Appendices

A.1 Equality Problems for Norm Inequalities

In this section, we will introduce the general theory of reproducing kernels that may be considered as the deepest theory in the general theory of reproducing kernels.

A.1.1 Introduction of the Results by Akira Yamada

We will introduce the quite general theory by A. Yamada [485] for some general conditions for norm inequalities derived from the theory of reproducing kernels.

In 1965, Lebedev and Milin [281] found the following inequality: If $f \in \mathscr{O}(\Delta(1))$ and e^f have Taylor expansion $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $e^{f(z)} = \sum_{n=0}^{\infty} b_n z^n$, respectively, then

$$\sum_{n=0}^{\infty} |b_n|^2 \leq \exp\left(\sum_{n=1}^{\infty} n|a_n|^2\right),$$

or equivalently

$$\sum_{n=1}^{\infty} |b_n|^2 \leq \exp\left(\sum_{n=1}^{\infty} n|a_n|^2\right) - 1, \tag{A.1}$$

where equality occurs if and only if there exists $\rho \in \Delta(1)$ with $a_n = \rho^n/n$ for all n. This is a prototype of the inequalities treated in Sect. A.1; that is, we can rewrite the above inequality as

$$\|e^f\|_{H^2}^2 \leq \exp(\|f\|_D^2), \tag{A.2}$$

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where

$$\|f\|_D = \sqrt{\frac{1}{\pi} \iint_{\Delta(1)} |f'(z)|^2 dxdy}$$

is the Dirichlet norm, and

$$\|e^f\|_{H^2} = \sqrt{\sum_{n=0}^{\infty} |b_n|^2}$$

is the Hardy H^2-norm of the function e^f. See Proposition 1.2. Also we remark that in this case we have the identity

$$k_{H^2}(z, w) = e^{k_D(z,w)} = \frac{1}{1 - z\bar{w}}, \; z, w \in \Delta(1), \tag{A.3}$$

where k_{H^2} and k_D are the reproducing kernels for the Hardy H^2-space and the Dirichlet space D on $\Delta(1)$ normalized by $f(0) = 0$, respectively. See Theorem 1.15 for the definition of k_D. Moreover, the above equality condition is equivalent to $f(z) = k_D(z, q)$ for some $q \in \Delta(1)$. Therefore, there exists a deep connection between the Lebedev Milin inequality and reproducing kernels.

Before we start to develop the problem, we recall a generality: Let $\psi(z) = \sum_{n=1}^{\infty} p_n z^n$ be an entire function such that $p_n \geq 0$ for all $n \in \mathbb{N}$. If $H_K(E)$ is a complex RKHS on a set E with the reproducing kernel K and the norm $\|\cdot\|_K$, then there exists a unique RKHS $H_{\psi^* K}(E)$ on E with the reproducing kernel $\psi^* K$, which is given by

$$\psi^* K = \psi(K(\cdot, \cdot)).$$

See Proposition 2.6 and Corollary 2.4 for the details.

Keeping this in mind, we consider another entire function φ such that $\varphi(0) = 0$. We expand

$$\varphi(z) = \sum_{n=1}^{\infty} c_n z^n$$

and we assume:

1. $p_n \geq 0$ for all n
2. $c_n = 0$ whenever $p_n = 0$.

Putting

$$\varphi_\psi(z) \equiv \sum_{p_n > 0} \frac{|c_n|^2}{p_n} z^n, \tag{A.4}$$

we have the following norm inequality which generalizes the Lebedev Milin inequality (A.1) [65, 388]: For all $f \in H_K(E)$,

$$\|\varphi(f)\|_{H_{\psi * K}(E)} \leq \sqrt{\varphi_\psi(\|f\|^2_{H_K(E)})}. \tag{A.5}$$

In the special case where $\varphi = \psi$, it is easy to see that if $f = K_q$ for some $q \in E$, then equality holds in (A.5) [65]. The converse of this fails in general. See [381] for counterexamples. However, by studying various special but important RKHSs, there are many papers [65–68, 373–376, 380, 381, 500] asserting that equality occurs in (A.5) if and only if $f = K_q$ for some $q \in E$. The most concrete case was dealt with [373, Section 31] in detail. In order to investigate the condition for equality, all these papers relied on case-by-case arguments. Yamada gave a general and satisfactory theory of equality conditions for such norm inequalities. In order to see his theory, we will introduce a class of RKHSs called *algebra-dense* and study relations between equality conditions and $\mathbb{C}$-algebra homomorphisms.

Let $m \geq 2$. Let $H_j(E) = H_{K^{(j)}}(E)$ be a complex RKHS on the set E with the reproducing kernel $K^{(j)}$ for $j = 1, 2, \ldots, m$. Then, according to Theorem 2.20, the Hilbert tensor product

$$H \equiv \otimes_{j=1}^m H_j(E)$$

is an RKHS on

$$E^m \equiv \prod_{j=1}^m E.$$

Definition A.1.

1. The set $E_{\rm d}^m$ denotes the *diagonal* $\{(x, x, \ldots, x) : x \in E\}$ of the set E^m.
2. Denote by H_0 the subspace of H defined by $\{f \in H : f|_{E_{\rm d}^m} = 0\}$.
3. For $f, g \in H$ define an equivalence relation "$\sim$" by $f \sim g$ if and only if f and g agree on the diagonal $E_{\rm d}^m$.

Definition A.2. An element $\phi \in H$ is said to be *extremal* if $\phi \in (H_0)^\perp$, or equivalently, ϕ is extremal if and only if $f \sim g$ implies $\langle f, \phi \rangle_H = \langle g, \phi \rangle_H$.

Remark A.1. The definition of the extremality above is closely related to equality conditions of norm inequalities for the tensor product. As we see easily, if H' denotes the unique RKHS with the kernel $\prod_{j=1}^m K_x^{(j)}$, $x \in E$, then H' consists of functions on E induced from the restrictions of functions in H to the diagonal $E_{\rm d}^m$. For $\phi_j \in H_j(E)$ with $j = 1, 2 \ldots, m$, we have

$$\|\phi_1 \phi_2 \cdots \phi_m\|_{H'} \leq \|\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_m\|_H = \|\phi_1\|_{H_1(E)} \|\phi_1\|_{H_2(E)} \cdots \|\phi_m\|_{H_m(E)}, \tag{A.6}$$

where equality occurs if and only if $\otimes_{j=1}^m \phi_j$ is extremal (cf. Sect. A.1.2 as well as Remark 2.7).

Here and below we denote $\mathbb{C}^E \equiv \mathscr{F}(E)$.

Lemma A.1. *Let R be a subalgebra of $\mathbb{C}^E$. For $j = 1, 2, \ldots, m$, suppose that we are given $f_j, g_j, \phi_j \in H_j(E)$. Assume that $\phi \equiv \otimes_{j=1}^m \phi_j \in (H_0)^\perp \setminus \{0\} \subset \otimes_{j=1}^m H_j(E)$ is nonzero extremal.*

Then

$$\prod_{j=1}^m \langle f_j, \phi_j \rangle_{H_j(E)} = \prod_{j=1}^m \langle g_j, \phi_j \rangle_{H_j(E)}, \tag{A.7}$$

if $\otimes_{j=1}^m f_j \sim \otimes_{j=1}^m g_j$, namely

$$\prod_{j=1}^m f_j = \prod_{j=1}^m g_j \tag{A.8}$$

on E.

Proof. Recall that the inner product of the tensor product satisfies

$$\langle \otimes_{j=1}^m \tilde{\psi}_j, \otimes_{j=1}^m \psi_j \rangle_H = \prod_{j=1}^m \langle \tilde{\psi}_j, \psi_j \rangle_{H_j(E)} \tag{A.9}$$

for all elements $\{\tilde{\psi}_j\}_{j=1}^m$, and $\{\psi_j\}_{j=1}^m \in H_1(E) \otimes H_2(E) \otimes \ldots \otimes H_m(E)$. Observe also that (A.8) implies $\otimes_{j=1}^m f_j \sim \otimes_{j=1}^m g_j$. Since ϕ is extremal, (A.7) follows.

In words of the paper [377], we recall several definitions:

Definition A.3. Let $H_j(E)$ be RKHSs on E for $j = 1, 2, \ldots, m$. Then the tensor product $H = \otimes_{j=1}^m H_j(E)$ is called *regular*, if for every nonzero extremal $\otimes_{j=1}^m \phi_j \in H$, there exists a point $q \in E$ such that $\phi_j \in \mathrm{Span}\{K_q^{(j)}\}$ for all $j = 1, 2, \ldots, m$. Also, H is called *weakly regular*, if for every nonzero extremal $\otimes_{j=1}^m \phi_j \in H$, there exists a point $q \in E$ such that for each $j = 1, 2, \ldots, m$, either of the following holds:

1. q is a common zero of the functions in $H_j(E)$,

$$f(q) = 0 \tag{A.10}$$

   for all $f \in H_j(E)$ with $j = 1, 2, \ldots, m$, or
2. $\phi_j \in \mathrm{Span}\{K_q^{(j)}\}$, that is, ϕ_j is a constant multiple of $K_q^{(j)}$.

For later references, we will call (A.10) above the *exceptional case*; this is independent of the functions ϕ_j.

In what follows, we always assume that R denotes a $\mathbb{C}$-subalgebra of $\mathbb{C}^E$, an algebra made up of complex-valued functions on E. The existence of the identity of R is not assumed. For a complex subspace H of $\mathbb{C}^E$, let $R^{-1}H$ denote the subspace of H defined by

$$R^{-1}H \equiv \{f \in H : rf \in H \text{ for all } r \in R\}.$$

Remark that under this definition $R^{-1}H$ is always a subset of H.

Lemma A.2. *Let R be a subalgebra of $\mathbb{C}^E$. Assume that $\phi = \otimes_{j=1}^m \phi_j \in (H_0)^\perp \setminus \{0\} \subset \otimes_{j=1}^m H_j(E)$ is nonzero extremal. If each $R^{-1}H_j(E)$ is dense in $H_j(E)$ ($j = 1, 2, \ldots, m$), then there exists a unique $\mathbb{C}$-algebra homomorphism $\Lambda_\phi : R \to \mathbb{C}$ satisfying, for every $j = 1, 2, \ldots, m$,*

$$\langle fu, \phi_j \rangle_{H_j(E)} = \Lambda_\phi(f) \langle u, \phi_j \rangle_{H_j(E)} \tag{A.11}$$

for every $f \in R$ and for every $u \in R^{-1}H_j(E)$ ($j = 1, 2, \ldots, m$).

Proof. Since $R^{-1}H_j(E)$ is assumed to be dense in $H_j(E)$ and $\phi_j \neq 0$, we can find an element $u_j \in R^{-1}H_j(E)$ with

$$\langle u_j, \phi_j \rangle_{H_j(E)} \neq 0. \tag{A.12}$$

Fixing such an element u_j for each j below, we defined $\Lambda_\phi(f)$ by

$$\Lambda_\phi(f) \equiv \frac{\langle fu_j, \phi_j \rangle_{H_j(E)}}{\langle u_j, \phi_j \rangle_{H_j(E)}} \tag{A.13}$$

for $f \in R$. Now we show that $\Lambda_\phi(f)$ is well defined, that is, Λ_ϕ is determined despite the ambiguity of the choice of j and u_j. For $f \in R$ define $f_k,\ g_k \in H_k(E)$ ($k = 1, 2, \ldots, m$) by

$$f_k \equiv \begin{cases} fu_i & (k = i) \\ u_k & (k \neq i) \end{cases}, \quad g_k \equiv \begin{cases} fu_j & (k = j) \\ u_k & (k \neq j) \end{cases}. \tag{A.14}$$

From (A.7) and (A.12), we have

$$\langle fu_i, \phi_i \rangle_{H_i(E)} \langle u_j, \phi_j \rangle_{H_j(E)} = \langle u_i, \phi_i \rangle_{H_i(E)} \langle fu_j, \phi_j \rangle_{H_j(E)} \tag{A.15}$$

from (A.12), since

$$u_1 \otimes \cdots \otimes u_{i-1} \otimes fu_i \otimes u_{i+1} \otimes \cdots \otimes u_m \sim u_1 \otimes \cdots \otimes u_{j-1} \otimes fu_j \otimes u_{j+1} \otimes \cdots \otimes u_m$$

and $\otimes_{j=1}^m \phi_j$ is extremal. Thus, for all $f \in R$ and i, j, from (A.12) and (A.15), we deduce that

$$\frac{\langle fu_i, \phi_i\rangle_{H_i(E)}}{\langle u_i, \phi_i\rangle_{H_i(E)}} = \frac{\langle fu_j, \phi_j\rangle_{H_j(E)}}{\langle u_j, \phi_j\rangle_{H_j(E)}}. \tag{A.16}$$

Similarly, for $f, g \in R$, setting

$$f_k \equiv \begin{cases} fu_i & (k = i) \\ gu_j & (k = j) \\ u_k & (k \neq i, j) \end{cases}, \quad g_k \equiv \begin{cases} fgu_i & (k = i) \\ u_k & (k \neq i) \end{cases},$$

we have

$$f_1 \otimes f_2 \otimes \cdots \otimes f_k \sim g_1 \otimes g_2 \otimes \cdots \otimes g_k$$

and hence

$$\langle fu_i, \phi_i\rangle_{H_i(E)} \langle gu_j, \phi_j\rangle_{H_j(E)} = \langle fgu_i, \phi_i\rangle_{H_i(E)} \langle u_j, \phi_j\rangle_{H_j(E)}.$$

Hence, from (A.16), we deduce that

$$\frac{\langle fgu_i, \phi_i\rangle_{H_i(E)}}{\langle u_i, \phi_i\rangle_{H_i(E)}} = \frac{\langle fu_i, \phi_i\rangle_{H_i(E)} \langle gu_j, \phi_j\rangle_{H_j(E)}}{\langle u_i, \phi_i\rangle_{H_i(E)} \langle u_j, \phi_j\rangle_{H_j(E)}} = \frac{\langle fu_i, \phi_i\rangle_{H_i(E)} \langle gu_i, \phi_i\rangle_{H_i(E)}}{\langle u_i, \phi_i\rangle_{H_i(E)} \langle u_i, \phi_i\rangle_{H_i(E)}}. \tag{A.17}$$

Therefore, we have proved by (A.17) that the linear functional Λ_ϕ on R is multiplicative and that its definition makes sense by (A.16). Since the right-hand side of (A.13) remains unchanged if we multiply ϕ_j by any nonzero constant, Λ_ϕ is dependent only on the tensor product ϕ. Hence $\Lambda_\phi : R \to \mathbb{C}$ is a well-defined $\mathbb{C}$-algebra homomorphism. The uniqueness of Λ_ϕ is clear from the definition and the assumption that $R^{-1}H_j$ is dense in H_j.

In view of the definition (A.13), (A.11) holds when $\langle u, \phi_j\rangle_{H_j(E)} \neq 0$. Thus, to show that the identity (A.11) holds for every $u \in R^{-1}H_j(E)$, it suffices only to show that if $\langle u, \phi_j\rangle_{H_j(E)} = 0$, then $\langle fu, \phi_j\rangle_{H_j(E)} = 0$. Indeed, from (A.15) we have, for $k \neq j$,

$$\langle fu, \phi_j\rangle_{H_j(E)} \langle u_k, \phi_k\rangle_{H_k} = \langle u, \phi_j\rangle_{H_j(E)} \langle fu_k, \phi_k\rangle_{H_k} = 0.$$

Thus $\langle fu, \phi_j\rangle_{H_j(E)} = 0$, since $\langle u_k, \phi_k\rangle_{H_k} \neq 0$.

Remark A.2. If R has the identity, then from (A.11) we have $\Lambda_\phi(1) = 1$.

Given complex subspaces R_1 and R_2 of R, let $R_1 \cdot R_2$ denote the complex subspace of R given by

$$R_1 \cdot R_2 \equiv \bigcup_{N=1}^{\infty} \left\{ \sum_{i=1}^{N} a_i b_i \; : \; a_i \in R_1 \text{ and } b_i \in R_2 \right\}. \tag{A.18}$$

Definition A.4. Let R be a subalgebra of $\mathbb{C}^E$. An RKHS H on E is called *R-dense* if $R \cdot (R \cap H)$ is densely contained in H. If H is R-dense for some $\mathbb{C}$-algebra R on E, H is called *algebra-dense*.

Remark A.3. If H is R-dense, then $R \cap H$ is both (1) a dense subspace of H and (2) an ideal of R. Moreover, if $1 \in R$, then H is R-dense if and only if (1) and (2) hold.

Indeed, if H is R-dense, then $R \cdot (R \cap H) \subset R \cap H \subset H$. Hence $R \cap H$ is dense in H. Also, since H is R-dense, we have $R \cdot (R \cap H) \subset H$ and since R is an algebra, we have $R \cdot (R \cap H) \subset R$. Thus, $R \cap H$ is an ideal of R. Conversely, suppose $1 \in R$ and that (1) and (2) hold. Then $R \cdot (R \cap H) = R \cap H$ is a dense subspace of H, which implies H is R-dense.

Definition A.5. Let R be a subalgebra of $\mathbb{C}^E$ and let H be an R-dense RKHS on E. Let $\chi : R \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism. If there exists a constant $C > 0$ such that

$$|\chi(f)| \leq C\|f\|_H \text{ for all } f \in R \cap H, \tag{A.19}$$

χ is called an *H-bounded homomorphism* of R. The set of nonzero H-bounded homomorphisms of R is called an *H-hull* of E and is denoted by $\hat{E}_H$.

For example, $\chi = \mathrm{ev}_q$ is an *H-bounded homomorphism* of R; see Lemma 2.1.

If $H_j(E)$ $(j = 1, 2, \ldots, m)$ is R-dense, then $R \cap H_j(E)$ is a dense R-invariant subspace of $H_j(E)$ and we can apply Lemma A.2 to the tensor product $\otimes_{j=1}^m H_j(E)$. In fact, $R^{-1}H_j(E)$ contains $R \cap H_j(E)$ and $R \cap H_j(E)$ is assumed dense in $H_j(E)$. Then, we have the main result:

Theorem A.1. *Let R be a subalgebra of $\mathbb{C}^E$. For each $j = 1, 2, \ldots, m$, assume that $H_j(E)$ is an R-dense RKHS on E. If $\phi = \otimes_{j=1}^m \phi_j \in \otimes_{j=1}^m H_j(E)$ is nonzero extremal, then there exists a unique nonzero $\mathbb{C}$-algebra homomorphism $\Lambda_\phi \in \bigcap_{j=1}^m \hat{E}_{H_j(E)}$ satisfying (A.11). Furthermore, for each $j = 1, 2, \ldots, m$, either of the following holds:*

1. $\Lambda_\phi | R \cap H_j(E) = 0$.
2. *For every $j = 1, 2, \ldots, m$, there exists a constant $C_j \neq 0$ such that*

$$\langle f, \phi_j \rangle_{H_j(E)} = C_j \Lambda_\phi(f) \tag{A.20}$$

for each $f \in R \cap H_j(E)$.

Proof. Let Λ_ϕ be a mapping from Lemma A.2. We claim this Λ_ϕ is the one we are looking for.

1. We must show that $\Lambda_\phi \neq 0$. Since $R \cdot (R \cap H_1(E))$ is dense in $H_1(E)$, there exists finite subsets $\{f_k\}_{k=1}^N \in R$ and $\{g_k\}_{k=1}^N \in R \cap H_1(E)$ such that

$$\left\langle \sum_{k=1}^N f_k g_k, \phi_1 \right\rangle_{H_1(E)} = \sum_{k=1}^N \langle f_k g_k, \phi_1 \rangle_{H_1(E)} \neq 0.$$

Thus there exists an index k_0 such that

$$\langle f_{k_0} g_{k_0}, \phi_1 \rangle_{H_1(E)} = \Lambda_\phi(f_{k_0}) \langle g_{k_0}, \phi_1 \rangle_{H_1(E)} \neq 0.$$

Hence $\Lambda_\phi(f_{k_0}) \neq 0$ and so $\Lambda_\phi \neq 0$.

2. We look for C_j satisfying (A.20) assuming that $\Lambda_\phi | R \cap H_j(E) \neq 0$. Then there exists an element $f_0^* \in R \cap H_j(E)$ such that $\Lambda_\phi(f_0^*) \neq 0$. Since $R \cdot R \cap H_j(E)$ is dense in $H_j(E)$, there exist elements $f_0 \in R$ and $g \in R \cap H_j(E) \subset R^{-1} H_j(E)$ with $\langle f_0 \cdot g, \phi_j \rangle_{H_j(E)} \neq 0$. For every $f \in R \cap H_j(E)$, Lemma A.2 implies the identity

$$\langle fg, \phi_j \rangle_{H_j(E)} = \Lambda_\phi(f_0) \langle g, \phi_j \rangle_{H_j(E)}. \tag{A.21}$$

Note that

$$\Lambda_\phi(g) \langle f, \phi_j \rangle_{H_j(E)} = \Lambda_\phi(f) \langle g, \phi_j \rangle_{H_j(E)} = \langle fg, \phi_j \rangle_{H_j(E)}$$

from (A.21) when f belongs to $R \cap H_j(E)$ as well. Putting $f = f_0 \in R$ above, we obtain $\Lambda_\phi(g) \neq 0$. Hence, (A.20) holds if we set

$$C_j \equiv \frac{\langle g, \phi_j \rangle_{H_j(E)}}{\Lambda_\phi(g)} \neq 0.$$

3. Now it is clear that Λ_ϕ is an $H_j(E)$-bounded homomorphism of R. For, if $\Lambda_\phi|_{R \cap H_j(E)} = 0$, then this is trivial, otherwise this follows from Schwarz's inequality. Thus, $\Lambda_\phi \in \hat{E}_{H_j(E)}$ for every $j = 1, 2, \ldots, m$.

Definition A.6. Let R be a subalgebra of $\mathbb{C}^E$ and let H be an R-dense RKHS on E. Then H is called *maximal* if every nonzero H-bounded homomorphism of R is a point evaluation of R at some point in E. If one needs to specify the algebra R when H is maximal, then H is called *R-maximal*.

We defined the generalized ℓ^2-space over any set E in Example 2.4. Here we can prove that this is maximal.

Example A.1. Let E be a set. Let $\ell^2(E)$ be the complex Hilbert space given by

$$\ell^2(E) \equiv \left\{ f \in \mathbb{C}^E \,:\, \sum_{x \in E} |f(x)|^2 < \infty \right\},$$

whose inner product is given by

$$\langle f, g \rangle_{\ell^2(E)} = \sum_{x \in E} f(x) \overline{g(x)},$$

for $f, g \in \ell^2(E)$. It is easy to see that $\ell^2(E)$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}^E$(without the identity). Putting $R \equiv \ell^2(E)$ we will show that $\ell^2(E)$ is R-dense and maximal. From $\delta_x^2 = \delta_x$ we see that $R \cdot R$ is dense in R, which implies that $\ell^2(E)$ is R-dense. Let $\chi: R \to \mathbb{C}$ be a nonzero $\mathbb{C}$-algebra homomorphism. From $\delta_x^2 = \delta_x$, $\chi(\delta_x)$ is equal to 0 or 1. Also, $\delta_x\delta_y = 0$ $(x \neq y)$ implies that $\chi(\delta_x)\chi(\delta_y) = 0$ $(x \neq y)$. From $\chi \not\equiv 0$, we conclude that there uniquely exists a point $q \in E$ such that $\chi(\delta_x) = \delta_x(q)$ for $x \in E$. Since the span of δ_x's is dense in R, $\chi(f) = f(q)$ for all $f \in R$. Thus $\ell^2(E)$ is maximal.

As a corollary to Theorem A.1, we have:

Corollary A.1. *Let $H_j(E)$ $(j = 1, 2, \ldots, m)$ be R-dense RKHSs on E. If $H_{j_0}(E)$ is maximal for some $j_0 = 1, 2, \ldots, m$, then their tensor product $\otimes_{j=1}^m H_j(E)$ is weakly regular.*

Proof. Let $\phi_1 \in H_1(E)$, $\phi_2 \in H_2(E)$, $\ldots$, $\phi_m \in H_m(E)$ and suppose that $\phi \equiv \otimes_{j=1}^m \phi_j$ is nonzero extremal in $\otimes_{j=1}^m H_j(E)$. Then by Theorem A.1 the algebra homomorphism Λ_ϕ is an $H_j(E)$-bounded homomorphism of R $(j = 1, 2, \ldots, m)$. Since $H_{j_0}(E)$ is maximal, Λ_ϕ is a point evaluation of R at some point $q \in E$; $\Lambda_\phi(f) = f(q)$.

We fix $j = 1, 2, \ldots, m$ until the end of the proof. Assume that q is not a common zero of $H_j(E)$, that is, $\tilde{f}(q) \neq 0$ for $\tilde{f} \in H_j(E)$. We prove $\phi_j \in \mathrm{Span}\{K_q^{(j)}\}_{q \in E}$. Since $R \cap H_j(E)$ is dense in $H_j(E)$, $\Lambda_\phi | R \cap H_j(E) \neq 0$. Thus, there exists a constant $C_j \neq 0$ such that $\langle f, \phi_j \rangle_{H_j(E)} = C_j f(q)$ for all $f \in R \cap H_j(E)$. Since $H_j(E)$ is R-dense, for each $f \in H_j(E)$ there exists a sequence $f_n \in R \cap H_j(E)$ $(n = 1, 2, \ldots)$ such that f_n converges strongly to f as $n \to \infty$. From $\langle f_n, \phi_j \rangle_{H_j(E)} = C_j f_n(q)$, letting $n \to \infty$ we have $\langle f, \phi_j \rangle_{H_j(E)} = C_j f(q)$ for $f \in H_j(E)$. Since $C_j \neq 0$, it follows that ϕ_j induces a constant multiple of the point evaluation of $H_j(E)$ at $q \in E$. Thus, ϕ_j is the reproducing kernel of $H_j(E)$ at q up to a nonzero multiplicative constant. Hence, $\otimes_{j=1}^m H_j(E)$ is weakly regular.

A.1.2 Equality Conditions for the Norm Inequalities

Before studying equality conditions we first recall some facts from the theory of reproducing kernels. Let $A : H_1 \to H_2$ be a linear map from a Hilbert space H_1 into a linear space H_2 with closed kernel $\ker(A) = A^{-1}(0)$. The *range norm* of $\mathrm{Ran}(A)$ is the norm which makes A a partial isometry from H_1 onto $\mathrm{Ran}(A)$:

$$\|Ax\|_{\mathrm{Ran}(A)} = \|x\|_{H_1}$$

for $x \in H_1 \cap \ker(A)^\perp$. In fact, the range $\mathrm{Ran}(A)$ equipped with this range norm is a Hilbert space isomorphic to $H_1 \ominus \ker(A)$ and is called the *operator range* of the map A. With these terminologies, the RKHS $H_{K_1+K_2}(E)$ is the operator range of the map

$$(f, g) \in H_{K_1}(E) \oplus H_{K_2}(E) \mapsto f + g \in H_{K_1+K_2}(E).$$

See (2.88). Hence, we have the Pythagorean inequality (see (2.95))

$$\|f + g\|^2_{H_{K_1+K_2}(E)} \leq \|f\|^2_{H_{K_1}(E)} + \|g\|^2_{H_{K_2}(E)}, \tag{A.22}$$

for all $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$, where equality holds if and only if

$$\langle f_1, h\rangle_{H_{K_1}(E)} = \langle f_2, h\rangle_{H_{K_2}(E)} \tag{A.23}$$

for all $h \in H_{K_1}(E) \cap H_{K_2}(E)$ according to (2.89). Also, the RKHS $H_{K_1K_2}(E)$ is the operator range of the mapping

$$H_{K_1}(E) \otimes H_{K_2}(E) \ni \sum_i f_i \otimes g_i \mapsto \sum_i f_i g_i \in H_{K_1K_2}(E)$$

which is induced by the restriction map from $E \times E$ to its diagonal. Hence, for all $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$

$$\|fg\|_{H_{K_1K_2}(E)} \leq \|f \otimes g\|_{H_{K_1 \otimes K_2}(E)} = \|f\|_{H_{K_1}(E)}\|g\|_{H_{K_2}(E)}, \tag{A.24}$$

where equality holds if and only if $f \otimes g \in (H_{K_1}(E) \otimes H_{K_2}(E))_0^{\perp}$. See Example 2.3 and (2.129). We note that if $\otimes_{j=1}^J \phi_j$ is extremal in $\otimes_{j=1}^J H_j$, then by the basic properties of the range norm and the tensor product, for any $\otimes_{j=1}^J g_j \in \otimes_{j=1}^J H_j$ we have

$$\langle \Pi_{j=1}^J g_j, \Pi_{j=1}^J \phi_j\rangle_{H_{\Pi_{j=1}^J K_j}(E^m)} = \left\langle \otimes_{j=1}^J g_j, \otimes_{j=1}^J \phi_j\right\rangle_{H_{\otimes_{j=1}^J K_j}(E)} = \Pi_{j=1}^J \langle g_j, \phi_j\rangle_{H_{K_j}(E)}$$

from the definition of the operator range. As a special case of what we have obtained we note that

$$\|f\|_{H_{cK}(E)} = \sqrt{c^{-1}}\|f\|_{H_K(E)} \tag{A.25}$$

for any positive constant c and hence

$$\sum_{p_n>0} \|c_n f^n\|^2_{H_{p_nK^n}(E)} = \sum_{p_n>0} \frac{|c_n|^2}{p_n}\|f^n\|^2_{H_{K^n}(E)}.$$

See Corollary 2.5.

Applying (A.22), (A.24) and (A.25), for $f \in H_K(E)$ we have

$$\|\varphi(f)\|^2_{H_{\psi*K}(E)} \leq \sum_{p_n>0} \|c_n f^n\|^2_{H_{p_nK^n}(E)} \leq \sum_{p_n>0} \frac{|c_n|^2}{p_n} \|f\|^{2n}_{H_K(E)}. \tag{A.26}$$

Thus we obtain the inequality (A.5) stated in the beginning of Sect. A.1.1.

Theorem A.2 below asserts that if the RKHS $H_K(E)$ is algebra-dense and maximal, then "usually" its reproducing kernels are, up to constants, the only functions which attain equality in (A.5), and hence the exceptional case does not occur.

Theorem A.2. *Let $H_K(E)$ be an RKHS on E which is R-dense and maximal. Assume that $\varphi(z) = \sum_{n=1}^{\infty} c_n z^n$ and $\psi(z) = \sum_{n=1}^{\infty} p_n z^n$ are entire functions with the properties:*

1. *$p_n \geq 0$ for all n.*
2. *$c_n = 0$ whenever $p_n = 0$.*
3. *At least two different elements c_i and c_j are nonzero.*

Then, equality in inequality (A.5) holds if and only if there exist a point $q \in E$ and constants C, C' such that $\varphi(Cz) = C'\psi(z)$ for all $z \in \mathbb{C}$ and that $f = CK_q$.

Proof. If $\varphi(Cz) = C'\psi(z)$ for all $z \in \mathbb{C}$, then $c_nC^n = C'p_n$ for all $n \geq 1$. Thus $\varphi_\psi(|C|^2 z) = |C'|^2\psi(z)$, where φ_ψ is given by (A.4). Observe also that

$$\|C'\psi(K_q)\|^2_{H_{\psi*K}(E)} = |C'|^2\|\psi(K_q)\|^2_{H_{\psi*K}(E)} = |C'|^2 \sum_n p_n K^n(q,q).$$

Hence we have

$$\|\varphi(CK_q)\|^2_{H_{\psi*K}(E)} = \|C'\psi(K_q)\|^2_{H_{\psi*K}(E)} = |C'|^2\psi(\|K_q\|^2_{H_K(E)}) = \varphi_\psi(\|CK_q\|^2_{H_K(E)}),$$

proving the sufficiency part of Theorem A.2.

To prove the necessity, assume that equality holds in (A.5) for $f \in H_K(E)$. Noting that the case $f = 0$ corresponds to the choice with $C = C' = 0$, we may assume that $f \neq 0$. By the hypotheses above there exist indices i and j with $p_ip_j \neq 0,\ 1 \leq i < j$. From the chain of inequalities (A.26) and since $j \geq 2, f^{\otimes j}$ must be nonzero extremal in $\otimes^j H_K(E)$. Meanwhile, by Corollary A.1 $\otimes^j H_K(E)$ is weakly regular. Thus, there exists a point $q \in E$ such that

$$\Lambda_{f^{\otimes j}}(g) = g(q) \tag{A.27}$$

for all $g \in R$, and that q is either a common zero of $H_K(E)$ or $f = CK_q$ for some nonzero constant C.

Next we disprove that q is a common zero of $H_K(E)$. Since $H_K(E)$ is R-dense and $f \neq 0$, there exists $u \in R \cap H_K(E)$ with $\langle f, u\rangle_{H_K(E)} \neq 0$ according to Remark A.3.

Since $f^{\otimes j}$ is extremal in $H_{K^{\otimes j}}(E)$, we have

$$\langle c_j f^j, u^j\rangle_{H_{p_j K^j}(E)} = \frac{c_j}{p_j}\langle f^j, u^j\rangle_{H_{K^j}(E)} = \frac{c_j}{p_j}\langle f, u\rangle^j_{H_K(E)} \tag{A.28}$$

from (A.25). On the other hand, if $i \geq 2$, $f^{\otimes i}$ is extremal in $\otimes^i H_K(E)$, and so by decomposing u^j as a product $u^{j-i+1} \cdot u^{i-1}$, we have

$$\langle c_i f^i, u^j\rangle_{H_{p_i K^i}(E)} = \frac{c_i}{p_i}\langle f, u^{j-i+1}\rangle_{H_K(E)}\langle f^{i-1}, u^{i-1}\rangle_{H_{K^{i-1}}(E)} \tag{A.29}$$

from (A.27) Obviously, (A.29) also holds for $i = 1$. Since $f^{\otimes j}$ is nonzero extremal, it follows from Lemma A.2

$$\langle u^{j-i+1}, f\rangle_{H_K(E)} = u^{j-i}(q)\langle u, f\rangle_{H_K(E)}. \tag{A.30}$$

Decomposing u^j as above we see that

$$u^j \in H_{p_i K^i}(E) \cap H_{p_j K^j}(E)$$

since $u \in R \cap H_K(E)$ and $H_K(E)$ is R-dense. The equality condition (A.23) implies, for any k, l with $p_k p_l \neq 0$,

$$\langle c_k f^k, u^l\rangle_{H_{p_k K^k}(E)} = \langle c_l f^l, u^l\rangle_{H_{p_l K^l}(E)}. \tag{A.31}$$

Combining (A.28), (A.29), (A.30), and (A.31), we have

$$\overline{u^{j-i}(q)} = \frac{c_j p_i}{c_i p_j}\langle f, u^{j-i}\rangle_{H_K(E)} \neq 0. \tag{A.32}$$

Since $j > i$, the point q is not a common zero of $H_K(E)$, as desired.

Thus, $f = CK_q$ for some constant $C \neq 0$, see Definition A.3. By (A.31) the reproducing property of f yields, for any k, l with $p_k p_l \neq 0$,

$$\frac{c_k C^k u(q)^k}{p_k} = \frac{c_l C^l u(q)^l}{p_l}. \tag{A.33}$$

Putting $C' = c_k C^k u(q)^k / p_k$, we immediately obtain the identity $\varphi(Cz) = C'\psi(z)$.

A.1.3 The Case of Polynomial Ring

As an important example, we first consider the case where E is a subset of the complex n-dimensional space $\mathbb{C}^n$, and R is a restriction to E of the polynomial ring $\mathbb{C}[z_1, z_2, \ldots, z_n]$. A power series with center at the origin is denoted by $\sum_\alpha a_\alpha z^\alpha$.

Definition A.7. Let H be an RKHS on a subset E of $\mathbb{C}^n$. If H is $\mathbb{C}[z_1, z_2, \ldots, z_n]|_E$-*dense*, then H is called *polynomially dense*.

Let $\chi\colon \mathbb{C}[z_1, z_2, \ldots, z_n] \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism with $\chi(1) = 1$. For any polynomial $f(z) = \sum_\alpha a_\alpha z^\alpha$, we have

$$\chi(f) = \sum_\alpha a_\alpha \chi(z)^\alpha = f(w), \tag{A.34}$$

where $w \equiv \chi(z) = (\chi(z_1), \chi(z_2), \ldots, \chi(z_n)) \in \mathbb{C}^n$. Hence we conclude that any nonzero $\mathbb{C}$-algebra homomorphism of $\mathbb{C}[z_1, z_2, \ldots, z_n]|_E$ is a point evaluation at some point of $\mathbb{C}^n$. Thus we have immediately:

Proposition A.1. *Let H be a polynomially dense RKHS on E. Then H is maximal if and only if the following holds: Given a point $q \in \mathbb{C}^n$, if there exists a constant $C > 0$ with $|f(q)| \le C\|f\|$ for all $f \in \mathbb{C}[z_1, z_2, \ldots, z_n] \cap H$, then $q \in E$.*

We next give an example of polynomially dense RKHSs and provide a sufficient condition for these RKHSs to be maximal.

Example A.2 ([68]). For $z, \zeta \in \mathbb{C}^n$ we put

$$z\zeta \equiv (z_1\zeta_1, z_2\zeta_2, \ldots, z_n\zeta_n) \in \mathbb{C}^n.$$

Fix a power series with positive coefficients $\eta(z) = \sum_\alpha c_\alpha z^\alpha$, $(c_\alpha > 0,\ \alpha \in \mathbb{Z}_+^n)$, and assume that the domain D of convergence of the function η is nonempty. A function f holomorphic in the domain D has a power series expansion $f(z) = \sum_\alpha a_\alpha z^\alpha$ on D. Define the norm $\|f\|$ of f by

$$\|f\| \equiv \sqrt{\sum_{\alpha\in\mathbb{Z}_+^n} \frac{|a_\alpha|^2}{c_\alpha}}, \tag{A.35}$$

and let $\mathscr{H}_\eta$ denote the space of holomorphic functions in D with $\|f\| < \infty$. Define an inner product of f and $g \in \mathscr{H}_\eta$ by

$$\langle f, g\rangle_{H_\eta} = \sum_\alpha \frac{a_\alpha \bar{b}_\alpha}{c_\alpha}, \quad g(z) = \sum_\alpha b_\alpha z^\alpha, \tag{A.36}$$

then $\mathscr{H}_\eta$ is a Hilbert space. For $\zeta \in D$ let $k_\zeta(z)$ denote the function $\eta(z\bar{\zeta})$. Then we easily see that $k_\zeta \in \mathscr{H}_\eta$ and that $f(\zeta) = \langle f, k_\zeta\rangle_{H_\eta}$ for all $f \in \mathscr{H}_\eta$. Thus, k_ζ is the reproducing kernel at ζ for the space $\mathscr{H}_\eta$, and hence $\mathscr{H}_\eta$ is an RKHS on D. By definition of the norm, $\mathscr{H}_\eta$ is clearly polynomially dense.

Proposition A.2. *If $\eta(z\bar{z}) = \infty$ for every $z \in \partial D$, then $\mathscr{H}_\eta$ is polynomially dense and maximal.*

Proof. Since $\eta(z\bar{z})$ is a series with nonnegative terms, if $\eta(z\bar{z}) < \infty$, then $\eta(tz\overline{tz}) < \infty$ for all t with $0 < t < 1$. Thus, from the hypothesis it is easy to see that $\eta(z\bar{z}) = \infty$ for every $z \notin D$. For $\zeta \notin D$ and $n \in \mathbb{N}$, let $k_\zeta^{(n)}(z) = \sum_{|\alpha|\leq n} c_\alpha \bar{\zeta}^\alpha z^\alpha \in \mathscr{H}_\eta$ be the n-th partial sum of $k_\zeta(z)$. Then

$$\frac{|k_\zeta^{(n)}(\zeta)|}{\|k_\zeta^{(n)}\|} = \sqrt{k_\zeta^{(n)}(\zeta)} \to \sqrt{\eta(\zeta\bar{\zeta})} = \infty \ (n \to \infty). \tag{A.37}$$

Thus, the point evaluation at $\zeta \notin D$ is not $\mathscr{H}_\eta$-bounded. In view of Proposition A.1 this implies that $\mathscr{H}_\eta$ is maximal.

Remark A.4. All the theorems in [68, Sections 5,6] are immediate consequences of our Theorem A.2 and Proposition A.2.

A.1.4 Algebra of Meromorphic Functions

Throughout Sect. A.1.4, let E be a regular subregion of a compact Riemann surface S. Here a proper subregion E of S is called *regular* if E and its exterior have the same boundary consisting of a finite number of analytic Jordan curves. Let $\mathscr{R}_E$ denote the complex algebra of meromorphic functions on S which are holomorphic on $\overline{E}$.

Definition A.8. An RKHS H on E is called *meromorphically dense* if H is $\mathscr{R}_E$-dense.

Lemma A.3. *For any $f \in \mathscr{R}_E$, $\chi(f) \in f(\overline{E})$.*

Proof. If $f = 0$, then this is obvious. If f never vanishes on $\overline{E}$, then $1/f \in \mathscr{R}_E$, and from the identity $\chi(f)\chi(1/f) = \chi(1) = 1$ we have $\chi(f) \neq 0$. Therefore, $\chi(f - \chi(f)) = 0$ implies that $f - \chi(f)$ vanishes for some point in $\overline{E}$. Thus we have $\chi(f) \in f(\overline{E})$.

Let $P_1, P_2, \ldots, P_m$ and $Q_1, Q_2, \ldots, Q_k$ be finite collections of points in S. Suppose that we are given positive integers $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_k$. Denote by g the genus of S. Assume that

$$1 - g - \sum_{j=1}^{m} p_j + \sum_{j=1}^{k} q_j \geq 1.$$

According to the Riemann Roch theorem, there exists a (nonconstant) meromorphic function f such that f vanishes P_j with order p_j for each $j = 1, 2, \ldots, m$ and that f has pole Q_j with order q_j for each $j = 1, 2, \ldots, k$. See [145] for details, which we admit in this book.

Lemma A.4. *Let $P \in S$ and $Q_1, Q_2, \ldots, Q_k \in S \setminus \{P\}$. Then there exists a function F such that $F(P) = 1$ and that $F(Q_1) = F(Q_2) = \cdots = F(Q_k) = 0$.*

Proof. Let $j = 1, 2, \ldots, k$. By the Riemann Roch theorem, we can find a function G_j such that G_j has a sole pole at Q_j and that G_j never vanishes at $\{Q_1, Q_2, \ldots, Q_k\} \setminus \{Q_j\}$. Define $H \equiv G_1 G_2 \cdots G_k - G_1(P) G_2(P) \cdots G_k(P) + 1$. Then $1/H = F$ has the desired result.

Lemma A.5. *Let $E \subset S$ be a compact set and $Q_1, Q_2, \ldots, Q_k \in S \setminus E$. Assume that we are given positive integers $q_1, q_2, \ldots, q_k$ such that*

$$\sum_{j=1}^{k} q_j > g,$$

where g denotes the genus of S. Then we can find a meromorphic function F such that $F^{-1}(0) = \{Q_1, Q_2, \ldots, Q_k\}$.

Proof. In fact, by the Riemann Roch theorem, we can find a function F_j such that F_j has a pole only at Q_j. If necessary, by adding a constant we can suppose that F_j never vanishes on $\{Q_1, Q_2, \ldots, Q_k\} \setminus \{Q_j\}$. Let $G \equiv F_1 \cdot F_2 \cdots F_k$. Then, the set of the poles of G is $\{Q_1, Q_2, \ldots, Q_k\}$. Since E is compact, we can find M such that $|G(z)| \leq M$ for all $z \in E$. Thus, if we set $F \equiv \dfrac{1}{G + 3M}$, then we have the desired function.

We prepare the following proposition which is useful for testing the maximality of meromorphically dense RKHSs.

Proposition A.3. *Let $\chi : \mathscr{R}_E \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism with $\chi(1) = 1$. Then there exists a unique point $q \in \overline{E}$ such that $\chi(f) = f(q)$ for all $f \in \mathscr{R}_E$.*

Proof. Choose any $f \in \mathscr{R}_E \setminus \mathbb{C}$ and set $\zeta = \chi(f)$. Let $f^{-1}(\zeta)$ consist of distinct points $q_1, q_2, \ldots, q_r$ $(1 \leq r \leq n)$, where n is the degree of the meromorphic function f on the compact Riemann surface S. From the above remark, $f^{-1}(\zeta) \cap \overline{E} \neq \phi$. Let us choose a point $\zeta_0 \in \mathbb{C}$ such that $\zeta_0 \neq \zeta$ and $f^{-1}(\zeta_0)$ consists of n distinct points. Applying Lemmas A.4 and A.5, one verifies easily that there exists $g_0 \in \mathscr{R}_E$ with the following properties:

1. g_0 takes different values at different points of $f^{-1}(\zeta_0)$.
2. $g_0(p) \neq 0$ for any $p \in \overline{E}$.
3. $g_0(q_j) = 0$ for all $q_j \notin \overline{E}$.

Indeed, by invoking Lemma A.5, we can construct a function g_1 such that $g_0(p) \neq 0$ for any $p \in \overline{E}$ and that $g_0(q_j) = 0$ for all $q_j \notin \overline{E}$. By using g_1 and Lemma A.4, we can find a function g_2 such that g_0 takes different values at different points of $f^{-1}(\zeta_0)$

and that $g_0(q_j) = 0$ for all $q_j \notin \overline{E}$. Since E is compact, if we choose M large enough, we see that $g_0 = g_2 + M$ is the desired function.

For sufficiently small $\varepsilon \neq 0$, the function $g \equiv 1/(g_0 + \varepsilon)$ is an element of $\mathscr{R}_E$ holomorphic at all points q_j $(j = 1, 2, \dots, r)$ and satisfies the inequality

$$\sup_{z \in \overline{E}} |g(z)| < |g(q_j)| \quad \text{for any } q_j \notin \overline{E}. \tag{A.38}$$

By the general theory of compact Riemann surfaces [145], there exist rational functions a_k $(k = 0, \dots, n)$ such that

$$\sum_{k=0}^{n} a_k(f) g^k = 0, \quad (a_n(z) = 1). \tag{A.39}$$

Since g is holomorphic at all points $q_1, q_2, \dots, q_r$, it is clear from well-known constructions of (A.39) that the rational functions $a_k(z)$ $(k = 0, 1, 2, \dots, n)$ are holomorphic at $z = \zeta$. Thus, we may assume that a_k is of the form b_k/c_k such that $b_k, c_k \in \mathbb{C}[z]$ and $c_k(\zeta) \neq 0$. Put $s = \prod_{k=0}^{n} c_k$ and multiply the identity (A.39) by $s(f)$. Since $g \in \mathscr{R}_E$ and each $sa_k (k = 0, \dots, n)$ is a polynomial, we can apply the homomorphism χ to the resulting identity. Thus

$$\sum_{k=0}^{n} sa_k(\zeta) \chi(g)^k = 0. \tag{A.40}$$

Since $sa_k(\zeta) = s(\zeta) a_k(\zeta)$ with $s(\zeta) \neq 0$, we have $\sum_{k=0}^{n} a_k(\zeta) \chi(g)^k = 0$. Hence by construction of (A.39), $\chi(g) = g(q_j)$ for some j $(1 \leq j \leq r)$. Therefore, we have proved that there exists a point $q = q_j$ such that $\chi(f) = f(q)$ and $\chi(g) = g(q)$. From (A.38) we conclude that $q \in \overline{E}$, since $\chi(g) \in g(\overline{E})$ and the algebra of functions $\mathscr{R}_E$ separates points of S.

We now show that the homomorphism χ is the point evaluation at q. By the property (i) above, the meromorphic functions f and g form a primitive pair [145, p. 233]. Thus, it is well known that for any meromorphic function h on S, there exist rational functions A_k $(k = 0, \dots, n-1)$ such that

$$h = \sum_{k=0}^{n-1} A_k(f) g^k. \tag{A.41}$$

First, consider the case where $h \in \mathscr{R}_E$ is holomorphic at each point of the set $f^{-1}(\zeta)$. Then, from well-known constructions of (A.41), every coefficient A_k is holomorphic at ζ. Again, by multiplying suitable polynomial as above, we obtain

$$\chi(h) = \sum_{k=0}^{n-1} A_k(\zeta) g(q)^k = h(q). \tag{A.42}$$

Second, for general $h \in \mathscr{R}_E$, choose a function $h_2 \in \mathscr{R}_E$ such that $h_2(q) \neq 0$ and that the product $h_2 h$ is holomorphic at each point of $f^{-1}(\zeta)$. This is possible if h_2 has sufficiently large order of zeros at every $q_j \notin \overline{E}$. From

$$h_2(q)h(q) = \chi(h_2 h) = \chi(h_2)\chi(h) = h_2(q)\chi(h), \tag{A.43}$$

we conclude that $\chi(h) = h(q)$ for all $h \in \mathscr{R}_E$, as desired. The uniqueness of the point q is obvious, since the algebra of functions $\mathscr{R}_E$ separates points of S.

From Definition A.6 and Proposition A.3, we immediately have

Proposition A.4. *Let H be a meromorphically dense RKHS on E. Then H is maximal if and only if the following holds: Given a point $q \in \overline{E}$, if there exists a constant $C > 0$ with $|f(q)| \leq C\|f\|_H$ for all $f \in \mathscr{R}_E \cap H$, then $q \in E$.*

A.1.5 Applications

As an application of the results obtained in the previous sections we study the regularity of tensor products of RKHSs consisting of analytic functions or analytic differentials on a compact bordered Riemann surface. Let E be the interior of a compact bordered Riemann surface $\overline{E} = E \cup \mathbf{E}$ with nonempty boundary $\mathbf{E}$. Let $\hat{E}$ be the Schottky double of $\overline{E}$ [425]. Then E can be viewed as a regular subregion of the compact Riemann surface $\hat{E}$. Define the $\mathbb{C}$-algebra $\mathscr{R}_E$ in this context. Consider the following RKHSs on E:

1. $\mathscr{H}_1(E, \rho)$: (Weighted Szegö space) Let ρ be a positive measurable function on E. The Hardy H^2 space of analytic functions f on E with norm

$$\|f\|_{\mathscr{H}_1(E,\rho)} \equiv \sqrt{\int_{\mathbf{E}} |f(z)|^2 \rho(z)|dz|} \tag{A.44}$$

   where $\rho|dz|$ is a positive continuous metric on $\mathbf{E}$. In the integrand f denotes the nontangential boundary value of f on $\mathbf{E}$.
2. $\mathscr{H}_2(E, \rho)$: (Weighted Dirichlet space) The space of analytic functions f on E with finite Dirichlet norm

$$\|f\|_{\mathscr{H}_2(E,\rho)} \equiv \sqrt{\frac{i}{2\pi} \iint_E \rho(z)\, df(z) \wedge \overline{df}(z)} \tag{A.45}$$

   satisfying $f(a) = 0$ for a fixed point $a \in E$, where ρ is a positive continuous function on $\overline{E}$.
3. $\mathscr{H}_3(E, \rho)$: (Weighted Bergman space) The *weighted Bergman space of analytic differentials* f on E with norm

$$\|f\|_{\mathscr{H}_3(E,\rho)} \equiv \sqrt{\frac{i}{2\pi}\iint_E \rho(z) f(z) \wedge \bar{f(z)}}, \tag{A.46}$$

where ρ is a positive continuous function on $\overline{E}$.

Theorem A.3. *The following hold:*

1. *The RKHSs* $\mathscr{H}_j(E,\rho)$ $(j = 1,2,3)$ *are meromorphically dense and maximal.*
2. *For any integer* $n \geq 2$, $\mathscr{H}_j(E,\rho)^{\otimes n}$ $(j \neq 2)$ *is regular and* $\mathscr{H}_2(E,\rho)^{\otimes n}$ *is weakly regular.*
3. *Let* $a = 0$. *Then* $\phi^{\otimes 2}$ *is extremal in* $\mathscr{H}_2(\Delta(1),1)^{\otimes 2}$ *if and only if* $\phi(z) = cz$ *or* $\phi = ck_q$ $(q \in \Delta(1) \setminus \{0\})$ *for some* $c \in \mathbb{C}$, *where*

$$k_q(z) \equiv -\log(1-\bar{q}z)$$

is the reproducing kernel of $\mathscr{H}_2(\Delta(1),1)$ *at* q. *Thus,* $\mathscr{H}_2(\Delta(1),1)^{\otimes 2}$ *is not regular.*

For the proof of Theorem A.3 above we need a weaker form of the result in [424, Theorem 8] on uniform approximation.

Proposition A.5 (S. Scheinberg). *Let* E *be a regular subregion of a compact Riemann surface* S. *For every holomorphic function* f *on* $\overline{E}$ *and for every positive constant* ε, *there exists a function* $g \in \mathscr{R}_E$ *such that* $\|f-g\|_\infty < \varepsilon$ *on* $\overline{E}$, *where* $\|\cdot\|_\infty$ *denotes the sup-norm on* $\overline{E}$.

Proof (of Theorem A.3).

1. First, we remark that the norm of the RKHS $\mathscr{H}_j(E,\rho)$ is equivalent for each weight ρ and, as a set, the space $\mathscr{H}_j(E,\rho)$ is independent of ρ, since ρ is positive and continuous on compact set $\overline{E}$. Thus we may assume $\rho = 1$ without loss of generality. Let $\mathscr{H}_j(E) = \mathscr{H}_j(E,1)$ for simplicity.

   Since $1 \in \mathscr{R}_E$, to prove that $\mathscr{H}_j(E)$ is meromorphically dense, it suffices to show that $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is dense in $\mathscr{H}_j(E)$ and is an ideal of $\mathscr{R}_E$. To establish that the space $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is dense in $\mathscr{H}_j(E)$ $(j = 1,2,3)$, we recall that the Szegö kernels, the exact Bergman kernels and the Bergman kernels are analytically continued to a neighborhood of $\overline{E}$ [388, 425]. Since the linear span of the reproducing kernels is dense, the set of functions holomorphic on $\overline{E}$ are dense in $\mathscr{H}_j(E)$, and by Proposition A.5 we see easily that $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is dense in $\mathscr{H}_j(E)$ $(j = 1,2,3)$. On the other hand, it is clear that $\mathscr{R}_E \cap \mathscr{H}_j(E)$ is an ideal of $\mathscr{R}_E$. Thus, $\mathscr{H}_j(E)$ $(j = 1,2,3)$ is meromorphically dense.

   Next, we show that all the $\mathscr{H}_j(E)$ $(j = 1,2,3)$ are maximal. In view of Proposition A.4 it suffices to show that, for fixed $b \in \mathbf{E}$, there exists a family of functions $\{f_p\}$, $f_p \in \mathscr{R}_E \cap \mathscr{H}_j(E)$ such that $|f_p(b)|/\|f_p\|$ tends to ∞ as $p \to b$. Consider the case of the Dirichlet space $\mathscr{H}_2(E)$. The proof of the other cases is similar but easier. Now we claim that we need only to show that there exists a family of functions $\{f_p\}$ holomorphic on $\overline{E}$ with the above property. This is seen as follows. Given a holomorphic function f on $\overline{E}$, there exists a regular subregion E_1 with $\overline{E} \subset E_1$ such that f is holomorphic on $\overline{E_1}$. Applying Cauchy's integral formula we see that there exists a constant $C > 0$ with $\|f\| \leq C\|f\|_{\infty,E_1}$ where $\|f\|_{\infty,E_1}$ denotes $\sup_{x\in E_1} |f(x)|$. From Proposition A.5 there exists $g \in \mathscr{R}_{E_1}$ such that $\|f - g\|_{\infty,E_1} \leq \varepsilon\|f\|$. Then

$$|g(b)| \geq |f(b)| - \|f - g\|_{\infty,E_1} \geq |f(b)| - \varepsilon\|f\|,$$
$$\|g\| \leq \|f\| + \|f - g\| \leq \|f\| + C\|f - g\|_{\infty,E_1} \leq (1 + \varepsilon C)\|f\|.$$

By choosing $\varepsilon(0, \min\{1, C^{-1}\})$, we have

$$\frac{|g(b)|}{\|g\|} \geq \frac{|f(b)|}{(1 + \varepsilon C)\|f\|} - \varepsilon > \frac{|f(b)|}{2\|f\|} - 1, \tag{A.47}$$

which implies our claim, as desired.
By definition we have the identity

$$k_B(x, y) = \frac{\partial^2 k_D}{\partial x \partial \bar{y}}(x, y), \tag{A.48}$$

where $k_D(x, y)$ is the reproducing kernel for the Dirichlet space $\mathscr{H}_2(E)$ and $k_B(x, y)$ is the exact Bergman kernel for E. Let ϕ be the canonical anticonformal involution for the double $\hat{E}$ fixing $\mathbf{E}$. It is well known [425, p. 118] that the exact Bergman kernel $k_B(x, y)$ is extended to a meromorphic bilinear differential on $\hat{E}$ with double pole only at $x = \phi(y)$, and that $k_B(x, y)$ has the expansion

$$k_B(x, y) = -\frac{1}{\pi(x - \phi(y))^2} + \text{regular terms} \tag{A.49}$$

for x, y in a coordinate neighborhood U centered at $b \in \mathbf{E}$. Integrating (A.49), we see that for $p \in E$ the Dirichlet kernel $k_D(x, p)$ is extended holomorphically to a neighborhood of $\overline{E}$ with the expansion

$$k_D(x, y) = \frac{1}{\pi} \log \frac{1}{x - \phi(y)} + \text{regular terms} \tag{A.50}$$

for $x, y \in U \cap E$. Setting $f_p(x) = k_D(x, p)$ for $p \in E$ near b, we have

$$\frac{|f_p(b)|}{\|f_p\|} = \frac{|k_D(b, p)|}{\sqrt{k_D(p, p)}}.$$

By (A.50) this tends to ∞ as $p \to b \in \mathbf{E}$ nontangentially. Thus $\mathscr{H}_2(E)$ is maximal.

2. By Corollary A.1 and (A.44) all the tensor products of these spaces are weakly regular. Moreover, since both $\mathscr{H}_1(E,\rho)$ and $\mathscr{H}_3(E,\rho)$ have no common zeros, the products $\mathscr{H}_j(E,\rho)^{\otimes n}$ $(j = 1, 3;\ n \geq 2)$ are regular.
3. By definition, if $\phi = ck_q$, it is clear that $\phi^{\otimes 2}$ is extremal. We will show that $z^{\otimes 2} \in (\mathscr{H}_2(\Delta(1),1)^{\otimes 2})_0^{\perp}$. Since the set of functions $\{z^i/\sqrt{i}\}_{i=1}^{\infty}$ is a complete orthonormal system (CONS) for $\mathscr{H}_2(\Delta(1),1)$, the set $\{z^i \otimes z^j/\sqrt{ij}\}_{i,j=1}^{\infty}$ is a CONS for the tensor product $\mathscr{H}_2(\Delta(1),1)^{\otimes 2}$. Hence, any $f \in \mathscr{H}_2(\Delta(1),1)^{\otimes 2}$ is given by

$$f = \sum_{i,j=1}^{\infty} \frac{c_{ij}}{\sqrt{ij}} z^i \otimes z^j, \quad \text{with} \quad \sum_{i,j=1}^{\infty} |c_{ij}|^2 < \infty. \tag{A.51}$$

Then we see that $f \in (\mathscr{H}_2(\Delta(1),1)^{\otimes 2})_0$ if and only if $\sum_{i+j=n} c_{ij}/\sqrt{ij} = 0$ for all $n \geq 2$. In particular, $f \in (\mathscr{H}_2(\Delta(1),1)^{\otimes 2})_0$ implies $c_{11} = 0$. Since $\langle f, z^{\otimes 2}\rangle_{H_\eta} = c_{11}$, $z^{\otimes 2} \in (\mathscr{H}_2(\Delta(1),1)^{\otimes 2})_0^{\perp}$, that is, $z^{\otimes 2}$ is extremal.

Contrarily, suppose that $\phi^{\otimes 2} \in (\mathscr{H}_2(\Delta(1),1)^{\otimes 2})_0^{\perp}$. We may assume without loss of generality $\phi \neq 0$. From Lemma A.2 there exists a point $q \in \Delta(1)$ such that

$$\langle zf, \phi\rangle_{H_\eta} = f(q)\langle z, \phi\rangle_{H_\eta}. \tag{A.52}$$

If $q \neq 0$, then q is not a common zero of $\mathscr{H}_2(\Delta(1),1)$. Since $\mathscr{H}_2(\Delta(1),1)^{\otimes 2}$ is weakly regular, $\phi = ck_q$ for some constant $c \in \mathbb{C}$. On the other hand, if $q = 0$, then by (A.52) $\phi \perp z^n$ for all $n \geq 2$. Therefore, ϕ is a constant multiple of the function z. Thus the first assertion of (A.46) is proved. Finally observe that $\mathscr{H}_2(\Delta(1),1)^{\otimes 2}$ is not regular, since the function z cannot be a constant multiple of the reproducing kernel k_q for any $q \in \Delta(1)$.

A.2 Generalizations of Opial's Inequality

In 1995, natural norm inequalities hold for a wide class of nonlinear maps between reproducing kernel Hilbert spaces in [386, 388]. This can be shown also by the theory of reproducing kernels. The method has proved to be very important for applications such as identifications of nonlinear systems [490]. Among several concrete inequalities, the most beautiful one is the following: For a real-valued function $f \in \mathrm{AC}[0,1]$ with $f(0) = 0$ and $\int_0^1 f'(x)^2 dx < 1$, where $\mathrm{AC}[0,1]$ is the set of all absolutely continuous functions on $[0,1]$, we have

$$\int_0^1 \left(\frac{f(x)}{1-f(x)}\right)'^2 (1-x)^2 dx \leq \frac{\int_0^1 f'^2(x)\,dx}{1-\int_0^1 f'^2(x)\,dx}. \tag{A.53}$$

Equality holds in (A.53), if f is of form $f(x) = \min(x,y)$, $x \in [0,1]$ for some $y \in [0,1)$. This is stated in [389, (22)]. But the equality condition was not completely determined.

Meanwhile, the following Opial's inequality [349] is famous and there are many papers extending it: For $f \in \mathrm{AC}[0,a]$ with $f(0) = 0$, we have

$$\int_0^a |f(x)f'(x)|\,dx \leq \frac{a}{2}\int_0^a |f'(x)|^2\,dx. \tag{A.54}$$

For Opial-type inequalities see, e.g., [38, 39, 307].

A function $f(x)$ positive and continuous on an interval $(0,R)$ is called *geometrically convex* if f satisfies the inequality, for all $x, y \in (0,R)$

$$f(\sqrt{xy}) \leq \sqrt{f(x)f(y)}, \tag{A.55}$$

that is, $\log \circ f \circ \exp$ is convex on $(-\infty, \log R)$ [290].

Following Akira Yamada [486], we will introduce a generalization of the inequality (A.53) with an elementary and direct proof, and for the generalization, we will be able to generalize Opial's inequality (Theorem A.6), by using geometrically convex functions (Theorem A.4): The tool is Hölder's inequality, and so the proof is elementary.

Very interestingly, similar, but many different generalizations of the results obtained by the theory of reproducing kernels were independently published by Nguyen Du Vi Nhan, Dinh Thanh Duc, and Vu Kim Tuan, in the same year, in [329].

A.2.1 Main Inequality

We now state the main theorem. Recall that $\mathrm{AC}[a,b]$ denotes the set of all absolutely continuous function on $[a,b]$.

Theorem A.4. *Let $p > 1$ and G be a function of class $C^1(-R,R)$ satisfying the conditions*

$$G(0) = 0, \tag{A.56}$$

$$|G'(x)| \leq G'(|x|) \tag{A.57}$$

for all $x \in (-R,R)$ and

$$0 < G'(x)^2 \leq G'(y)G'(z) \tag{A.58}$$

for all $x, y, z \in (0,R)$ satisfying $x^2 \leq yz$. Assume that functions F, $f \in \mathrm{AC}[a,b]$ with $F(a) = f(a) = 0$ satisfy

$$F'(x) > 0, F(b) \leq R, \tag{A.59}$$

for almost all $x \in [a, b]$ *and*

$$\int_a^b |f'(t)|^p F'(t)^{1-p}\, dt < R. \tag{A.60}$$

Then,

$$\int_a^b \frac{|(G \circ f)'(x)|^p}{(G \circ F)'(x)^{p-1}}\, dx \le G\Big(\int_a^b \frac{|f'(x)|^p}{F'(x)^{p-1}}\, dx\Big). \tag{A.61}$$

If $f(x) = F(\min(x, y))$ *for some* $y \in (a, b]$*, then equality holds in* (A.61).

Before the proof, a helpful remark may be in order.

Remark A.5. Under the assumptions above, one can obtain $|f(x)| < R$. In fact, $f(a) = 0$. Thus, applying Hölder's inequality with conjugate exponent $1/p + 1/q = 1$ to the identity $f(x) = \int_a^x f'(t)\, dt$, we have

$$\begin{aligned} |f(x)| &\le \sqrt[q]{F(x) - F(a)}\, \sqrt[p]{\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt} \\ &= \sqrt[q]{F(x)}\, \sqrt[p]{\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt}. \end{aligned} \tag{A.62}$$

Thus $|f(x)| < R$ from (A.60).

Proof. Since G' is continuous, we remark that from (A.57) and (A.58), the function $G'(x)$ is positive, monotone increasing and geometrically convex on the interval $(0, R)$. Hence we see that for $0 \le x, y, z < R$,

$$x \le \sqrt[p]{y}\sqrt[q]{z} \implies G'(x) \le \sqrt[p]{G'(y)}\,\sqrt[q]{G'(z)}. \tag{A.63}$$

Hence, by (A.63) and (A.62), we obtain for $a \le x < b$

$$G'(|f(x)|) \le \sqrt[q]{G'(F(x))}\, \sqrt[p]{G'\left(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt\right)}.$$

Multiplying $|f'(x)|^p/(G \circ F)'(x)^{p-1}$ (≥ 0) to the p-th power of the above inequality, we have

$$\frac{|(G \circ f)'(x)|^p}{(G \circ F)'(x)^{p-1}} \le \frac{d}{dx}\Big\{G\Big(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt\Big)\Big\} = \frac{|f'(x)|^p}{F'(x)^{p-1}} G'\Big(\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\, dt\Big). \tag{A.64}$$

Integrating both sides of (A.64) on the interval $[a, b]$, we obtain the desired inequality (A.61) in view of (A.62). When $f(x)$ is of the form $F(\min(x, y))$, $y \in (a, b]$, it is obvious that equality holds since $F \geq 0$.

Remark A.6. As a special case of $G(x) \equiv x/(1-x)$, $F(x) \equiv x$, $p = 2$, $a = 0$ and $b = R = 1$ in Theorem A.4, (A.53) yields the prototype inequality. Note that $G'(x) = (1-x)^{-2}$ is geometrically convex on $(0, 1)$. Equality holds if and only if $f(x) = \min(x, y)$, $y \in [0, 1)$, which is seen from Theorem A.5 in the next section.

Remark A.7. Let H be a real vector space of functions $f \in \mathrm{AC}[a, b]$ with $f(a) = 0$. Then H becomes a reproducing kernel Hilbert space if we assign $f \in H$ the norm

$$\sqrt{\int_a^b |f'(t)|^2 \rho(t)\, dt}, \tag{A.65}$$

where the weight ρ is a positive continuous function on $[a, b]$. The reproducing kernel k of H is given by

$$k(x, y) = F(\min(x, y)) \tag{A.66}$$

with $F(x) = \displaystyle\int_a^x \rho(t)^{-1} dt$. Since the kernel determines its reproducing kernel Hilbert space uniquely, denote this RKHS by $H_k(E)$ and the above norm by $\|f\|_k$. Then, for $p = 2$ we can rewrite formally the inequality (A.61) of Theorem A.4 as

$$\|G \circ f\|_{G(k)}^2 \leq G(\|f\|_k^2) \quad \text{for any } f \in H_k[a, b]. \tag{A.67}$$

Thus, it may seem that the inequality (A.61) is merely an example of general norm inequalities for RKHSs. This, however, is not the case, since the inequality does not require the real analyticity of the function G, while we need to assume this in general norm inequalities for RKHSs.

A.2.2 Equality Condition

For most cases, equality in the Main Inequality is attained only for functions stated in Theorem A.4, i.e., the function 0 or $F(\min(x, y))$. We are able to see this by adding further assumptions on the function $G(x)$.

Theorem A.5. *Under the same hypothesis as in Theorem A.4, assume, moreover, that G' is strictly monotone increasing on* $(0, R)$;

$$G'(t) > G'(s) \quad 0 < s < t < R. \tag{A.68}$$

Then, if equality holds in inequality (A.61)*, then there exist constants C and y* ($a < y \le b$) *such that*

$$f(x) = C \cdot F(\min(x, y)). \tag{A.69}$$

If, in addition, we assume that

1. *for some* $x \in (0, F(y))$*,*

$$|G'(-x)| \ne G'(x) \tag{A.70}$$

   and that
2. *there exist no constants* $\alpha > 0$ *and* $\beta \ge 0$ *such that*

$$G'(x) = \alpha x^{\beta} \tag{A.71}$$

   on $(0, F(y))$*,*

then equality holds in inequality (A.61) *if and only if f is of the form* (A.69) *with* $C \in \{0, 1\}$.

Proof. First, we remark that equality occurs on the right-hand side inequality of (A.63) if and only if $x = \sqrt[p]{y}\sqrt[q]{z}$. For, since G' is strictly monotone increasing, we have

$$G'(x) \le G'(\sqrt[p]{y}\sqrt[q]{z}) \le \sqrt[p]{G'(y)}\sqrt[q]{G'(z)} = G'(x), \tag{A.72}$$

which implies $x = \sqrt[p]{y}\sqrt[q]{z}$.

When $f = 0$ it suffices to take $C = 0$. Hence, we may assume $f \ne 0$. Putting $y = \operatorname{ess\,sup}\{x{:}f'(x+0) \ne 0,\ a \le x < b\}$, we have $a < y \le b$. If equality holds in (A.61), then y must be a cluster point of the set $\{x;\ f'(x) \ne 0\}$. Hence, by continuity

$$G'(|f(y)|) = \sqrt[q]{G'\big(F(y)\big)}\sqrt[p]{G'\Big(\int_a^y \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt\Big)}, \tag{A.73}$$

and from the remark above we obtain

$$|f(y)| = \sqrt[q]{F(y)}\sqrt[p]{\int_a^y \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt}. \tag{A.74}$$

From the equality condition of Hölder's inequality, there exists a constant $C \ne 0$ such that $f(x) = CF(x),\ (a \le x \le y)$. Since $f'(x) = 0(y \le x \le b$ almost every) by definition of y, we conclude that f is of the form

$$f(x) = C \cdot F(\min(x, y)). \tag{A.75}$$

Thus we obtain (A.69). Now we prove the latter half of the assertion of Theorem A.4. For all $x \in [a, y]$ we have

$$\int_a^x \frac{|f'(t)|^p}{F'(t)^{p-1}}\,dt = |C|^p F(x), \tag{A.76}$$

and hence, the following identities must hold simultaneously: for all $x,\ a \le x \le y$,

$$|C|F(x) = \sqrt[q]{F(x)}\,\sqrt[p]{|C|^p F(x)},$$
$$G'(|C|F(x)) = \sqrt[q]{G'(F(x))}\,\sqrt[p]{G'(|C|^p F(x))}.$$

If $|C| \neq 1$, then $F(x) \neq |C|^p F(x)$ for $x > a$. Since the equality condition for Jensen's inequality for two distinct points imply linearity of the function on the interval between these points, one verifies easily that $G'(x)$ is of the form αx^β, $(\alpha > 0,\ \beta \ge 0)$ on the interval $(0, F(y))$. But we can exclude this by our assumption (A.71). Finally, if $C = -1$, then we must have equality, which contradicts the condition (A.70).

Remark A.8. If $G(x) \equiv \alpha|x|^\beta$ $(\alpha > 0,\ \beta > 1)$, then equality holds in inequality (A.61) for every $f(x)$ of the form $C \cdot F(\min(x, y))$ $(C \in \mathbb{R}, a < y \le b)$.

A.2.3 Applications

The main inequality allows us immediately to derive Opial-type inequalities [38, 39, 75, 307]. For brevity we restrict ourselves to the case that the constant R in Theorem A.4 is infinity.

Theorem A.6. *Let $p > 1$ and $q, r > 0$ satisfy $1/p + 1/r = 1/q$. Let s and t be nonnegative, measurable functions on $[a, b]$ such that*

$$\int_a^b \frac{1}{\sqrt[p-1]{t(x)}}\,dx < +\infty \tag{A.77}$$

for some $p > 1$. Set

$$F(x) \equiv \int_a^x \frac{1}{\sqrt[p-1]{t(y)}}\,dy \quad x \in [a, b]$$

and assume that the functions G, F and f satisfy the same conditions as Theorem A.4 with $R = +\infty$. We assume

$$K \equiv \sqrt[r]{\int_a^b \sqrt[p]{(G \circ F)'(x)^{r(p-1)}}\,\sqrt[q]{s(x)^r}\,dx} < \infty. \tag{A.78}$$

Then we have

$$\sqrt[q]{\int_a^b |(G\circ f)'(x)|^q s(x)\,dx} \le K\cdot \sqrt[p]{G\Big(\int_a^b |f'(x)|^p t(x)\,dx\Big)}. \tag{A.79}$$

If, in addition, we assume the conditions (A.68), (A.70) *and* (A.71) *in Theorem A.5, then equality holds in the inequality* (A.79) *if and only if either* $f = 0$ *or there exist constants* $C \in [0,\infty)$ *and* $y \in (a,b]$ *such that* $f(x) \equiv F(\min(x,y))$ *and that* $s(x) \equiv C\cdot (G\circ F)'(x)^{1-q}$.

Proof. Rewrite the integrand on the left-hand side of (A.79) as

$$|(G\circ f)'|^q\cdot s = \frac{|(G\circ f)'|^q}{(G\circ F)'^{\alpha}}\cdot (G\circ F)'^{\alpha} s, \quad \alpha \equiv \frac{q(p-1)}{p}, \tag{A.80}$$

use Hölder's inequality with conjugate exponents p/q and r/q, and then apply Theorem A.4. We obtain equality condition immediately from Theorem A.5.

Remark A.9. The existence of the multiplicative constant K is a merit of the inequality (A.79). See [75, 188].

Remark A.10. Let $-\infty < a < b < \infty$. Let $G(x) \equiv |x|^{p/q}$, $p > q$, $k > 1$ and $k > q > 0$. If

$$\int_a^b \frac{1}{\sqrt[k-1]{t(x)}}dx < +\infty,$$

then Theorem A.6 yields the Opial-type inequality

$$\int_a^b |f(x)|^{p-q}|f'(x)|^q s(x)\,dx \le K\cdot \sqrt[k]{\Big\{\int_a^b |f'(x)|^k t(x)\,dx\Big\}^p}, \tag{A.81}$$

where the constant

$$K \equiv \sqrt[k]{\Big(\frac{q}{p}\Big)^q\Big\{\int_a^b \sqrt[k-q]{\frac{s(x)^k}{t(x)^q}}\Big(\int_a^x \frac{dy}{\sqrt[k-1]{t(y)}}\Big)^{\frac{(p-q)(k-1)}{k-q}}dx\Big\}^{k-q}} \tag{A.82}$$

is finite. Note that this is exactly the same inequality as the equality condition that can be obtained easily as above. Remark that Beesack and Das employed a fundamental technique to prove (A.82) in [39].

A.3 Explicit Integral Representations of Implicit Functions

Here, we will refer to the explicit integral representation of implicit functions derived from Implicit Function Theory that may be considered as one of the master pieces in mathematical analysis.

A.3.1 2-Dimensional Case

Auxiliary Material and Framework

We first state in a precise way the representations of inverses of nonlinear mappings for the 2-dimensional and n dimensional spaces cases.

Let $D \subset \mathbb{R}^2$ be a bounded domain with a finite number of piecewise C^1-class boundary components. Let f be one-to-one C^1-class mapping from $\overline{D}$ into $\mathbb{R}^2$ and we assume that its Jacobian $J(x)$ is positive on D. We represent f in the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \tag{A.83}$$

and the inverse mapping f^{-1} of f as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (f^{-1})_1(y) \\ (f^{-1})_2(y) \end{pmatrix} = \begin{pmatrix} f_1^{-1}(y_1, y_2) \\ f_2^{-1}(y_1, y_2) \end{pmatrix}, \tag{A.84}$$

where $y = (y_1, y_2)$ and f_1^{-1} and f_2^{-1} abbreviate $(f^{-1})_1$ and $(f^{-1})_2$, respectively. Then, we would like to represent ${}^t((f^{-1})_1(y), (f^{-1})_2(y))$ in terms of the direct mapping (A.83).

Additionally, we are also interested in some numerical and practical solutions of the nonlinear simultaneous equations (A.83).

We recall Theorem 8.10. For the mappings (A.83) and (A.84), we obtain the representation, for any $y^* = (y_1^*, y_2^*) \in f(D)$,

$$\begin{aligned} \begin{pmatrix} f_1^{-1}(y^*) \\ f_2^{-1}(y^*) \end{pmatrix} &= \frac{1}{2\pi} \oint_{\partial D} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\mathrm{Arctan} \left(\frac{f_2(x_1, x_2) - y_2^*}{f_1(x_1, x_2) - y_1^*} \right) \\ &\quad - \frac{1}{2\pi} \iint_D \frac{\mathrm{adj}J(x_1, x_2)}{|f(x_1, x_2) - y^*|^2} \begin{pmatrix} f_1(x_1, x_2) - y_1^* \\ f_2(x_1, x_2) - y_2^* \end{pmatrix} dx_1 dx_2. \end{aligned} \tag{A.85}$$

We can consider the singular integral in (A.85) (as well as all the others in the sequel) in the sense of the Cauchy principal value.

In order to obtain a general version, we recall the following general representation formula of real-valued functions of class C^1.

Let us discuss the general case for the time being. We denote by $*$ the *Hodge star operator*, by G_n the *fundamental solution* of the *Laplacian* $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. We also denote by $(\cdot, \cdot)$ the *inner product* of the *vector space* $A^n(D)$ consisting of the n *order differential forms* over D with finite L^2-norms, that is

$$(\omega, \eta) = \int_D \omega \wedge *\eta = \int_D \eta \wedge *\omega \quad (\omega, \eta \in A^n(D)).$$

We write

$$f^{-1}(y) = (f_1^{-1}(y), f_2^{-1}(y), \ldots, f_n^{-1}(y)).$$

We recall Theorem 8.8: Let D be a bounded domain in $\mathbb{R}^n$ whose boundary ∂D is made up of a finite number of C^1-class boundary components. Let f be a C^1-class real-valued function on $\overline{D}$. For any $\hat{x} \in D$, we have the representation

$$f(\hat{x}) = -c_n\big(df(x), dG_n(x-\hat{x})\big) + c_n \int_{\partial D} f(x) * dG_n(x - \hat{x}), \tag{A.86}$$

where for $c_n = \max(1, n-2)$.

By using the pullback f^* for the integral representations of the inversions, we obtain the following result: Let us consider the situation of Theorem 8.8 and, furthermore, assume that f is a preserving C^1-class function on $\overline{D}$ in $\mathbb{R}^n$ with a single-valued inverse. Then, for $\hat{y} \in f(D)$, we have the representation

$$f_i^{-1}(\hat{y}) = -\int_D dx_i \wedge f^*[*dG_n(\cdot - \hat{y})](x) + \int_{\partial D} x_i f^*[*dG_n(\cdot - \hat{y})](x). \tag{A.87}$$

Here, f_i^{-1} denotes the i component of f^{-1}.

Meanwhile, for $\hat{y}$ in the outside of the image $f(D)$, the right-hand sides of the integrals are zero for all i.

For $n = 2$, we have (A.85) and we can represent (A.87) as follows. For any $\hat{y} \in f(D)$, we have

$$f_i^{-1}(\hat{y}) = \frac{1}{2\pi}\left(\oint_{\partial D} x_i d\theta_i - \int_D dx_i \wedge d\theta_i\right), \quad i = 1, 2.$$

Here,

$$\theta_1 = -\mathrm{Arctan}\left(\frac{f_1(x) - \hat{y}_1}{f_2(x) - \hat{y}_2}\right), \quad \theta_2 = \mathrm{Arctan}\left(\frac{f_2(x) - \hat{y}_2}{f_1(x) - \hat{y}_1}\right).$$

In particular, furthermore, when $f(D)$ is a convex domain, we have the representation

$$f_i^{-1}(\hat{y}) = \frac{\hat{x}_i^{min} + \hat{x}_i^{max}}{2} + \frac{1}{2\pi}\left(\oint_{\partial D} \theta_i dx_i - \int_D dx_i \wedge d\theta_i\right), \quad i = 1, 2.$$

Here, we can determine $\hat{x}_i^{min}$ and $\hat{x}_i^{max}$ by $\hat{y}$ as the two points of ∂D [491].

A.3.2 Representations of Implicit Functions

We will now pay attention to the Implicit Function Theorem, and explain the main goal of the present session. For simplifying the statement, we assume some global properties: On a smooth bounded domain $U \subset \mathbb{R}^{n+k}$ surrounded by finite number of C^1-class and simple closed surfaces, for k functions

$$f_i(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k}), \quad i = 1, 2, \ldots, k, \tag{A.88}$$

we assume that for some point on U, it holds that

$$f_i(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = 0 \tag{A.89}$$

and on U we have

$$\det \frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x) > 0. \tag{A.90}$$

Then, we assume globally that there exist k C^1-class functions $g_j(x_1, x_2, \ldots, x_n)$ for $j = 1, 2, \ldots, k$ on $U \cap \mathbb{R}^n$ satisfying the properties.

We followed [90] when we write this section.

$$f_i(x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_k) = 0, \quad i = 1, 2, \ldots, k, \tag{A.91}$$

and

$$x_{n+j} = g_j(x_1, x_2, \ldots, x_n), \quad j = 1, 2, \ldots, k. \tag{A.92}$$

The main purpose is to represent the explicit functions g_j explicitly, in terms of the implicit functions $\{f_i\}$.

We are now in the position to present the main result.

Theorem A.7. *Let $U \subset \mathbb{R}^{n+k}$ be a smooth bounded domain surrounded by a finite number of C^1-class and simple closed hypersurfaces. For k functions*

$$f_i(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k}), \quad (i = 1, 2, \ldots, k), \tag{A.93}$$

we assume that for some point on U it holds that

$$f_i(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = 0, \quad i = 1, 2, \ldots, k, \tag{A.94}$$

and that, on U, we have

$$\det\left(\frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x)\right) > 0. \tag{A.95}$$

In this way we also assume that C^1-functions $g_j(x_1, x_2, \ldots, x_n)$ on V globally satisfies

$$f_i(x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_k) = 0, \quad i = 1, 2, \ldots, k, \tag{A.96}$$

and

$$x_{n+j} = g_j(x_1, x_2, \ldots, x_n), \quad j = 1, 2, \ldots, k \tag{A.97}$$

for each $j = 1, 2, \ldots, k$, where

$$V \equiv \left\{ (x_1, x_2, \ldots, x_n) \;:\; {}^t\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \cdots \\ g_k(x_1, x_2, \ldots, x_n) \end{pmatrix} \in U \right\}.$$

Then, for $j = 1, 2, \ldots, k$, it holds that

$$\begin{aligned} &g_j(x_1, x_2, \ldots, x_n) \\ &= \sum_{i=1}^{n+k} \frac{(-1)^{n+j+i+1}}{c_{n+k}\omega_{n+k}} \\ &\quad \times \int_U \frac{(\eta - \eta_0)_i}{|\eta - \eta_0|^{n+k}} \det \frac{\partial(\eta_1, \eta_2, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{n+k})}{\partial(\xi_1, \xi_2, \ldots, \xi_{n+j-1}, \xi_{n+j+1}, \ldots, \xi_{n+k})}(\xi)\, d\xi \\ &\quad + \int_{\partial U} \xi_{n+j} F^*[*dG_{n+k}(\eta - \eta_0)], \end{aligned}$$

where $\omega_n \equiv 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ *is the surface measure of the n dimensional unit disk and F is a mapping from U into* $\mathbb{R}^{n+k}$ *such that*

$$F(x_1, x_2, \ldots, x_{n+k}) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f_1(x_1, x_2, \ldots, x_{n+k}) \\ f_2(x_1, x_2, \ldots, x_{n+k}) \\ \vdots \\ f_k(x_1, x_2, \ldots, x_{n+k}) \end{pmatrix}. \tag{A.98}$$

Proof. In order to apply Theorem 8.8, first we fix the direct mapping in Theorem 8.8 for our situation.

We will consider the C^1-class mapping F from U into $\mathbb{R}^{n+k}$ introduced in (A.98). It is clear that the Jacobian of this mapping is not vanishing on U as in

$$\det F'(x_1, x_2, \ldots, x_{n+k}) = \det\left(\frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x)\right) > 0. \tag{A.99}$$

By assumption, since the mapping F is injective on U, we can consider its inversion on its image domain. In particular, for any $(x_1, x_2, \ldots, x_n, 0, \ldots, 0)$ of the image domain that is the restriction to the domain U, we have from the situation in Sect. A.3.1, on $U \cap \mathbb{R}^n$ that

$$\begin{aligned} &F|_U^{-1}(x_1, x_2, \ldots, x_n, 0, \ldots, 0) \\ &= F|_U^{-1}\Big(x_1, x_2, \ldots, x_n, f_1(\mathbf{x}, \mathbf{g}), f_2(\mathbf{x}, \mathbf{g}), \ldots, f_k(\mathbf{x}, \mathbf{g})\Big) \\ &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ g_1 \\ g_2 \\ \vdots \\ g_k \end{pmatrix}, \end{aligned} \tag{A.100}$$

where

$$(\mathbf{x}, \mathbf{g}) = (x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_k).$$

Therefore, by the representation in Theorem 8.8, we obtain the identities for the explicit functions g_i,

$$g_i(x_1, x_2, \ldots, x_n) = \int_{\partial U} \xi_{n+i} F^*[*dG_{n+k}](\eta - \eta_0) - \int_U d\xi_{n+i} \wedge F^*[*dG_{n+k}](\eta - \eta_0), \tag{A.101}$$

for $(x_1, x_2, \ldots, x_n) \in U \cap \mathbb{R}^n$. Here,

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+i}, \ldots, \xi_{n+k}), \quad (i = 1, 2, \ldots, k), \tag{A.102}$$

and

$$\eta - \eta_0 = (\xi_1 - x_1, \xi_2 - x_2, \ldots, \xi_n - x_n, f_1(\xi_1, \xi_2 \ldots, \xi_{n+k}), \ldots, f_k(\xi_1, \xi_2 \ldots, \xi_{n+k})). \tag{A.103}$$

Recalling that ω_n is the surface measure of the n dimensional unit disk, we have

$$G_n(x) = \frac{1}{c_n \omega_n} \begin{cases} |x|, & n = 1 \\ \log|x|, & n = 2 \quad \text{(logarithmic kernel)} \\ -|x|^{-n+2}, & n \geq 3 \quad \text{(Newton kernel).} \end{cases}$$

Hence, on $\mathbb{R}^n \setminus U_\varepsilon(0)$ we have

$$dG_n(x) = \frac{1}{c_n \omega_n |x|^n} \sum_{i=1}^n x_i dx_i.$$

Therefore, by definition,

$$* \, dG_n(x) = \frac{1}{c_n \omega_n |x|^n} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n, \tag{A.104}$$

for $x = (x_1, x_2 \ldots, x_n)$.

As a consequence, it holds that

$$*dG_{n+k}(\eta - \eta_0) = \sum_{i=1}^{n+k} \frac{(\eta - \eta_0)_i}{c_{n+k}\omega_{n+k}|\eta - \eta_0|^{n+k}} * d\eta_i.$$

Then, we can compute the pull back $F^*[*dG_{n+k}(\eta - \eta_0)]$ needed in the representation of the explicit functions by the following general formula and the Jacobian:

$$\begin{aligned} &y^*(dy_1 \wedge \ldots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \ldots \wedge dy_n) \\ &= \sum_{k=1}^n (-1)^{k+} \det\left(\frac{\partial(y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)}(x) \right) * dx_k. \end{aligned}$$

Indeed,

$$F^* * dG_{n+k}(\eta - \eta_0)$$

$$= \frac{1}{c_{n+k}\omega_{n+k}|\eta - \eta_0|^{n+k}}$$

$$\sum_{i=1}^{n+k}\sum_{p=1}^{n+k}(-1)^{i+p}(\eta - \eta_0)_i \det\left(\frac{\partial(\eta_1, \eta_2, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{n+k})}{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \xi_{p+1}, \ldots, \xi_{n+k})}(\xi)\right) * d\xi_p$$

$$= \frac{1}{c_{n+k}\omega_{n+k}|\eta - \eta_0|^{n+k}}$$

$$\sum_{p=1}^{n+k}\Big(\sum_{i=1}^{n+k}(-1)^{i+p}(\eta - \eta_0)_i \cdot \det\frac{\partial(\eta_1, \eta_2, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{n+k})}{\partial(\xi_1, \xi_2, \ldots, \xi_{p-1}, \xi_{p+1}, \ldots, \xi_{n+k})}(\xi)\Big) * d\xi_p.$$

Therefore, we obtain the desired representation of g_j for $j = 1, 2, \ldots, k$.

A.3.3 The 2-Dimensional Case: $n = 1, k = 1$

We will state the concrete representation formula for the 2-dimensional case. Realizing that

$$F^* * dG_2(\eta - \eta_0) = \frac{1}{\pi c_2((\xi_1 - x_1)^2 + f_1(\xi_1, \xi_2)^2)}$$
$$\times \left\{(\xi_1 - x_1)\frac{\partial f_1}{\partial \xi_1}d\xi_1 + (\xi_1 - x_1)\frac{\partial f_1}{\partial \xi_2}d\xi_2 - f_1(\xi_1, \xi_2)d\xi_1\right\},$$

we obtain

Theorem A.8. *For a C^1-class function $f(x_1, x_2)$ on a domain U in $\mathbb{R}^2$, we assume that for a point $x^0 = (x_1^0, x_2^0)$,*

$$f(x_1^0, x_2^0) = 0,$$
$$\frac{\partial f}{\partial x_2}(x_1^0, x_2^0) \neq 0.$$

- *There exist a neighbourhood* $U_1 \times U_2$ ($\subset U$) *around the point* x^0 *and an explicit function* $g : U_1 \to U_2$ *determined by the implicit function* $f = 0$ *as* $f(x_1, g(x_1)) = 0$ *and, furthermore, it is represented as follows:*

$$g(x_1^*) = \frac{1}{2\pi}\left\{\oint_{\partial(U_1\times U_2)} x_2 d\theta - \int_{U_1\times U_2} dx_2 \wedge d\left[\mathrm{Arctan}\left(\frac{f(x_1,x_2)}{x_1 - x_1^*}\right)\right]\right\},$$

for any $x_1^* \in U_1$.

- *For any* $x_1^* \in U_1$, *it holds that*

$$x_1^* = \frac{1}{2\pi}\left\{\oint_{\partial(U_1\times U_2)} x_1 d\left[\mathrm{Arctan}\left(\frac{f(x_1,x_2)}{x_1 - x_1^*}\right)\right]\right.$$

$$\left. - \int_{U_1\times U_2} dx_1 \wedge d\left[\mathrm{Arctan}\left(\frac{f(x_1,x_2)}{x_1 - x_1^*}\right)\right]\right\}.$$

Corollary A.2 (Representations of exact differential equations). *Let* $f(x, y)$ *be the* C^1*-class solution of the differential equation* $f_x dx + f_y dy = 0$ *on some domain* $I_1 \times I_2$ *on* $\mathbb{R}^2$ *satisfying* $\dfrac{\partial f(x,y)}{\partial y} \neq 0$ *and* $y(x_0) = y_0, (x_0, y_0) \in I_1 \times I_2$. *Then, we obtain the representation of the explicit function* $y = y(x)$ *which is determined by the implicit function* $f(x, y) - f(x_0, y_0) = 0$ *for any* $x^* \in I_1$,

$$y(x^*) = \frac{1}{2\pi}\oint_{\partial(I_1\times I_2)} yd\left[\mathrm{Arctan}\left(\frac{f(x,y) - f(x_0,y_0)}{x - x^*}\right)\right]$$

$$- \frac{1}{2\pi}\iint_{I_1\times I_2} dy \wedge d\left[\mathrm{Arctan}\left(\frac{f(x,y) - f(x_0,y_0)}{x - x^*}\right)\right].$$

Corollary A.3 (Representations of the inverse functions). *On an open interval* $[a, b]$, *for a* C^1*-class function* f *satisfying* $f' > 0$ *on* $[a, b]$, *its inverse function* f^{-1} *on* $[f(a), f(b)]$ *can be represented as follows:*

$$f^{-1}(y^*) = \frac{1}{2\pi}\iint_{[a,b]\times[f(a),f(b)]} dx \wedge d\left[\mathrm{Arctan}\left(\frac{y - f(x)}{y - y^*}\right)\right]$$

$$-\frac{1}{2\pi}\oint_{\partial([a,b]\times[f(a),f(b)])} xd\left[\mathrm{Arctan}\left(\frac{y - f(x)}{y - y^*}\right)\right]$$

for any $y^* \in [f(a), f(b)]$.

A.3.4 The 3-Dimensional Cases: $n = 1, k = 2$ or $n = 2, k = 1$

Finally, we will compute the 3-dimensional cases. Suppose $n = 2$ and $k = 1$. From

$$F^*[*dG_3(\eta - \eta_0)] = \frac{1}{4\pi c_3((\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + f_1(\xi_1, \xi_2, \xi_3)^2)^{3/2}}$$

$$\cdot\Big((\xi_1 - x_1)\frac{\partial f_1}{\partial \xi_3} d\xi_2 \wedge d\xi_3 - (\xi_2 - x_2)\frac{\partial f_1}{\partial \xi_3} d\xi_1 \wedge d\xi_3$$

$$+\big(f_1(\xi_1, \xi_2, \xi_3)\big)d\xi_1 \wedge d\xi_2 - (\xi_1 - x_1)\frac{\partial f_1}{\partial \xi_1} - (\xi_2 - x_2)\frac{\partial f_1}{\partial \xi_2}\Big),$$

we have:

$$g_1(x_1, x_2) = \frac{1}{4\pi c_3}\int_U \frac{1}{((\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + f_1(\xi_1, \xi_2, \xi_3)^2)^{3/2}}$$

$$\cdot\left((\xi_1 - x_1)\frac{\partial f_1}{\partial \xi_1} + (\xi_2 - x_2)\frac{\partial f_1}{\partial \xi_2} - f_1(\xi_1, \xi_2, \xi_3)\right) d\xi_1 \wedge d\xi_2 \wedge d\xi_3$$

$$+ \int_{\partial U} \xi_3 F^* * dG_3(\eta - \eta_0).$$

Next, suppose $n = 1$ and $k = 2$. From

$$F^* * dG_3(\eta - \eta_0) = \frac{1}{c_3\omega_3((\xi_1 - x_1)^2 + f_1(\xi_1, \xi_2, \xi_3)^2 + f_2(\xi_1, \xi_2, \xi_3)^2)^{3/2}}$$

$$\cdot\Big[(\xi_1 - x_1)\det\frac{\partial(f_1, f_2)}{\partial(\xi_2, \xi_3)} d\xi_2 \wedge d\xi_3 + (\xi_1 - x_1)\det\frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_3)} d\xi_1 \wedge d\xi_3$$

$$+\Big((\xi_1 - x_1)\det\frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_2)} - f_1(\xi_1, \xi_2, \xi_3)\frac{\partial f_2}{\partial \xi_2} + f_2(\xi_1, \xi_2, \xi_3)\frac{\partial f_1}{\partial \xi_2}\Big)d\xi_1 \wedge d\xi_2\Big],$$

we have:

$$g_1(x_1) = \frac{1}{c_3\omega_3}\int_U \frac{(\xi_1 - x_1)}{((\xi_1 - x_1)^2 + f_1(\xi_1, \xi_2, \xi_3)^2 + f_2(\xi_1, \xi_2, \xi_3)^2)^{3/2}}$$

$$\cdot\det\frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_3)} d\xi_1 \wedge d\xi_2 \wedge d\xi_3 + \int_{\partial U} \xi_2 F^* * dG_3(\eta - \eta_0),$$

and

$$g_2(x_1) = \int_{\partial U} \xi_3 F^* * dG_3(\eta - \eta_0)$$

$$-\frac{1}{4\pi c_3} \int_U \frac{d\xi_1 \wedge d\xi_2 \wedge d\xi_3}{((\xi_1 - x_1)^2 + f_1(\xi_1, \xi_2, \xi_3)^2 + f_2(\xi_1, \xi_2, \xi_3)^2)^{3/2}}$$

$$\cdot \left((\xi_1 - x_1) \det \frac{\partial(f_1, f_2)}{\partial(\xi_1, \xi_2)} - f_1(\xi_1, \xi_2, \xi_3) \frac{\partial f_2}{\partial \xi_2} + f_2(\xi_1, \xi_2, \xi_3) \frac{\partial f_1}{\partial \xi_2} \right).$$

Furthermore, for the singular integrals appeared in the above formulas, simple regularizations and their error estimates were given in [420]. By their regularizations, the above singular integrals may be calculated, easily by computers. We can see also numerical experiments there.

A.4 Overview on the Theory of Reproducing Kernels

For the theory of reproducing kernels in one-variable complex analysis, see the fundamental books [46, 146, 202]. For the advanced and profound theory, see the books D. A. Hejhal [202] and J. D. Fay [146] in connection with the Riemann theta functions and the Klein prime form. For their applications, see A. Yamada [484]. Their theory now seems to be, however, too complicated and advanced to deal with for any mathematician. For the theory of reproducing kernels in complex analysis with general variables, see the classical books [167, 168]. For the old history of reproducing kernels, see [28, 382]. In the proceedings [392] of an international conference, we can find various results on the theory of reproducing kernels. See [16] for wider topics on reproducing kernels.

We find many results in **learning theory** where applications of the theory of reproducing kernels are important; for example, the estimation of covering numbers by the disks of reproducing kernel Hilbert spaces as subspaces of a family of continuous functions, some detailed smoothness relationships between reproducing kernels, and reproducing kernel Hilbert spaces, and approximations of functions by Sobolev spaces. See, for example, S. Smale and Zhou [117, 502]. Indeed, we have many references for learning theory, see [471].

For **Support Vector Machines** that are favourable for engineers, see [116] and its research center at MIT.

We would like to note the active research results of K. Fukimizu [119, 169, 170] on statistic theory and reproducing kernels.

The article [15] summarizes various applications of the theory in connection with operator theory on Hilbert spaces whose research center is in Israel with a great group; see for example, D. Alpay at Ben-Gurion University of the Negev.

For the connection with stochastic theory and reproducing kernels, see Berlinet [48, 49] who was very active in the 5th ISAAC Catania Congress with his colleagues and we see many results in this field.

For some more recent general discretization principles with many concrete applications, see [92, 93]. A new global theory combining the fundamental relations among eigenfunctions, initial value problems in general linear partial differential operators, and reproducing kernels, see [96, 97].

References

1. L.D. Abreu, The reproducing kernel structure arising from a combination of continuous and discrete orthogonal polynomials into Fourier systems. Constr. Approx. **28**(2), 219–235 (2008)
2. L.D. Abreu, A q-sampling theorem related to the q-Hankel transform. Proc. Am. Math. Soc. **133**, 1197–1203 (2005)
3. L.D. Abreu, Sampling theory associated with q-divergence equations of the Sturm-Liouville type. J. Phys. A: Math. Gen. **38**, 10311–10319 (2005)
4. L.D. Abreu, A. Bandeira, Landau's necessary density conditions for the Hankel transform. J. Funct. Anal. **162**, 1845–1866 (2012)
5. L.D. Abreu, O. Ciaurri, J.L. Varona, Bilinear biorthogonal expansions and the Dunkl kernel on the real line. Expo. Math. **30**, 32–48 (2012)
6. A.A. Adamov, On the asymptotic expansion of the polynomials $e^{ax^2/2}d^n[e^{(-ax^2/2)}]/dx^n$ for large values of n (in Russian). Ann. Polytech. Inst. St. Petersb. **5**, 127–143 (1906)
7. H. Aikawa, Infinite order sobolev spaces, analytic continuation and polynomial expansions. Complex Var. **18**, 253–266 (1992)
8. H. Aikawa, N. Hayashi, S. Saitoh, Isometrical identities for the Bergman and the Szegö spaces on a sector. J. Math. Soc. Jpn. **43**, 195–201 (1991)
9. J. Agler, Nevanlinna-Pick interpolation on Sobolev space. Proc. Am. Math. Soc. **108**, 341–351 (1990)
10. J. Agler, J.E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*. Graduate Studies in Mathematics, vol. 44 (American Mathematical Society, Providence, 2002)
11. G.D. Allen, F.J. Narcowich, J.P. Williams, An operator version of a theorem of Kolmogorov. Pac. J. Math. **61**(2), 305–312 (1975)
12. D. Alpay, P. Bruinsma, A. Dijksma, H.d. Snoo, Interpolation problems, extensions of symmetric operators and reproducing kernel spaces I. Integr. Equ. Oper. Theory **14**, 465–500 (1991)
13. D. Alpay, P. Bruinsma, A. Dijksma, H.d. Snoo, Interpolation problems, extensions of symmetric operators and reproducing kernel spaces II. Integr. Equ. Oper. Theory **14**, 378–388 (1992)
14. D. Alpay, H. Dym, On applications of reproducing kernel spaces to the Schur algorithm and rational J unitary factorization. Oper. Theory Adv. Appl. **18**, 89–159 (1986)
15. D. Alpay, *The Schur Algorithm, Reproducing Kernel Spaces and System Theory* (Translation of book: *Algorithme de Schur, espaces noyau reproduisant et théorie des systémes*. Panoramas et Synthéses, vol. 6 (Société Mathématique de France, 1998)). SMF/AMS Texts and Monographs of the Société Mathématique de France, vol. 5 (American Mathematical Society, Providence, 2001)

S. Saitoh, Y. Sawano, *Theory of Reproducing Kernels and Applications*,
Developments in Mathematics 44, DOI 10.1007/978-981-10-0530-5

16. D. Alpay, *Reproducing Kernel Spaces and Applications*. Operator Theory, Advances and Applications, vol. 143 (Birkhäuser, Basel, 2003)
17. L.V. Ahlfors, *Complex Analysis* (McGraw-Hill Science/Engineering/Mathematics, New York, 1978)
18. I.A. Aizenberg, A.P. Yuzhakov, *Integral Representations and Residues in Multidimensional Complex Analysis*, translated from the Russian by H.H. McFaden, translation ed. by Lev J. Leifman. Translations of Mathematical Monographs, vol. 58 (American Mathematical Society, Providence, 1983)
19. K. Amano, S. Saitoh, M. Yamamoto, Error estimates of the real inversion formulas of the laplace transform. Integr. Trans. Spec. Funct. **10**(3–4), 165–178 (2000)
20. K. Amano, D. Okano, H. Ogata, M. Sugihara, Numerical conformal mappings onto the linear slit domains. Jpn. J. Ind. Appl. Math. **29**, 165–186 (2012)
21. M.H. Annaby, A.I. Zayed, On the use of Green's function in sampling theory. J. Integr. Equ. Appl. **10**(2), 117–139 (1998)
22. B.D.O. Anderson, T. Kailath, Fast algorithms for the integral equations of the inverse scattering problem. Integr. Equ. Oper. Theory **1**(1), 132–136 (1978)
23. G.E. Andrews, R. Askey, R. Roy, *Special Functions*. Encyclopedia of Mathematics and Its Applications, vol. 71 (Cambridge University Press, Cambridge, 2009)
24. D.D. Ang, D.N. Thanh, V.V. Thanh, Regularized solutions of a Cauchy problem for the Laplace equation in an irregular strip. J. Integr. Equ. Appl. **5**, 429–441 (1993)
25. M.H. Annaby, G. Freiling, Sampling expansions associated with Kamke problems. Math. Z. **234**(1), 163–189 (2000)
26. J. Arazy, S. Fisher, J. Peetre, Hankel operators on weighted Bergman spaces. Am. J. Math. **110**, 989–1054 (1988)
27. G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists*, 5th edn. (Harcourt/Academic Press, Burlington, 2001)
28. N. Aronszajn, Theory of reproducing kernels. Trans. Am. Math. Soc. **68**, 337–404 (1950)
29. N. Aronszajn, Green's functions and reproducing kernels, in *Proceedings of the Symposium on Spectral Theory and Differential Problems* (Oklahoma Agricultural and Mechanical College, Oklahoma) (1951), pp. 164–187
30. N. Aronszajn, K.T. Smith, Theory of Bessel potentials. Part I. Ann. Inst. Fourier Grenoble **11**, 385–475 (1961)
31. M. Asaduzzaman, T. Matsuura, S. Saitoh, *Constructions of Approximate Solutions for Linear Differential Equations by Reproducing Kernels and Inverse Problems*. Advances in Analysis, Proceedings of the 4th International ISAAC Congress (World Scientific, Singapore/Hackensack, 2005), pp. 335–343
32. K.T. Atkinson, The numerical solution of the eigenvalue problem for compact integral operators. Trans. Am. Math. Soc. **129**, 458–465 (1967)
33. S. Axler, Bergman spaces and their operators, in *Survey of Some Recent Results in Operator Theory, Volume I*, ed. by J.B. Conway, B.B. Morrel. Pitman Research Notes in Mathematics, vol. 171 (Longman Scientific & Technical/Wiley, New York, 1988), pp. 1–50
34. S. Azizi, D. Cochran, Reproducing kernel structure and sampling on time-warped spaces with application to warped wavelets. IEEE Trans. Inf. Theory **48**(3), 789–790 (2002)
35. J.A. Ball, V. Vinnikov, *Formal Reproducing Kernel Hilbert Spaces: The Commutative and Noncommutative Settings. Reproducing Kernel Spaces and Applications*. Operator Theory: Advances and Applications, vol. 143 (Birkhäuser, Basel, 2003), pp. 77–134
36. R.B. Bapat, *Graphs and Matrices*. Universitext (Springer/Hindustan Book Agency, London/New Delhi, 2010). x+171pp.
37. M.G. Beaty, M.M. Dodson, S.P. Eveson, J.R. Higgins, On the approximate form of Kluvanek's theorem. J. Approx. Theory **160**, 281–303 (2009)
38. P.R. Beesack, Elementary proofs of some Opial-type integral inequalities. J. Anal. Math. **36**(1980), 1–14 (1979)
39. P.R. Beesack, K.M. Das, Extensions of Opial's inequality. Pac. J. Math. **26**, 215–232 (1968)

40. H. Begehr, R.P. Gilbert, *Transformations, Transmutations, and Kernel Functions*, vol. 1. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 58 (Longman Scientific and Technical/Copublished in the United States with John Wiley and Sons, Harlow/New York, 1992)
41. H. Begehr, R.P. Gilbert, *Transformations, Transmutations, and Kernel Functions*, vol. 2. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 59 (Longman Scientific and Technical/Copublished in the United States with John Wiley and Sons, Harlow/New York, 1993)
42. Ju.M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*. Translations of Mathematical Monographs, vol. 17 (American Mathematical Society, Providence, 1968)
43. F.A. Berezin, Quantization. Math. USSR IZv. **8**, 1109–1165 (1974)
44. F.A. Berezin, General concept of quantization. Commun. Math. Phys. **40**, 153–174 (1975)
45. S. Bergman, Über die Entwicklung der harmischen Funktionen der Ebene und des Rümes nach Orthogonal Functionen. Math. Ann. **86**, 238–271 (1922)
46. S. Bergman, *The Kernel Function and Conformal Mapping* (American Mathematical Society, Providence, 1950/1970)
47. B. Bergman, M.M. Schiffer, *Kernel Functions and Differential Equations* (Academic, New York, 1953)
48. A. Berlinet, C. Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics* (Kluwer Akademic Publishers, Boston/Dordrecht/London, 2004)
49. A. Berlinet, Reproducing kernels in probability and statistics, in *More Progress In Analysis, Proceedings of the 5th International ISAAC Congress*, Catania (2010), pp. 153–162
50. F. Berntsson, L. Elden, Numerical solution of a Cauchy problem for the Laplace equation. Special issue to celebrarate Pierre Sabatier's 65th birthday (Montpellier, 2000). Inverse Probl. **17**, 10–87 (2005)
51. M. Bertero, Linear inverse and ill-posed problems. Adv. Electron. Electron Phys. **75**, 2–120 (1989)
52. R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, vol. 169 (Springer, New York, 1997)
53. G. Bogveradze, L.P. Castro, Toeplitz plus Hankel operators with infinite index. Integr. Equ. Oper. Theory **62**, 43–63 (2008)
54. B.E. Boser, I.M. Guyon, V.N. Vapnik, A training algorithm for optimal margin classifiers, in *Fifth Annual ACM Workshop on Computational Learning Theory*, ed. by D. Haussler (ACM Press, Pittsburgh, 1992), pp. 144–152
55. A. Boumenir, T.K. Tuan, Sampling eigenvalues in Hardy spaces. SIAM J. Numer. Anal. **45**, 473–483 (2007)
56. A. Boumenir, T.K. Tuan, The computation of eigenvalues of singular Sturm-Liouville operators. Adv. Appl. Math. **39**, 222–236 (2007)
57. A. Boumenir, T.K. Tuan, Representation and sampling of Hardy functions. Math. Methods Appl. Sci. **33**, 485–492 (2010)
58. L. Bourgeois, A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation. Inverse Probl. **21**, 1087–1104 (2005)
59. L. de Branges, J. Rovnyak, *Square Summable Power Series* (Holt, Rinehart and Winston, New York, 1966)
60. L. de Branges, *Hilbert Spaces of Entire Functions* (Prentice-Hall, Englewood Cliffs, 1968)
61. L. de Branges, *Underlying Concepts in the Proof of the Bieberbach Conjecture* (Proceedings of International Congress of Mathematics, Berkeley, 1986), pp. 25–42
62. J. Buescu, A.C. Paixao, On differentiability and analyticity of positive definite functions. J. Math. Anal. Appl. **375**(1), 336–341 (2011)
63. J. Buescu, A.C. Paixao, Algebraic, differential, integral and spectral properties of Mercer-like-kernels, in *More Progress in Analysis. Proceedings of the 5th International ISAAC Congress*, Catania, ed. by H.G.W. Begehr et al., 25–30 July 2005 (World Scientific, Hackensack, 2009), pp. 175–188
64. J. Burbea, Total positivity of certain reproducing kernels. Pac. J. Math. **67**, 101–130 (1976)

65. J. Burbea, A Dirichlet norm inequality and some inequalities for reproducing kernel spaces. Proc. Am. Math. Soc. **83**(2), 279–285 (1981)
66. J. Burbea, Norm inequalities of exponential type for holomorphic functions. Kodai Math. J. **5**(2), 339–354 (1982)
67. J. Burbea, Inequalities for reproducing kernel spaces. Ill. J. Math. **27**(1), 130–137 (1983)
68. J. Burbea, Inequalities for holomorphic functions of several complex variables. Trans. Am. Math. Soc. **276**(1), 247–266 (1983)
69. J. Burbea, P. Masani, Banach and Hilbert spaces of vector-valued functions. in *Their General Theory and Applications to Holomorphy*. Research Notes in Mathematics, vol. 90 (Pitman (Advanced Publishing Program), Boston, 1984)
70. T. Byczkowski, RKHS for Gaussian measures on metric vector spaces. Bull. Polish Acad. Sci. Math. **35**(1–2), 93–103 (1987)
71. D.W. Byun, S. Saitoh, A real inversion formula for the Laplace transform. Z. Anal. Anw. **12**, 597–603 (1993)
72. D.W. Byun, S. Saitoh, Approximation by the solutions of the heat equation. J. Approx. Theory **78**, 226–238 (1994)
73. D.W. Byun, S. Saitoh, Best approximation in reproducing kernel Hilbert spaces, in *Proceedings of the 2th International Colloquium on Numerical Analysis* (VSP-Holland, 1994), pp. 55–61
74. P.L. Butzer, P.J.S.G. Ferreira, J.R. Higgins, S. Saitoh, G. Schmeisser, R.L. Stens, Interpolation and Sampling: E.T. Whittaker, K. Ogura and their followers. J. Fourier Anal. Appl. **17**, 320–354 (2011)
75. J. Calvert, Some generalizations of Opial's inequality. Proc. Am. Math. Soc. **18**, 72–75 (1967)
76. J.R. Cannon, P. DuChateau, Approximating the solution to the Cauchy problem for Laplace's equation. SAM J. Numer. Anal. **14**, 473–483 (1977)
77. L.P. Castro, R. Duduchava, F.O. Speck, Singular integral equations on piecewise smooth curves in spaces of smooth functions. Oper. Theory Adv. Appl. **135**, 107–144 (2002)
78. L.P. Castro, Q. Chen, S. Saitoh, Source inversion of heat conduction from a finite number of observation data. Appl. Anal. **89**, 801–813 (2010)
79. L.P. Castro, E.M. Rojas, Reduction of singular integral operators with flip and their Fredholm property. Lobachevskii J. Math. **29**(3), 119–129 (2008)
80. L.P. Castro, E.M. Rojas, Similarity transformation methods for singular integral operators with reflection on weighted Lebesgue spaces. Int. J. Mod. Math. **3**, 295–313 (2008)
81. L.P. Castro, E.M. Rojas, A collocation method for singular integral operators with reflection. RIMS Kokyuroku **1719**, 155–167 (2010–2011)
82. L.P. Castro, E.M. Rojas, S. Saitoh, Inversion from different kinds of information and real inversion formulas of the Laplace transform from a finite number of data. MESA Math. Eng. Sci. Aerosp. **1**, 181–190 (2010)
83. L.P. Castro, S. Saitoh, N.M. Tuan, Convolutions, integral transforms and integral equations by means of the theory of reproducing kernels. Opusc. Math. **32**(4), 633–646 (2012)
84. L.P. Castro, S. Saitoh, Natural outputs and global inputs of linear systems with a finite number of input data. Appl. Anal. **91**(2), 225–236 (2012)
85. L.P. Castro, S. Saitoh, Y. Sawano, A. Simões, General inhomogeneous discrete linear partial differential equations with constant coefficients on the whole spaces. Complex Anal. Oper. Theory **6**(1), 307–324 (2012)
86. L.P. Castro, S. Saitoh, Y. Sawano, A.S. Silva, Discrete linear differential equations. Analysis **32**(3), 181–191 (2012)
87. L.P. Castro, S. Saitoh, New convolutions and norm inequalities. Math. Inequal. Appl. **15**(3), 707–716 (2012)
88. L.P. Castro, S. Saitoh, Fractional functions and their representations. Complex Anal. Oper. Theory **7**(4), 1049–1063 (2013)
89. L.P. Castro, H. Itou, S. Saitoh, Numerical solutions of linear singular integral equations by means of Tikhonov regularization and reproducing kernels. Houst. J. Math. **38**(4), 1261–1276 (2012)

90. L.P. Castro, K. Murata, S. Saitoh, M. Yamada, Explicit integral representations of implicit functions. Carpath. J. Math. **29**(2), 141–148 (2013)
91. L.P. Castro, H. Fujiwara, M.M. Rodrigues, S. Saitoh, A new discretization method by means of reproducing kernels, in *Interactions Between Real and Complex Analysis*, ed. by L.H. Son, W. Tutscheke (Science and Technics Publication House, Ha Noi, 2012), pp. 185–223
92. L.P. Castro, H. Fujiwara, M.M. Rodrigues, S. Saitoh, V.K. Tuan, Aveiro discretization method in mathematics: a new discretization principle, in *Mathematics Without Boundaries: Surveys in Pure Mathematics*, ed. by P. Pardalos, T.M. Rassias (Springer, New York, 2014), pp. 37–92
93. L.P. Castro, H. Fujiwara, T. Qian and S. Saitoh, How to catch smoothing properties and analyticity of functions by computers? in *Mathematics Without Boundaries: Surveys in Interdisciplinary Research* (Springer, New York, 2014), pp. 101–116
94. L.P. Castro, S. Saitoh, Moore-Penrose generalized solutions of convolution integral equations. Numer. Funct. Anal. Optim. (in press)
95. L.P. Castro, A. Silva, S. Saitoh, A reproducing kernel Hilbert space constructive approximation for integral equations with Toeplitz and Hankel kernels. Libertus Math. (New Ser.) **34**(1), 1–22 (2014)
96. L.P. Castro, M.M. Rodorigues, S. Saitoh, Initial value problems in linear integral operators equations, in *Topics in Mathematical Analysis and Applications*, vol. 94, ed. by L. Toth, T.M. Rassias (Springer, Cham, 2014), pp. 175–188
97. L.P. Castro, M.M. Rodorigues,S. Saitoh, A fundamental theorem on initial value problems by using the theory of reproducing kernels. Complex Anal. Oper. Theory **9**(1), 87–98 (2014)
98. A.L. Cauchy, Cours d'Analyse de l'École Royale Polytechnique, Irére partie: Analyse Algébrique, Paris, Debure frères, 1821, Also in Oeuvres Complètes d'Augustin Cauchy, Séric II, T. E (Gauthier-Villars, Paris, 1987)
99. K. Chadan, P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd edn. Texts and Monographs in Physics (Springer, New York, 1989)
100. B.L. Chalmers, Subspace kernels and minimum problems in Hilbert spaces with kernel function. Pac. J. Math. **31**, 619–628 (1969)
101. Y.-S. Chan, G.H. Paulino, A.C. Fannjiang, The crack problem for nonhomogeneous materials under antiplane shear loading: a displacement based formulation. Int. J. Solids Struct. **38**, 2989–3005 (2001)
102. Y.-S. Chan, A.C. Fannjiang, G.H. Paulino, Integral equations with hypersingular kernels–theory and applications to fracture mechanics. Int. J. Eng. Sci. **41**, 683–720 (2003)
103. Z. Chena, Y. Zhoub, An efficient algorithm for solving Hilbert type singular integral equations of the second kind. Comput. Math. Appl. **58**, 632–640 (2009)
104. J. Cheng, Y.C. Hon, T. Wei, M. Yamamoto, Numerical computation of a Cauchy problem for Laplace's equation. Z. Angew. Math. Mech. **81**, 665–674 (2001)
105. V.G. Cherednichenko, *Inverse Logarithmic Potential Problem* (VSP, Utrecht, 1996)
106. T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978)
107. D. Colton, R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (Springer, Berlin, 1992)
108. A.M. Cohen, *Numerical Methods for Laplace Transform Inversion* (Springer, New York, 2007)
109. D. Constales, R.S. Krausshar, Szegö and polymonogenic Bergman kernels for half-space and strip domains, and single-periodic functions in Clifford analysis. Complex Var. Theory Appl. **47**(4), 349–360 (2002)
110. D. Constales, R.S. Krausshar, Bergman kernels for rectangular domains and multiperiodic functions in Clifford analysis. Math. Meth. Appl. Sci. **25**(4), 1509–1526 (2002)
111. T. Constantinescu, A. Gheondea, Representations of Hermitian kernels by means of Krein spaces. II. Invariant kernels. Commun. Math. Phys. **216**(2), 409–430 (2001)
112. T. Constantinescu, A. Gheondea, Invariant Hermitian kernels and their Kolmogorov decompositions (English. Abridged French version). C. R. Acad. Sci. Paris Ser. I Math. **331**(10), 797–802 (2000)

113. T. Constantinescu, A. Gheondea, Representations of Hermitian kernels by means of Krein spaces. Publ. RIMS Kyoto Univ. **33**, 917–951 (1997)
114. C. Corduneanu, *Integral Equations and Applications* (Cambridge University Press, Cambridge, 1991)
115. B. Cotterell, J.R. Rice, Slightly curved or kinked cracks. Int. J. Fract. **16**, 155–169 (1980)
116. N. Cristianini, J. Shawe-Taylor, *An Introduction to Support Vector Machines and Other Kernel-Based Learning Methods* (Cambridge University Press, Cambridge, 2001)
117. F. Cucker, S. Smale, On the mathematical foundations of learning. Bull. Am. Math. Soc. **39**, 1–49 (2002)
118. M. Cui, Y. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space* (Nova Science Publisher, New York, 2009)
119. M. Cuturi, K. Fukumizu, J.P. Vert, Semigroup kernels on measures. J. Mach. Learn. Res. **6**, 1169–1198 (2005)
120. C.J. Dalzell, J.O. Ramsay, Computing reproducing kernels with arbitrary boundary constraints. SIAM J. Sci. Comput. **14**(3), 511–518 (1993)
121. I. Daubechies, *Ten Lectures on Wavelets and Applications* (Society for Industrial and Applied Mathematics, Philadelphia, 1992)
122. P.J. Davis, *Interpolation and Approximation* (Blaisdell, New York, 1963)
123. A. Devinatz, Integral representations of positive definite functions. Trans. Am. Math. Soc. **74**, 56–76 (1953)
124. A. Devinatz, On the extensions of positive definite functions. Acta Math. **102**, 109–134 (1959)
125. N.H. Dinh, M.H. Pham, Stability results for the Cauchy problems for the Laplace equation in a strip. Inverse Probl. **19**, 833–844 (2003)
126. G. Doetsch, *Handbuch der Laplace Transformation*, vol. 1 (Birkhäuser Verlag, Basel, 1950)
127. M.F. Driscoll, The reproducing kernel Hilbert space structure of the sample paths of a Gaussian process. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **26**, 309–316 (1973)
128. J. Du, M. Cui, Solving the forced duffing equation with integral boundary conditions in the reproducing kernel space. Int. J. Comput. Math. **87**, 2088–2100 (2010)
129. D.T. Duc, N.D.V. Nhan, On some convolution norm inequalities in weighted $L_p(\mathbb{R}^n, \rho)$ spaces and their applications. Math. Inequal. Appl. **11**(3), 495–505 (2008)
130. D.T. Duc, N.D.V. Nhan, Some applications of convolution inequalities in weighted L_p spaces. Integr. Transf. Spec. Funct. **19**(7–8), 471–480 (2008)
131. D.T. Duc, N.D.V. Nhan, On some reverse weighted $L_p(\mathbb{R}^n)$-norm inequalities in convolutions and their applications. Math. Inequal. Appl. **12**(1), 67–80 (2009)
132. D.T. Duc, N.D.V. Nhan, Generalize some norm inequalities of Saitoh. Kodai Math. J. **34**(2), 191–201 (2011)
133. D.T. Duc, N.D.V. Nhan, Norm inequalities for new convolutions and their applications. Appl. Anal. Discret. Math. **9**, 168–179 (2015)
134. J. Duoandikoetxea, *Fourier Analysis*. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Graduate Studies in Mathematics, vol. 29 (American Mathematical Society, Providence, 2001)
135. D.L. Duttweiler, T. Kailath, RKHS approach to detection and estimation problems. V. Parameter estimation. IEEE Trans. Inf. Theory **IT-19**(1), 29–36 (1973)
136. D.L. Duttweiler, T. Kailath, RKHS approach to detection and estimation problems. IV. Non-Gaussian detection. IEEE Trans. Inf. Theory **IT-19**(1), 19–28 (1973)
137. D.L. Duttweiler, T. Kailath, An RKHS approach to detection and estimation problems III. Generalized innovations representations and a likelihood-ratio formula. IEEE Trans. Inf. Theory **IT-18**, 730–745 (1972)
138. H. Dym, *Hermitian Block Toeplitz Matrices, Orthogonal Polynomials, Reproducing Kernel Pontryagin Spaces, Interpolation And Extension. Orthogonal Matrix-Valued Polynomials and Applications (Tel Aviv, 1987–1988)*. Operator Theory: Advances and Applications, vol. 34 (Birkhäuser, Basel, 1988), pp. 79–135
139. H. Dym, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*. CBMS Regional Conference Series in Mathematics, vol. 71 (Published for the

Conference Board of the Mathematical Sciences/American Mathematical Society, Washington, DC/Providence, 1989)
140. I. Feldman, I. Gohberg, N. Krupnik, Convolution equations on finite intervals and factorization of matrix functions. Integr. Equ. Oper. Theory **36**, 201–211 (2000)
141. H.W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*. Mathematics and Its Applications, vol. 376 (Kluwer Academic Publishers, Dordrecht/Boston, 2000)
142. R. Estrada, R.P. Kanwal, *Singular Integral Equations* (Birkhäuser, Boston, 2000)
143. W.N. Everitt, G. Nasri–Roudsari, Sturm–Liouville problems with coupled boundary conditions and Lagrange interpolation series. J. Comput. Anal. Appl. **1**, 319–347 (1999)
144. A.C. Fannjiang, Y.-S. Chan, G.H. Paulino, Strain gradient elasticity for antiplane shear cracks: a hypersingular integrodifferential equation approach. SIAM J. Appl. Math. **62**, 1066–1091 (2002)
145. H.M. Farkas, I. Kra, *Riemann Surfaces*, 2nd edn. (Springer, New York, 1992)
146. J.D. Fay, *Theta Functions on Riemann Surfaces*. Lecture Notes in Mathematics, vol. 352 (Springer, Berlin/New York, 1973)
147. R.J.P. de Figueirredo, G. Chen, *Nonlinear Feedback Control Systems: An Operator Theory Approach* (Academic Press, Boston/San Diego/New York, 1993)
148. H.G. Feichtinger, Modulation spaces on locally compact Abelian groups. Technical report, University of Vienna (1983)
149. H.G. Feichtinger, Atomic characterization of modulation spaces through Gabor-type representation, in *Proceedings of the Conference Constructive Function Theory*, Edmonton, pp. 113–126 (1989)
150. H.G. Feichtinger, Gewichtsfunktionen auf lokalkompakten Gruppen. Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **188**, 451–471 (1979)
151. H.G. Feichtinger, K. Gröchenig, *A Unified Approach to Atomic Decompositions via Integrable Group Representations. Function Spaces and Applications*, Lund, 1986. Lecture Notes in Mathematics, vol. 1302 (Springer, Berlin, 1988), pp. 52–73
152. H.G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, I. J. Funct. Anal. **21**, 307–340 (1989)
153. H.G. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, II. Monatsh. Math. **108**, 129–148 (1989)
154. H.G. Feichtinger, K. Gröchenig, Gabor wavelets and the Heisenberg group: Gabor expansions and short time fourier transform from the group theoretical point of view, in *Wavelets A Tutorial in Theory and Applications*, ed. by C.K. Chui. Wavelet Analysis and Its Applications, vol. 2 (Academic Press, Boston, 1992), pp. 359–397
155. H.G. Feichtinger, K. Gröchenig, Gabor frames and time-frequency analysis of distributions. J. Funct. Anal. **146**(2), 464–495 (1997)
156. M. Frazier, B. Jawerth, *φ-Transform and Applications to Distribution Spaces, Function Spaces and Applications*, (Lund 1986). Lecture Notes in Mathematics, vol. 1302 (Springer, Berlin, 1988), pp. 223–246
157. M. Frazier, B. Jawerth, A discrete transform and decompositions of distribution spaces. J. Funct. Anal. **93**(1), 34–170 (1990)
158. H. Fujiwara, High-accurate numerical method for integral equation of the first kind under multiple-precision arithmetic. Theor. Appl. Mech. Jpn. **52**, 192–203 (2003)
159. H. Fujiwara, exflib – multiple-precision arithmetic library (2005). http://www-an.acs.i.kyoto-u.ac.jp/~fujiwara
160. H. Fujiwara, T. Matsuura, S. Saitoh, Y. Sawano, The real inversion of the Laplace transform by numerical singular value decomposition. J. Anal. Appl. **6**, 55–68 (2008)
161. H. Fujiwara, Numerical real inversions of the Laplace transform by multiple-precision arithmetic. RIMS Kokyuroku **1566**, 181–195 (2007)
162. H. Fujiwara, Applications of reproducing kernel spaces to real inversions of the Laplace transform. RIMS Kokyuroku **1618**, 188–209 (2008)

163. H. Fujiwara, T. Matsuura, S. Saitoh, Y. Sawano, *Numerical Real Inversion of the Laplace Transform by Using a High-Accuracy Numerical Method*. Further Progress in Analysis (World Scientific Publication, Hackensack, 2009), pp. 574–583
164. H. Fujiwara, Numerical real inversion of the Laplace transform by reproducing kernel and multiple-precision arithmetic, in *Progress in Analysis and Its Applications, Proceedings of the 7th International ISAAC Congress* (World Scientific, London, 2010), pp. 289–295
165. H. Fujiwara, N. Higashimori, Numerical inversion of the laplace transform by using multiple-precision arithmetic. Libertas Math. (new Ser.) **34**(2), 5–21 (2014)
166. H. Fujiwara, S. Saitoh, The general sampling theory by using reproducing kernels, in *Contributions in Mathematics and Engineering. Honor of Constantin Caratheodory*, ed. by T.M. Rassias, P. Pardalos (Springer, 2016)
167. B.A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*. Translations of Mathematical Monographs, vol. 8 (American Mathematical Society, Providence, 1963)
168. B.A. Fuks, *Special Chapters in the Theory of Analytic Functions of Several Complex Variables*. Translations of Mathematical Monographs, vol. 14 (American Mathematical Society, Providence, 1965)
169. K. Fukumizu, F.R. Bach, M.I. Jordan, Dimensionality reduction for supervised learning with reproducing kernel Hilbert spaces. J. Mach. Learn. Res. **5**, 73–99 (2004)
170. K. Fukumizu, F.R. Bach, A. Gretton, Statistical consistency of kernel canonical correlation analysis. J. Mach. Learn. Res. **8**, 361–383 (2007)
171. P.R. Garabedian, *Partial Differential Equations* (Wiley, New York, 1964)
172. A.G. Garcia, M.A. Hernandez-Medina, The discrete Kramer sampling theorem and indeterminate moment problems. J. Comput. Appl. Math. **134**, 13–22 (2001)
173. A.G. Garcia, A brief walk through sampling theory. Adv. Imaging Electron Phys. **124**, 63–137 (2002)
174. A.G. Garcia, F.H. Szafraniec, A converse of the Kramer sampling theorem. Sampl. Theorem Image Proc. **1**, 53–61 (2002)
175. J.B. Garnett, *Bounded Analytic Functions*. Pure and Applied Mathematics, vol. 96 (Academic Press Inc., New York, 1981)
176. A. Gasmi, F. Soltani, Fock spaces for the Bessel-Stuve kernel. J. Anal. Appl. **3**, 91–106 (2005)
177. F. Geng, M. Cui, Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space. Appl. Math. Comput. **192**, 389–398 (2007)
178. F. Geng, M. Cui, A reproducing kernel method for solving nonlocal fractional boundary value problems. Appl. Math. Lett. **25**, 818–823 (2012)
179. F. Geng, M. Cui, Homotopy perturbation-reproducing kernel method for nonlinear systems of second order boundary value problems. J. Comput. Appl. Math. **235**, 2405–2411 (2011)
180. F. Geng, M. Cui, A novel method for nonlinear two-point boundary value problems: combination of ADM and RKM. Appl. Math. Comput. **217**, 4676–4681 (2011)
181. F. Geng, M. Cui, Solving singular nonlinear boundary value problems by combining the homotopy perturbation method and reproducing kernel Hilbert space method. Int. J. Comput. Math. **87**, 2024–2031 (2010)
182. B.T. Giang, N.V. Mau, N.M. Tuan, Operational properties of two integral transforms of Fourier type and their convolutions. Integr. Equ. Oper. Theory **65**(3), 363–386 (2009)
183. B.T. Giang, N.V. Mau, N.M. Tuan, Convolutions for the Fourier transforms with geometric variables and applications. Math. Nachr. **283**, 1758–1770 (2010)
184. B.T. Giang, N.M. Tuan, Generalized convolutions for the Fourier integral transforms and applications. J. Sib. Federal Univ. **1**(4), 371–379 (2008)
185. R.P. Gilbert, G.N. Hile, Hilbert function modules with reproducing kernels. Nonlinear Anal. **1**(2), 135–150 (1977)
186. R.P. Gilbert, R.J. Weinacht, Reproducing kernels for elliptic systems. J. Approx. Theory **15**, 243–255 (1975)
187. C.D. Godsil, *Algebraic Combinatorics* (Chapman and Hall Mathematics, New York, 1993)

188. E.K. Godunova, V.I. Levin, An inequality of Maroni (in Russian). Mat. Zametki **2**, 221–224 (1967)
189. I.S. Gradshlein, L.M. Ryzhik, *Table of Integrals, Series, and Products* (Elsevier/Academic Press, Amsterdam, 2007)
190. B.L. Granovsky, H.G. Müller, On the optimality of a class of polynomial kernel functions. Stat. Decis. **7**(4), 301–312 (1989)
191. A. Gretton, K.M. Borgwardt, M. Rasch, B. Schölkopf, A. Smola, A kernel method for the two-sample-problem, in *Advances in Neural Information Processing Systems*, vol. 19, ed. by B. Schölkopf, J. Platt, T. Hoffman (MIT Press, Cambridge, 2007)
192. C.W. Groetsch, *Inverse Problems in the Mathematical Sciences* (Vieweg Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden, 1993)
193. D. Han, M.Z. Nashed, Q. Sun, Sampling expansions in reproducing kernel Hilbert and Banach spaces. Numer. Funct. Anal. Optim. **30**(9–10), 971–987 (2009)
194. P.C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve. SIAM Rev. **34**, 561–580 (1992)
195. D.N. Hao, N. Dinh, D. Lesnic, The Cauchy problem for Laplace's equation via the conjugate gradient method. IMA J. Appl. Math. **65**, 199–217 (2000)
196. N. Hayashi, Global existence of small analytic solutions to nonlinear Schrödinger equations. Duke Math. J. **60**, 717–727 (1990)
197. N. Hayashi, Solutions of the (generalized) Korteweg-de Vries equation in the Bergman and the Szegö spaces on a sector. Duke Math. J. **62**, 575–591 (1991)
198. N. Hayashi, K. Kato, Regularity of solutions in time to nonlinear Schrödinger equations. J. Funct. Anal. **128**, 255–277 (1995)
199. N. Hayashi, S. Saitoh, Analyticity and smoothing effect for the Schrödinger equation. Ann. Inst. Henri Poincaré **52**, 163–173 (1990)
200. N. Hayashi, S. Saitoh, Analyticity and global existence of small solutions to some nonlinear Schrödinger equation. Commun. Math. Phys. **139**, 27–41 (1990)
201. H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman Spaces*. Graduate Texts in Mathematics, vol. 199 (Springer, New York, 2000)
202. D.A. Hejhal, *Theta Functions, Kernel Functions and Abel Integrals*, vol. 129 (Memoirs American Mathematical Society, Providence, 1972)
203. T. Hida, N. Ikeda, Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral, in *1967 Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley. Contributions to Probability Theory, Part 1, vol. II (University California Press, Berkeley, 1965/1966), pp. 117–143
204. J.R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations* (Clarendon Press, Oxford, 1996)
205. J.R. Higgins, R.L. Stens, *Sampling Theory in Fourier and Signal Analysis: Advanced Topics* (Clarendon Press, Oxford, 1999)
206. J.R. Higgins, A sampling principle associated with Saitoh's fundamental theory of linear transformations, in *Analytic Extension Formulas and Their Applications*. International Society for Analysis, Applications, and Computation, vol. 9 (Kluwer Academic Publishers, Dordrecht/Boston, 2001), pp. 73–86
207. J.R. Higgins, Converse sampling theorems in a reproducing kernel theory setting. Sampl. Theorem Signal Image Proc. **14**(2), 145–152 (2015)
208. J.W. Hilgers, On the equivalence of regularization and certain reproducing kernel Hilbert space approaches for solving first kind problems. SIAM J. Numer. Anal. **13**, 172–184 (1976)
209. E. Hille, Introduction to general theory of reproducing kernels. Rocky Mt. J. Math. **2**(3), 321–368 (1972)
210. D.A. Hills, D. Nowell, A. Sackfield, *Mechanics of Elastic Contacts* (Butterworth–Heinemann, Oxford, 1993)
211. D.A. Hills, P.A. Kelly, D.N. Dai, A.M. Korsunsky, *Solution of Crack Problems: The Distributed Dislocation Technique* (Kluwer Academic Publishers, Dordrecht, 1996)

212. I.I. Hirschman, D.V. Widder, *The Convolution Transform* (Princeton University Press, Princeton, 1955)
213. Y. Hishikawa, Fractional calculus on parabolic Bergman spaces. Hiroshima Math. J. **38**, 471–488 (2008)
214. Y. Hishikawa, The reproducing formula with fractional orders on the parabolic Bloch space. J. Math. Soc. Jpn. **62**, 1219–1255 (2010)
215. Y. Hishikawa, M. Nishio, M. Yamada, A conjugate system and tangential derivative norms on parabolic Bergman spaces. Hokkaido Math. J. **39**, 85–114 (2010)
216. Y. Hishikawa, Representing sequences on parabolic Bergman spaces. J. Korean Math. Soc. **48**, 1017–1041 (2011)
217. H. Hochstadt, *Integral Equations* (Wiley, New York, 1973)
218. Y.C. Hon, T. Wei, Backus-Gilbert algorithm for the cauchy problem of the laplace equation. Inverse Probl. **17**(2), 261–271 (2001)
219. L. Hörmander, *An Introduction to Complex Analysis in Several Variables* (D. Van Nostrand Company, Inc. Princeton, 1967)
220. L. Hörmander, L^2 estimates and existence theorems for the $\overline{\partial}$-operator. Acta Math. **113**, 89–152 (1965)
221. http://library.wolfram.com/inforcenter/MathSource/4738/
222. http://www.columbia.edu/ww2040/abate.html
223. L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*. Translations of Mathematical Monographs, vol. 6 (American Mathematical Society, Providence, 1963)
224. T.J.R. Hughes, *The Finite Element Method* (Prentice Hall, Englewood Cliffs, 1987)
225. H. Imai, T. Takeuchi, M. Kushida, On numerical simulation of partial differential equations in infinite precision. Adv. Math. Sci. Appl. **9**, 1007–1016 (1999)
226. H. Imai, Multipy-precision arithmetic (in Japanese). Sūgaku Math. Soc. Jpn. Iwanami-Shoten Co. **55**, 316–325 (2003)
227. S.W. Indratno, A.G. Ramm, Inversion of the laplace transform from the real axis using an adaptive iterative method. Int. J. Math. Math. Sci. **2009**, 38. Article ID 898195
228. V. Isakov, *Inverse Problems for Partial Differential Equation* (Springer, New York, 1998)
229. P.D. Iseger, Numerical transform inversion using Gaussian quadrature. Probab. Eng. Inf. Sci. **20**, 1–44 (2006)
230. H. Itou, S. Saitoh, Analytical and numerical solutions of linear singular integral equations. Int. J. Appl. Math. Stat. **12**, 76–89 (2007)
231. K. Iwamura, T. Matsuura, S. Saitoh, A numerical construction of a natural inverse of any matrix by using the theory of reproducing kernels with the Tikhonov regularization. Far East J. Math. Edu. **4**, 141–149 (2010)
232. W. Jiang, M. Cui, Y. Lin, Anti-periodic solutions for Rayleigh-type equations via the reproducing kernel Hilbert space method. Commun. Nonlinear Sci. Numer. Simul. **15**, 1754–1758 (2010)
233. W. Jiang, M Cui, Solving nonlinear singular pseudoparabolic equations with nonlocal mixed conditions in the reproducing kernel space. Int. J. Comput. Math. **87**, 3430–3442 (2010)
234. A.J. Jerri, The Shannon sampling theorem – its various extensions and applications – a tutorial review. Proc. IEEE **65**, 1565–1596 (1977)
235. X.Q. Jin, L.M. Keer, Q. Wang, A practical method for singular integral equations of the second kind. Eng. Fract. Mech. **75**, 1005–1014 (2008)
236. P.E.T. Jorgensen, Integral representations for locally defined positive definite functions on Lie group. Int. J. Math. **2**, 257–286 (1991)
237. T. Kailath, Some integral equations with nonrational kernels. IEEE Trans. Inf. Theory **IT-12**, 442–447 (1966)
238. T. Kailath, D. Duttweiler, An RKHS approach to detection and estimation problems, III. Generalized innovations representations and a likelihood-ratio formula. IEEE Trans. Inf. Theory **IT-18**, 730–745 (1972)

239. T. Kailath, R. Geesey, H. Weinert, Some relations among RKHS norms, Fredholm equations, and innovations representations. IEEE Trans. Inf. Theory **IT-18**, 341–348 (1972)
240. T. Kailath, RKHS approach to detection and estimation problems, I. Deterministic signals in Gaussian noise. IEEE Trans. Inf. Theory **IT-17**, 530–549 (1971)
241. J.P. Kaipio, E. Somersalo, *Statistical and Computational Inverse Problems* (Springer, New York, 2004)
242. V.A. Kakichev, On the convolution for integral transforms. Izv. ANBSSR Ser. Fiz. Mat. **2**(2), 48–57 (1967) (in Russian)
243. G. Kallianpur, Abstract Wiener processes and their reproducing kernel Hilbert spaces. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **17**, 113–123 (1971)
244. G. Kallianpur, *The Role of Reproducing Kernel Hilbert Spaces in the Study of Gaussian Processes*. Advances in Probability and Related Topics, vol. 2 (Dekker, New York, 1970), pp. 49–53
245. Y. Kametaka, K. Watanabe, A. Nagai, The best constant of Sobolev inequality in an n dimensional Euclidean space. Proc. Jpn. Acad. Ser. A **81**, 57–60 (2005)
246. Y. Kametaka, K. Watanabe, A. Nagai, S. Pyatkov, The best constant of Sobolev inequality in an n dimensional Euclidean space. Sci. Math. Jpn. **e-2004**, 295–303 (2004); Sci. Math. Jpn. **61**(1), 15–23 (2005)
247. Y. Kametaka, Y. Oshime, K. Watanabe, H. Yamagishi, A. Nagai, K. Takemura, The best constant of L^p Sobolev inequality corresponding to the periodic boundary value problem for $(-1)^M(d/dx)^{2M}$. Sci. Math. Jpn. **e-2007**, 269–281 (2007)
248. Y. Kametaka, K. Watanabe, A. Nagai, H. Yamagishi, K. Takemura, The best constant of Sobolev inequality which corresponds to a bending problem of a string with periodic boundary condition. Sci. Math. Jpn. **66**(2), 151–168 (2007)
249. Y. Kametaka, H. Yamagishi, K. Watanabe, A. Nagai, K. Takemura, The best constant of Sobolev inequality corresponding to Dirichlet boundary value problem for $(-1)^M(d/dx)^{2M}$. Sci. Math. Jpn. **e-2008**, 439–451 (2008)
250. Y. Kametaka, H. Yamagishi, K. Watanabe, A. Nagai, K. Takemura, M. Arai, The best constant of Sobolev inequality which corresponds to Schrödinger operator with Dirac delta potential. Sci. Math. Jpn. **69**(2), 211–225 (2009)
251. Y. Kametaka, H. Yamagishi, K. Watanabe, A. Nagai, K. Takemura, M. Arai, H. Yama-gishi, The best constant of Sobolev inequality corresponding to Dirichlet-Neumann boundary value problem for $(-1)^M(d/dx)^{2M}$. Hiroshima Math. J. **39**, 421–442 (2009)
252. A.A. Karelin, Applications of operator equalities to singular integral operators and to Riemann boundary value problems. Math. Nachr. **280**, 1108–1117 (2007)
253. K. Kataoka, Quasi-positivity for pseudodiŁerential operators and microlocal energy methods, in *Proceedings of Taniguchi Symposium HERT*, Katata (1984), pp. 125–141
254. K. Kataoka, Microlocal energy methods and pseudo-differential operators. Invent. Math. **81**, 305–340 (1985)
255. K. Kataoka, Some applications of microlocal energy methods to analytic hypoellipticity. Prospect Algebr. Anal. **1**, 287–303 (1988)
256. G. Kimeldorf, G. Wahba, Some results on Tchebycheffian spline functions. J. Math. Anal. Appl. **33**, 82–95 (1971)
257. M.V. Klibanov, F. Santosa, A computational quasi-reversibility method for Cauchy problems for Laplace's equation. SIAM J. Appl. Math. **51**(6), 1653–1675 (1991)
258. D. Klusch, The sampling theorem, Dirichlet series and Hankel transforms. J. Comput. Appl. Math. **44**, 261–273 (1992)
259. M. Kobayashi, Y. Sawano, Molecular decomposition of the modulation spaces $M^{p,q}$ and its application to the pseudo-differential operators. Osaka J. Math. **47**(4), 1029–1053 (2010)
260. A.N. Kolmogoroff, Stationary sequences in Hilbert's space (in Russian). Bull. Math. Univ. Mosc. **2**, 40 (1941)
261. M.G. Kreĭn, Hermitian positive kernels on homogeneous spaces. I. Am. Math. Soc. Transl. (2) **34**, 69–108 (1963)

262. H. Körezlioğlu, Reproducing kernels in separable Hilbert spaces. Pac. J. Math. **25**, 305–314 (1968)
263. H. Kramer, A generalized sampling theorem. J. Math. Phys. **38**, 68–72 (1959)
264. S. Krantz, *Function Theory of Several Complex Variables*. Pure and Applied Mathematics. A Wiley-Interscience Publication (Wiley, New York, 1982)
265. S. Krantz, *Function Theory of Several Complex Variables*. Reprint of the 1992 edition (AMS Chelsea Publishing, Providence, 2001)
266. V.G. Kravchenko, G.S. Litvinchuk, *Introduction to the Theory of Singular Integral Operators with Shift* (Kluwer Academic Publishers, Dordrecht, 1994)
267. R. Kress, *Linear Integral Equations* (Springer, New York, 1998)
268. V.V. Kryzhniy, Regularized inversion of integral transformations of Mellin convolution type. Inverse Probl. **19**, 573–583 (2003)
269. V.V. Kryzhniy, Numerical inversion of the Laplace transform: analysis via regularized analytic continuation. Inverse Probl. **22**, 579–597 (2006)
270. A.B. Kuijlaas, M. Vanlessen, Universality for eigenvalue correlations from the modified Jacobi unitary ensemble. Int. Math. Res. Not. **30**, 1575–1600 (2002)
271. J.L. Lagrange, (Nouv. Mém. Acad., Berlin, 1773); Also in Oeuvres, T. 3 (Gauthier-Villars, Paris, 1869), p. 662f
272. T.L. Lai, Reproducing kernel Hilbert spaces and the law of the iterated logarithm for Gaussian processes. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **29**, 7–19 (1974)
273. I.E. Lagaris, A. Likas, D.I. Fotiadis, Artificial neural networks for solving ordinary and partial differential equations. IEEE Trans. Neural Netw. **9**, 987–1000 (1998)
274. I.E. Lagaris, A. Likas, D.I. Fotiadis, Neural-network methods for boundary value problems with irregular boundaries. IEEE Trans. Neural Netw. **11**, 1041–1049 (2000)
275. F.M. Larkin, Optimal approximation in Hilbert spaces with reproducing kernel functions. Math. Comput. **24**, 911–921 (1970)
276. C.L. Lawson, R.J. Hanson, *Solving Least Squares Problems* (Prentice-Hall, Englewood Cliffs, 1974)
277. J.Y. Li, S.W. Luo, Y.J Qi, Y.P. Huang, Numerical solution of elliptic partial differential equation using radial basis function neural networks. Neural Netw. **16**, 729–734 (2003)
278. A. Van Der Linde, Interpolation of regression functions in reproducing kernel Hilbert spaces. Statistics **16**, 351–361 (1985)
279. M.A. Lytrides, N.S. Stylianopoulos, Error analysis of the Bergman kernel method with singular based functions. Comput. Methods Funct. Theory **11**(2), 487–529 (2011)
280. R. Le Page, Subgroups of paths and reproducing kernels. Ann. Probab. **1**, 345–347 (1973)
281. N.A. Lebedev, I.M. Milin, An inequality. Vestnik Leningrad. Univ. **20**(19), 157–158 (1965) (Russian)
282. J.Y. Lee, J.R. Yoon, A numerical method for the Cauchy problem using singular decompositon, in *Second Japan-Korea Joint Seminar on Inverse Problems and Related Topics*, Seoul, 2001. Commun. Korean Math. Soc. **16**, 487–508 (2001)
283. Y. Lin, J. Niu, M. Cui, A numerical solution to nonlinear second order three-point boundary value problems in the reproducing kernel space. Appl. Math. Comput. **218**, 7362–7368 (2012)
284. Y. Lin, M. Cui, A numerical solution to nonlinear multi-point boundary value problems in the reproducing kernel space. Math. Methods Appl. Sci. **34**, 44–47 (2011)
285. G.S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift* (Kluwer Academic, Dordrecht, 2000)
286. D. Lubinsky, A new approach to universality limits involving orthogonal polynomials. Ann. Math. **170**(2), 915–939 (2009)
287. D. Lubinsky, Universality for arbitrary measures on compact sets. www.math.gatech.edu/news/conferences/banff/abstracts.html-19k[O]
288. W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third enlarged edition (Springer, Berlin, 1966)
289. B.N. Mandal, G.H. Bera, Approximate solution of a class of singular integral equations of second kind. J. Comput. Appl. Math. **206**, 189–195 (2007)

290. J. Matkowski, L^p-like paranorms, in *Selected Topics in Functional Equations and Iteration Theory*, Graz, 1991. Grazer Math. Ber. **316**, 103–138 (1992)
291. J.T. Marti, *Introduction to the Theory of Bases* (Springer, New York/Berlin/Heidelberg, 1969)
292. T. Matsuura, S. Saitoh, Analytical and numerical solutions of the inhomogeneous wave equation. Aust. J. Math. Anal. Appl. **1**(1), Article 7, 18 (2004)
293. T. Matsuura, S. Saitoh, D.D. Trong, Numerical solutions of the Poisson equation. Appl. Anal. **83**, 1037–1051 (2004)
294. T. Matsuura, S. Saitoh, Analytical and numerical solutions of linear ordinary differential equations with constant coefficients. J. Anal. Appl. **3**, 1–17 (2005)
295. T. Matsuura, S. Saitoh, Numerical inversion formulas in the wave equation. J. Comput. Math. Optim. **1**, 1–19 (2005)
296. T. Matsuura, S. Saitoh, D.D. Trong, Approximate and analytical inversion formulas in heat conduction on multidimensional spaces. J. Inverse Ill-Posed Probl. **13**, 479–493 (2005)
297. T. Matsuura, S. Saitoh, Dirichlet's principle using computers. Appl. Anal. **84**, 989–1003 (2005)
298. T. Matsuura, S. Saitoh, Analytical and numerical inversion formulas in the Gaussian convolution by using the Paley-Wiener spaces. Appl. Anal. **85**, 901–915 (2006)
299. T. Matsuura, A. Al-Shuaibi, H. Fujiwara, S. Saitoh, Numerical real inversion formulas of the Laplace transform by using a Fredholm integral equation of the second kind. J. Anal. Appl. **5**, 123–136 (2007)
300. T. Matsuura, A. Al-Shuaibi, H. Fuijiwara, S. Saitoh, M. Sugihara, Numerical real inversion formulas of the Laplace transform by a sinc method. Far East J. Math. Sci. **27**, 1–14 (2007)
301. R. Meidan, Reproducing-kernel Hilbert spaces of distributions and generalized stochastic processes. SIAM J. Math. Anal. **10**(1), 62–70 (1979)
302. H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, (German) Die Grundlehren der mathematischen Wissenschaften, vol. 113 (Springer, Berlin/Göttingen/Heidelberg, 1962), viii+256pp
303. M. Mboup, *On the Structure of Self-Similar Systems: A Hilbert Space Approach*. Reproducing Kernel Spaces and Applications, Operator Theory: Advances and Applications, vol. 143 (Birkhäuser, Basel, 2003), pp. 273–302
304. R. Megginson, *A Introduction to Banach Space Theory*. Graduate Texts in Mathematics, vol. 183 (Springer, New York, 1998)
305. S.G. Mikhlin, K.L. Smolitskiy, *Approximate Methods for Solution of Differential and Integral Equations* (American Elsevier, New York, 1967)
306. S.G. Mikhlin, S. Prössdorf, *Singular Integral Operators* (Springer, Berlin, 1986)
307. D.S. Mitrinović, *Analytic Inequalities*. Die Grundlehren der mathematischen Wissenschaften, vol. 165 (Springer, New York/Berlin, 1970), xii+400pp
308. Y. Mo, T. Qian, Support vector machine adapted Tikhonov regularization method to solve Dirichlet problem. Appl. Math. Comput. **245**, 509–519 (2014)
309. B. Mohar, The Laplacian spectrum of graphs, in *Graph Theory, Combinatorics, and Applications*, ed. by Y. Alavi, G. Chartrand, O.R. Oellermann, A.J. Schwenk, vol. 2 (Wiley, 1991), pp. 871–898
310. E.H. Moore, *General Analysis*. Memoirs of the American Philosophical Society Part I; Memoirs of the American Philosophical Society Part II (The American Philosophical Society, Philadelphia, 1935–1939)
311. P.M. Morse, H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company Inc., New York, 1953)
312. N.I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1972)
313. A. Nagai, K. Takemura, Y. Kametaka, K. Watanabe, H. Yamagishi, Green function for boundary value problem of $2M$-th order linear ordinary equations with free boundary condition. Far East J. Appl. Math. **26**, 393–406 (2007)
314. K. Nakade, T. Ohwada, K.S. Saito, Kolmogorov's factorization theorem for von Neumann algebras. J. Math. Anal. Appl. **401**(1), 289–292 (2013)

315. M. Nakai, L. Sario, Square integrable harmonic functions on plane regons. Ann. Acad. Scxi. Fenn. **4**, 193–201 (1978/1979)
316. M.Z. Nashed, G. Wahba, Convergence rates of approximate least squares solutions of linear integral and operator equations of the first kind. Math. Comput. **28**, 69–80 (1974)
317. M.Z. Nashed, G. Wahba, Regularization and approximation of linear operator equations and reproducing kernel spaces. Bull. Am. Math. Soc. **80**, 1213–1218 (1974)
318. M.Z. Nashed, G. Wahba, Generalized inverses in reproducing kernel spaces: an approach to regularization of linear operator equations. SIAM J. Math. Anal. **5**, 974–987 (1974)
319. M.Z. Nashed, G.G. Walter, General sampling theorems for functions in reproducing kernel Hilbert spaces. Math. Control Signals Syst. **4**, 363–390 (1991)
320. M.Z. Nashed, Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. J. Funct. Anal. **258**(7), 2422–2452 (2010)
321. F. Natterer, *The Mathematics of Computerized Tomography*. Classics in Applied Mathematics, vol. 32 (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2001)
322. Z. Nehari, *Conformal Mapping* (McGraw-Hill Book Company Inc., New York, 1952)
323. N.D.V. Nhan, D.T. Duc, V.K. Tuan, Weighted L_p-norm inequalities for various convolution type transformations and their applications. Armen. J. Math. **1**(4), 1–18 (2008)
324. N.D.V. Nhan, D.T. Duc, Fundamental inequalities for the iterated Laplace convolution in weighted L_p spaces and their applications. Integr. Trans. Spec. Funct. **19**(9–10), 655–664 (2008)
325. N.D.V. Nhan, D.T. Duc, Reverse weighted L_p-norm inequalities and their applications. J. Math. Inequal. **2**(1), 57–73 (2008)
326. N.D.V. Nhan, D.T. Duc, Weighted L_p-norm inequalities in convolutions and their applications. J. Math. Inequal. **2**(1), 45–55 (2008)
327. N.D.V. Nhan, D.T. Duc, V.K. Tuan, Reverse weighted L_p-norm inequalities for convolution type integrals. Armen. J. Math. **2**(3), 77–93 (2009)
328. N.D.V. Nhan, D.T. Duc, Fundamental iterated convolution inequalities in weighted L_p spaces and their applications. Math. Inequal. Appl. **12**(3), 487–498 (2009)
329. N.D.V. Nhan, D.T. Duc, V.K. Tuan, Weighted norm inequalities for a nonlinear transform. Comput. Math. Appl. **61**, 832–839 (2011)
330. N.D.V. Nhan, D.T. Duc, Convolution inequalities in l_p weighted spaces. Canad. Math. Bull. **55**(2), 355–367 (2012)
331. N.D.V. Nhan, D.T. Duc, Various inequalities in reproducing kernel Hilbert spaces. Taiwan. J. Math. **17**(1), 221–237 (2012)
332. M. Nishio, N. Suzuki, M. Yamada, Schatten class Toeplitz operators on the parabolic Bergman space. II. Kodai Math. J. **35**, 52–77 (2012)
333. M. Nishio, N. Suzuki, M. Yamada, Interpolating sequences of parabolic Bergman spaces. Potential Anal. **28**, 357–378 (2008)
334. M. Nishio, N. Suzuki, M. Yamada, Carleson inequalities on parabolic Bergman spaces. Tohoku Math. J. **62**(2), 269–286 (2010)
335. M. Nishio, N. Suzuki, M. Yamada, Weighted Berezin transformations with application to Toeplitz operators of Schatten class on parabolic Bergman spaces. Kodai Math. J. **32**, 501–520 (2009)
336. M. Nishio, N. Suzuki, M. Yamada, Compact Toeplitz operators on parabolic Bergman spaces. Hiroshima Math. J. **38**, 177–192 (2008)
337. M. Nishio, K. Shimomura, N. Suzuki, α-parabolic Bergman spaces. Osaka J. Math. **42**, 133–162 (2005)
338. H. Ogawa, N.-E. Berrached, Theory of extended pseudo-biorthogonal bases. IEICE Trans. Inf. Syst. **E76-D**(8), 890–897 (1993)
339. H. Ogawa, N.-E. Berrached, A theory of extended pseudo-biorthogonal bases and its application to generalized sampling theorem, in *Mathematical Analysis, Wavelets, and Signal Processing. An International Conference on Mathematical Analysis and Signal Processing*, Cairo University, Cairo, ed. by M.E.H. Ismail et al., Jan 1994. Contemporary Mathematics, vol. 190 (American Mathematical Society, Providence, 1995), pp. 305–321

340. H. Ogawa, Theory of pseudo-biorthogonal bases and its application. The theory of reproducing kernels and their applications (Japanese) (Kyoto, 1998). RIMS Kokyuroku No. **1067**, 24–38 (1998)
341. H. Ogawa and A. Hirabayashi, Sampling theorem with optimum noise suppression. Sampl. Theory Signal Image Process. **6**(2), 167–184 (2007)
342. H. Ogawa, *Functional Analysis for Engineers* (*in Japanese*) (Morikita Co., 2009), 292pp
343. K. Oguiso, *Daisu kyokusen ron* (*in Japanese*) (Asakura, 2002)
344. T. Ohe, K. Ohnaka, Uniqueness and convergence of numerical solution of the Cauchy problem for the Laplace equation by a charge simulation method. Jpn. J. Indust. Appl. Math. **21**, 339–559 (2004)
345. T. Ohsawa, T. Takegoshi, On the extension of L^2 holomorphic functions. Math. Zeit. **195**, 197–204 (1987)
346. T. Ohsawa, A remark on Kazhdan's theorem on sequences of Bergman metrics. Kyushu J. Math. **63**, 133–137 (2009)
347. B. Okutmustur, A. Gheondea, *Reproducing Kernel Hilbert Spaces* (Lambert Academic Publishing, Saarbücken, 2010)
348. K. Onishi, Q. Wang, *Numerical Solutions of the Cauchy Problem in Potential and Elastostatics*. Inverse problems and related topics (Koebe, 1998). Chapman & Hall/CRC Research Notes in Mathematics, vol. 419 (Chapman & Hall/CRC, Boca Raton, 2000), pp. 115–132
349. Z. Opial, Sur une inégalité. Ann. Polon. Math. **8**, 29–32 (1960)
350. Y. Oshime, Y. Kametaka, H. Yamagishi, The best constant of L^p Sobolev inequality corresponding to Dirichlet boundary value problem for $(d/dx)^{4m}$. Sci. Math. Jpn. **e-2008**, 461–469 (2008)
351. Y. Oshime, K. Watanabe, The best constant of L^p Sobolev inequality corresponding to Dirichlet boundary value problem II. Tokyo J. Math. **34**, 115–133 (2011)
352. Y. Oshime, H. Yamagishi, K.Watanabe, The best constant of L^p Sobolev inequality corresponding to Neumann boundary value problem for $(-1)^M(d/dx)^{2M}$. Hiroshima Math. J. **42**, 1–7 (2012)
353. K.R. Parthasarathy, K. Schmidt, *Positive Definite Kernels, Continuous Tensor Products, and Central Limit Theorems of Probability Theory*, vol. 107 (Springer Lecture Notes in Mathematics, Berlin, 1972)
354. E. Parzen, *Probability Density Functionals and Reproducing Kernel Hilbert Spaces*. 1963 Proceedings of Symposium on Time Series Analysis (Wiley, New York, Brown University, 1962), pp. 155–169
355. E. Parzen, Extraction and detection problems and reproducing Kernel Hilbert spaces. J. SIAM control Ser. A **1**, 35–62 (1962)
356. E. Parzen, Statical inference on time series by RKHS methods, in *Proceedings of 12th Biennial Seminar of the Canadian Mathematical Congress* (American Mathematical Society, Providence, 1971), pp. 1–37
357. J. Peng, S.K. Chung, Laplace transforms and generators of semigroups of operators. Proc. Am. Math. Soc. **126**(8), 2407–2416 (1998)
358. A.D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC, Boca Raton, 2002)
359. A.D. Polyanin, V.F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, Boca Raton, 2003)
360. A.D. Polyanin, A.V. Manzhirov, *Handbook of Integral Equations* (CRC Press, Boca Raton, 2008)
361. E.L. Post, Generalized differentiation. Trans. Am. Math. Soc. **32**(4), 723–781 (1930)
362. A.P. Prudnikov, Yu.A. Brychkov, O.L. Marichev, *More Special Functions*. Integrals and Series, vol. 3 (Gordon and Breach Publisher, New York, 1990)
363. T. Qian, L. Tan, Backward shift invariant subspaces with applications to band preserving and phase retrival problems. Math. Math. Appl. Sci. 7/2015. doi:10.1002/mma3591
364. A.G. Ramm, *Random Fields Estimation Theory*. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 48 (Longman Scientific and Technical, Harlow, 1990)

365. A.G. Ramm, *Multidimensional Inverse Scattering Problems*. Monographs and Surveys in Pure and Applied Mathematics, vol. 51 (Longman Scientific and Technical, Harlow, 1992)
366. T. Ransford, *Potential Theory in the Complex Plane*. London Mathematical Society, Student Texts, vol. 28 (Press Syndicate of the University of Cambridge, Cambridge/New York, 1995)
367. Th.M. Rassias, S. Saitoh, The Pythagorean theorem and linear mappings. PanAmerican Math. J. **12**, 1–10 (2002)
368. H. Rauhut, T. Ullrich, Generalized coorbit space theory and inhomogeneous function spaces of Besov-Lizorkin-Triebel type. J. Funct. Anal. **260**(11), 3299–3362 (2011)
369. M. Reimer, *Constructive Theory of Multivariate Functions with an Application to Tomography* (Wissenschaftsverlag, Mannheim/Wien/Zürich, 1990)
370. H.J. Reinhardt, H. Han, N.H. Dinh, Stability and regularization of a discrete approximation to the Cauchy problem for the Laplace's equation. SIAM J. Numer. Anal. **36**(3), 890–905 (1999)
371. E.M. Rocha, A reproducing kernel Hilbert discretization method for linear PDEs with nonlinear right-hand side. Lib. Math. (New Ser.) **34**(2), 91–104 (2014)
372. W. Rudin, *Functional Analysis*, 2nd edn. (McGraw Hill, New York, 1991)
373. S. Saitoh, The Bergman norm and the Szegö norm. Trans. Am. Math. Soc. **249**(2), 261–279 (1979)
374. S. Saitoh, The Dirichlet norm and the norm of Szegö type. Trans. Am. Math. Soc. **254**, 355–364 (1979)
375. S. Saitoh, Some inequalities for analytic functions with a finite Dirichlet integral on the unit disc. Math. Ann. **246**(1), 69–77 (1979)
376. S. Saitoh, Some inequalities for entire functions. Proc. Am. Math. Soc. **80**(2), 254–258 (1980)
377. S. Saitoh, Reproducing kernels of the direct product of two Hilbert spaces. Riazi J. Karachi Math. Assoc. **4**, 1–20 (1982)
378. S. Saitoh, Hilbert spaces induced by Hilbert space valued functions. Proc. Am. Math. Soc. **89**, 74–78 (1983)
379. S. Saitoh, The Weierstrass transform and an isometry in the heat equation. Appl. Anal. **16**, 1–6 (1983)
380. S. Saitoh, A fundamental inequality in the convolution of L_2 functions on the half line. Proc. Am. Math. Soc. **91**(2), 285–286 (1984)
381. S. Saitoh, Hilbert spaces admitting reproducing kernels on the real line and related fundamental inequalities. Riazi J. Karachi Math. Assoc. **6**, 25–31 (1984)
382. S. Saitoh, *Theory of Reproducing Kernels and Its Applications*. Pitman Research Notes in Mathematics Series, vol. 189 (Longman Scientific & Technical, Harlow, 1988)
383. S. Saitoh, Interpolation problems of Pick-Nevanlinna type. Pitman Res. Notes Math. Ser. **212**, 253–262 (1989)
384. S. Saitoh, Representations of the norms in Bergman-Selberg spaces on strips and half planes. Complex Variab. Theory Appl. **19**(4), 231–241 (1992)
385. S. Saitoh, One approach to some general integral transforms and its applications. Integr. Trans. Spec. Funct. **3**, 49–84 (1995)
386. S. Saitoh, Natural norm inequalities in nonlinear transforms, in *General Inequalities, 7*, Oberwolfach, 1995. International Series of Numerical Mathematics, vol. 123 (Birkhäuser, Basel, 1997), pp. 39–52
387. S. Saitoh, Representations of inverse functions. Proc. Am. Math. Soc. **125**, 3633–3639 (1997)
388. S. Saitoh, *Integral Transforms, Reproducing Kernels and Their Applications*. Pitman Research Notes in Mathematics Series, vol. 369 (Addison Wesley Longman Ltd, 1998). CRC Press/Taylor & Francis Group, Boca Raton/London/New York (in hard cover)
389. S. Saitoh, *Nonlinear Transforms and Analyticity of Functions*. Nonlinear Mathematical Analysis and Applications (Hadronic Press, Palm Harbor, 1998), pp. 223–234
390. S. Saitoh, Various operators in Hilbert space introduced by transforms. Int. J. Appl. Math. **1**, 111–126 (1999)
391. S. Saitoh, Applications of the general theory of reproducing kernels, *Reproducing Kernels and Their Applications* (Kluwer Academic Publishers, 1999), pp. 165–188

392. S. Saitoh, D. Alpay, J.A. Ball, T. Ohsawa (eds), *Reproducing Kernels and Their Applications* (Kluwer Academic Publishers, 1999)
393. S. Saitoh, M. Yamamoto, Integral transforms involving smooth functions, in *Reproducing Kernels and Their Applications* (Kluwer Academic Publishers, 1999), pp. 149–164
394. S. Saitoh, Linear integro-differential equations and the theory of reproducing kernels, in *Volterra Equations and Applications*, ed. by C. Corduneanu, I.W. Sandberg (Gordon and Breach Science Publishers, Amsterdam, 2000)
395. S. Saitoh, Weighted L_p-norm inequalities in convolutions, in *Survey on Classical Inequalities*. Mathematics and Its Applications, vol. 517 (Kluwer Academic Publisher, Dordrecht, 2000), pp. 225–234
396. S. Saitoh, V.K. Tuan, M. Yamamoto, Reverse weighted L_p-norm inequalities in convolutions. J. Inequal. Pure Appl. Math. **1**(1), Article 7, 7 (2000)
397. S. Saitoh, Analytic extension formulas, integral transforms and reproducing kernels, in *Analytic Extension Formulas and Their Applications* (Kluwer Academic Publishers, Dordrecht/Boston, 2001), pp. 207–232
398. S. Saitoh, Applications of the reproducing kernel theory to inverse problems. Comm. Korean Math. Soc. **16**, 371–383 (2001)
399. S. Saitoh, Principle of telethoscope,in *Functional-Analytic and Complex Methods, Their Interaction and Applications to Partial Differential Equations – Proceedings of the International Graz Workshop*, Graz, 12–16 Feb 2001 (World Scientific, 2001), pp. 101–117
400. S. Saitoh, N. Hayashi, M. Yamamoto (eds.), *Analytic Extension Formulas and Their Applications* (Kluwer Academic Publishers, Dordrecht/Boston, 2001)
401. S. Saitoh, V.K. Tuan, M. Yamamoto, Conditional stability of a real inverse formula for the laplace transform. Z. Anal. Anw. **20**, 193–202 (2001)
402. S. Saitoh, V.K. Tuan, M. Yamamoto, Reverse convolution inequalities and applications to inverse heat source problems. J. Inequal. Pure Appl. Math. **3**(5), Article 80, 11 (2002)
403. S. Saitoh, *Introduction to the Theory of Reproducing Kernels* (in Japanese) (Makino-Shoten Co., 2002)
404. S. Saitoh, T. Matsuura, M. Asaduzzaman, Operator equations and best approximation problems in reproducing kernel Hilbert spaces. J. Anal. Appl. **1**, 131–142 (2003)
405. S. Saitoh, Constructions by reproducing kernels of approximate solutions for linear differential equations with L_2 integrable coefficients. Int. J. Math. Sci. **2**, 261–273 (2003)
406. S. Saitoh, Approximate real inversion formulas of the Laplace transform. Far East J. Math. Sci. **11**, 53–64 (2003)
407. S. Saitoh, Reproducing kernels and a family of bounded linear operators. Oper. Theory Adv. Appl. **143**, 303–312 (2003)
408. S. Saitoh, Generalizations of the triangle inequality. J. Inequal. Pure Appl. Math. **4**(3), Article 62 (2003)
409. S. Saitoh, Approximate real inversion formulas of the Gaussian convolution. Appl. Anal. **83**, 727–733 (2004)
410. S. Saitoh, Best approximation, Tikhonov regularization and reproducing kernels. Kodai. Math. J. **28**, 359–367 (2005)
411. S. Saitoh, Tikhonov regularization and the theory of reproducing kernels, in *Finite or Infinite Dimensional Complex Analysis and Applications (Proceedings of the 12th ICFIDCAA)*, (Kyushu University Press, Fukuoka-shi, 2005), pp. 291–298
412. S. Saitoh, Applications of reproducing kernels to best approximations, Tikhonov regularizations and inverse problems, in *Advances in Analysis, Proceedings of the 4th International ISAAC Congress* (World Scientific, Singapore/Hackensack, 2005), pp. 439–445
413. S. Saitoh, T. Matsuura, M. Asaduzzaman, *Operator Equations and Best Approximation Problems in Reproducing Kernel Hilbert Spaces with Tikhonov Regularization* (Advances in Analysis World Scientific Publication, Hackensack, 2005), pp. 99–107
414. S. Saitoh, M. Yamada, Inversion formulas for a linear system determined by input and response relations, by using suitable function spaces. Hokkaido Univ. Tech. Report Ser. Math. **118**, 18–21 (2007)

415. S. Saitoh, Theory of reproducing kernels; applications to approximate solutions of bounded linear operator equations on Hilbert spaces. Am. Math. Soc. Transl. **230**, 107–134 (2010)
416. S. Saitoh, Generalized inversions of Hadamard and tensor products for matrices. Adv. Linear Algebra Matrix Theory **4**(2), 87–95 (2014)
417. S. Saitoh, Y. Sawano, *Generalized Delta Functions as Generalized Reproducing Kernels* (manuscript)
418. S. Saitoh, Y. Sawano, *General Initial Value Problems Using Eigenfunctions and Reproducing Kernels* (manuscript)
419. Y. Sawano, H. Fujiwara, S. Saitoh, Real inversion formulas of the Laplace transform on weighted function spaces. Complex Anal. Oper. Theory **2**, 511–521 (2008)
420. Y. Sawano, M. Yamada, S. Saitoh, Singular integrals and natural regularizations. Math. Inequal. Appl. **13**, 289–303 (2010)
421. Y. Sawano, Pasting reproducing kernel Hilbert spaces. Jaen J. Approx. **3**(1), 135–141 (2011)
422. Y. Sawano, H. Tanaka, Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces. Math. Z. **257**(4), 871–905 (2007)
423. Y. Sawano, A quick introduction to vector analysis, Kyoritsu (2014, in Japanese)
424. S. Scheinberg, Uniform approximation by meromorphic functions having prescribed poles. Math. Ann. **243**(1), 83–93 (1979)
425. M. Schiffer, D.C. Spencer, Functionals of finite Riemann surfaces (Princeton University Press, Princeton, 1954)
426. E. Schmidt, Über die Charlier–Jordansche Entwicklung einer willkürlichen Funktion nach der Poissonschen Funktion und ihren Ableitungen. Z. Angew. Math. Mech. **13**, 139–142 (1933)
427. B. Schölkopf, A.J. Smola, *Learning with Kernels* (MIT Press, Cambridge, 2002)
428. L. Schwartz, Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants). J. Anal. Math. **13**, 115–256 (1964)
429. C.E. Shannon, A mathematical theory of communication. Bell Syst. Tech. J. **27**, 379–423, 623–656 (1948)
430. T. Shirai, Fermion measure and related topics (in Japanese). www.math.kyoto-u.ac.jp/probability/sympo/shirai.pdf
431. H.S. Shapiro. A.L. Shields, On the zeros of functions with finite Dirichlet integral and some related function spaces. Math. Z. **80**, 217–229 (1962)
432. B. Simon, Weak convergence of CD kernels and applications. Duke Math. J. **146**, 305–330 (2009)
433. B. Simon, *The Christoffel-Darboux Kernel. Perspectives in Partial Differential Equations, Harmonic Analysis and Applications*. Proceedings of Symposia in Pure Mathematics, vol. 79 (American Mathematical Society, Providence, 2008), pp. 295–335
434. G.D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Methods* (Clarendon Press, Oxford, 1978)
435. F. Soltani, Inversion formulas in the Dunkl-type heat conduction on $\mathbb{R}^d$. Appl. Anal. **84**, 541–553 (2005)
436. F. Soltani, Practical inversion formulas in a quantum mechanical system. Appl. Anal. **84**, 759–767 (2005)
437. F. Soltani, Multiplication and translation operators on the Fock spaces for the q-modified Bessel function. Adv. Pure Math. (APM) **1**, 221–227 (2011)
438. F. Soltani, Toeplitz and translation operators on the q-Fock spaces. Adv. Pure Math. (APM) **1**, 325–333 (2011)
439. F. Soltani, Best approximation formulas for the Dunkl L^2-multiplier operators on $\mathbb{R}^d$. Rocky Mt. J. Math. (RMJM) **42**, 305–328 (2012)
440. F. Soltani, Fock spaces for the q-Dunkl kernel. Adv. Pure Math. (APM) **2**, 169–176 (2012)
441. F. Soltani, Multiplier operators and extremal functions related to the dual Dunkl-Sonine operator. Acta Math. Sci. **33**(2), 430–442 (2013)
442. F. Soltani, Extremal functions on Sturm-Liouville hypergroups. Complex Anal. Oper. Theory **8**(1), 311–325 (2014)

443. E.M. Stein, *Singular Integral and Differential Property of Functions* (Princeton University Press, Princeton, 1970)
444. E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* (Princeton University Press, Princeton, 1993)
445. I. Steinwart, A. Christmann, *Support Vector Machine* (Springer, New York, 2008)
446. F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*. Springer Series in Computational Mathematics, vol. 20 (Springer, New York, 1993)
447. J. Stewart, Positive definite functions and generalizations, an historical survey. Rocky Mt. J. Math. **6**(3), 409–434 (1976)
448. K. Stroethoff, The Berezin transform and operators on spaces of analytic functions. Linear Oper. Banach Center Publ. **38**, 361–380 (1997). Institute of Mathematics Polish Academy of Sciences, Warsaw
449. S. Szegö, Über orthogonale Polynome, die zu einer gegebenen Kurve der Komplexen Ebene gehören. Math. Z. **9**, 218–270 (1921)
450. F.H. Szafraniec, Multipliers in the reproducing kernel Hilbert space, subnormality and noncommutative complex analysis, in *Reproducing Kernel Spaces and Applications*. Operator Theory: Advances and Applications, vol. 143 (Birkhäuser, Basel, 2003), pp. 313–331
451. G. Szegö, Orthogonal polynomials (American Mathematical Society, Providence, 1959)
452. H. Takahasi, M. Mori, Double exponential formulas for numerical integration. Publ. RIMS. Kyoto Univ. **10**, 721–741 (1974)
453. M. Seto, S. Suda, T. Taniguchi, Gram matrics of reproducing kernel Hilbert spaces over graphs. Linear Algebra Appl. **445**, 56–68 (2014)
454. S.-E. Takahasi, J.M. Rassias, S. Saitoh, Y. Takahashi, Refined generalizations of the triangle inequality on Banach spaces. Math. Inequal. Appl. **13**(4), 733–741 (2010)
455. K. Takemura, H. Yamagishi, Y. Kametaka, K. Watanabe, A. Nagai, The best constant of Sobolev inequality correspondence to a bending problem of a beam on an interval. Tsukuba J. Math. **33**(2), 253–280 (2009)
456. K. Takemura, The best constant of Sobolev inequality corresponding to clamped-free boundary value problem for $(-1)^M(d/dx)^{2M}$. Proc. Jpn. Acad. Ser. A Math. Sci. **85**(8), 112–117 (2009)
457. K.Takemura, Y. Kametaka, K. Watanabe, A. Nagai, H. Yamagishi, The best constant of Sobolev inequality corresponding to a bending problem of a beam on a half line. Far East J. Appl. Math. **51**(1), 45–71 (2011)
458. H. Tanabe, *Functional Analysis* (in Japanese) (Jikkyo, Tokyo, 2003)
459. C. Thomas-Agnan, Computing a family of reproducing kernels for statistical applications. Numer. Algorithms **13**(1–2), 21–32 (1996)
460. A.N. Tikhonov, V.Y. Arsenin, *Solutions of Ill-Posed Problems*, Translated from the Russian. Preface by translation editor Fritz John. Scripta Series in Mathematics (V.H. Winston & Sons/Wiley, Washington, DC/New York/Toronto/London, 1977)
461. V. Totik, Asymptotics for Christoffel functions for general measures on the real line. J. Anal. Math. **81**, 282–303 (2000)
462. H. Triebel, *The Structure of Functions* (Birkhäuser, Basel, 2000)
463. J.N. Tsitsiklis, B.C. Levy, Integral equations and resolvents of Toeplitz plus Hankel kernels. Technical report LIDS-P-1170, Laboratory for Information and Decision Systems, M.I.T., Silver Edition (1981)
464. N.M. Tuan, N.T.T. Huyen, The solvability and explicit solutions of two integral equations via generalized convolutions. J. Math. Anal. Appl. **369**, 712–718 (2010)
465. N.M. Tuan, N.T.T. Huyen, The Hermite functions related to infinite series of generalized convolutions and applications. Complex Anal. Oper. Theory **6**, 219–236 (2012)
466. N.M. Tuan, P.D. Tuan, Generalized convolutions relative to the Hartley transforms with applications. Sci. Math. Jpn. **70**, 77–89 (2009)
467. V.K. Tuan, D.T. Duc, A new real inversion formula of the Laplace transform and its convergence rate. Dedicated to the 60th anniversary of Prof. Francesco Mainardi. Fract. Calc. Appl. Anal. **5**(4), 387–394 (2002)

468. V.K. Tuan, N.T. Hong, Interpolation in the Hardy space. Integr. Transf. Spec. Funct. **24**(8), 664–671 (2013)
469. W. Ulmer, W. Kaissl, The inverse problem of a Gaussian convolution and its application to the finite size of measurement chambers/detectors in photon and proton dosimetry. Phys. Med. Biol. **48**, 707–727 (2003)
470. W. Ulmer, Inverse problem of linear combinations of Gaussian convolution kernels (deconvolution) and some applications to proton/photon dosimetry and image processing. Inverse Probl. **26**(8), Article ID 085002 (2010)
471. V. Vapnik, *Statistical Learning Theory* (Wiley, New York, 1998)
472. G. Wahba, On the approximate solution of Fredholm integral equations of the first kind. Technical summit report 990, Mathematics Research Center, University of Wisconsin-Madison (1969)
473. G. Wahba, Convergence rates of certain approximate solutions to Fredholm integral equations of the first kind. J. Approx. Theory **7**, 167–185 (1973)
474. G. Wahba, A class of approximate solutions to linear operator equations. J. Approx. Theory **9**, 61–77 (1973)
475. G. Wahba, Practical approximate solutions to linear operators equations when the data are noisy. SIAM J. Numer. Anal. **14**, 651–667 (1977)
476. G. Wahba, *Spline Models for Observational Data*. CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 59 (SIAM, Philadelphia, 1990)
477. W. Wang, B. Han, M. Yamamoto, Inverse heat problem of determining time-dependent source parameter in reproducing kernel space. Nonlinear Anal. Real World Appl. **14**(1), 875–887 (2013)
478. K. Watanabe, Y. Kametaka, A. Nagai, K. Takemura, H. Yamagishi, The best constant of Sobolev inequality on a bounded interval. J. Math. Anal. Appl. **340**, 699–706 (2008)
479. K. Watanabe, Y. Kametaka, H. Yamagishi, A. Nagai, K. Takemura, The best constant of Sobolev inequality corresponding to clamped boundary value problem. Bound. Value Probl. **2011**(Article ID 875057), 17
480. K. Watanabe, Y. Kametaka, A. Nagai, K. Takemura, H. Yamagishi, The best constants of Sobolev and Kolmogorov type inequalities on a half line. Far East J. Appl. Math. **52**(2), 101–129 (2011)
481. T. Wei, Y.C. Hon, J. Cheng, Computation for multidimensional Cauchy problem. SIAM J. Control Optim. **42**(2), 381–396 (2003) (electronic)
482. D.V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, 1972)
483. K. Yajima, *Lebesgue Integral and Functional Analysis* (in Japanese) (Asakura Shoten, 2002)
484. A. Yamada, Fay's trisecant formula and Hardy H^2 reproducing kernels, in *Reproducing Kernels and Their Applications* (Kluwer Academic Publishers, 1999), pp. 165–188
485. A. Yamada, Equality conditions for general norm inequalities in reproducing kernel Hilbert spaces, in *Advances in Analysis* (World Scientific, 2005), pp. 447–455
486. A. Yamada, Saitoh's inequality and Opial's inequality. Math. Inequal. Appl. **14**, 523–528 (2011)
487. A. Yamada, Oppenheim's inequality and RKHS. Math. Inequal. Appl. **15**(2), 449–456 (2012)
488. A. Yamada, Inequalities for Gram matrices and their applications to reproducing kernel Hilbert spaces. Taiwan. J. Math. **17**(2), 427–430 (2013)
489. M. Yamada, T. Matsuura, S. Saitoh, Representations of inverse functions by the integral transform with the sign kernel. Fract. Calc. Appl. Anal. **10**, 161–168 (2007)
490. M. Yamada, S. Saitoh, Identification of non-linear systems. J. Comput. Math. Optim. **4**, 47–60 (2008)
491. M. Yamada, S. Saitoh, Numerical solutions of two non-linear simultaneous equations. Appl. Anal. **88**, 151–160 (2009)
492. M. Yamada, S. Saitoh, Explicit and direct representations of the solutions of nonlinear simultaneous equations, in *Progress in Analysis and Its Applications Proceedings of the 7th International ISAAC Congress* (World Scientific, Singapore/Hackensack/London, 2010), pp. 372–378

493. K. Yao, Application of reproducing kernel Hilbert spaces – bandlimited signal models. Inf. Control **11**, 429–444 (1967)
494. L. Yingzhen, Z. Yongfang, Solving nonlinear pseudoparabolic equations with nonlocal boundary conditions in reproducing kernel space. Numer. Algor. **52**, 173–186 (2009)
495. N. Young, *An Introduction to Hilbert Space*. Cambridge Mathematical Textbooks (Cambridge University Press, Cambridge/New York, 1988)
496. K. Yoshida, Functional Analysis (Springer, Berlin, 1974)
497. S. Zaremba, *L'equation biharminique et une class remarquable de fonctions foundamentals harmoniques*. Bull. Int. de l'Academie des Sci. de Cracovie **39**, 147–196 (1907)
498. A.I. Zayed, On Kramer's sampling theorem associated with general Sturm–Liouville problems and Lagrange interpolation. SIAM J. Appl. Math. **51**, 575–604 (1991)
499. A.I. Zayed, *Advances in Shannon's Sampling Theory* (CRC Press, Boca Raton, 1993)
500. M. Zedek, On the Lebedev-Milin inequality. Proc. Am. Math. Soc. **33**, 395–397 (1972)
501. X.D. Zhang, Vertex degrees and doubly stochastic graph matrices. J. Graph Theory **66**(2), 104–114 (2011)
502. D.X. Zhou, Capacity of reproducing kernel spaces in learning theory. IEEE Trans. Inform. Theory **49**, 1743–1752 (2003)

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