Appendix

In order to establish the formula (19) of p. 32 we introduce a function f^* which serves essentially the same purpose as f_D defined in (16), p. 31, but has the property that it is continuous in R.

Let D be the region given by

$$D:\quad a \leqslant x \leqslant b, \quad \psi(x) \leqslant y \leqslant \phi(x),$$

which is normal relative to the x-axis and suppose that f is continuous in D. Choose an arbitrary (small) positive number ε and construct the additional arcs $y = \phi(x) + \varepsilon, y = \psi(x) - \varepsilon$. Thus D is extended by two bands B_1, B_2. Enclose the extended

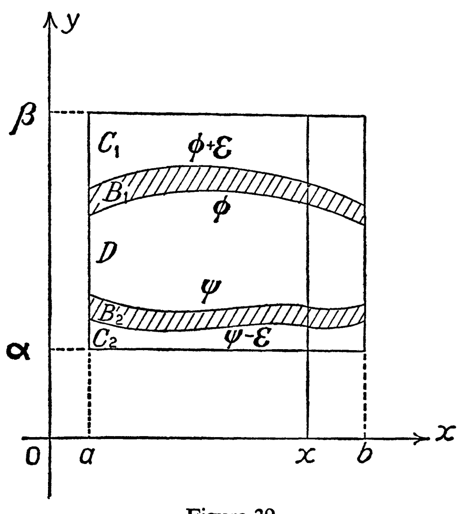

Figure 39.

region in a rectangle R: $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$. The rectangle R now appears as the union of the non-overlapping regions D, B_1, B_2, C_1, C_2 as shown in Fig. 38. Define the function f^* by the following rule

$$f^* = \begin{cases} f, \text{ if } (x,y) \in D \\ \frac{1}{\varepsilon}(\varepsilon + \phi(x) - y)f, \text{ if } (x,y) \in B_1 \\ \frac{1}{\varepsilon}(\varepsilon - \psi(x) + y)f, \text{ if } (x,y) \in B_2 \\ \qquad 0 \qquad\quad , \text{ if } (x,y) \in C_1 \cup C_2. \end{cases}$$

It is easy to verify that f^* is continuous in R; indeed there are no inconsistencies on the boundary between any two regions. For example, on the arc $y = \phi(x)$, $f = f^*$ as in D and, on the arc $y = \phi(x) + \varepsilon$, $f^* = 0$ as in C_1. In fact, the discontinuities of f_D have been smoothed out across the bands B_1 and B_2. The area of B_1 is given by

$$|B_1| = \int_a^b \{\phi + \varepsilon - \phi\} dx = \int_a^b \varepsilon\, dx = \varepsilon(b - a), \tag{1}$$

and similarly

$$|B_2| = \varepsilon(b - a). \tag{2}$$

It follows from our assumption that f is bounded, say

$$|f| \leqslant M \text{ in } D,$$

and this implies that f^* is bounded by the same number, because the factors $\frac{1}{\varepsilon}(\varepsilon + \phi(x) - y)$ and $\frac{1}{\varepsilon}(\varepsilon - \psi(x) + y)$ have values between 0 and 1 in the regions B_1 and B_2 respectively. We may therefore state that

$$|f^*| \leqslant M \text{ in } R.$$

Now compare the integrals of f^* and f. Clearly

$$\iint_R f^* = \iint_D f + \iint_{B_1} f^* + \iint_{B_2} f^* + \iint_{C_1} f^* + \iint_{C_2} f^*.$$

The last two integrals are zero, since f^* vanishes in C_1 and C_2. On using (1) and (2) we find that

$$\left|\iint_{B_1} f^*\right| \leqslant \varepsilon M(b-a), \left|\iint_{B_2} f^*\right| \leqslant \varepsilon M(b-a).$$

Thus

$$\left|\iint_R f^* - \iint_D f\right| \leqslant 2\varepsilon M(b-a). \tag{3}$$

Next, by Theorem 1,

$$\iint_R f^* = \int_a^b dx \int_\alpha^\beta f^* dy = \int_a^b dx \int_{\psi-\varepsilon}^{\psi} f^* dy$$
$$+ \int_a^b dx \int_\psi^\phi f\,dy + \int_a^b dx \int_\phi^{\phi+\varepsilon} f^* dy,$$

where the inner integral has been broken up corresponding to the segments in which the vertical line with abscissa x meets the various subregions. We recall that $f^* = 0$ when $\alpha \leqslant y \leqslant \psi - \varepsilon$ and $\phi + \varepsilon \leqslant y \leqslant \beta$ and that $f^* = f$ when $\psi \leqslant y \leqslant \phi$. Since

$$\left|\int_a^b dx \int_{\psi-\varepsilon}^{\psi} f^* dy\right| \leqslant \int_a^b dx \int_{\psi-\varepsilon}^{\psi} M\,dy = M\int_a^b \varepsilon\,dx = M\varepsilon(b-a)$$

and similarly,

$$\left|\int_a^b dx \int_\phi^{\phi+\varepsilon} f^* dy\right| \leqslant M\varepsilon(b-a),$$

we infer that

$$\left|\iint_R f^* - \int_a^b dx \int_\varphi^\phi f\,dy\right| \leqslant 2M\varepsilon(b-a). \tag{4}$$

Combining the inequalities (3) and (4) we deduce that

$$\left| \iint_D f - \int_a^b dx \int_\varphi^\phi f\, dy \right| \leqslant 4M\varepsilon(b-a).$$

Since the right-hand side can be made arbitrary small, the two expressions on the left are in fact equal. This proves the assertion.

Solutions to Exercises

Chapter I

1. $x = t^2, y = t^3\ (-1 < t < 1)$.
2. $x = t^2\ (-1 \leqslant t \leqslant 1)$ and $x = 1\ (1 \leqslant t \leqslant 2)$; $y = t\ (-1 \leqslant t \leqslant 1)$ and $y = 3 - 2t\ (1 \leqslant t \leqslant 2)$.
3. $(5\sqrt{5} - 1)/12$.
4. 0
5. $2 \log 2 - 1$.
6. Let $P = \int p\, dx$, $Q = \int q\, dy$, $f = P + Q$ and use equation (8) on p. 11.
8. The function $\tan^{-1}(y/x)$ is not single-valued.
9. $2\pi\,(a^2 + \pi b^2)$.
11. Rotate the axes so as to bring the equation of the ellipse into the normal form $(\lambda\xi)^2 + (\mu\eta)^2 = 1$ and observe that the area is $\pi/\lambda\mu$.
12. $4ab/3$.

Chapter II

1. $\pi a^4/4$.
2. $\sin y$.
3. The region of integration is the triangle with vertices (0,0), $(\pi/2, \pi/2)$, $(0, \pi/2)$.
4. The rule for interchanging the order of integration does not apply, because the integrand has a singularity at (0,0).
5. The region of integration is bounded by the parabola $y^2 = ax$ and the straight line $y = x$.
6. $a^3(3\pi - 4)/9$.
7. Use (43), p. 55. ANS.: $(ab)^{-1}$.
8. Put $u = x^2 - y^2$, $v = xy$; ANS.: 3/4.
9. Put $u = x^2 + 2y^2$, $v = y/x$. ANS.: $2^{-\frac{1}{2}}\{\tan^{-1}2\sqrt{2} - \tan^{-1}\tfrac{1}{2}\sqrt{2}\}$.

10. Change to polar coordinates; the ensuing integral

$$\int_0^{\pi/2} (1 + \cos\alpha \sin 2\theta)^{-1} d\theta$$

may be evaluated by putting $t = \tan\theta$.

11. 84.
12. $128a^3/15$.
13. Region of integration: $x^2 + y^2 \leqslant 15a^2/16$; boundary surfaces: $z = (a^2 - x^2 - y^2)^{\frac{1}{2}}, z = 2a - (4a^2 - x^2 - y^2)^{\frac{1}{2}}$. ANS.: $13\pi^3/24$.
14. $a^2b(9\pi - 16)/36$.
15. Take for region of integration $x^2 + y^2 \leqslant a^2$; on account of symmetry it suffices to consider the portion for which $0 \leqslant y \leqslant x$, where the bounding surfaces are $z = \pm(a^2 - x^2)^{\frac{1}{2}}$. ANS.: $16a^3(1 - 2^{-\frac{1}{2}})$.

Chapter III

2. The two surfaces intersect in a curve whose projection on the (x,y)-plane are the lines $x + y - a = 0$ and $x + y + a = 0$ (eliminate z). Since $z^2 = 2xy$ passes through the lines $x = 0$ and $y = 0$, the region of integration is the union of the triangles with vertices (0.0), $(a,0)$, $(0,a)$ and (0,0), $(-a,0)$, $(0,-a)$ respectively. Use II, (43). ANS.: $\sqrt{2}\pi a^2$.
3. $4\pi^2 ab$.
4. $4\pi a^4/3$. Observe that, by symmetry,

$$\iint_S x^2 dS = \iint_S y^2 dS = \iint_S z^2 dS.$$

5. $37\pi/10$.
6. Note that $(a^2 - 2ap + p^2)^{\frac{1}{2}} = |a - p|$.

Chapter IV

1. 2.
2. Put $x = a\xi$, $y = b\eta$, $z = c\zeta$ and then use polar coordinates. ANS.: $4\pi abc(e - 2)$.
3. Put $\xi = x - a$, $\eta = y - a$, $\zeta = z - a$. Note that the integral of an odd power of ξ or η or ζ over the interior of the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$ is zero. ANS.: $32\pi a^6/5$.

6. By symmetry reduce the region to the positive octant, then put $x = a\xi^{\frac{1}{2}}$, $y = b\eta^{\frac{1}{2}}$, $z = c\zeta^{\frac{1}{2}}$ and use Dirichlet's formula (exercise 4). ANS.: $4\pi(abc)^3/945$.
7. Use the formulae $px = a^2 \cos \alpha, \ldots$ of p. 88 and apply the divergence theorem (p. 86).
9. Use spherical polar coordinates. After integration with respect to ϕ and θ, the integral reduces to $2\pi \int_0^a \frac{r}{\rho}\{r + \rho - |\rho - r|\}dr$ (care is needed when $\rho < a$).
11. $k = 2$.
12. Apply the divergence theorem.

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