

# Subject Index

## Symbols

$(b, B)$ -coboundaries, 83  
 $(b, B)$ -cocycle, 83  
 $B$  (differential), 81  
 $b$  (differential), 81  
 $C(X)$ -algebra, 115, 117  
 $C^*$ -algebra, 64, 104, 115  
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# Notation Index

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- $\|\cdot\|_{\text{Lip}}$ , 62
- $\|\cdot\|_s$ , 85
- $\|\cdot\|$ , 12
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