Appendix A
Notational Conventions

Throughout the book we use Einstein's implicit summation convention: repeated indices in an expression are automatically summed over.

We work in natural units where the Planck constant is $\hbar = 1$ and the speed of light is $c = 1$. In these units, energy and mass have the same mass dimension equal to +1 and time and length have the same mass dimension equal to -1. A useful relation is $1 = 197\,\text{MeVfm}$, which allows the conversion between energy in units of MeV[1] and length in units of fm.[2]

In Lattice Quantum Chromodynamics the fields representing the quarks and the gluons are defined on a Euclidean lattice in four dimensions with lattice spacing a. The directions on the lattice are labelled by $\mu = 0, 1, 2, 3$. The lattice points have coordinates $x = (x_0, x_1, x_2, x_3)$ which are integer multiples of a: $x_\mu = n_\mu a$, $n_\mu \in \mathbb{N}$ for $\mu = 0, 1, 2, 3$. The coordinate x_0 is referred to as the Euclidean time and the spatial coordinates are collectively denoted by $\underline{x} = (x_1, x_2, x_3)$. The nearest neighbor of a lattice point x in positive μ direction is denoted by $x + a\hat{\mu}$, where $\hat{\mu}$ is the unit vector in direction μ. If the lattice has a finite volume $V = T \times L^3$ then the ranges of the coordinates are $x_0/a = 0, 1, \ldots, T/a - 1$ and $x_k = 0, 1, \ldots, L/a - 1$ for $k = 1, 2, 3$. Unless otherwise specified, in finite volume we impose periodic boundary conditions, which mean that the points $x + T\hat{0}$ and $x + L\hat{k}$ for $k = 1, 2, 3$ are identified with x.

[1] MeV means Mega electronvolt; 1 eV is the potential energy that an electron aquires when a tension of 1 V is applied.

[2] fm means fermi; $1\,\text{fm} = 10^{-15}\,\text{m}$.

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A.1 $SU(N)$ Notation

An $SU(N)$ gauge field on the lattice assign an element $U_\mu(x)$ of $SU(N)$ to the link connecting the point $x + a\hat{\mu}$ with x. $SU(N)$ is the special unitary group. In the fundamental representation, the elements U are $N \times N$ complex matrices which satisfy

$$U \in SU(N) \Leftrightarrow U^{-1} = U^\dagger \equiv (U^T)^* \,,\ \det(U) = 1 \,. \tag{A.1}$$

The Lie algebra $\mathfrak{su}(N)$ of $SU(N)$ may be identified with the linear space of all anti-hermitian traceless $N \times N$ matrices X:

$$X^\dagger = -X \,, \quad \text{and} \quad \mathrm{tr}\, X = 0 \,. \tag{A.2}$$

We denote the generators (basis) of $\mathfrak{su}(N)$ by T^i, $i = 1, \ldots, N^2 - 1$. Any element X of $\mathfrak{su}(N)$ can written as

$$X = X^i T^i \,, \quad X^i = -2\mathrm{tr}\, \left\{ X T^i \right\} \,,$$

in terms of real components X^i. The normalisation of the generators is given by

$$\mathrm{tr}\, \left\{ T^i T^j \right\} = -\frac{1}{2} \delta^{ij} \,. \tag{A.3}$$

The natural scalar product in $\mathfrak{su}(N)$ is

$$\langle X, Y \rangle = X^i Y^i = -2\mathrm{tr}\, \{XY\} \,. \tag{A.4}$$

The structure constants f^{ijk} defined by the commutation relation

$$[T^i, T^j] = f^{ijk} T^k \tag{A.5}$$

are real and totally anti-symmetric in the indices i, j, k. They satisfy $f^{ikl} f^{jkl} = N \delta^{ij}$. The exponential map

$$\mathrm{e}^X = I + \sum_{k=1}^{\infty} \frac{X^k}{k!} \,,$$

maps an element $X \in \mathfrak{su}(N)$ onto the group $SU(N)$. Differentiation of a function of the lattice gauge field $f(U)$ with respect to a link $U_\mu(x)$ is defined by the link differential operators [1]

$$\partial_{x,\mu} f(U) = T^i \partial^i_{x,\mu} f(U) \quad \text{with} \quad \partial^i_{x,\mu} f(U) = \left. \frac{\mathrm{d} f(U_s)}{\mathrm{d}s} \right|_{s=0} \,. \tag{A.6}$$

In the derivative with respect to s the gauge field U_s is given by

$$(U_s)_\nu(y) = \begin{cases} e^{sT^i} U_\mu(x) & \text{if } (y, \nu) = (x, \mu) \\ U_\nu(y) & \text{otherwise} \end{cases} . \tag{A.7}$$

Useful formulae are

$$\left.\frac{\mathrm{d}(U_s)_\mu(x)}{\mathrm{d}s}\right|_{s=0} = T^i U_\mu(x)\,, \quad \text{and} \quad \left.\frac{\mathrm{d}(U_s)_\mu^{-1}(x)}{\mathrm{d}s}\right|_{s=0} = -U_\mu^{-1}(x) T^i\,,$$

where the second expression follows from $0 = \left.\frac{\mathrm{d}(U_s)_\mu^{-1}(x)(U_s)_\mu(x)}{\mathrm{d}s}\right|_{s=0}$.

A.2 Fermions

A fermion field assigns Dirac spinors $\psi_j^\alpha(x)$ and $\bar{\psi}_j^\alpha(x)$ to each lattice point x. The spinor index is $\alpha = 1, 2, 3, 4$ and the colour index in the fundamental representation of $SU(N)$ is $j = 1, 2, \ldots, N$. The components of the fermion field are Grassmann numbers.

The Grassmann algebra of q Grassmann numbers $\{\eta_1, \eta_2, \ldots, \eta_q\}$ is defined as

$$\{\eta_i, \eta_j\} = 0\,, \quad i, j = 1, 2, \ldots, q\,. \tag{A.8}$$

For a single Grassmann variable, the Berezin integral is

$$\int d\eta = 0, \quad \text{and} \quad \int d\eta\ \eta = 1.$$

These are not integrals in our usual intuitive sense of the area under a curve but should be considered as abstract and simple rules that define the fermion path integral in a physically sensible way in analogy to the bosonic path integral. Since $\eta^2 = 0$ for a Grassmannian, any function of this single variable can be written

$$f(\eta) = f_0 + f_1\eta,$$

with f_0 and f_1 two constants and so the integral of any function is then

$$\int d\eta\ f(\eta) = f_1$$

and we see the first peculiar property of a Grassmann variable; integration is equivalent to differentiation. Now consider integrating q Grassmannians. The rules above apply straightforwardly, but care is needed to take the order of integration appropriately since Grassmann variables anti-commute. We have

$$\int \mathscr{D}[\eta] \;\; \eta_1 \eta_2 \ldots \eta_q = 1 \tag{A.9}$$

with $\mathscr{D}[\eta] = \prod_{k=1}^{q} d\eta_k$. All other integrals, which have at least one of the variables missing, vanish. A function of q Grassmannians will have 2^N coefficients in its most general form and can be written in a binary notation. For example, with $q = 3$

$$f(\eta_1, \eta_2, \eta_3) =$$
$$f_{000} + f_{100}\eta_1 + f_{010}\eta_2 + f_{001}\eta_3 + f_{110}\eta_1\eta_2 + f_{101}\eta_1\eta_3 + f_{011}\eta_2\eta_3 + f_{111}\eta_1\eta_2\eta_3,$$

and the integral result follows easily;

$$\int \mathscr{D}[\eta] \;\; f(\eta_1, \eta_2, \ldots, \eta_q) = f_{11\ldots1}.$$

An important role is played by "Gaussian" integrals. The Matthews–Salam formula is

$$\int \mathscr{D}[\eta]\mathscr{D}[\bar{\eta}] \;\; \mathrm{e}^{\bar{\eta}_i M_{ij} \eta_j} = \det(M)\,, \tag{A.10}$$

where $\{\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_q\}$ is a second set of independent Grassmann numbers. Consider a linear transformation of the Grassmann variables $\eta = A\eta'$, where A is a complex matrix. Using the chain of equalities

$$\det(M) = \int \mathscr{D}[A\eta']\mathscr{D}[\bar{\eta}]\mathrm{e}^{\bar{\eta}M(A\eta')} = J(A)\int \mathscr{D}[\eta']\mathscr{D}[\bar{\eta}]\mathrm{e}^{\bar{\eta}(MA)\eta'} = J(A)\det(MA) \tag{A.11}$$

we conclude that the Jacobian of the transformation is $J(A) = \det(A)^{-1}$. Similarly, the transformation $\bar{\eta} = \bar{\eta}'B$ leads to a Jacobian $\mathscr{D}[\bar{\eta}'B] = J(B)\mathscr{D}[\bar{\eta}']$ with $J(B) = \det(B)^{-1}$.

On an infinite lattice we can represent the fermion fields as integrals

$$\psi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{ipx} \tilde{\psi}(p)\,, \tag{A.12}$$

$$\bar{\psi}(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-ipx} \tilde{\bar{\psi}}(p)\,. \tag{A.13}$$

The momenta $p = (p_\mu)$ are defined over the Brillouin zone $p_\mu \in [-\pi/a, \pi/a]$. The Fourier components can be computed by Fourier transformation as

$$\tilde{\psi}(p) = a^4 \sum_x \mathrm{e}^{-ipx} \psi(x)\,, \tag{A.14}$$

$$\tilde{\bar{\psi}}(p) = a^4 \sum_x \mathrm{e}^{ipx} \bar{\psi}(x)\,. \tag{A.15}$$

The Wilson–Dirac operator acts on fermion fields according to Eq. (1.83). There, the 4×4 Dirac matrices γ_μ, $\mu = 0, 1, 2, 3$ act in spinor space. They are Hermitian $(\gamma_\mu)^\dagger = \gamma_\mu$ and satisfy the anti-commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\,. \tag{A.16}$$

Since γ_μ is Hermitian, it follows $(\gamma_\mu^*)^{\alpha\beta} = \gamma_\mu^{\beta\alpha}\ \forall\mu$. An explicit choice is given by the chiral representation of the Dirac matrices, where

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ (e_\mu)^\dagger & 0 \end{pmatrix}. \tag{A.17}$$

A possible choice for the 2×2 matrices e_μ is

$$e_0 = -I\,, \qquad e_k = -i\sigma_k\,,\ k = 1, 2, 3\,, \tag{A.18}$$

where σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ with the properties

$$(\gamma_5)^\dagger = \gamma_5\,, \qquad (\gamma_5)^2 = I\,, \qquad \{\gamma_\mu, \gamma_5\} = 0\ \forall\mu\,. \tag{A.19}$$

In the chiral representation Eq. (A.17) of the Dirac matrices we have

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and for the matrices $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$

$$\sigma_{0k} = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \qquad \sigma_{ij} = -\varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix},$$

where ε_{ijk} is the totally anti-symmetric tensor with $\varepsilon_{123} = 1$.

A pseudofermion field is defined like the fermion field to have space x, spin α and color i indices but it takes complex values instead of being Grassmann-valued. The scalar product of two pseudofermion fields ϕ and ψ is defined as

$$\langle \phi, \psi \rangle = \sum_{x,\alpha,j} \phi_j^{*\alpha}(x)\, \psi_j^{\alpha}(x)\,. \tag{A.20}$$

We remind the useful properties $\langle \phi, \psi \rangle^* = \langle \psi, \phi \rangle$ and $\langle \phi, A\,\psi \rangle = \langle A^\dagger \phi, \psi \rangle$ for any matrix A acting on the pseudofermion fields. The scalar product in Eq. (A.20) can be rewritten as

$$\langle \phi, \psi \rangle = \sum_{x} \mathrm{tr}_{\,\sigma,c} \left[\psi(x) \phi^\dagger(x) \right]\,. \tag{A.21}$$

Here, $M = \psi(x)\phi^\dagger(x)$ is a matrix in colour and spinor space with elements $M_{jk}^{\alpha\beta} = \psi_j^{\alpha}(x)\phi_k^{*\beta}(x)$ and $\mathrm{tr}_{\,c}$ means the trace over the colour indices and $\mathrm{tr}_{\,\sigma}$ the trace over the spinor indices.

A.3 Probability Spaces

A *probability space* $(\Omega, \mathscr{F}, P)$ consists of a non-empty set Ω, a σ*-algebra* $\mathscr{F}$ and a *probability measure* P mapping $\mathscr{F}$ onto [0, 1]. Hereby a set of subsets of Ω is called σ-algebra, if the following three conditions hold:

1. $\Omega \in \mathscr{F}$.
2. $A \in \mathscr{F} \Rightarrow \Omega \setminus A \in \mathscr{F}$.
3. $A_i \in \mathscr{F}$ for $i = 1, 2, \ldots \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathscr{F}$.

For P being a probability measure, $P(\Omega) = 1$ has to hold, as well as $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for A_i being pairwise disjoint.

A *random variable* $X : \Omega \to \mathbb{R}^n$ is a measurable function mapping Ω onto $\mathbb{R}^n$, i.e., $f^{(-1)}(B) \in \mathscr{F}$ for all open sets $B \in \mathbb{R}^n$. Any measurable function f with the property

$$P(A \in \Omega) = \int_{X^{-1}A} dP = \int_A f\, dx$$

for all open set $A \in \mathbb{R}^n$ is called a probability density function.

Reference

1. M. Lüscher, Commun. Math. Phys. **293**, 899 (2010). doi:10.1007/s00220-009-0953-7