Appendix

Category Theory

In this book, we introduced and studied several mathematical structures,viz. semigroups, groups, rings, and fields. We noticed some common features in the study of these structures. The category theory gives a unified general and abstract setting for all these and many more mathematical structures such as modules, vector spaces, and topological spaces. Quite often, in mathematics, the concrete results are expressed in the language of category theory. This appendix introduces the very basics in category theory for the purpose. The Gödel–Bernays axiomatic system for sets is the most suitable axiomatic system for category theory. As described in Chap. 2, class is a primitive term in this axiomatic system instead of sets. Indeed, sets are simply the members of classes. The classes which are not sets are termed as proper classes.

Definition A category Σ consists of the following:

1. A class $Obj\Sigma$ called the class of objects of Σ.

2. For each pair A, B in $Obj\Sigma$, we have a set $Mor_{\Sigma}(A, B)$ called the set of morphisms from the object A to the object B. Further,

$$Mor_{\Sigma}(A, B) \bigcap Mor_{\Sigma}(A', B') \neq \emptyset \text{ if and only if } A = A' \text{ and } B = B'.$$

3. For each triple A, B, C in $Obj\Sigma$, we have a map $\cdot$ from $Mor_{\Sigma}(B, C) \times Mor_{\Sigma}(A, B)$ to $Mor_{\Sigma}(A, C)$ called the law of composition. We denote the image $\cdot(g, f)$ by gf. Further, the law of composition is associative in the sense that if $f \in Mor_{\Sigma}(A, B),\ g \in Mor_{\Sigma}(B, C)$ and $h \in Mor_{\Sigma}(C, D)$, then $(hg)f = h(gf)$.

4. For each $A \in Obj\Sigma$, there is an element I_A in $Mor_{\Sigma}(A, A)$ such that $fI_A = f$ for all morphisms from A, and $gI_A = g$ for all morphisms to A.

Clearly, for each object A of Σ, I_A is unique morphism, and it is called the identity morphism on A. The category Σ is called a **small category** if $Obj\Sigma$ is a set.

Examples 1. We have the category SET of sets whose objects are sets, and the morphisms from a set A to a set B are precisely the maps from A to B.

2. There is a category GP of groups whose objects are groups, and the morphisms from a group H to a group K are homomorphisms from H to K.

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3. *SG* denotes the category of semigroups all of whose objects are semigroups, and morphisms are semigroup homomorphisms.

4. *AB* denotes the category of abelian groups whose objects are abelian groups, and morphisms are group homomorphisms.

5. *RING* denotes the category of rings whose objects are rings, and morphisms are ring homomorphisms.

6. *TOP* denotes the category of topological spaces whose objects are topological spaces, and the morphisms are continuous maps.

7. A group G can also be treated as a category having a single object G. The elements of the group can be taken as morphisms from G to G, and the composition of the morphisms is the binary operation of G.

Let Σ be a category. A morphism f from A to B is said to be a monomorphism (epimorphism) if it can be left (right) canceled in the sense that $fg = fh$ $(gf = hf)$ implies that $g = h$. A morphism f from A to B is said to be an isomorphism if there is a morphism g from B to A such that $gf = I_A$ and $fg = I_B$. Clearly, such a morphism g is unique, and it is called the inverse of f. The inverse of f, if exists, is denoted by f^{-1}.

Remark In the category *SET* of sets, a morphism f from a set A to a set B is a monomorphism (epimorphism) if and only if it is an injective (surjective) map. Also in the category *GP*, a morphism f from a group H to a group K is a monomorphism if and only if f is injective (prove it). Clearly, an isomorphism is a monomorphism as well as an epimorphism. However, a morphism which is a monomorphism as well as epimorphism need not be an isomorphism. For example, consider the category *Haus* of Hausdorff topological spaces. The inclusion map from $\mathbb{Q}$ to $\mathbb{R}$ is a monomorphism as well as an epimorphism (it is an epimorphism because $\mathbb{Q}$ is dense in $\mathbb{R}$) but it is not an isomorphism.

Let Σ be a category and A be an object of Σ. Then, $Mor_{\Sigma}(A, A)$ is a monoid with respect to the composition of morphisms. The members of $Mor_{\Sigma}(A, A)$ are called the endomorphisms of A. The monoid $Mor_{\Sigma}(A, A)$ is denoted by $End(A)$. An isomorphism from A to A is called an automorphism of A. The set of all automorphisms of A is denoted by $Aut(A)$ which is a group under the composition of morphisms.

Let Σ be a category. We say that a category Γ is a subcategory of Σ if (i) $Obj\Gamma \subseteq \Sigma$, (ii) for each pair $A, B \in Obj\Gamma$, $Mor_{\Gamma}(A, B) \subseteq Mor_{\Sigma}(A, B)$, and (iii) the law of composition of morphisms in Γ is the restriction of the law of composition of morphisms in Σ to Γ. The subcategory Γ is said to be a full subcategory if $Mor_{\Gamma}(A, B) = Mor_{\Sigma}(A, B)$ for all $A, B \in Obj\Gamma$. *AB* is a full subcategory of *GP*.

Functors

Definition Let Σ and Γ be categories. A functor F from Σ to Γ is an association which associates with each member $A \in Obj\Sigma$, a member $F(A)$ of Γ, and to each morphism $f \in Mor_{\Sigma}(A, B)$, a morphism $F(f) \in Mor_{\Gamma}(F(A), F(B)$ such that the following two conditions hold:

(i) $F(gf) = F(g)F(f)$ whenever the composition gf is defined.
(ii) $F(I_A) = I_{F(A)}$ for all $A \in \Sigma$.

Let Σ be a category. Consider the category Σ^o whose objects are same as the objects of Σ, $Mor_{\Sigma}(A, B) = Mor_{\Sigma^o}(B, A)$, and the composition fg in Σ^o is same as gf in Σ. The category Σ^o is called the opposite category of Σ. A functor from Σ^o to the category Γ is called a contravariant functor from Σ to Γ.

If Σ is a category, then the identity map $I_{Obj\Sigma}$ from $Obj\Sigma$ to itself defines a functor called the identity functor. Composition of any two functors is again a functor.

Examples 1. Let H be a group. Denote its abelianizer $H/[H, K]$ by $Ab(H)$. Let f be a homomorphism from H to a group K. Then f induces a homomorphism $Ab(f)$ from $Ab(H)$ to $Ab(K)$ defined by $Ab(f)(h[H, H]) = f(h)[K, K]$. This defines a functor Ab from the category GP of groups to the category AB. This functor is called the abelianizer functor.

2. Let H be a group. Denote the commutator $[H, H]$ of H by $Comm(H)$. If f be a homomorphism from H to K, it induces a homomorphism $Comm(f)$ from $Comm(H)$ to $Comm(K)$ defined by $Comm(f)([a, b]) = [f(a), f(b)]$. This defines a functor $Comm$ from the category GP to itself. This functor is called the commutator functor.

3. We have a functor Ω from the category GP of groups to the category SET of sets which simply forgets the group structure and retains the set part of the group. More explicitly, $\Omega(G, o) = G$. Such a functor is called a forgetful functor. There is another such functor from the category $RING$ of rings to the category AB of abelian groups which forgets the ring structure, but retains the additive group part of the ring. There is still another forgetful functor from the category TOP to the category SET which forgets the topological structure, and retains the set part of the space.

4. Let f be a map from a set X to a set Y. Then, f induces a unique homomorphism $F(f)$ from the free group $F(X)$ to the free group $F(Y)$. This gives us the functor F from the category SET to the category GP. This functor is called the free group functor.

5. Let Σ be a category and A be an object of Σ. For each $B \in Obj\Sigma$, we put $Mor_{\Sigma}(A, -)(B) = Mor_{\Sigma}(A, B)$, and for each morphism f from B to C, we have a map $Mor_{\Sigma}(A, -)(f)$ from $Mor_{\Sigma}(A, B)$ to $Mor_{\Sigma}(A, C)$ defined by $Mor_{\Sigma}(A, -)(f)(g) = fg$. It is easily verified that $Mor_{\Sigma}(A, -)$ defined above is a functor from Σ to the category SET of sets. Similarly, we have a contravariant functor $Mor_{\Sigma}(-, A)$ from the category Σ to the category SET of sets.

Remark There is a useful important functor, viz, homotopy group functor π_1 from the category $TOP^\star$ of pointed topological spaces to the category GP of groups. It has tremendous application in geometry and topology. This functor will be discussed in Algebra 3.

Let F be a functor from a category Σ to a category Γ. The functor F is said to be faithful if for each pair $A, B \in Obj\Sigma$, the induced map $f \mapsto F(f)$ from $Mor_\Sigma(A, B)$ to $Mor_\Gamma(F(A), F(B))$ is injective. The functor F is said to be a full functor if these induced maps are surjective. The forgetful functor from GP to SET is faithful but it is not full. The abelianizer functor Ab is not faithful. A functor F is said to be an isomorphism from the category Σ to the category Γ if there is a functor G from Γ to Σ such that $GoF = I_\Sigma$ and $FoG = I_\Gamma$.

Natural Transformations

Definition Let F and G be functors from a category Σ to a category Γ. A natural transformation η from F to G is a family $\{\eta_A \in Mor_\Gamma(F(A), G(A)) \mid A \in Obj\Sigma\}$ of morphisms in Γ such that the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"] & G(B)
\end{tikzcd}$$

is commutative for all morphisms f in Σ.

Examples 1. Let ν_G denote the quotient homomorphism from G to $G/[G, G]$. Then, the family $\{\nu_G \mid G \in ObjGP\}$ defines a natural transformation ν from the identity functor I_{GP} to the abelianizer functor Ab.

2. For each set X, we have the inclusion map i_X from X to $\Omega(F(X))$ where F is the free group functor, and Ω is the forgetful functor. Evidently, the family $\{i_X \mid X \in ObjSET\}$ of maps defines a natural transformation from the identity functor I_{SET} to the functor ΩoF on the category SET.

Let F and G be two functors from a category Σ to a category Γ. A natural transformation η from F to G is called a natural equivalence if η_A is an isomorphism from $F(A)$ to $G(A)$ for all $A \in Obj\Sigma$. This is equivalent to say that there is a natural transformation ρ from G to F such that $\rho_A o\eta_A = I_{F(A)}$ and $\eta_A o\rho_A = I_{G(A)}$ for all objects A of Σ. A functor F from Σ to Γ is called an equivalence from Σ to Γ if there is a functor G from Γ to Σ such that FoG and GoF are naturally equivalent to the corresponding identity functors. Notice that there is a difference between isomorphism and equivalence between categories. An equivalence need not be an isomorphism.

Let Σ and Γ be categories. Then, $\Sigma \times \Gamma$ represents the category whose objects are pairs $(A, B) \in Obj\Sigma \times Obj\Gamma$, and a morphism from (A, B) to (C, D) is a pair (f, g), where f is a morphism from A to C in Σ, and g is a morphism from B to D in Γ. The composition law is coordinate-wise. This category is called the product category.

Let F be a functor from a category Σ to a category Γ. Let f be a morphism from C to A in Σ, and g be a morphism from B to D in Γ. This defines a map $Mor_\Gamma(F(f), g)$ from the set $Mor_\Gamma(F(A), B)$ to $Mor_\Gamma(F(C), D)$ given by $Mor_\Gamma(F(f), g)(h) = ghF(f)$. This defines a functor $Mor_\Gamma(F(-), -)$ from the product category $\Sigma^o \times \Gamma$ to the category SET of sets. Similarly, given a functor G from Γ to Σ, we have a functor $Mor_\Sigma(-, G(-))$ to the category SET of sets. We say that F is left adjoint to G, or G is right adjoint to F if there is a natural isomorphism η from the functor $Mor_\Gamma(F(-), -)$ to the functor $Mor_\Sigma(-, G(-))$. More explicitly, for each object A in Σ and each object B in Γ, we have a bijective map $\eta_{A,B}$ from $Mor_\Gamma(F(A), B)$ to the set $Mor_\Sigma(A, G(B))$ such that $Mor_\Sigma(f, G(g))o\eta_{A,B} = \eta_{C,D}oMor_\Gamma(F(f), g)$ for all morphisms (f, g) in $\Sigma^o \times \Gamma$ (look at the corresponding commutative diagram).

Examples 1. Consider the category SET of sets and the category GP of groups. We have the free group functor F from the category SET to the category GP. More explicitly, for each set X, we have the free group $F(X)$ on the set X. We also have the forgetful functor Ω from GP to SET. From the universal property of free group, every group homomorphism f from $F(X)$ to G determines and is uniquely determined by its restriction to X. This gives us a bijective map $\eta_{X,G}$ from $Hom(F(X), G)$ to $Map(X, \Omega(G))$. It is easy to observe (using the universal property of a free group) that η, thus obtained, is a natural equivalence. Hence, the free group functor F is left adjoint to the forgetful functor Ω.

2. We have the forgetful functor Ω from the category AB of abelian groups to the category GP of groups. We also have the abelianizer functor Ab from GP to AB. It can be easily verified that Ab is left adjoint to Ω.

Products and Coproducts in a Category

Definition Let A and B be objects in a category Σ. A product of A and B in Σ is a triple (P, f, g), where P is an object of the category Σ, f is morphisms from P to A, and g is a morphism from P to B such that given any such triple (P', f', g'), there is a unique morphism ϕ from P' to P such that $f\phi = f'$ and $g\phi = g'$.

It is easily observed from the definition that if (P, f, g) and (P', f', g') are two products of A and B, then there is an isomorphism ϕ from P' to P with $f\phi = f'$ and

$g\phi = g'$. Thus, if the product exists, then it is unique up to natural isomorphism. The product of A and B is usually denoted by $A \times B$.

In the category *SET* of sets, the Cartesian product $A \times B$ with the corresponding projection maps is the product in the category *SET*. Similarly, the direct product $H \times K$ of the groups H and K together with the corresponding projection maps is the product of H and K in the category *GP*.

Dually, we have the following:

Definition Let A and B be objects in a category Σ. A coproduct of A and B in Σ is a triple (U, f, g), where U is an object of the category Σ, f is morphisms from A to U, and g is a morphism from B to U such that given any such triple (U', f', g'), there is a unique morphism ϕ from U to U' such that $\phi f = f'$ and $\phi g = g'$.

It is easily observed from the definition that if (U, f, g) and (U', f', g') are two coproducts of A and B, then there is an isomorphism ϕ from U to U' with $\phi f = f'$ and $\phi g = g'$. Thus, if the co-product exists, then it is unique up to natural isomorphism. The coproduct of A and B is denoted by $A \coprod B$.

In the category *SET* of sets, the disjoint union $(A \times \{0\}) \bigcup (B \times \{1\})$ of A and B with the natural inclusion maps is the coproduct of A and B in the category *SET*. Similarly, the free product $H \star K$ of the groups H and K together with the natural inclusion maps is the coproduct of H and K in the category *GP*.

PullBack and Push-out Diagrams

Definition Let Σ be a category. Let $f \in Mor_{\Sigma}(A, C),$ *and* $g \in Mor_{\Sigma}(B, C)$. A commutative diagram

$$\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

is said to be a pullback diagram if given any commutative diagram

$$\begin{tikzcd}
D \arrow[r, "\mu"] \arrow[d, "\nu"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

there exists a unique morphism ϕ from D to P such that $h\phi = \mu$ and $k\phi = \nu$.

Dually, a push-out diagram can be defined by reversing the arrows in the definition. The reader is advised to formulate the definition of push-out diagram.

In general, pullback and push out need not exist in a category. However, they exist in the category SET of sets and also in the category GP of groups: Let f be a morphism from A to C, and g be a morphism from B to C in the category SET/GP. Consider the product $A \times B$ in the category SET/GP. Let $P = \{(a, b) \in A \times B \mid f(a) = g(b)\}$. Let h denote first projection from P to A, and k denotes the second projection from P to B. This gives us a pullback diagram in SET/GP. Similarly, push out also exists in the category SET, and also it exists in the category GP.

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Index

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