APPENDIX I

PADÉ AND SHANKS TRANSFORMS

PADÉ APPROXIMANTS:

The objective here is to find a solution in the large, i.e., in the range $(0,\infty)$ from the decomposition series which normally has a finite circle of convergence for initial-value problems. The procedure is to seek a rational function for the series. Given a function f(z) expanded in a Maclaurin series $f(z)=\sum_{n=0}^{\infty} c_n z^n$, we can use the coefficients of the series to represent the function by a ratio of two polynomials

$$\frac{a_0 + a_1 z + \cdots + a_L z^L}{b_0 + b_1 z + \cdots + b_m z^M}$$

symbolized by [L/M] and called the Padé approximant. The basic idea is to match the series coefficients as far as possible. Even though the series has a finite region of convergence, we can obtain the limit of the function as $x \to \infty$ if $L = M$.

Notice that if we are satisfied with [1/1], we will have

$$(a_0 + a_1 z) = (b_0 + b_1 z)(c_0 + c_1 z + c_2 z^2 + \cdots)$$

so that coefficients of z^2 are zero, i.e.,

$$b_1 c_1 + b_0 c_2 = 0$$

Taking $b_0 = 1$, we have

$$b_1 c_1 + c_2 = 0$$

Now consider [2 / 2] or

$$(a_0 + a_1 z + a_2 z^2) = (b_0 + b_1 z + b_2 z^2)(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots)$$

Clearly the coefficients of z^3 are zero, so that we can write

$$b_2 c_1 + b_1 c_2 + b_0 c_3 = 0$$

In general, we note that there are $L + 1$ independent coefficients in the numerator and $M + 1$ coefficients in the denominator. To make the system determinable, it is customary to let $b_0 = 1$. We then have M independent coefficients in the denominator and $L + M + 1$ independent coefficients in all. Now the [L/M] approximant can fit the power series through orders $1, z, z^2, \ldots, z^{L+M}$ with an error of $O(z^{L+M+1})$. For example, for

$$f(z) = 1 - \frac{1}{2}z + \frac{1}{3}z^2 + \cdots$$

we have

$$[1/1] = \frac{1 + (1/6)z}{1 + (2/3)z} = f(z) + O(z^3)$$

Consequently,

$$(a_0 + a_1 z + \cdots + a_L z^L) = (b_0 + b_1 z + \cdots + b_M z^M)(c_0 + c_1 z + \cdots)$$

Equating coefficients of $z^{L+1}, z^{L+2}, \ldots, z^{L+M}$ in turn, we can write

$$\begin{aligned} b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + \cdots + b_0 c_{L+1} &= 0 \\ b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + \cdots + b_0 c_{L+2} &= 0 \\ &\vdots \\ b_M c_L + b_{M-1} c_{L+1} + \cdots + b_0 c_{L+M} &= 0 \end{aligned}$$

Setting $b_0 = 1$, we have M linear equations for the M coefficients in the denominator.

$$\begin{bmatrix} c_{L-M+1} & c_{L-M+2} & \cdots c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots c_{L+1} \\ \vdots & & \\ c & c_L & \cdots c_{L+m-1} \end{bmatrix} \cdot \begin{bmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} c_{L+1} \\ c_{L+2} \\ \vdots \\ c_{L+M} \end{bmatrix}$$

We invert the matrix on the left and solve for the b_i for $i = 1, \ldots, M$. Since we know the $c_0, c_1, c_2, \ldots$, we can equate coefficients of $1, z, z^2, \ldots, z^L$ to get $a_0, a_1, \ldots, a_L$. Thus

$$\begin{aligned} a_0 &= c_0 \\ a_1 &= c_1 + b_1 c_0 \\ a_2 &= c_2 + b_1 c_1 + b_2 c_0 \\ &\vdots \\ a_L &= c_L + \sum_{i=1}^{\min(L,m)} b_i c_{L-i} \end{aligned}$$

Thus the numerator and denominator of the Padé approximant are determined and we have agreement with the original series through order z^{L+M}. From the matrix equation, we can write the lower-order approximants. (For higher orders, one can use symbolic programs.)

1) $[L/M] = [1/1]$

2) $[L/M] = [2/2]$ $\quad b_1 = -c_2$

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = -\begin{pmatrix} c_3 \\ c_4 \end{pmatrix}$$

or

$$\begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_3/D & -c_2/D \\ -c_2/D & c_1/D \end{pmatrix} \cdot \begin{pmatrix} -c_3 \\ -c_4 \end{pmatrix}$$

where $D = c_1c_3 - c_2^2$ so that we have

$$b_2 = \frac{c_2c_4 - c_3^2}{c_1c_3 - c_2^2}$$

$$b_1 = \frac{c_2c_3 - c_1c_4}{c_1c_3 - c_2^2}$$

For $f(z) = c_0 + c_1z + c_2z^2 + \cdots$ we have

$$[1/1] = \frac{a_0 + a_1z}{b_0 + b_1z} \qquad \lim_{z\to\infty}[1/1] = a_1/b_1$$

$$[2/2] = \frac{a_0 + a_1z + a_2z^2}{b_0 + b_1z + b_2z^2} \qquad \lim_{z\to\infty}[2/2] = a_2/b_2$$

$$[3/3] = \frac{a_0 + a_1z + a_2z^2 + a_3z^3}{b_0 + b_1z + b_2z^2 + b_3z^3} \qquad \lim_{z\to\infty}[3/3] = a_3/b_3$$

$$\vdots$$

$$\lim_{z\to\infty}[m/m] = a_m/b_m$$

EXAMPLE: Find the limit for $e^{-x/(1+x)}$.

$[1/1] = .333...$
$[2/2] = .368...$
$[3/3] = .368...$

EXAMPLE: For e^x, we have

$$[1/1] = \frac{2+x}{2-x}$$

$$[2/2] = \frac{12+6x+x^2}{12-6x+x^2}$$

$$[3/3] = \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3}$$

Note that if we let $x = 1$ to consider the series for e, we get the correct limit.

$$[1/1] = 3$$
$$[2/2] = 2.714$$
$$[3/3] = 2.718$$
$$[4/4] = 2.718$$

PROBLEM: Noting that the limit is correct for $x = 1$ but the limit at ∞ for [1/1], [2/2], [3/3] fluctuates between ± 1 for both e^x and e^{-x}, i.e., $a_m/b_m = \pm 1$ as m increases, explain the lack of convergence to a limit since we know $e^{-x} \to 0$ as $x \to \infty$ and $e^x \to \infty$ as $x \to \infty$. Try the Shanks (or Wynn*) transformation. These transformations are effective in accelerating convergence of many slowly convergent series.

For cases where the Padé approximant appears inapplicable, we can sometimes use transformations of the series. Consider

$$f(z) = c_0 + c_3 z^3 + c_6 z^6 + \cdots$$

i.e., $c_{3m} \neq 0$ but $c_{3m+1} = c_{3m+2} = 0$. We let $y = z^3$ so that

$$f(y) = c_0 + c_1 y + c_2 y^2 + \cdots$$

is now in a form for Padé transformation. In general for a series

$$f(z) = \sum_{m=0}^{\infty} c_{Nm} z^{Nm}$$

where N is a positive integer (or "skip factor") with $c_{Nm} \neq 0$ but $c_{Nm+\upsilon} = 0$ for $1 \leq \upsilon \leq N-1$. We substitute $y = Z^N$ to get

* Other useful transforms used to accelerate convergence are the Euler, Wynn, and Van Wijngaarden transforms.

$$f(y) = c_0 + c_N y + c_{2N} y^2 + \cdots$$

or

$$f(y) = \sum_{n=0}^{\infty} c_n y^n$$

where $n = Nm$.

EXAMPLE: $$f(x) = \left(\frac{1+(1/2)x}{1+2x}\right)^{1/2} = 1 - \frac{3}{4}x + \frac{39}{32}x^2 - \cdots$$

To approximate by [1/1] we have

$$c_1 b_1 = -c_2$$

$$-\frac{3}{4}b_1 = -\frac{39}{32}$$

$$b_1 = 13/8$$

$$a_0 = c_0 = 1$$

$$a_1 = c_1 + b_1 c_0 = -\frac{3}{4} + \frac{13}{8} \cdot 1 = 7/8$$

Consequently

$$[1/1] = \frac{1+(7/8)x}{1+(13/8)x} = 0.54$$

(which is within 8% of the correct limit of 0.5 for the function).

VERIFICATION:

$$\left(1 + \frac{13}{8}x\right)\left(1 - \frac{3}{4}x + \frac{39}{32}x^2\right) = 1 + \frac{7}{8}x + 0\left(x^3\right)$$

EXERCISE: Calculate [2/2] and [3/3] to show that the limit approaches 0.5 more and more closely as we go to higher-order [L/M].

EXAMPLE: $f(x) = 1 - x + x^2/2! - x^3/3! + \cdots$

$$[1/1] = \frac{1-(1/2)x}{1+(1/2)x} \to -1 \text{ as } x \to \infty$$

EXAMPLE: $f(x) = e^x$

$$[1/1] = \frac{2+x}{2-x}$$

$$[2/2] = \frac{12+6x+x^2}{12-6x+x^2}$$

EXAMPLE:

$$f(x) = \left[\frac{1+2x}{1+x}\right]^{1/2} = 1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{13}{16}x^3 - \frac{141}{128}x^4 + \cdots$$

We note that even though the series has a limited region of convergence ($x \le 1/2$), the function is smooth for $0 \le x < \infty$. If we write a ratio

$$(a+bx)\ /(c+dx)$$

it is clear that we get a finite limit as x approaches ∞. Calculating [L/M] = [1/1], we have

$$[1/1] = \frac{1+(7/4)x}{1+(5/4)x} \to 1.4 \text{ as } x \to \infty$$

and carrying out the division, we get

$$[1/1] = 1 + \frac{1}{2}x - \frac{5}{8}x^2 + \frac{25}{32}x^3 - \frac{125}{128}x^4 + \cdots$$

which exactly matches the first three terms of the Taylor series. If we go as high as [5/5], we match the first 11 terms of the Taylor series

$$[2/2] = \frac{1+(13/4)x+(41/16)x^2}{1+(11/14)x+(29/16)x^2} \to 1.4137 \text{ as } x \to \infty$$

which is the correct limit of $2^{1/2}$ to three-place accuracy.

EXERCISE: Since $\cos x = \sum_{n=0}^{\infty} (-1)^n x \big/ (2n)!$, show that

$$[2/2] = (12-5x^2)\,/\,(12+x^2)$$

uses the first five terms of the Taylor series. Show that [2 / 2] is closer to the exact value of cos x than the sum of the first 5 terms.

SERIES NOT SUITABLE FOR PADÉ TRANSFORM:

Sometimes the series is not in a convenient form for the Padé transform which is designed for a series $\sum_{n=0}^{\infty} c_n x^n$ with non-zero c_0, c_1, c_2.

If we have missing terms, e.g., $f(x) = \sum_{n=0}^{\infty} c_{Nn} x^{Nn}$ where N is a positive integer and $c_0 \neq 0$, e.g., $\sum_{n=0}^{\infty} c_0 x^0 + c_3 x^3 + c_6 x^6 + \cdots$ then we use the transform $z = x^N$ (or $z = x^3$ in this specific case) and let $b_n = c_{Nn}$ to write $f(z) = \sum_{n=0}^{\infty} b_n z^n$ and $b_0 \neq 0$ which is now suitable for the Padé transform.

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and $c_0 = 0$, we can apply a translation $z = x - \xi$ with $\xi < \rho$ with ρ symbolizing the radius of convergence. If we use $|\xi| < 1$, the result will be simpler for manual computation because the resulting series for each new coefficient in $f(z) = \sum_{n=0}^{\infty} b_n z^n$ will converge rapidly. The result for b_n will be $b_0 \neq 0$ and

$$b_n = \sum_{\nu=0}^{\infty} \binom{\nu + n}{n} c_{\nu+n} \xi^{\nu}$$

which is equivalent to an analytic continuation.

THE SHANKS TRANSFORM:

This is a nonlinear transform which can be very effective, particularly in accelerating convergence of slowly converging series. It has even been applied to diverging series which seems contradictory. However, if a power series has been obtained by dividing out a rational function, this nonlinear transform is a means of inverting the procedure to obtain the rational function.

The Shanks transform is related to the Padé approximant. It is more accurate; however, the Padé approximant is more explicitly expressed in terms of the coefficients of the original series. Let s write the sequence of partial sums $\{S_n\}$ for a series and define the Shanks transform by

$$T\{S_n\} = \frac{S_{n+1} S_{n-1} - S_n^2}{S_{n+1} + S_{n-1} - 2S_n}$$

We often want repeated transforms, called the iterated Shanks transform, so it is convenient to write $\{A_n\}$ for the $\{S_n\}$. The iterated transforms will often lead to an extremely accurate solution. The first-order transform is written as:

$$B_n = T\{A_n\} = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}$$
$$C_n = T\{B_n\}$$
$$D_n = T\{C_n\}$$

We now write the sequence simply as

$$\begin{array}{llll} A_0 & & & \\ A_1 & B_1 & & \\ A_2 & B_2 & C_2 & \\ A_3 & B_3 & C_3 & D_3 \\ \bullet & \bullet & \bullet & \bullet \quad \bullet \end{array}$$

where $A_n \equiv S_n$.

EXAMPLE : In the continued fraction representing $\sqrt{2}$, the sequence of partial sums $\{S_n\}$ is $S_1 = 1$, $S_2 = 3/2$, $S_3 = 7/5$. Find the limit. Calculation yields:

n	A_n	B_n	C_n
1	1		
2	3/2	17/12	
3	7/5	99/70	19601/13860
4	17/12	577/408	
5	41/29		

We note that A_5 = 1.4137 which is correct to 3 figures, while C_3 = 1.414213564 is correct to 9 figures.

EXERCISE: Verify all results.

EXAMPLE: The Leibnitz series for π is $\pi = 4 - 4/3 + 4/5 - 4/7 + \cdots$. The results are:

n	A_n	B_n	C_n	D_n	E_n
0	4.0000000				
1	2.6666667	3.1666667			
2	3.4666667	3.1333333	3.1421053		
3	2.8952381	3.1452381	3.1414502	3.1415993	
4	3.3396825	3.1396825	3.1416433	3.1415909	3.1415928
5	2.9760462	3.1427129	3.1415713	3.1415933	3.1415927
6	3.2837385	3.1408814	3.1416029	3.1415925	
7	3.0170718	3.1420718	3.1415873		
8	3.2523659	3.1412548			
9	3.0418396				

We see that the tenth partial sum A_9, or S_9, is correct only to one figure. Shanks [1] points out that to give an answer correct to eight figures would require n = 40 million in S_n while we note that $e_5(S_n)$, or E_n, is already correct to eight figures.

EXERCISE: In the example

$$f(x) = 1 - \frac{3}{4}x + \frac{39}{32}x^2 - \cdots$$

in the previous section on Padé transforms, we obtained [1/1] = .54 which was close to the correct limit of .5. Show that the Shanks transform $T(S_2)$ = .54 also. Investigate the relationship between the two procedures.

Another related transform (which can also be iterated like the Shanks transform) is the Aitken transform defined by

$$T\{S_n\} = S_n - \{(S_{n+1} - S_n)^2 / (S_{n+2} - 2S_{n+1} + S_n)\}$$

The first-order Shanks transform is equivalent to the Aitken (δ^2) process and the mth order Shanks transform of the nth partial sum is equivalent to the [m/n] Padé approximant. We can view the Shanks transform as a unifying concept subsuming the Aitken process and the Padé approximent.

SUGGESTED READING

1. Daniel Shanks, Nonlinear Transformations of Divergent and Slowly Converging Series, *J. Math. and Physics,* **34**, (1-42) (1953).
2. A.C. Aitken, On Bernoulli's Numerical Solution of Algebraic Equations, *Proc. Roy. Soc.*, Edinburgh, (289-305) (1926).

APPENDIX II

ON STAGGERED SUMMATION OF DOUBLE DECOMPOSITION SERIES

$$
\begin{aligned}
u_0 &= u_0^{(0)} \\
u_1 &= u_0^{(1)} + u_1^{(0)} \\
u_2 &= u_0^{(2)} + u_1^{(1)} + u_2^{(0)} \\
u_3 &= u_0^{(3)} + u_1^{(2)} + u_2^{(1)} + u_3^{(0)} \\
u_4 &= u_0^{(4)} + u_1^{(3)} + u_2^{(2)} + u_3^{(1)} + u_4^{(0)} \\
&\vdots
\end{aligned}
\tag{1}
$$

The first column on the right of the equation is equal to u_0. The second column is equal to u_1. The third column is equal to u_2, etc. These sums of each column will be denoted by u_0^i, u_1^i, u_2^i,... respectively where i indicates the initial-value format. The sum of the column to the left of the equal sign is denoted by $\sum_{n=0}^{\infty} u_m^b$ for boundary-value format. We have

$$u = \sum_{n=0}^{\infty} u_m^b = u_0^i + u_1^i + \ldots = \sum_{n=0}^{\infty} u_n^i$$

Thus in writing the approximants ϕ_m we can write

$$
\begin{aligned}
\phi_1[u^b] &= \phi_1[u_0^i] \\
\phi_2[u^b] &= \phi_2[u_0^i] + \phi_1[u_1^i] \\
\phi_3[u^b] &= \phi_3[u_0^i] + \phi_2[u_1^i] + \phi_1[u_2^i] \\
&\vdots \\
\phi_m[u^b] &= \phi_m[u_0^i] + \phi_{m-1}[u_1^i] + \cdots + \phi_1[u_{m-1}^i]
\end{aligned}
$$

which can be written

$$\phi_m[u^b] = \sum_{n=0}^{m-1} \phi_{m-n}[u_n^i] \tag{2}$$

Referring again to (1), we have

$$u_m = u_0^{(m)} + u_1^{(m-1)} + u_2^{(m-1)} + \cdots + u_{m-2}^{(2)} + u_{m-1}^{(1)} + u_m^{(0)}$$

or $u_m = \sum_{n=0}^{m} u_n^{m-n}$ in our boundary-value format. Returning to (1) we can write in initial-value format,

$$u_0 = u_0^{(0)} + u_0^{(1)} + u_0^{(2)} + \cdots$$
$$u_1 = u_1^{(0)} + u_1^{(1)} + u_1^{(2)} + \cdots$$
$$\vdots$$
$$u_n = u_n^{(0)} + u_n^{(1)} + u_n^{(2)} + \cdots$$

or $u_n = \sum_{n=0}^{\infty} u_n^{(m)}$ (in initial-value format). By decomposition $u = \sum_{n=0}^{\infty} u_n$. By double decomposition $u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$.

FORMULAS OF INTEREST:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} u_n^{(m)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

$$\phi_m[u^b] = \sum_{n=0}^{m-1} \phi_{m-n}[u_n^i]$$

$$u_m^b \equiv \sum_{n=0}^{m} u_n^{(m-n)}$$

$$u_n^i \equiv \sum_{m=0}^{\infty} u_n^{(m)}$$

$$u^b = \sum_{m=0}^{\infty} \sum_{n=0}^{m} u_n^{(m-n)}$$

$$u^i = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_n^{(m)}$$

$$u^b = u^i$$

$$\lim_{m\to\infty} \phi_m[u^b] = u^b$$

$$\lim_{m\to\infty} \sum_{n=0}^{m-1} \phi_{m-n}[u_n^i] = \sum_{m=0}^{\infty} u_n^i = u^i$$

APPENDIX III

CAUCHY PRODUCTS OF INFINITE SERIES

In one dimension for $u=\sum_{\mu=0}^{\infty} a_{\mu}x^{\mu}$ and $\beta=\sum_{m=0}^{\infty} \beta_m x^m$, we can write

$$\beta u=\sum_{m=0}^{\infty} x^m \sum_{\mu=0}^{m} \beta_{m-\mu}\, a_{\mu}$$

In two dimensions,

$$\beta=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \beta_{m_1,m_2}\, x_1^{m_1}\, x_2^{m_2}$$

$$u=\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{\infty} a_{\mu_1,\mu_2}\, x_1^{\mu_1}\, x_2^{\mu_2}$$

$$\beta u=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} x_1^{m_1}\, x_2^{m_2} \sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2} \beta_{m_1-\mu_1,m_2-\mu_2}\, a_{\mu_1,\mu_2}$$

In three dimensions,

$$\beta=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_2=0}^{\infty} \beta_{m_1,m_2,m_3}\, x_1^{m_1}\, x_2^{m_2}\, x_3^{m_3}$$

$$u=\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{\infty}\sum_{\mu_3=0}^{\infty} a_{\mu_1,\mu_2,\mu_3}\, x_1^{\mu_1}\, x_2^{\mu_2}\, x_3^{\mu_3}$$

$$\beta u=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_2=0}^{\infty} x_1^{m_1}\, x_2^{m_2}\, x_3^{m_3}$$

$$\sum_{\mu_1=0}^{m_1}\sum_{\mu_2=0}^{m_2}\sum_{\mu_3=0}^{m_3} \beta_{m_1-\mu_1,m_2-\mu_2,m_3-\mu_3}\, a_{\mu_1,\mu_2,\mu_3}$$

In four dimensions,

$$\beta=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\sum_{m_4=0}^{\infty} \beta_{m_1,m_2,m_3,m_4}\, x_1^{m_1}\, x_2^{m_2}\, x_3^{m_3}\, x_4^{m_4}$$

$$u=\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{\infty}\sum_{\mu_3=0}^{\infty}\sum_{\mu_4=0}^{\infty} a_{\mu_1,\mu_2,\mu_3,\mu_4}\, x_1^{\mu_1}\, x_2^{\mu_2}\, x_3^{\mu_3}\, x_4^{\mu_4}$$

$$\beta u = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}$$

$$\sum_{\mu_1=0}^{m_1} \sum_{\mu_2=0}^{m_2} \sum_{\mu_3=0}^{m_3} \sum_{\mu_4=0}^{m_4} \beta_{m_1-\mu_1, m_2-\mu_2, m_3-\mu_3, m_4-\mu_4} \, a_{\mu_1,\mu_2,\mu_3,\mu_4}$$

In N dimensions,

$$\beta = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \beta_{m_1,\ldots,m_N} \, x_1^{m_1} \cdots x_N^{m_N}$$

$$u = \sum_{\mu_1=0}^{\infty} \cdots \sum_{\mu_N=0}^{\infty} a_{\mu_1,\ldots,\mu_N} \, x_1^{\mu_1} \cdots x_N^{\mu_N}$$

$$\beta u = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} x_1^{m_1} \cdots x_N^{m_N} \sum_{\mu_1=0}^{m_1} \cdots \sum_{\mu_N=0}^{m_2} \beta_{m_1-\mu_N,\ldots,m_N-\mu_N} \, a_{\mu_1,\ldots,\mu_N}$$

These can all be programmed using "do-loops" and stopping rules. These product rules should be useful in programming solutions where the system input and system coefficients are known only as power series.

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