Conclusion

Differential equations plays major role in applications of sciences and engineering. It arises in wide variety of engineering applications for e.g. electromagnetic theory, signal processing, computational fluid dynamics, etc. These equations can be typically solved using either analytical or numerical methods. Since many of the differential equations arising in real life application cannot be solved analytically or we can say that their analytical solution does not exist. For such type of problems certain numerical methods exists in the literature. In this book, our main focus is to present an emerging meshless method based on the concept of neural networks for solving differential equations or boundary value problems of type ODE's as well as PDE's. Here in this book, we have started with the fundamental concept of differential equation, some real life applications where the problem is arising and explanation of some existing numerical methods for their solution. We have also presented some basic concept of neural network that is required for the study and history of neural networks. Different neural network methods based on multilayer perceptron, radial basis functions, multiquadric functions and finite element etc. are then presented for solving differential equations. It has been pointed out that the employment of neural network architecture adds many attractive features towards the problem compared to the other existing methods in the literature. Preparation of input data, robustness of methods and the high accuracy of the solutions made these methods highly acceptable. The main advantage of the proposed approach is that once the network is trained, it allows evaluation of the solution at any desired number of points instantaneously with spending negligible computing time.

Moreover, different hybrid approaches are also available and the work is in progress to use better optimization algorithms. People are also working in the combination of neural networks to other existing methods to propose a new method for construction of a better trail solution for all kind of boundary value problems. Such a collection could not be exhaustive; indeed, we can hope to give only an indication of what is possible.

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Appendix

Matlab Pseudo Code for the Solution of Differential Equation Using MLP Neural Network

```
1. *************** Training *************************************************

function [G yT u v w] = DE3(u,v,w,xk,eta,F,g)
h = length(w);
[yT dyT d2yT sig sig1 sig2 sig3 sig4] = DE1(u,v,w,xk);

************************** Preparation************************************
G=0;
G = (d2yT + F*dyT - g);
for j=1:h
   dNdp        =          sig (j,:);
   d2Ndxdp     =     w(j)  *sig1(j,:);
   d3Ndx2dp    =     w(j)^2*sig2(j,:);
   d4Ndx3dp    =     w(j)^3*sig3(j,:);
   d5Ndx4dp    =     w(j)^4*sig4(j,:);
   dGdv        =     -2.*dNdp + 2.*(12-2.*xk).*d2Ndxdp + (12.*xk-xk.^2).*d3Ndx2dp +...
                      F.*((12-2.*xk).*dNdp + (12*xk-xk.^2).*d2Ndxdp)- g;


   dNdp        =      v(j)       *sig1(j,:);
   d2Ndxdp     =     v(j)   *w(j)  *sig2(j,:);
   d3Ndx2dp    =     v(j)   *w(j)^2*sig3(j,:);
   d4Ndx3dp    =     v(j)   *w(j)^3*sig3(j,:);
   d5Ndx4dp    =     v(j)   *w(j)^4*sig4(j,:);
   dGdu         =     -2.*dNdp + 2.*(12-2.*xk).*d2Ndxdp + (12.*xk-xk.^2).*d3Ndx2dp +...
                       F.*((12-2.*xk).*dNdp + (12*xk-xk.^2).*d2Ndxdp)- g;


   dNdp        =      v(j)*xk      .*sig1(j,:);
   d2Ndxdp     =      v(j)*xk*w(j)  .*sig2(j,:); +   v(j)     * sig1(j,:);
   d3Ndx2dp    =      v(j)*xk*w(j)^2.*sig3(j,:) + 2*v(j)*w(j)*sig2(j,:);
   d4Ndx3dp    =      v(j)*xk*w(j)^3.*sig3(j,:) + 3*v(j)*w(j)^2*sig3(j,:);
   d5Ndx4dp    =      v(j)*xk*w(j)^4.*sig4(j,:)+ 4*v(j)*w(j)^3*sig4(j,:);
   dGdw        =      -2.*dNdp + 2.*(12-2.*xk).*d2Ndxdp + (12.*xk-xk.^2).*d3Ndx2dp +...
                       F.*((12-2.*xk).*dNdp + (12*xk-xk.^2).*d2Ndxdp)- g;
```


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```
**************************Updation**************************************

  v(j)        =        v(j) - eta*sum(2*G.*dGdv);
   u(j)       =        u(j) - eta*sum(2*G.*dGdu);
   w(j)       =        w(j) - eta*sum(2*G.*dGdw);
end
```

*************End*****************

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