSS) closed-loop system will be obtained when either a memoryless casual control law or a control sequence from an optimisation-based control algorithm is applied such that the GDC inequalities are fulfilled. The IpSS is firstly defined below.

**Definition A.6** The controlled system $\mathscr{S}$ (A.1) is said to be IpSS stabilised if there are two functions $\alpha_i$ of class $\mathscr{KL}$, $i = \{0, 1\}$, a finite initial supply rate $\xi_{(k_0)}$ and a function $\gamma$ of class $\mathscr{K}$, such that for each initial state $x(k_0) = x_{k_0}$, the following inequality is satisfied for all $k > k_0$:

$$\|x(k, x_{k_0}, d)\| \leqslant \alpha_0(\|x_{k_0}\|, k - k_0) + \alpha_1(|\xi_{(k_0)}|, k - k_0) + \gamma(\|d\|_\infty), \tag{A.10}$$

with some admissible control sequences $\{u(k) \in \mathbb{U}\}$.

It is noted here that $\|x(k)\|$ may diverge more aggressively during certain time intervals when the term $\alpha_1(|\xi_{(k_0)}|, k - k_0)$ is additionally included in (A.10).

The stability condition is stated in the next theorem. For a real-valued non-negative function $V(k, x)$, $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_0^+$, denote

$$\Delta_{\boldsymbol{\tau}} V\big(k, x(k), x(k-1)\big) := V\big(k, x(k)\big) - \boldsymbol{\tau} V\big(k, x(k-1)\big),\ 0 < \boldsymbol{\tau} < 1. \tag{A.11}$$

**Theorem A.1** *Consider the nominal system $\mathscr{S}$ (A.1) with vanishing $d(k)$, and a real-valued piecewise-continuous supply rate $\xi\big(k, x(k), u(k)\big)$, $\xi : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Let $\tau \in \mathbb{R}^+$, $\tau < 1$. Suppose that the bounded condition (A.2) on $\xi\big(k, x(k), u(k)\big)$ holds true and there are two $\mathscr{K}_\infty$ functions $\underline{\alpha}(\|x\|)$, $\overline{\alpha}(\|x\|)$ and a real-valued, piecewise-continuous, non-negative, and radially unbounded (in $x$), function $V(k, x(k))$, $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_0^+$, $V\big(k_0, x(k_0)\big)$ is finite, such that for each $k_0 \geqslant 0$ and each $x(k_0) \in \mathbb{X}$, the following conditions hold for all $k > k_0$:*

1. $\underline{\alpha}\big(\|x(k)\|\big) \leqslant V\big(k, x(k)\big) \leqslant \overline{\alpha}\big(\|x(k)\|\big)$,
2. $\Delta_\tau V\big(k, x(k), x(k-1)\big) \leqslant \big|\xi\big(k, x(k), u(k)\big)\big|$,
3. $V\big(k, x(k-1)\big) \leqslant V\big(k-1, x(k-1)\big)$;

*with some admissible control sequences $\{u(k) \in \mathbb{U}\}$.*

*Then $x(k)$ of $\mathscr{S}$ (A.1) is quadratically attractive, i.e. $\|x(k)\| \to 0$ as $k \to 0$.*

*Proof* *The evolution of* $V\big(k, x(k)\big)$—from the conditions 2 and 3 in Theorem A.1, the following inequality is obtained for all $k > k_0$:

$$\begin{aligned}
&V\big(k, x(k)\big)\\
\leqslant&\tau V\big(k, x(k-1)\big) + \big|\xi\big(k, x(k), u(k)\big)\big| \qquad\qquad (A.12)\\
\leqslant&\tau V\big(k-1, x(k-1)\big) + \big|\xi\big(k, x(k), u(k)\big)\big|\\
\leqslant&\tau\big[\tau V\big(k-1, x(k-2)\big) + \big|\xi\big(k-1, x(k-1), u(k-1)\big)\big|\big] + \big|\xi\big(k, x(k), u(k)\big)\big|\\
\leqslant&\tau^2 V\big(k-2, x(k-2)\big) + \big[\tau\,\big|\xi\big(k-1, x(k-1), u(k-1)\big)\big| + \big|\xi\big(k, x(k), u(k)\big)\big|\big].
\end{aligned} \tag{A.13}$$

Continuing in this way, we get

$$V\big(k, x(k)\big) \leqslant \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \sum_{i=k_0+1}^{k-k_0-1} \tau^i \big|\xi\big(i, x(k-k_0-i), u(k-k_0-i)\big)\big|. \quad \text{(A.14)}$$

Applying the convolution sum and (1) of Lemma A.1 to the second term on the right-hand side of (A.14), we have both the first and second terms on the right-hand side of (A.14) goes to zero as $k \to +\infty$, due to $0 < \tau < 1$. Accordingly, using $\underline{\alpha}(\|x(k)\|) \leqslant V(k, x(k)) \leqslant \overline{\alpha}(\|x(k)\|)$ in condition 1 in Theorem A.1, it can be concluded that $\|x(k)\| \to 0$ as $k \to 0$. The radially unbounded function $V(x)$ ensures that the stated condition is not restricted to some local conditions. The proof is complete ■

The next theorem states a sufficient stability condition with the GDC.

**Theorem A.2** *Consider the system $\mathscr{S}$ (A.1) and a real-valued piecewise-continuous supply rate $\xi\big(k, x(k), u(k)\big)$, $\xi : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with $\xi\big(k, x(k), u(k)\big)$ is finite for each $k > k_0$. Let $\tau \in \mathbb{R}^+$, $\tau < 1$. Suppose there are two $\mathscr{K}_\infty$ functions $\underline{\alpha}(\|x\|)$, $\overline{\alpha}(\|x\|)$ and a real-valued, piecewise-continuous, non-negative, and radially unbounded (in x), function $V(k, x(k))$, $V : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}_0^+$, $V\big(k_0, x(k_0)\big)$ is finite, such that for each $k_0 \geqslant 0$ and each $x(k_0) \in \mathbb{X}$, the following conditions hold for all $k > k_0$:*

1. $\underline{\alpha}\big(\|x(k)\|\big) \leqslant V\big(k, x(k)\big) \leqslant \overline{\alpha}\big(\|x(k)\|\big)$,
2. $\Delta_\tau V\big(k, x(k), x(k-1)\big) \leqslant \big|\xi\big(k, x(k), u(k)\big)\big| + \sigma\, d(k)^T d(k), \;\; \sigma \in \mathbb{R}^+$,
3. *$\xi\big(k, x(k), u(k)\big)$ is $\mathscr{K}\mathscr{L}$ bounded, and*
4. $V\big(k, x(k-1)\big) \leqslant V\big(k-1, x(k-1)\big)$;

*with some admissible control sequences $\{u(k) \in \mathbb{U}\}$.*

*Then $\mathscr{S}$ (A.1) is IpSS stabilised and the nominal controlled system is quadratically attractive.*

*Proof* (1) *Attractiveness*: From (2) in Lemma A.1, the boundedness of (A.2) is obtained from the condition 3 in Theorem (A.2). The attractiveness of $\mathscr{S}$ is thus obtained as a redirect result of Theorem A.1 when $d = 0$.
(2) *IbSS stabilisability when $d(k) \neq 0$*: From the conditions 2 and 4 in Theorem A.2, the following inequality is obtained for all $k > k_0$:

$$\begin{aligned}
V\big(k, x(k)\big) &\leqslant \tau V\big(k, x(k-1)\big) + \big|\xi\big(k, x(k), u(k)\big)\big| + \sigma d(k)^T d(k) \\
&\leqslant \tau V\big(k-1, x(k-1)\big) + \big|\xi\big(k, x(k), u(k)\big)\big| + \sigma d(k)^T d(k) \\
&\leqslant \tau\big[\tau V\big(k-1, x(k-2)\big) + \big|\xi\big(k-1, x(k-1), u(k-1)\big)\big| \\
&\quad + \sigma d(k-1)^T d(k-1)\big] + \big|\xi\big(k, x(k), u(k)\big)\big| \sigma d(k)^T d(k) \\
&\leqslant \tau^2 V\big(k-2, x(k-2)\big) + \big[\tau\,\big|\xi\big(k-1, x(k-1), u(k-1)\big)\big| \\
&\quad + \big|\xi\big(k, x(k), u(k)\big)\big|\big] + \sigma\big[\tau\, d(k-1)^T d(k-1) + d(k)^T d(k)\big].
\end{aligned}$$

Continuing in this way, and using $d^T(k)d(k) \leqslant \theta$ for each $k$ and, from condition 3 of Theorem A.2, that

$$\left|\xi\big(k, x(k), u(k)\big)\right| \leqslant \alpha\left(\left|\xi\big(k_0, x(k_0), u(k_0)\big)\right|, k - k_0\right),$$

we get

$$V\big(k, x(k)\big) \leqslant \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \sum_{i=0}^{k-k_0-1} \tau^i \alpha\left(\left|\xi\big(k_0, x(k_0), u(k_0)\big)\right|, k - k_0 - i\right) + \sigma\theta \sum_{i=0}^{k-k_0-1} \tau^i. \tag{A.15}$$

Applying equality (A.5) to the second term on the right-hand side of (A.15), we have

$$\begin{aligned} V\big(k, x(k)\big) &\leqslant \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \alpha\left(\left|\xi\big(k_0, x(k_0), u(k_0)\big)\right|, 0\right) \times \sum_{i=0}^{k-k_0-1}\left(\tau^i \prod_{j=1+k_0}^{k-i} \varepsilon_j\right) \\ &\quad + \sigma\theta \sum_{i=0}^{k-k_0-1} \tau^i, \quad \text{where } \varepsilon_j \in \mathscr{E} \\ &= \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \alpha\big(|\xi_{k_0}|, 0\big) \sum_{i=0}^{k-k_0-1}\left(\tau^i \prod_{j=1+k_0}^{k-i} \varepsilon_j\right) + \sigma\theta \sum_{i=0}^{k-k_0-1} \tau^i. \end{aligned} \tag{A.16}$$

<u>*Case i*</u>: $0 < \tau < \varepsilon_k \leqslant 1$ for all $\varepsilon_k \in \mathscr{E}$.

Denote $\tau^i_{(k)} := \dfrac{\tau^i}{\prod_{j=k-i+1}^{k} \varepsilon_j}$, it follows from (A.16) that

$$V\big(k, x(k)\big) \leqslant \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \alpha\big(|\xi_{k_0}|, 0\big) \prod_{j=1+k_0}^{k} \varepsilon_j \sum_{i=0}^{k-k_0-1} \tau^i_{(k)} + \sigma\theta \sum_{i=0}^{k-k_0-1} \tau^i. \tag{A.17}$$

By the sum of powers, we have

$$V\big(k, x(k)\big) \leqslant \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \alpha\big(|\xi_{k_0}|, 0\big) \frac{1 - \tau_m^{k-k_0}}{1 - \tau_m} \prod_{j=1+k_0}^{k} \varepsilon_j + \sigma\theta \frac{1 - \tau^{k-k_0}}{1 - \tau}, \tag{A.18}$$

for all $k > k_0$, where $\tau_m := \max_{1 \leqslant i \leqslant k-1} \tau_{(i)}$, $0 < \tau_m < 1$.

*Case ii*: $1 > \tau > \varepsilon_k$ for all $\varepsilon_k \in \mathscr{E}$.

Denote $\varepsilon_{(k)} := \dfrac{\varepsilon_k}{\tau}$, it follows from (A.16) and the sum of powers that

$$V\big(k, x(k)\big) \leqslant \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \tau^{k-k_0-1}\alpha\big(|\xi_{k_0}|, 0\big)\frac{1-\varepsilon_m^{k-k_0}}{1-\varepsilon_m} + \sigma\theta\frac{1-\tau^{k-k_0}}{1-\tau}, \tag{A.19}$$

for all $k > k_0$, where $\varepsilon_m := \max\limits_{1\leqslant i\leqslant k-1} \varepsilon_{(i)}$, $0 < \varepsilon_m < 1$.

*Case iii*: $1 > \tau > \varepsilon_i \,\wedge\, 1 \geqslant \varepsilon_j > \tau > 0$ for some $\varepsilon_i \in \mathscr{E}$ and $\varepsilon_j \in \mathscr{E}$.

It follows from (A.18) and (A.19) that

$$\begin{aligned} V\big(k, x(k)\big) \leqslant{}& \tau^{k-k_0} V\big(k_0, x(k_0)\big) + \sigma\theta\frac{1-\tau^{k-k_0}}{1-\tau} \\ &+\alpha\big(|\xi_{k_0}|, 0\big) \times \max\left(\frac{1-\tau_m^{k-k_0}}{1-\tau_m}\prod_{j=1+k_0}^{k}\varepsilon_j,\ \tau^{k-k_0-1}\frac{1-\varepsilon_m^{k-k_0}}{1-\varepsilon_m}\right). \end{aligned} \tag{A.20}$$

Now, applying the inequality

$$\underline{\alpha}^{-1}(y_1+y_2+y_3) \leqslant \underline{\alpha}^{-1}(2y_1) + \underline{\alpha}^{-1}(4y_2) + \underline{\alpha}^{-1}(4y_3)$$

Sontag [144], we obtain from (A.18) and condition 1

$$\begin{aligned} \|x(k)\| \leqslant{}& \underline{\alpha}^{-1}\big(2\tau^{k-k_0}\overline{\alpha}(\|x(k_0)\|)\big) + \underline{\alpha}^{-1}\left(4\alpha\big(|\xi_{k_0}|, 0\big)\frac{1-\tau_m^{k-k_0}}{1-\tau_m}\prod_{j=1+k_0}^{k}\varepsilon_j\right) \\ &+\underline{\alpha}^{-1}\left(4\sigma\frac{1-\tau^{k-k_0}}{1-\tau}\theta\right), \end{aligned}$$

as well as from (A.19) and condition 1

$$\begin{aligned} \|x(k)\| \leqslant{}& \underline{\alpha}^{-1}\big(2\tau^{k-k_0}\overline{\alpha}(\|x(k_0)\|)\big) + \underline{\alpha}^{-1}\left(4\tau^{k-k_0-1}\alpha\big(|\xi_{k_0}|, 0\big)\frac{1-\varepsilon_m^{k-k_0}}{1-\varepsilon_m}\right) \\ &+\underline{\alpha}^{-1}\left(4\sigma\frac{1-\tau^{k-k_0}}{1-\tau}\theta\right). \end{aligned}$$

And further,

$$\tau^{k-k_0} \to 0,\ \frac{1-\tau_m^{k-k_0}}{1-\tau_m}\prod_{j=1+k_0}^{k}\varepsilon_j \to 0,\ \tau^{k-k_0-1}\frac{1-\varepsilon_m^{k-k_0}}{1-\varepsilon_m} \to 0,$$

$$\text{and } \sigma \frac{1-\tau^{k-k_0}}{1-\tau} \to \frac{\sigma}{1-\tau}, \text{ as } k \to +\infty,$$

with $0 < \tau < 1$, $0 < \tau_m < 1$, $\theta < +\infty$ and $\sigma < +\infty$, the following inequality is obtained:

$$\|x(k)\| \leqslant \alpha_1(\|x(k_0), k\|) + \alpha_2(\left|\xi_{(k_0)}\right|, k-k_0) + \gamma(\|d\|_\infty), \; k_0 \geqslant 0,$$

where $\alpha_1(s,k) \geqslant \underline{\alpha}^{-1}\big(2\tau^k \overline{\alpha}(s)\big)$, $\gamma(s) \geqslant \underline{\alpha}^{-1}\left(\frac{\sigma}{1-\tau}\beta(s)\right)$, and

$$\alpha_2(s,k) \geqslant \underline{\alpha}^{-1}\left(\max\left(4\varepsilon_0 \alpha\big(|\xi_{k_0}|, 0\big), \; \frac{\tau^{k-1}}{1-\varepsilon_m} 4\alpha\big(|\xi_{k_0}|, 0\big)\right)\right),$$

in which $\varepsilon_0 := \frac{\prod_{j=1+k_0}^{k} \varepsilon_j}{1-\tau_m}$, $\beta(s) = 4s$, $\alpha_1$ and $\alpha_2$ are the two class $\mathscr{K}\mathscr{L}$ functions and $\gamma$ is a class $\mathscr{K}$ function. The IpSS inequality (A.10) is thus obtained. The proof is complete ■

The condition for IpSS in Theorem A.2 is restated in a corollary below when

$$V(k, x(k)) = x(k)^T P(k) x(k), \; P(k) = P(k)^T \succ 0,$$

$$W(k, x(k-1)) := x(k-1)^T P(k-1) x(k-1), \text{ and}$$

$$\Delta_\tau W\big(k, x(k), x(k-1)\big) := V(k, x(k)) - \tau W(k, x(k-1)) \tag{A.21}$$

are employed in condition 2 in Theorem A.2.

**Corollary A.1** *Consider the system $\mathscr{S}$ (A.1) and a real-valued piecewise-continuous supply rate $\xi\big(k, x(k), u(k)\big)$, $\xi : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Let $\tau \in \mathbb{R}^+$ with $\tau < 1$. Suppose there are two real-valued, piecewise-continuous, non-negative, and radially unbounded, functions $V\big(k, x(k)\big) := x(k)^T P(k) x(k)$, $P(k) \succ 0$, and $W(k, x(k-1)) = x(k-1)^T P(k-1) x(k-1)$, $P(k-1) \succ 0$, such that for each $x(0) \in \mathbb{X}$ the following conditions hold for all $k > 0$:*

1. $\Delta_\tau W(k, x(k), x(k-1)) \leqslant \left|\xi\big(k, x(k), u(k)\big)\right| + \sigma d(k)^T d(k)$, *$\sigma \in \mathbb{R}^+$, and*

2. $\xi(k, x(k), u(k))$ *is $\mathscr{K}\mathscr{L}$-bounded,*

*with some admissible control sequences $\{u(k) \in \mathbb{U}\}$.*

*Then, $\mathscr{S}$ (A.1) is IpSS stabilised.*

*Proof* Since $P(k) \succ 0$, the condition (1) in Theorem A.2 is fulfilled. From the definition of $\Delta_\tau W(k, x(k), x(k-1))$, which is $\Delta_\tau V(k, x(k), x(k-1))$ in Theorem A.2 with $V(k, x(k-1))$ is replaced by $W(k, x(k-1))$, we deem obtain

$W\big(k, x(k-1)\big) = V\big(k-1, x(k-1)\big) = x(k-1)^T P(k-1)\, x(k-1)$, which is a special case of condition 4 in Theorem A.2 $\big(V\big(k, x(k-1)\big) \leqslant V\big(k-1, x(k-1)\big)\big)$. Condition 1 is thus a special case of condition 2 in Theorem A.2. The proof is then similar to that for Theorem A.2 with the initial time step $k_0 = 0$. ■

*Remark A.1* The storage functions in the GDC method are not the ISS Lyapunov functions since $\big|\xi\big(k, x(k), u(k)\big)\big| \geqslant 0$.

## A.3 Stability Analysis

The stabilisation with the GDC is governed by the non-negativeness of $\triangle V(x, k)$ along the trajectories (i.e. $\triangle V(x, k) \geqslant 0$), in which $V(x, k) \geqslant 0$ is a storage function. In this section, we analyse the stability that is obtained from the stabilisation with the GDC in the context of Lyapunov stability, Lagrange stability and asymptotic stability. The GDC provides a type of stability that is similar to the Lyapunov stability starting from a future time instant $k^* > 0$. The GDC also provides a boundedness property that is similar to the Lagrange uniform boundedness, but with a feasible condition. As a result, the convergence property for every initial state within the region of interest with the GDC is different to the asymptotic stability in the Lyapunov sense.

There are two main notions of stability in the control literature, namely Lyapunov's and Lagrange's. Several research works in the systems and control field have been centred around the stability theorems of Lyapunov's methods and Lasalle's invariance principle. There have been recent developments for the computerised and networked control systems that extend the traditional methods with non-monotonic Lyapunov functions. Another cornerstone in the classical control literature is the Lagrange stability. The Lagrange stability had been introduced before the time the Lyapunov's methods were becoming well known. However, the Lagrange stability has not been widely used as it only provides a boundedness property or a uniformly bounded system, see, e.g. [96]. Furthermore, the Lyapunov stability defines the stability of a system around an equilibrium point, in other words stability of the equilibria.

An autonomous system is called asymptotically stable around its equilibrium point at the origin if it satisfies the following two conditions:

1. Given any $\varepsilon > 0$, $\exists \delta_1 > 0$ such that if $\|x(t_0)\| < \delta_1$, then $\|x(t)\| < \varepsilon \ \forall t > t_0$.

2. $\exists \delta_2 > 0$ such that if $\|x(t_0)\| < \delta_2$, then $x(t) \to 0$ as $t \to \infty$.

The first condition requires that the state trajectory can be confined to an arbitrarily small ball centred at the equilibrium point, and of radius $\varepsilon$, when released from an arbitrary initial condition in a ball of sufficiently small radius $\delta_1$. This is called stability in the sense of Lyapunov. It is possible to have stability in the sense of Lyapunov without having asymptotic stability.

The key conditions in the GDC method consist of the dissipation inequality with a storage function, as in the dissipative system theory [172], of the form

$$V(x_k) - \tau V(x_{k-1}) \leqslant \left|(x_k^T\ u_k^T)N(x_k^T\ u_k^T)^T\right|,\ V(x_k) \geqslant 0,$$

plus a dissipation-based inequality of the form

$$\left|(x_k^T\ u_k^T)N(x_k^T\ u_k^T)^T\right| \leqslant \gamma\ \left|(x_{k-1}^T\ u_{k-1}^T)N(x_{k-1}^T\ u_{k-1}^T)^T\right|,$$

where $N := \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$, $\gamma \in (0, 1)$ and $\tau \in (0, 1)$, $x_k = x(k),\ u_k = u(k)$.

We have proved that $\|x_k\| \to 0$ as $k \to \infty$ with the two above inequalities if $V(x_k) = x_k^T P x_k$, $P \succ 0$. Furthermore, a more general form of dissipation-based inequality with a $\mathscr{KL}$−bounded supply rate has been employed. For $\triangle V(x_k) := V(x_{k+1}) - \tau V(x_k)$, we always have $\triangle V(x_k) \geqslant 0\ \forall k \geqslant 0$ and $\triangle V(x_k)$ is decreasing (not necessarily monotonically) along the trajectories in this GDC method. This means $V(x_k)$ may increase during some time intervals. This is significantly different to the traditional approach of Lyapunov methods, in which $\triangle V(x_k) \leqslant 0$ for the exponentially stable discrete-time systems. We will show here that the stabilisation with every $x(0) \in \mathbb{X} \subset R^n$, $X$ is compact, employing the two above conditions will not lead to the Lyapunov stability in its original form. One may claim that the Lyapunov stability will eventually be obtained in some future time $k > k^* > 0$. Formally speaking, however, that line of thought is not quite correct since it is not the Lyapunov stability by definition.

**a. General Dissipativity Constraint with Lyapunov Stability**

The extension of the traditional Lyapunov stability is not new in the control literature. For example, in [179], 'all regularity assumptions on traditional Lyapunov function are removed', and the property of Lyapunov function '$V$ along the system trajectories is non-increasing' is replaced with '$V$ along the system trajectories may increase its value during some proper time intervals'.

In the GDC approach, the storage function acts like a relaxed non-monotonic Lyapunov function, but is different to the above relaxed Lyapunov function for switched systems. Here, we consider the following property: '$V$ along the system trajectories may increase its value during some initial time intervals and then eventually decrease, but not necessarily monotonically, after a certain time instant'. The storage function $V(x(k))$ is a relaxed non-monotonic Lyapunov function only in the GDC method, since $\Delta_\tau(V) := V(x_k) - \tau V(x_{k-1}) \geqslant 0$, and $\Delta_\tau(V)$ is decreasing, not necessarily monotonically.

The GDC stability in the discrete-time domain can be stated as: given any $\varepsilon > 0$, $\exists \delta_0 > 0$ and $\exists k_s > k_0 \geqslant 0$, such that if $\|x(k_0)\| < \delta_0$, then $\|x(k)\| < \varepsilon\ \forall k > k_s$ (instead of $\forall k > k_0 \geqslant 0$). Also, $\exists \delta_c > 0$ such that if $\|x(k_0)\| < \delta_c$, then $x(k) \to 0$ as $k \to \infty$.

**b. General Dissipativity Constraint with Lagrange Uniformly Boundedness**

The Lagrange stability provides the boundedness property which is different than the Lyapunov stability. It states that a motion of a dynamical system is bounded if $\exists\, \varepsilon_L$ such that $\|x(t)\| \leqslant \varepsilon_L\ \forall k > k_0$. A dynamical system is *uniformly bounded* if for every $\delta_L > 0$ and for every $t_0$, there exists an $\varepsilon_L = \varepsilon_L(\delta_L)$, independent of $t_0$, such that $\|x(0)\| < \delta_L \Rightarrow \|x(t)\| < \varepsilon_L \; \forall t > t_0$.

This is fundamentally different to the Lyapunov stability since the Lyapunov stability starts with every $\varepsilon$, but not $\delta_L$. It is not difficult to show that the stabilisation with the GDC provides a closed-loop or controlled system that has the motions uniformly bounded, provided that the GDC is recursively feasible for every $x(0) \in \mathbb{X}$. Furthermore, the stabilisation with the GDC also provides the converged motion, i.e. $\|x(k)\| \to 0$ as $k \to \infty$, together with the feasibility condition such that this convergence incurs for every $x(0) \in \mathbb{X}$.

### c. GDC Stability and IpSS

The question here is whether the stabilisation with the GDC will provide a closed-loop or controlled system, that is (i) 'futurely' Lyapunov stable and converged with a quasi-Lyapunov function of $\|x(k)\| \to 0$ as $k \to \infty$ or (ii) 'conditionally' Lagrange uniformly bounded and converged. Or we should give a new stability concept, such as *stability in the GDC sense*, or simply *GDC stability*, and the *stabilisation in the GDC sense* method, and the GDC method, as an alternative or not. In [162], we have chosen the latter alternative to respect the original notions and definitions. However, the input-to-state stability (ISS) as stated in [162] will only be obtained with an extra condition on the continuity of $u(k)$, and that has been an assumption. In the same spirit with regard to the original notion of ISS [144, 147], we have also introduced the *input-to-power-and-state stability* (IpSS) (and IpSS stabilisabilty) in [158]. This means the 'ISS stabilisability in the GDC sense' will lead to a closed-loop or controlled system that is IpSS without any assumptions on the continuity of $u(k)$ or $\xi(k, u(k), s(k))$ that may restrict the implementation of the GDC method.

Similar to the results in [147] and [65], we have IpSS $\Rightarrow$ GDC stability, but not reversely. The IpSS is defined locally here to ensure that the existence of a solution is practically feasible, since $V(k, x)$ may increase during certain time intervals when the term $\alpha_2(|\xi_0|, k-1)$, a second $\mathscr{KL}$ function, is included in the IpSS inequality.

## A.4 Model Predictive Control with GDC

The control $u(k)$ is computed online by the model predictive control algorithm that employs the open-loop model of $\mathscr{S}$ (A.1). The traditional objective function of the following [88] is considered:

$$\mathscr{J}(k) = \|x(k+N)\|^2_{P_o} + \sum_{\ell=1}^{N} \|x(k+\ell)\|^2_{W_x} + \|u(k+\ell-1)\|^2_{W_u},$$

where $W_x,\ W_u$ are weighting matrices, and $N$ is the predictive (and control) horizon. The weighting coefficients in $W_x,\ W_u$ are tuning parameters. The term $\|x(k+N)\|^2_{P_o}$ is called 'terminal cost'. The notation $\|x\|_{W_x}$ is the weighted $\ell_2-$norm of $x,\ W_x \geqslant 0$. The current state vector $x(k)$ is assumed to be known. The optimisation problem of minimising $\mathscr{J}(k)$ subject to the open-loop model of $\mathscr{S}$, the state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$, respectively, and the GDC $|\xi(k, x(k), u(k))| \leqslant \alpha(|\xi(0, x(0), u(0))|, k)$, formulated in the following:

$$\begin{aligned} &\min_{\{\hat{\mathbf{u}}\}} \ \mathscr{J}(k) \\ &\text{subject to} \quad (A.1),\ x \in \mathbb{X},\ u \in \mathbb{U},\ \text{and the GDC}, \end{aligned} \tag{A.22}$$

is then solved for the optimising vector sequence $\{\hat{\mathbf{u}}\}$ which consists of $N$ elements of $u^*(k+\ell), \ell = 0, 1, \ldots, N-1$. Only the first element $u^*(k)$ of the sequence is applied to $\Sigma$. This rolling process is repeated at the next time step and continues thereon. The GDC has the form of $u^*(k) \in \mathbb{V} \subset \mathbb{R}$, in which $0 \in \mathbb{V}$. If $P_o$ is from the Riccati equation as in the control literature, the solution to (A.22) without the GDC for the linearised system will be identical to that of the LQR problem.

**Stability Condition**

The closed-loop system stability will be formed by the following conditions: (i) the state convergence; (ii) the GDC feasibility; (iii) the recursive feasibility with invariant sets delineated in the following:

(1) $\mathbb{X}_{\mathtt{f}}$—Terminal constraint set: The state evolution beyond the predictive horizon should belong to a *terminal constraint set*, which is a positively invariant set, denoted as $\mathbb{X}_{\mathtt{f}}$, [93], to guarantee the recursive feasibility of MPC.

The terminal constraint set for linear systems is often chosen as the *maximal output admissible set*, $O_\infty$ [42] of the closed-loop system of the form $x(k+1) = \left(A + BK_f\right) x(k)$, where the control law $u = K_f\, x$ is the optimal controller for the unconstrained infinite horizon LQ problem in the linear system cases. From the maximal output admissible set $\mathbb{X}_{\mathtt{f}}$, the initial feasible set is then computed.

(2) $\mathbb{X}_{\mathtt{r}}$—Initial feasible set: The set $\mathbb{X}_{\mathtt{r}}$ is the initial feasible set w.r.t $\mathbb{X}_{\mathtt{f}}$ for system $x(k+1) = A(k) + Bu(k)$, with constraints $x \in \mathbb{X},\ u \in \mathbb{U}$ [68], if and only if there exists an admissible control law that will drive the state of the system into $\mathbb{X}_{\mathtt{f}}$ in $N$ steps or less from $\mathbb{X}_{\mathtt{r}}$, while keeping the evolution of $x$ inside $\mathbb{X}$, i.e.

$$\mathbb{X}_{\mathtt{r}} := \{x(k) \in \mathbb{X} |\ \exists \{u(k) \in \mathbb{U}\}_{k=0}^{N-1} : \{x(k) \in \mathbb{X}\}_{k=0}^{N-1} \ \wedge \ x(N) \in \mathbb{X}_{\mathtt{f}}\}.$$

The above initial feasible set $\mathbb{X}_{\mathtt{r}}$ is determined by computing backward from the terminal constraint set $\mathbb{X}_{\mathtt{f}}$ using set operations. The representation of $\mathbb{X}_{\mathtt{r}}$ in relation to $\mathbb{X}_{\mathtt{f}}$ using the Minkowski sum for discrete-time systems can be found in [122]. The constrained optimisation of MPC is then recursively feasible (for all time $k \geqslant 0$) if and only if the initial state $x(0)$ belongs to the initial feasible set $\mathbb{X}_{\mathtt{r}}$.

(3) $\mathbb{V}$—One-step admissible control set: Given the current state $x(k) \in \mathbb{X}_{\mathtt{r}}$ and the past control $u(k-1)$, the one-step admissible control set $\mathbb{V}$ is defined as

$$\mathbb{V}\big(x(k)\big) := \{u(k) \in \mathbb{U} |\, Ax(k) + Bu(k) \in \mathbb{X}_{\mathtt{r}}\}\,. \tag{A.23}$$

(4) Recursive feasibility: The condition for assuring the recursive feasibility of the MPC is to have the intersection of $\mathbb{V}_k$ and the GDC ellipsoid denoted as $\mathbb{E}_k$ (governed by the GDC) non-empty, i.e. $\mathbb{V}_k \cap \mathbb{E}_k \neq \emptyset$, and

$$u(k) \in \mathbb{V}_k \cap \mathbb{E}_k. \tag{A.24}$$

The stability condition is then stated below.

**Proposition A.1** *Let $0 < \tau < 1$. Consider the nominal system $\mathscr{S}$ (A.1) and the MPC problem with the predictive horizon $N$ (A.22). Suppose there are two $\mathscr{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, a real-valued non-negative function $V(k, x)$ with finite $V\big(0, x(0)\big)$ and a real-valued supply-rate function $\xi(k, x(k), u(k))$, such that for each $x(k_0) \in \mathbb{X}_{\mathtt{r}}$, $\xi(0, x(0), u(0))$ is finite and the following hold for all $k > 0$:*

1. $\underline{\alpha}\big(\|x(k)\|\big) \leqslant V\big(k, x(k)\big) \leqslant \overline{\alpha}\big(\|x(k)\|\big)$;
2. $V(k, x(k)) - \tau V(k, x(k-1)) \leqslant |\xi(k, x(k), u(k))|$;
3. $|\xi(k, x(k), u(k))| \leqslant \alpha(|\xi(0, x(0), u(0))|, k)$, *where $\alpha$ is some $\mathscr{K}\mathscr{L}$ function;*
4. $V\big(k, x(k-1)\big) \leqslant V\big(k-1, x(k-1)\big)$;
5. *The one-step admissible control set $\mathbb{V}_k$ (A.23) determined from the initial feasible set $\mathbb{X}_{\mathtt{r}}$ (A.23) intersects the feasible region $\mathbb{E}_k$ governed by the GDC in 3), i.e. $\mathbb{V}_k \cap \mathbb{E}_k \neq \emptyset$, and*
$$\boldsymbol{u}(k) \in \mathbb{V}_k \cap \mathbb{E}_k; \tag{A.25}$$

*Then, the controlled system $\mathscr{S}$ is asymptotically stable.* □

*Implementation notes*: To obtain non-conservative sets $\mathbb{F}_k = \mathbb{V}_k \cap \mathbb{E}_k$ for assuring the recursive feasibility, it is possible to have a sufficiently long predictive horizon $N$ to determine the initial feasible set $\mathbb{X}_{\mathtt{r}}$ off-line. The MPC problem (A.22) can have a shorter predictive horizon for online computation. In the case of $\mathbb{V}_k \cap \mathbb{E}_k = \emptyset$, the supply-rate function will be changed; in other words, $\mathbb{E}_k$ should be adjusted online such that it intersects $\mathbb{V}_k$. This can be performed with the quadratic supply-rate function by having a new set of coefficient matrices $Q,\ S$ and $R$ at certain time steps $k > 0$.

In the following, the detailed derivations for linear matrix inequalities (LMIs) with the QDC for linear-time-invariant (LTI) systems are provided.

## A.5 Linear Matrix Inequalities for Quadratic Dissipativity Constraint

The state feedback design with the QDC is developed in this section. The system in consideration is a single LTI system $\Sigma$,

$$\Sigma:\ x(k+1) = Ax(k) + Bu(k). \tag{A.26}$$

The LMI for an open-loop dissipative system is derived from the dissipation inequality as follows:

The dissipation inequality of the form

$$x_{k+1}^T P x_{k+1} - \tau x_k^T P x_k \leqslant [x_k^T\ u_k^T] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} [x_k^T\ u_k^T]^T \tag{A.27}$$

is equivalent to the following inequality:

$$(*) \begin{bmatrix} A^T PA - \tau P - Q & A^T PB - S \\ * & B^T PB - R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \leqslant 0, \tag{A.28}$$

which is then fulfilled by the following LMI for every $x(k)$ and $u(k)$:

$$\begin{bmatrix} A^T PA - \tau P - Q & A^T PB - S \\ * & B^T PB - R \end{bmatrix} \prec 0. \tag{A.29}$$

Now, with the arrangement of

$$\begin{aligned} \begin{bmatrix} A^T PA - \tau P - Q & A^T PB - S \\ * & B^T PB - R \end{bmatrix} &= \begin{bmatrix} A^T PA & A^T PB \\ * & B^T PB \end{bmatrix} - \begin{bmatrix} \tau P + Q & S \\ * & R \end{bmatrix} \\ &= \begin{bmatrix} A^T \\ B^T \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} \tau P + Q & S \\ * & R \end{bmatrix}, \end{aligned}$$

and applying the Schur complement [14], we obtain from (A.29) the following matrix inequality:

$$\begin{bmatrix} P^{-1} & A & B \\ * & \tau P + Q & S \\ * & * & R \end{bmatrix} \succ 0, \tag{A.30}$$

which is equivalent to the following LMI by pre- and post-multiplying with the symmetric matrix $\mathsf{diag}[P,\ I,\ I]$:

$$\begin{bmatrix} P & PA & PB \\ * & \tau P + Q & S \\ * & * & R \end{bmatrix} \succ 0. \tag{A.31}$$

We can rewrite this LMI with the decision variables in bold typeface, as follows:

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{P}A & \boldsymbol{P}B \\ * & \tau \boldsymbol{P} + \boldsymbol{Q} & \boldsymbol{S} \\ * & * & \boldsymbol{R} \end{bmatrix} \succ 0. \tag{A.32}$$

This result is well known in the control literature, e.g. [17].

**a. State Feedback Synthesis**

(i) With the state feedback of $u(k) = Kx(k)$, the dissipation inequality is equivalent to the following inequality:

$$x(k)^T[(A^T + K^T B^T)P(A + BK) - \tau P - M]x(k) \leqslant 0, \tag{A.33}$$

where $M := Q + 2SK + K^T RK$,

which is fulfilled by the following LMI for every $x(k)$:

$$(A^T + K^T B^T)P(A + BK) - \tau P - M \prec 0. \tag{A.34}$$

Then, applying the Schur complement, we obtain the following LMI:

$$\begin{bmatrix} P^{-1} & A + BK \\ * & \tau P + M \end{bmatrix} \succ 0, \tag{A.35}$$

which is equivalent to

$$\begin{bmatrix} P^{-1} & AP^{-1} + BX \\ * & \tau P^{-1} + W \end{bmatrix} \succ 0, \tag{A.36}$$

where $K = XP$ and $W = P^{-1}MP^{-1}$, by multiplying both side with $\mathsf{diag}[I,\ P^{-1}]$, $P \succ 0$.

And with decision variables in bold typeface:

$$\begin{bmatrix} \mathscr{P} & A\mathscr{P} + BX \\ * & \tau\mathscr{P} + \boldsymbol{W} \end{bmatrix} \succ 0, \tag{A.37}$$

where $\mathscr{P} := \boldsymbol{P}^{-1}$.

(ii)The dissipation-based constraint of the form

$$0 \leqslant [x_{k+1}^T \; u_{k+1}^T] N [x_{k+1}^T \; u_{k+1}^T]^T \leqslant \gamma \, [x_k^T \; u_k^T] N [x_k^T \; u_k^T]^T,$$

where $N = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, $\gamma \in (0, 1)$, is also obtained for every $x(k)$, if the following matrix inequality holds:

$$\begin{bmatrix} M^{-1} & A + BK \\ * & \gamma M \end{bmatrix} \succ 0. \tag{A.38}$$

The two matrix inequalities (A.36) and (A.38) are then solved for the feasible $K = XP^{-1}$ by having the variable transformation of $M = YP$, as delineated in the following.

**b. Solving Two Matrix Inequalities**

The task is to solve a system of two matrix inequalities in the following:

$$\begin{bmatrix} P^{-1} & A + BK \\ * & \tau P + M \end{bmatrix} \succ 0, \; P \succ 0, \tag{A.39}$$

$$\begin{bmatrix} M^{-1} & A + BK \\ * & \gamma M \end{bmatrix} \succ 0, \; M \succ 0. \tag{A.40}$$

Suppose that $K = XP$ and $M = YP$, then pre- and post-multiplying (A.39) and (A.40) with $\mathsf{diag}[I, \; P^{-1}]$, we obtain the two following matrix inequalities:

$$\begin{bmatrix} P^{-1} & AP^{-1} + BX \\ * & \tau P^{-1} + W \end{bmatrix} \succ 0, \; P \succ 0, \; W \succ 0, \text{ and} \tag{A.41}$$

$$\begin{bmatrix} P^{-1}Y^{-1} & AP^{-1} + BX \\ * & \gamma P^{-1} Y \end{bmatrix} \succ 0, \; Y \succ 0, \tag{A.42}$$

where $M = YP$, $W = P^{-1}MP^{-1}$.

These two matrix inequalities have the decision variables of $\mathscr{P} = P^{-1}, \; X, \; W, \; Y$, in which the second matrix inequality is not linear.

The solution will be obtained in two consecutive steps, as follows:

(1) First, find $X$, $W$ and $P$ from LMI (A.41) only.

Denote the solution as $X_0$, $W_0$ and $P_0$. Thus, $M_0 = P_0 W P_0$, and $Y_0 = M_0 P_0^{-1}$.

(2) Second, substituting $M = Y_0 P$, $X = X_0$ and $Y = Y_0$ to (A.41) and (A.42), we obtain the following two LMIs in $\mathscr{P}$:

$$\begin{bmatrix} \mathscr{P} & A\mathscr{P} + BX_0 \\ * & \tau\mathscr{P} + \mathscr{P}Y_0 \end{bmatrix} \succ 0, \;\; \mathscr{P} \succ 0, \tag{A.43}$$

$$\begin{bmatrix} \mathscr{P}Y_0^{-1} & A\mathscr{P} + BX_0 \\ * & \gamma\mathscr{P}Y_0 \end{bmatrix} \succ 0, \tag{A.44}$$

where $\mathscr{P} := \boldsymbol{P}^{-1}$.

Then, we can recover the state feedback gain $K = X_0\boldsymbol{P}$ and the matrix $M = Y_0\boldsymbol{P}$.

A generalised dissipation-based constraint has been introduced in this Appendix—the general dissipativity constraint (GDC). The asymptotically positive realness constraint (APRC) and the quadratic dissipativity constraint (QDC) whose supply functions are quadratic are special cases of the GDC. Neither the Lyapunov stability in its original form, nor the Lagrange uniform boundedness is assured by the GDC. The associated storage function in the GDC method is a relaxed non-monotonic Lyapunov function. The GDC stability and the input-to-power-and-state stability and stabilisation (IpSS) have been defined, and the corresponding sufficient condition for use with the MPC has been stated.

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