Appendix A
Algebraic Graph Theory

The necessary results from algebraic graph theory (Merris 1994; Chung 1997) are introduced in this section to address the decentralized formation keeping and cooperative attitude synchronization problems. A directed communication topology can be described by directed graph. A directed graph $G_n$ consists of a finite set of vertices, denoted $V$, and a set of arcs $A \subseteq V^2$, where $a = (\bar{a}, \bar{b}) \in A$ and $\bar{a}, \bar{b} \in V$. The arc $(\bar{a}, \bar{b})$ denotes that vertex $\bar{b}$ can obtain the information of vertex $\bar{a}$. In decentralized control problem for satellite formation, the arc $a; b$ denotes that satellite $\beta$ can measure relative state of satellite $\alpha$ with respect to satellite $\beta$. In attitude synchronization application, the arc $(\bar{a}, \bar{b})$ denotes that satellite $\bar{b}$ can obtain attitude information of satellite $\bar{a}$. It is assumed that the graph has no self-loops, meaning that $(\bar{a}, \bar{b}) \in A$ implies $\bar{a} \neq \bar{b}$. If every possible arc exists, the graph is said to be complete.

A path on $G_n$ of length $N$ from $\bar{x}_1$ to $\bar{x}_{N+1}$ is an ordered set of distinct vertices $\{\bar{x}_1, \ldots, \bar{x}_{N+1}\}$ such that $(\bar{x}_{i-1}, \bar{x}_i) \in A$ for all $i \in [2, \ldots, N+1]$. A graph in which a path exists from any vertex to any other vertex is said to be strongly connected. A spanning tree in $G_n$ is a graph in which exists a vertex $\bar{a}$ such that there is a directed path from vertex $\alpha$ to every other vertex in $G_n$.

The adjacency matrix of $G_n$ denoted $A$ is a square matrix of size $n$ with entries

$$
\begin{align*}
    a_{i,j} &= 0 & \text{if } (\bar{z}_j, \bar{z}_i) \in A \\
    a_{i,j} &= 0 & \text{otherwise}
\end{align*}
$$

where the non-negative $a_{i,j}$ is subsequently chosen to be the control weight parameter between the $i$th and $j$th satellite. For decentralized formation keeping, $w_{i,j} a_{i,j}$ can be used to balance fuel consumption among satellites. Note that $a_{i,i} = 0$ from (A.1).

The in-degree matrix of $G_n$ is the diagonal matrix $D$ with diagonal entries

$$
d_{i,i} = \sum_{j=1, j\neq i}^{n} a_{i,j}, \ i = 1, \ldots, n
$$
Following Merris (1994), the Laplacian \( L \in \mathbb{R}^{n \times n} \) of the graph \( G_n \) is defined as follows:

\[
L = D - A
\]  

(A.3)

Note that a graph with the property that for any \( (\vec{a}, \vec{b}) \in A \), the arc \( (\vec{b}, \vec{a}) \in A \) as well is said to be undirected. In the satellite formation application, this corresponds to having bidirectional relative measurement or communication. It is valid to assume \( a_{i,j} = a_{j,i} \) in the case of the undirected communication-sensing topology. Under this assumption, the Laplacian \( L \) is a symmetrical matrix, which simplifies the stability analysis of cooperative control system. However, in the case of directed communication-sensing topology, \( L \) is generally not symmetric because \( a_{i,j} = a_{j,i} \) does not hold.

The following results are used in Chaps. 6 and 7 to derive stability proof for the proposed controllers design.

**Proposition A.1** (Merris 1994) Zero is an eigenvalue of \( L \). The associated eigenvector is all the ones vector \( 1^T \).

**Proposition A.2** (Merris 1994) For a digraph with \( N \) vertices, all the eigenvalues of \( L \) have nonnegative real part less than or equal to \( 2(N - 1) \) (use Gershgorin’s theorem). Moreover, except for the eigenvalue zero, the real part of all other eigenvalues is positive.

**Proposition A.3** (Merris 1994) If \( G_n \) is undirected, then all the eigenvalues of \( L \) are real. Moreover, the least nonzero eigenvalue \( \lambda_2 \) of \( L \) grows monotonically with the number of arcs. More precisely, adding edges never decreases \( \lambda_2 \).

**Proposition A.4** (Ren et al. 2005) If and only if \( G_n \) has a spanning tree, zero is an eigenvalue of algebraic multiplicity one for the Laplacian \( L \). The associated eigenvector is all the ones vector \( 1^T_n \). The associated left eigenvector is \( [v_1, v_2, \ldots, v_n] \), where \( \sum_{i=1}^{n} v_i = 1 \), and \( v_i \geq 0, i = 1, 2, \ldots, n \).
Appendix B
Optimal Guaranteed Cost Control

This section introduces a control method called the optimal guaranteed cost control, which is applied to a linear norm-bound uncertain system; details are given in Li (1985).

Consider a class of linear uncertain systems described by the following state-space equation:

\[
\dot{x}(\theta) = (\bar{A} + \Delta \bar{A})x(\theta) + (\bar{B} + \Delta \bar{B})\bar{u}(\theta), \quad \bar{x}(0) = \bar{x}_0
\]  

(A.4)

where \(\theta\) is the free variable, \(\dot{x}(\theta)\) is the derivative of the state variable \(\bar{x}\) with respect to \(\theta\), \(\bar{A}\) and \(\bar{B}\) are known constant real matrices of appropriate dimensions, \(\Delta \bar{A}\), and \(\Delta \bar{B}\) are matrix-valued functions representing time-varying parameter uncertainties in the system model. The parameter uncertainties considered here are assumed to be norm-bounded and of the form:

\[
[\Delta \bar{A} \quad \Delta \bar{B}] = D F(\theta) [E_1 \quad E_2]
\]  

(A.5)

where \(D, E_1\) and \(E_2\) are known constant real matrices of appropriate dimensions, which represent the structure of uncertainties. Suppose the uncertain matrix \(F(\theta)\) is of the following block diagonal form:

\[
F(\theta) = \text{diag}\{F_1(\theta), F_2(\theta), \ldots, F_l(\theta)\}
\]  

(A.6)

where \(F_k(\theta) \in R^{j_k \times j_k}\) and satisfies

\[
F_k^T(\theta)F_k(\theta) \leq I_{j_k \times j_k}, \quad k = 1, 2, \ldots, l.
\]  

(A.7)

where \(I_{j_k \times j_k}\) denotes the identity matrix of dimension \(j_k\). For

\[
\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l], \quad \varepsilon_k > 0, \quad k = 1, 2, \ldots, l
\]  

(A.8)
Define

\[ \tilde{M} = \text{diag}\{ \varepsilon_1 I_{i_1 \times i_1}, \varepsilon_2 I_{i_2 \times i_2}, \ldots, \varepsilon_l I_{i_l \times i_l} \} \]  
(A.9)

\[ \tilde{N} = \text{diag}\{ \varepsilon_1^{-1} I_{j_1 \times j_1}, \varepsilon_2^{-1} I_{j_2 \times j_2}, \ldots, \varepsilon_l^{-1} I_{j_l \times j_l} \} \]  
(A.10)

Then

\[ \text{DF}(\theta)[E_1 \ E_2] = D\tilde{M}F(\theta) [\tilde{N}E_1 \ \tilde{N}E_2] \]  
(A.11)

Associated with the system in (A.4) is the following cost function, where \( \bar{Q} \) and \( \bar{R} \) are given positive definite symmetric matrices.

\[ J = \int_0^\infty \left[ \bar{x}^T(\theta)\bar{Q}\bar{x}(\theta) + \bar{u}^T(\theta)\bar{R}\bar{u}(\theta) \right] d\theta \]  
(A.12)

**Definition A.1** Consider the uncertain system in (A.4), if there exists a control law \( \bar{u}^*(\theta) \) and a positive scalar \( J^* \) such that for all admissible uncertainties, the closed-loop system is stable and the closed-loop value of the cost function in (A.12) satisfies \( J \leq J^* \), then \( J^* \) is said to be a guaranteed cost and \( \bar{u}^*(\theta) \) is said to be a guaranteed cost control law for the uncertain system in (A.4).

The objective of this section is to design a state-feedback guaranteed cost control law \( \bar{u}(\theta) = K\bar{x}(\theta) \) for the uncertain system in (A.4).

**Theorem A.1** (Li 1985) Given the system described (A.4) and the associated cost function in (A.12), if the following optimization problem

\[ \min_{\varepsilon, \ W, \ X, \ M} \text{Trace}(M) \]  
(A.13)

subject to

\[ \begin{bmatrix} \bar{A}X + \bar{B}W + (\bar{A}X + \bar{B}W)^T & (E_1 X + E_1 W)^T & X & W^T & D\tilde{M}^- \ \\ E_1 X + E_1 W & -\bar{N}^{-1} & 0 & 0 & 0 \\ X & 0 & -\bar{Q}^{-1} & 0 & 0 \\ W & 0 & 0 & -\bar{R}^{-1} & 0 \\ MD^T & 0 & 0 & 0 & -\bar{M} \end{bmatrix} < 0 \]  
(A.14)

\[ \begin{bmatrix} M & \ I \\ \ I & \ X \end{bmatrix} > 0 \]  
(A.15)
has a solution $\varepsilon^*, W^*, X^* > 0$, $M^* > 0$, then the state-feedback control law is the optimal guaranteed cost controller as follows:

$$
\bar{u}(\theta) = W^*(X^*)^{-1}\bar{x}(\theta)
$$

(A.16)

and the associated guaranteed cost is as follows:

$$
\bar{J} \leq \text{trace}(X^{-1}) = \bar{J}^*
$$

(A.17)

where $W$ is matrix variable, and $X, M$ are positive definite matrix variables.

The above problem is a convex optimization problem subject to linear matrix inequalities constraints and can be solved by using LMI toolbox in MATLAB.
Appendix C
Nomenclature

Subscripts and superscript

\( j \) The \( j \)th deputy satellite
\( f \) Final value
\( \tilde{r} \) Scaled variable
\( \ddot{r} \) Secular part of orbit element variables
\( \dot{r} \) Time-varying variables
\( \dot{r} \) First derivative
\( \ddot{r} \) Second derivative
\( v \) The \( v \)th phase in the trajectory
\( 0 \) Initial value
\( 0 \) Reference or chief satellite

The following list of symbols is alphabetical—lowercase and then upper case; Arabic; and then Greek letters.

\( a \) Semimajor axis
\( \mathbf{a} \) Control acceleration vector
\( a_x \) Control acceleration in \( x \) direction in LVLH frame
\( a_y \) Control acceleration in \( y \) direction in LVLH frame
\( a_z \) Control acceleration in \( z \) direction in LVLH frame
\( c_o \) \( \cos(o) \)
\( \dot{d} \) Disturbance torque
\( d \) Distance between satellites
\( \dot{d}_{\text{max}} \) Possible maximum relative distance during the maneuver
\( \dot{d}_{\text{safe}} \) Assumed safe distance between satellites
\( e \) Eccentricity
\( \mathbf{e} \) Euler axis
\( f \) True anomaly
\( g_0 \) Sea-level acceleration due to gravity
\( h \) Angular momentum vector
\( \vec{h} \) Magnitude of angular momentum vector
\( i \) Inclination
\( i \) Index (general)
\( j \) Index (general)
\( m \) Satellite mass
\( q \) Relative position vector in LVLH frame
\( \vec{q} \) Error quaternion
\( q \) Vector part of error quaternion
\( q_0 \) Scalar part of error quaternion
\( r \) Position vector of reference or chief satellite
\( r \) Geocentric distance of reference or chief satellite
\( s_o \) \( \sin(o) \)
\( s_j \) The \( j \)th component of multi-satellite sliding vector
\( t \) General time
\( u \) Thrust direction
\( u \) Control torque
\( v_{jx} \) Component of velocity vector of the \( j \)th member satellite in ECI frame
\( v_{jy} \) Component of velocity vector of the \( j \)th member satellite in ECI frame
\( v_{jz} \) Component of velocity vector of the \( j \)th member satellite in ECI frame
\( x \) Radial difference between two objects in LVLH frame
\( \dot{x} \) Unit vector along radial direction in LVLH frame
\( y \) Along-track difference between two objects in LVLH frame
\( \dot{y} \) Unit vector along along-track direction in LVLH frame
\( z \) Cross-track difference between two objects in LVLH frame
\( \dot{z} \) Unit vector along cross-track direction in LVLH frame
\( D \) Reference frame for satellite reference attitude
\( E \) Orbital energy
\( F_j \) Magnitude of constant thrust of the \( j \)th member satellite
\( H \) Polar component of the orbital angular momentum
\( I_{sp} \) The specific impulse of the engine
\( J \) Inertia matrix of satellite
\( J_2 \) Geopotential coefficient representing Earth’s oblateness
\( K_j \) Kinetic energy of the \( j \)th member satellite
\( R \) Rotation Matrix
\( r_c \) Radius of circular formation
\( R_e \) Earth equatorial radius
\( S \) Multi-satellite sliding vector
\( S_0 \) Reference satellite or chief satellite
\( S_j \) The \( j \)th member satellite
\( L_j \) Lagrangian of the \( j \)th member satellite
\( L \) Laplacian matrix in algebraic graph theory
\( M \) Satellite mean anomaly
\( N \) Number of LGL points
\( N_s \) Number of deputy satellites

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Appendix C: Nomenclature

$T$ Nodal orbital period
$T$ Magnitude of constant thrust
$T_c$ Period of chief satellite
$U$ Gravitational potential
$\hat{X}$ Unit vector in Earth-centered inertial (ECI) frame
$\hat{Y}$ Unit vector in ECI frame
$\hat{Z}$ Unit vector in ECI frame
$\alpha_x$ Satellite steering acceleration
$\alpha_z$ Satellite orbital acceleration
$\rho$ Position vector in LVLH frame
$\rho$ Magnitude of position vector in LVLH frame
$\theta$ True latitude
$\phi$ Geocentric latitude of satellite
$\varphi$ Euler angle
$\eta$ Quaternion
$\eta$ Vector part of quaternion
$\eta_0$ Scalar part of quaternion
$\phi$ Geocentric latitude of satellite
$\Omega$ Right ascension of ascending node
$\omega$ Vector of orbital angular velocity
$\omega$ Argument of perigee
$\omega$ Angular velocity
$\omega_d$ Desired angular velocity
$\dot{\omega}$ Angular velocity error
$\omega_x$ Component of orbital angular velocity
$\omega_y$ Component of orbital angular velocity
$\omega_z$ Component of orbital angular velocity
$\mu$ Earth gravitational constant
$\lambda$ Eigenvalue

References

Li Y (1985) Robust control: an LMI approach. Tsinghua University, Beijing, China