Appendix A
Derivation of Two Formulae

A.1 Quantization of the Hall Conductance

In this section we present a proof showing that the Hall conductance is quantized to be $\nu e^2/h$ (ν is an integer) in (4.54). For simplicity, we first drop the band index first. Given the definition of the Berry curvature, the Hall conductance is expressed as

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\int_0^{2\pi} dk_x \int_0^{2\pi} dk_y [\nabla_{\mathbf{k}} \times \mathbf{A}(k_x, k_y)]_z, \tag{A.1}$$

where the lattice constant is taken to be the unit. Therefore the conductance is determined by the Berry curvature integrated over the reduced Brillouin zone.

To evaluate the surface integral, the Stokes's theorem can be applied with the condition that the surface is simply connected. To this end, we illustrate the formation of the torus from a rectangle with the periodic boundary condition, as shown in Fig. A.1. In this way the surface integral can be reduced to a line integral around the first Brillouin zone:

$$\begin{aligned}\sigma_{xy} &= \frac{e^2}{h}\frac{1}{2\pi}\int_0^{2\pi} dk_x \int_0^{2\pi} dk_y \left[\partial_{k_x}\mathbf{A}_y(k_x, k_y) - \partial_{k_y}\mathbf{A}_x(k_x, k_y)\right] \\ &= \frac{e^2}{h}\frac{1}{2\pi}\int_0^{2\pi} dk_y \left[\mathbf{A}_y(2\pi, k_y) - \mathbf{A}_y(0, k_y)\right] \\ &\quad -\frac{e^2}{h}\frac{1}{2\pi}\int_0^{2\pi} dk_x \left[\mathbf{A}_x(k_x, 2\pi) - \mathbf{A}_x(k_x, 0)\right].\end{aligned} \tag{A.2}$$

Recalling that $|u(k_x, 0)\rangle$ and $|u(k_x, 2\pi)\rangle$ actually represent the same physical state due to the periodicity in the reciprocal vector space, which can only differ by a phase factor, $|u(k_x, 2\pi)\rangle = \exp[i\theta_x(k_x)]|u(k_x, 0)\rangle$, one has

$$\begin{aligned}\mathbf{A}_x(k_x, 2\pi) &= \langle u(k_x, 2\pi)|i\boldsymbol{\partial}_{k_x}|u(k_x, 2\pi)\rangle \\ &= -\boldsymbol{\partial}_{k_x}\theta_x(k_x) + \mathbf{A}_x(k_x, 0).\end{aligned} \tag{A.3}$$

S.-Q. Shen, *Topological Insulators*, Springer Series in Solid-State Sciences 187,
DOI 10.1007/978-981-10-4606-3

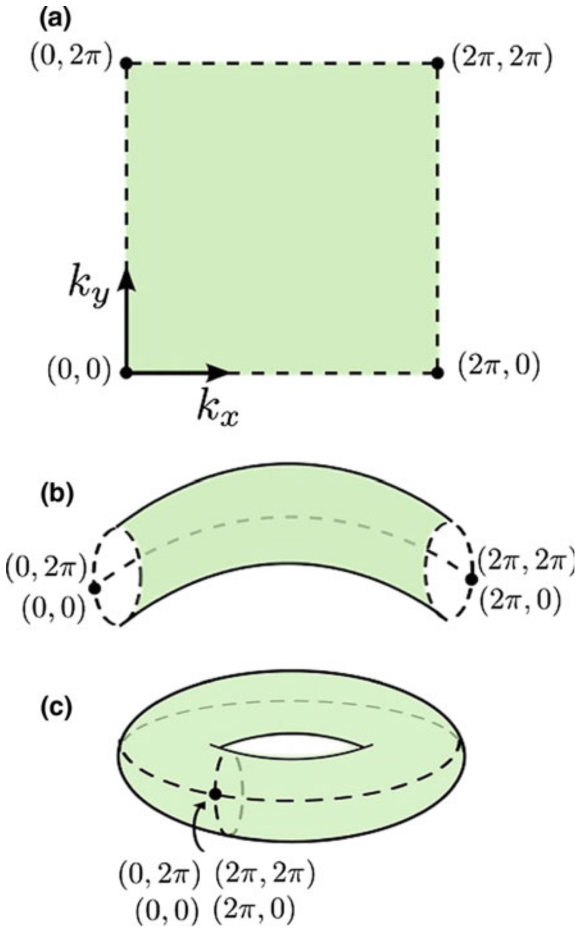

Fig. A.1 Equivalence of the first Brillouin zone and a torus: **a** *rectangle* of the first Brillouin zone with periodic boundary conditions; **b** the *rectangle* is rolled into a tube along the k_y direction; **c** the tube is rolled into a torus along the k_x direction. The four corners of the *rectangle* are actually one point in the torus surface

Similarly, taking $|u(2\pi, k_y)\rangle = \exp[i\theta_y(k_y)]|u(0, k_y)\rangle$, one obtains

$$\mathbf{A}_y(2\pi, k_y) = -\boldsymbol{\partial}_{k_y}\theta_y(k_y) + \mathbf{A}_y(0, k_y). \tag{A.4}$$

$\theta_x(k_x)$ and $\theta_y(k_y)$ are smooth functions. Using these two relations, the integral is reduced to

$$\begin{aligned}\sigma_{xy} &= \frac{e^2}{h}\frac{1}{2\pi}\int_0^{2\pi} dk_y\left[-\boldsymbol{\partial}_{k_y}\theta_y(k_y)\right] + \frac{e^2}{h}\frac{1}{2\pi}\int_0^{2\pi} dk_x\left[\boldsymbol{\partial}_{k_x}\theta_x(k_x)\right] \\ &= \frac{e^2}{h}\frac{1}{2\pi}\left[\theta_y(0) - \theta_y(2\pi) + \theta_x(2\pi) - \theta_x(0)\right].\end{aligned} \tag{A.5}$$

On the torus surface of the first Brillouin zone, the four wave states $|u(0,0)\rangle$, $|u(0,2\pi)\rangle$, $|u(2\pi,0)\rangle$, and $|u(2\pi,2\pi)\rangle$ actually represent the same states (see in Fig. A.1). Using the phase matching relations of these states,

$$e^{i\theta_x(0)}|u(0,2\pi)\rangle = |u(0,0)\rangle, \tag{A.6}$$
$$e^{i\theta_x(2\pi)}|u(2\pi,2\pi)\rangle = |u(2\pi,0)\rangle, \tag{A.7}$$
$$e^{i\theta_y(0)}|u(2\pi,0)\rangle = |u(0,0)\rangle, \tag{A.8}$$
$$e^{i\theta_y(2\pi)}|u(2\pi,2\pi)\rangle = |u(0,2\pi)\rangle, \tag{A.9}$$

one obtains

$$|u(0,0)\rangle = e^{i[\theta_x(0)+\theta_y(2\pi)-\theta_x(2\pi)-\theta_y(0)]}|u(0,0)\rangle. \tag{A.10}$$

The single-valuedness of $|u(0,0)\rangle$ requires that the exponent must be an integer multiple of 2π, i.e.,

$$\theta_x(0) + \theta_y(2\pi) - \theta_x(2\pi) - \theta_y(0) = 2\nu\pi \tag{A.11}$$

with an integer ν (including 0). Therefore the Hall conductance must be quantized when the band is fully filled. This integer ν is called the Thouless-Kohmoto-Nightingale-Nijs (TKNN) number or the first Chern number, and it characterizes the topological structure of the Bloch states $|u(k_x, k_y)\rangle$ in the parameter space (k_x, k_y).

A.2 A Simple Formula for the Hall Conductance

A simple two-band model has a general form in terms of the Pauli matrices σ_α,

$$H(\mathbf{k}) = \epsilon(\mathbf{k}) + \sum_{\alpha=1,2,3} d_\alpha(\mathbf{k})\sigma_\alpha. \tag{A.12}$$

The energy spectra of the model are

$$E_\pm(\mathbf{k}) = \epsilon(\mathbf{k}) \pm d(\mathbf{k}) \tag{A.13}$$

with $d(\mathbf{k}) = \sqrt{\sum_{\alpha=1,2,3} |d_\alpha(\mathbf{k})|^2}$, and the corresponding eigenstates are

$$|\mathbf{k},+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \tag{A.14}$$

and

$$|\mathbf{k},-\rangle = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}, \tag{A.15}$$

where $\theta = \arccos\frac{d_z(\mathbf{k})}{d(\mathbf{k})}$ and $\phi = \arctan\frac{d_x(\mathbf{k})}{d_y(\mathbf{k})}$.

In electric conduction, the conductivity $\sigma_{\alpha\beta}$ is defined as

$$J_\alpha(\mathbf{r}, t) = \sum_\beta \sigma_{\alpha\beta}(\mathbf{q}, \omega)\Xi_\beta \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)], \tag{A.16}$$

where $J_\alpha(\mathbf{r}, t)$ is the electric current and $\Xi_\beta \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]$ is the electric field. In the linear response theory, the Kubo formula for the Hall conductance gives

$$\sigma_{xy}(\mathbf{q}, \omega) = +\frac{i}{\omega}\Pi_{xy}(\mathbf{q}, \omega) \tag{A.17}$$

with the retarded correlation function of the current operator $J_x(\mathbf{q}, t)$ and $J_y(\mathbf{q}, t')$

$$\Pi_{xy}(\mathbf{q}, \omega) = -\frac{i}{V}\int_{-\infty}^{+\infty} dt\theta(t - t')e^{i\omega(t-t')}\left\langle\psi\right|[J_x(\mathbf{q}, t), J_y(\mathbf{q}, t')]\left|\psi\right\rangle, \tag{A.18}$$

where V is the volume of the system. The dc conductivity is obtained by taking the limit $\mathbf{q} \to \mathbf{0}$ and then $\omega \to 0$,

$$\sigma_{xy} = \lim_{\omega\to 0}\lim_{q\to 0}\sigma_{xy}(\mathbf{q}, \omega). \tag{A.19}$$

Usually the retarded correlation function can be calculated in the Matsubara formalism

$$\Pi^M_{xy}(i\omega_\nu) = \frac{1}{V}\frac{1}{\beta}\sum_{k,\nu'} Tr\left\{J_x(\mathbf{k})G[\mathbf{k}, i(\omega_\nu + \omega_{\nu'})]J_y(k)G[\mathbf{k}, i\omega_{\nu'}]\right\} \tag{A.20}$$

with frequencies $\omega_\nu = 2\nu\pi/\beta$ and $\omega_{\nu'} = (2\nu' + 1)\pi/\beta$ ($\beta = k_B T$). The Matstubara-Green function is given by

$$\begin{aligned} G(\mathbf{k}, i\omega_\nu) &= [i\omega_\nu - H(\mathbf{k})]^{-1} \\ &\equiv \frac{P_+}{i\omega_\nu - E_+(\mathbf{k})} + \frac{P_-}{i\omega_\nu - E_-(\mathbf{k})} \end{aligned} \tag{A.21}$$

with

$$P_\pm = \frac{1}{2}\left[1 \pm \sum_{\alpha=1,2,3}\frac{d_\alpha(\mathbf{k})\sigma_\alpha}{d}\right]. \tag{A.22}$$

Using the frequency summation over $i\omega_{\nu'}$,

$$\frac{1}{\beta}\sum_{\nu'}\frac{1}{i(\omega_\nu + \omega_{\nu'}) - E_n}\frac{1}{i\omega_{\nu'} - E_m} = \frac{f_{\mathbf{k},m} - f_{\mathbf{k},n}}{i\omega_\nu + E_m(\mathbf{k}) - E_n(\mathbf{k})}, \tag{A.23}$$

where the Dirac–Fermi distribution function $f_{\mathbf{k},n} = 1/\{1+\exp[\beta(E_n(\mathbf{k})-\mu)]\}$, one obtains

$$\Pi^{\mathrm{M}}_{xy}(\omega_\nu) = \frac{1}{V}\sum_{\mathbf{k},n,n'} \langle\mathbf{k},n|\, J_x(\mathbf{k})\left|\mathbf{k},n'\right\rangle\left\langle\mathbf{k},n'\right|\, J_y(\mathbf{k})\,|\mathbf{k},n\rangle \frac{f_{\mathbf{k},n}-f_{\mathbf{k},n'}}{i\omega_\nu + E_n(\mathbf{k}) - E_{n'}(\mathbf{k})}. \tag{A.24}$$

Its analytical continuation to the retarded function is realized by replacing $i\omega_n \rightarrow \hbar\omega + i\epsilon$,

$$\Pi^{\mathrm{M}}_{xy}(\omega_\nu) \rightarrow \Pi^{\mathrm{R}}_{xy}(\omega). \tag{A.25}$$

Using the L'Hospital's rule,

$$\lim_{\omega\rightarrow 0}\frac{\mathrm{Im}\left(\Pi^{R}_{xy}(\omega)\right)}{\omega} = \mathrm{Im}\left(\frac{d\Pi^{R}_{xy}(\omega)}{d\omega}\right)_{\omega=0} \tag{A.26}$$

and

$$\lim_{\omega\rightarrow 0}\frac{d}{\hbar d\omega}\left[\frac{1}{\hbar\omega + i\epsilon + E_n - E_{n'}}\right] = -\frac{1}{(E_n - E_{n'})(E_n - E_{n'} + i\epsilon)}, \tag{A.27}$$

the Kubo formula for the dc Hall conductivity can be written as

$$\sigma_{xy} = \frac{\hbar}{V}\lim_{\epsilon\rightarrow 0^+}\sum_{k,n\neq n'}\frac{(f_{\mathbf{k},n} - f_{\mathbf{k},n'})Im\left(\langle\mathbf{k},n|\, J_x(\mathbf{k})\left|\mathbf{k},n'\right\rangle\left\langle\mathbf{k},n'\right|\, J_y(\mathbf{k})\,|\mathbf{k},n\rangle\right)}{(E_n(\mathbf{k}) - E_{n'}(\mathbf{k}))(E_n(\mathbf{k}) - E_{n'}(\mathbf{k}) + i\epsilon)}. \tag{A.28}$$

From the model in (A.12), the current operator $J_i(\mathbf{k}) = -ev_i(\mathbf{k})$ is given by

$$J_i(\mathbf{k}) = -\frac{e}{\hbar}\partial_{k_i}H(\mathbf{k}) = -\frac{e}{\hbar}\left(\partial_{k_i}\epsilon(\mathbf{k}) + \sum_{\alpha=1,2,3}\partial_{k_i}d_\alpha(\mathbf{k})\sigma_\alpha\right). \tag{A.29}$$

For $n \neq n'$, one has

$$\langle\mathbf{k},n|\, J_i(\mathbf{k})\left|\mathbf{k},n'\right\rangle = -\frac{e}{\hbar}\sum_{\alpha=1,2,3}\partial_{k_i}d_\alpha(\mathbf{k})\,\langle\mathbf{k},n|\,\sigma_\alpha\left|\mathbf{k},n'\right\rangle. \tag{A.30}$$

Furthermore,

$$Im\left(\langle\mathbf{k},n|\,\sigma_\alpha\,|\mathbf{k},-n\rangle\,\langle\mathbf{k},-n|\,\sigma_\beta\,|\mathbf{k},n\rangle\right) = n\epsilon_{\alpha\beta\gamma}\frac{d_\gamma(\mathbf{k})}{d(\mathbf{k})}. \tag{A.31}$$

We limit our discussion in the case that two levels do not cross in the whole momentum space such that $\epsilon \rightarrow 0^+$ can be taken before the integral of $\mathbf{k}$. Thus, the conductance can be expressed as

$$\sigma_{xy} = \frac{1}{2\Omega}\frac{e^2}{\hbar}\sum_k \epsilon_{\alpha\beta\gamma}\frac{\left[\partial_{k_x} d_\alpha(\mathbf{k})\right]\left[\partial_{k_y} d_\beta(\mathbf{k})\right] d_\gamma(\mathbf{k})}{d^3(\mathbf{k})}(f_{\mathbf{k},+} - f_{\mathbf{k},-}). \tag{A.32}$$

If there exists an energy gap between the upper and lower bands, and the lower band is fully filled, i.e., $E_{\mathbf{k},-} < \mu < E_{\mathbf{k},+}$, then $f_{\mathbf{k},+} = 0$ and $f_{\mathbf{k},-} = 1$ at zero temperature. The Hall conductance has the form

$$\sigma_{xy} = -\frac{e^2}{h}\frac{1}{4\pi}\int dk_x dk_y \frac{\left(\partial_{k_x}\mathbf{d}(\mathbf{k})\times\partial_{k_y}\mathbf{d}(\mathbf{k})\right)\cdot\mathbf{d}(\mathbf{k})}{d^3(\mathbf{k})}. \tag{A.33}$$

It is noted that the conductance may not be quantized in some continuous models in which the Brillouin zone is not finite. For example a massive Dirac model has a half-quantized conductance. This case does not occur in a lattice model.

Appendix B
Time Reversal Symmetry

The time reversal symmetry demonstrates the invariance of physical laws under time reversal transformation. The terminology was first introduced by E. Wigner in 1932.

B.1 Classical Case

Let us first consider a classic case: the motion of a particle subjected to a certain force. Its trajectory is given by the Newtonian equation of motion,

$$m\frac{d^2\mathbf{r}}{dt^2} = -\nabla V(r). \tag{B.1}$$

If $\mathbf{r}(t)$ is the solution of the equation, then $\mathbf{r}(-t)$ is also the solution of the equation. In other word, when we make the transformation $t \to -t$, the Newtonian equation of motion is unchanged. However, we should note any changes in the boundary condition or initial conditions of the problem.

Maxwell's equations and the Lorentz force $\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ are invariant under the time reversal provided that

$$\mathbf{v} \to -\mathbf{v}, \mathbf{j} \to -\mathbf{j}, \rho \to \rho \tag{B.2}$$

and

$$\mathbf{B} \to -\mathbf{B}, \mathbf{E} \to \mathbf{E}. \tag{B.3}$$

Maxwell's equations are

$$\nabla \cdot \mathbf{D} = \rho, \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{B.4}$$

$$\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = I, \tag{B.5}$$

S.-Q. Shen, *Topological Insulators*, Springer Series in Solid-State Sciences 187,
DOI 10.1007/978-981-10-4606-3

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$. Therefore, the magnetic field changes its sign and the electric field remains unchanged under time reversal.

B.2 Quantum Case

In quantum mechanics, the Schrödinger equation is written as

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \Psi(x,t), \tag{B.6}$$

in which the Hamiltonian in the right hand side is invariant under the time reversal. If $\Psi(x,t)$ is a solution of the equation, $\Psi(x,-t)$ is not a solution of the equation because of the first order time derivative and the imaginary sign of the left hand side. However, $\Psi^*(x,-t)$ is a solution. One can check it by using the solution of a free particle, $\Psi(x,t) = ce^{i(p\cdot x - Et)/\hbar}$. The $\Psi(x,-t) = ce^{i(p\cdot x + Et)/\hbar}$ is also a solution of the Schrödinger equation. However, the momentum is still p, NOT $-p$.

Definition: the transformation θ

$$|\alpha\rangle \to |\tilde{\alpha}\rangle = \theta |\alpha\rangle, \, |\beta\rangle \to \left|\tilde{\beta}\right\rangle = \theta |\beta\rangle \tag{B.7}$$

is said to be anti-unitary if

$$\left\langle \tilde{\beta} | \tilde{\alpha} \right\rangle = \langle \beta | \alpha \rangle^* ; \tag{B.8}$$

$$\theta (c_1 |\alpha\rangle + c_2 |\beta\rangle) = c_1^* \theta |\alpha\rangle + c_2^* \theta |\beta\rangle . \tag{B.9}$$

In this case the operator θ is an anti-unitary operator. Usually, an anti-unitary operator can be written as

$$\theta = UK, \tag{B.10}$$

where U is a unitary operator and K is the complex conjugation operator, which is defined as

$$K\varphi = \varphi^* K.$$

Here, φ can be either a function or an operator.

B.3 Time Reversal Operator Θ

Let us denote the time reversal operator by Θ. Consider

$$|\alpha\rangle \to \Theta |\alpha\rangle , \tag{B.11}$$

where $\Theta\,|\alpha\rangle$ is the time reversed state. More appropriately, $\Theta\,|\alpha\rangle$ should be called the motion-reversed state. For a momentum eigenstate $|\mathbf{p}\rangle$, $\Theta\,|\mathbf{p}\rangle$ should be $|-\mathbf{p}\rangle$ up to a possible phase factor. Θ is an anti-unitary operator. We can see this property from the Schrödinger equation of a time reversal invariant system,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t), \tag{B.12}$$

provided that $\Theta i\Theta^{-1} = -i$ and $\Theta\frac{\partial}{\partial t}\Theta^{-1} = \frac{\partial}{\partial(-t)}$. The transformed momentum operator $\mathbf{p}$, the position $\mathbf{x}$, and the angular momentum $\mathbf{J}$ are

$$\Theta\mathbf{p}\Theta^{-1} = -\mathbf{p}, \tag{B.13}$$

$$\Theta\mathbf{x}\Theta^{-1} = \mathbf{x}, \tag{B.14}$$

$$\Theta\mathbf{J}\Theta^{-1} = -\mathbf{J}. \tag{B.15}$$

Note that for $\mathbf{p} = -i\hbar\frac{d}{dx}$, $\Theta\mathbf{p}\Theta^{-1} = -\mathbf{p}$.

From the spherical harmonic $Y_l^m(\theta,\phi)$, one has

$$Y_l^m(\theta,\phi) \to \left(Y_l^m(\theta,\phi)\right)^* = (-1)^m Y_l^{-m}(\theta,\phi). \tag{B.16}$$

Therefore the eigenstate $|l,m\rangle$ of the orbital angular momentum and its z-component has the relation,

$$\Theta\,|l,m\rangle = (-1)^m\,|l,-m\rangle\,. \tag{B.17}$$

B.4 Time Reversal for a Spin $\frac{1}{2}$ System

Under the time reversal, $t \to -t$. Does applying the time reversal operation twice return us to the original states? Yes, but Θ^2 is not always equal to 1. For a spin $\frac{1}{2}$ system,

$$\Theta\sigma_\alpha\Theta^{-1} = -\sigma_\alpha, \tag{B.18}$$

where $\alpha = x, y, z$. Note that

$$\sigma_y\sigma_x\sigma_y = -\sigma_x, \tag{B.19}$$

$$\sigma_y\sigma_y\sigma_y = +\sigma_y, \tag{B.20}$$

$$\sigma_y\sigma_z\sigma_y = -\sigma_z. \tag{B.21}$$

By convention, σ_y is taken to be purely imaginary, as in (2.6), and σ_x and σ_z are real. We have $K\sigma_y = -\sigma_y K$ and $K\sigma_{x,z} = \sigma_{x,z}K$. Therefore the time reversal operator can be constructed by combining σ_y and the complex conjugation operator K,

$$\Theta = i\sigma_y K. \tag{B.22}$$

Its inverse matrix is

$$\Theta^{-1} = -\Theta = -i\sigma_y K. \tag{B.23}$$

One can check the relation,

$$\Theta^2 = -1. \tag{B.24}$$

Consider the eigenstate $|n, +\rangle$ of $\mathbf{S} \cdot \mathbf{n}$ with the eigenvalue $+\hbar/2$,

$$|n, +\rangle = e^{-iS_z\alpha/\hbar} e^{-iS_y\beta/\hbar} |+\rangle, \tag{B.25}$$

$$\Theta |n, +\rangle = \Theta e^{-iS_z\alpha/\hbar} e^{-iS_y\beta/\hbar} \Theta^{-1} \Theta |+\rangle. \tag{B.26}$$

As $\Theta S_\alpha \Theta^{-1} = -S_\alpha$ and $\Theta i \Theta^{-1} = -i$,

$$\Theta |n, +\rangle = e^{-iS_z\alpha/\hbar} e^{-iS_y\beta/\hbar} \Theta |+\rangle = e^{-iS_z\alpha/\hbar} e^{-iS_y\beta/\hbar} |-\rangle = |n, -\rangle. \tag{B.27}$$

where $\Theta |+\rangle = |-\rangle$ with an eigenvalue $-\frac{1}{2}$. On the other hand,

$$|n, -\rangle = e^{-iS_z\alpha/\hbar} e^{-iS_y(\pi+\beta)/\hbar} |+\rangle = e^{-iS_z\alpha/\hbar} e^{-iS_y\beta/\hbar} e^{-iS_y\pi/\hbar} |+\rangle. \tag{B.28}$$

Noting that K acting on $|+\rangle$ gives $|+\rangle$. We have

$$\Theta = e^{-i\pi S_y/\hbar} K = i\sigma_y K. \tag{B.29}$$

In general, for a system with the angular momentum operator of the eigenvalue j, the time reversal operator is

$$\Theta = i e^{-i\pi J_y} K, \tag{B.30}$$

where J_y is the y-component of orbital angular momentum operator. The operator satisfies the relation

$$\Theta^2 = (-1)^{2j}. \tag{B.31}$$

Kramers degeneracy: the energy states for an odd number of electrons in a time reversal invariant system has at least a double degeneracy.

This theorem is determined by the fact that the total spin of an odd number of electrons is always half of an odd number of $\hbar$. The time reversal operator has always the relation $\Theta^2 = -1$.

Appendix C
The Dirac Matrices and the Dirac Gamma Matrices

In the Dirac representation, the four Dirac matrices are

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \ \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \ \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{C.1}$$

The four Dirac Gamma matrices have the form

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}, \ \gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{C.2}$$

The relation between the Dirac matrices and the Dirac Gamma matrices are

$$\gamma^i = \beta\alpha_i; \gamma^0 = \beta. \tag{C.3}$$

The gamma matrices satisfy the anticommutation relation,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} I_4, \tag{C.4}$$

where $\eta^{\mu\nu}$ is the Minkowski metric with signature $(+, -, -, -)$ and I_4 is the identity matrix. The product of the four Gamma matrices defines

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}. \tag{C.5}$$

γ^5 anticommutes with the four Gamma matrices, and is useful in discussion of quantum mechanical chirality. It is not one of the gamma matrices of $Cl_{1,3}(\mathbf{R})$. The number 5 is a relic of old notation in which γ^0 was called γ^4.

Under the time reversal symmetry $\Theta = i\alpha_x\alpha_z K$, the four Dirac matrices obey

$$\alpha_x \to -\alpha_x, \ \beta \to \beta. \tag{C.6}$$

S.-Q. Shen, *Topological Insulators*, Springer Series in Solid-State Sciences 187,
DOI 10.1007/978-981-10-4606-3

Index

A
Adiabatic condition, 62
Adiabatic evolution, 59
Aflack-Khomoto-Lieb-Tasaki state, 194
Aharonov-Bohm effect, 120
Ampere's law, 164
Anderson-Brinkman-Morel state, 9, 183
Anderson insulator, 237
Angle-resolved photoemission spectroscopy, 7
Anomalous Hall conductance, 4
Anomalous Hall effect, 2
Antiparticle, 18
ARPES, 127, 148, 212
Axion equation, 7

B
Büttiker formula, 103
Band insulators, 8
Band inversion, 10
BCS theory, 173
Berry connection, 53
Berry curvature, 4, 54, 55, 58, 60, 72, 97
Berry phase, 2, 53, 60, 83, 133
$Bi_{1-x}Sb_x$, 11, 126, 127, 210
Bi_2Se_3, 6, 96, 127, 129, 148
Bi_2Te_3, 6, 95, 127, 129
Bloch state, 51
Bloch's theorem, 51
Bloch wave, 51
Bogoliubov-de Gennes (BdG) Hamiltonian, 173
Bogoliubov-de Gennes equation, 10, 176
Bogoliubov transformation, 175
Born approximation, 227
Bose-Einstein condensation, 174
Bott periodicity, 249
Bound states, 19, 157
Brillouin zone, 35, 52, 59, 92, 127, 157
Bulk-boundary correspondence, 212
Bulk-edge correspondence, 4, 214
Bulk-edge relation, 28

C
Cd_3As_2, 11, 212
Cd_2Sb_3, 13
CdTe, 109
Charge pump, 61
Chern number, 4, 28, 52, 55, 62, 91, 92, 94, 95, 137
Chiral anomaly, 11, 222
Chiral edge state, 9, 177
Chirality, 207, 208
Chirality symmetry, 83
Chiral magnetic effect, 226
Chiral symmetry, 81
Clifford algebra, 17
Complex conjugate operator, 199
Composite fermions, 4
Compressibility, 179
Continuity condition, 20, 21
Continuous model, 35
Cooper pairs, 9, 10
$Cu_xBi_2Se_3$, 185

D
Debye energy, 174
Dimensional modified Dirac equation, 31
Dirac Gamma Matrices, 261
Dirac cone, 7, 30, 75
Dirac equation, 10–12, 17, 19, 22, 23, 30, 44

S.-Q. Shen, *Topological Insulators*, Springer Series in Solid-State Sciences 187,
DOI 10.1007/978-981-10-4606-3

Dirac Gamma matrices, 99
Dirac matrix, 17, 18, 37, 261
Dirac model, 83
Dirac semimetal, 11, 13, 208, 210
Dirichlet boundary condition, 19, 25, 113, 143, 158
Drude conductivity, 228

E
Edge state, 3, 101, 104, 107, 112
Effective Medium Theory, 237
Eigensystem, 13
Einstein relation, 228
Electric polarization, 59
End states, 25
Equal Spin Pairing, 182
Extrinsic spin Hall effect, 3

F
Fano factor, 236
Faraday law, 66
Fermi arc, 11, 216
Fermi-Dirac distribution function, 59
Fermi liquid theory, 179
Fermi surface, 11
Ferromagnetic insulator, 92
Filling factor, 3
Finite size effect, 148
First Bessel function, 167
First Brillouin zone, 28
Fractional quantum Hall effect, 4, 9

G
GaAs, 3
Gauss' law, 164
Gauss's theorem, 209
GdBiPt, 9
Green function, 77
Group Z_2, 51
Group Z, 51

H
Haldane model, 5, 92, 96, 98, 99
Half quantized hall conductance, 138
Half quantized vortex, 189
Half quantum vortex, 192
Hall conductance, 3, 4, 6, 28
Hall effect, 1–3, 6, 13
Hall resistance, 1, 2
^{3}He, 178
^{4}He, 178
Helical edge states, 8, 27, 31, 51, 112
Helicity, 207, 208
Heusler compounds, 9
Hexagonal warping effect, 133
$HgCr_2Se_{13}$, 11
HgTe, 5, 109, 127, 129
HgTe/CdTe, 8, 109, 117, 118, 128
Hubbard model, 9

I
InAs/GaAs, 117, 118
InAs/GaSb, 5
InGaAs, 3
In-gap bound state, 169
Integer quantum Hall effect, 3, 28, 40, 42, 66, 67
Intrinsic spin Hall effect, 3
Inversion symmetry, 72, 210
Ising model, 88

J
Jackiw-Rebbi solution, 19
Jordan–Wigner transformation, 88, 89
Josephson effect, 202
Josephson junction, 202

K
Kane-Mele model, 98–100
Kitaev model, 87
Kitaev's Toy Model, 193
Knight shift, 185
Kramers degeneracy, 6, 65, 68, 260
Kramers pair, 69, 108
Kubo formula, 59, 138

L
Landau bands, 223
Landau degeneracy, 223
Landauer-Büttiker formalism, 139
Landauer-Büttiker formula, 101, 105, 116, 118, 122
Landau gauge, 141
Landau level, 3, 4, 92
Landau Theory, 12
Lattice model, 33, 35, 37, 40, 42, 48
Laughlin argument, 66
Laughlin wave function, 4
Liquid helium ^{3}He, 9, 173
Lorentz force, 1, 2

M

Magnetic flux, 161
Magnetic flux quantization, 173
Magnetic monopole, 55, 165, 208
Magneto-electric effect, 7
Magnetoresistance, 224
Magnetotransport, 223
Majorana bound state, 199
Majorana fermion, 10, 11, 87, 178, 190
Massless Dirac cones, 145
Matsubara formalism, 254
Maxwell's equation, 7, 59, 89
Meissner effect, 173
Metal-insulator transition, 8
Metamaterial, 90
Mobility gap, 5, 231
Modified Bessel function, 163
Modified Dirac equation, 25, 75, 111, 155
Moore-Read quantum Hall state, 9, 174
Mott insulator, 9
Multi-scattering processes, 121

N

Na_3Bi, 11, 212
Negative magnetoresistance, 11
No-go theorem, 209
Non-Abelion Statistics, 203
Nuclear magnetic resonance, 9

O

Ohm's law, 115
Open boundary condition, 21, 35

P

Pair annihilation, 240
Parity, 44, 50
Particle-hole symmetry, 8, 10, 87, 173, 244
Particle-hole transformation, 88
Pauli exclusion principle, 18, 51
Pauli matrices, 18, 30
$Pb_xSn_{1-x}Te$, 125
PbTe, 125
Peierls instability, 8
Peierls substitution, 112
Periodic boundary condition, 34, 35, 75
Pfaffian, 70
Polyacetylene, 8, 81, 85, 90
p-wave pairing superconductor, 87
p-wave superconductor, 10

Q

Quadratic correction, 24
Quantum adiabatic theorem, 57
Quantum anomalous Hall effect, 1, 5, 7, 91, 95, 97, 218
Quantum computation, 205
Quantum conductance, 3, 5
Quantum diffusive regime, 223
Quantum Hall conductance, 4, 55
Quantum Hall effect, 3, 4, 8, 98, 118
Quantum percolation, 169
Quantum phase transition, 13
Quantum spin Hall system, 67, 104
Quantum Spin Hall effect, 1, 5, 6, 10, 12, 91, 98, 107, 109, 115, 118–120, 128

R

Rashba spin-orbit coupling, 10, 202
Reciprocal lattice vector, 34
Resonant spin Hall effect, 3
Reversal symmetry, 133
Rice–Mele model, 63

S

Sau–Lutchyn–Tewari–Das Sarma Model, 199
Sb_2Te_3, 95, 130
Scanning tunneling microscopy, 199
Schrödinger equation, 52, 56, 57
Shubnikov-de haas oscillation, 136, 228
Skew-symmetric matrix, 72
Skymion, 25
SmB_6, 8
SnTe, 125
Spherical harmonic Bessel function, 156
Spin current, 15, 48, 105, 120
Spin Hall conductance, 5, 6
Spin Hall effect, 2
Spin-orbit coupling, 5, 8, 10, 13, 65, 86, 92, 125, 213
Spin-orbit force, 2
Spin pump, 64
Spin susceptibility, 179
Spin transverse force, 2, 4
Spin-triplet Superconductor, 10, 184
Sr_2RuO_4, 10, 184
STM, 134
Stokes' theorem, 54, 66, 251
Streda formula, 94
Strong topological insulators, 126
Structure inversion asymmetry, 146
Superconducting proximity effect, 10, 198

Surface quantum hall effect, 137
Surface states, 5, 27, 29–31, 51, 125, 131, 140
Su–Schrieffer–Heeger model, 63, 81, 86, 88, 90, 214
SU(2) symmetry, 13
s-wave superconductor, 10
Symmetry classes, 243

T
TaAs, 11, 13
The modified Bessel function of the second kind, 159
Thouless charge, 61
Thouless-Kohmoto-Nightingale-Nijs (TKNN), 28, 59
Tight binding approximation, 33
Tight-binding model, 33
Time reversal invariant momenta, 44, 49, 50, 71
Time reversal polarization, 71
Time reversal symmetry, 4–6, 8, 10, 23, 68, 91, 244, 257
TKNN number, 91
Topological Anderson insulator, 8, 231
Topological band insulators, 8
Topological classification, 243
Topological defect, 163
Topological Dirac semimetal, 211
Topological insulator, 5, 6, 12, 13
Topological invariant, 3, 51
Topological Mott insulators, 9
Topological phase, 3
Topological phases of matter, 13
Topological phase transition, 13
Topological photonic crystals, 13
Topological quantum phase, 4, 12
Topological quantum phase transition, 33
Topological quantum transition, 84
Topological superconductor, 10, 173
Topological Weyl semimetal, 207, 212
Transport relaxation time, 228
Tunneling spectroscopy measurements, 10
Two-node model, 214

U
Unitary transformation, 23
UPt_3, 184
U(1) symmetry, 13

W
Wannier state, 60
Weak antilocalization, 135, 223
Weak localization, 224
Weak topological insulators, 125
Weyl equations, 11, 207
Weyl fermions, 11, 207
Weyl semimetals, 11, 13
Wigner-Seitz unit cell, 93
Witten Effect, 164
Wormhole Effect, 163

Y
Yu-Shiba state, 153

Z
Z_2 classification, 109
Z_2 index, 46, 68
Z_2 invariant, 73
Zeeman field, 10
Zero energy mode, 38, 161, 190
Zero mode solution, 21