APPENDIX A

The Fourier integral theorem

In equation (1.7) of Section 1.3 we gave a description of a signal defined on an infinite range in the form of a double integral, with no explanation as to how that result was obtained. Here we offer an outline derivation, taking as starting point the Fourier *series* representation of a function defined on a *finite* range $[-l, l]$.

From (1.6) we have

$$x(t) = \frac{1}{2l} \sum_{n=-\infty}^{\infty} \left\{ \int_{-l}^{l} x(t')\,\mathrm{e}^{-jn\pi t'/l}\,\mathrm{d}t' \right\} \mathrm{e}^{jn\pi t/l}$$

in which n is a discrete integer variable. (It is assumed that $x(t)$ satisfies the usual conditions of continuity and differentiability except, possibly, at a limited number of points.)

It is necessary that we be able to represent functions $x(t)$ defined for *all* values of t and which are not necessarily periodic. We can proceed (in a non-rigorous manner) from the expression above as follows. On putting $u = \pi/l, \Delta(nu) = u$ is the frequency spacing between consecutive harmonics in the Fourier series. Since n is integer the variable (nu) is discrete, and the previous equation can be written

$$x(t) = \frac{1}{2\pi} \sum_{nu=-\infty}^{\infty} \left\{ \int_{-\pi/u}^{\pi/u} x(t')\,\mathrm{e}^{-jnut'}\,\mathrm{d}t' \right\} \mathrm{e}^{jnut}\Delta\,(nu)$$

Now consider what happens if $l \to \infty$. We may argue, first, that the interval $(-l, l)$, which implied a period $T = 2l$, becomes $(-\infty, \infty)$ and $u \to 0$, and second, that the discrete variable nu, whose values extend over a doubly infinite range, can be replaced by a continuous variable, say $nu \to w$, so that $\Delta\,(nu) \to \mathrm{d}w$ and the range of w is $(-\infty, \infty)$. Using the Riemann definition of an integral as the limit of a sum, we then have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(t')^{-jwt'}\,\mathrm{d}t'\right\}\mathrm{e}^{jwt}\,\mathrm{d}w$$

The factor $1/2\pi$ can be removed by substituting $w = 2\pi f$, $\mathrm{d}w = 2\pi\mathrm{d}f$. The limit process is valid if $x(t)$ *does* have a Fourier series and if $\int_{-\infty}^{\infty}|x(t')|\,\mathrm{d}t'$ exists.

Either in the above form (which is equation (1.7)) or rewritten in terms of f (equation (1.8)), the Fourier integral theorem is the fundamental theorem underlying all integral transform pairs (and their discrete equivalents). The various transform pairs so validated and discussed in this text are the more significant examples of what is available.

APPENDIX B

The Hartley transform

As long as it is confirmed that a proposed integral (or discrete) transform pair leads to correct recovery of the time-domain signal on inversion, one is at liberty to design transforms with either particular applications or computational implications in mind, or both. For example, once fast algorithms were developed for processing the DFT (as described in Chapters 6 and 7) there were prompted not only considerations of improved FFT algorithms but also of modifications and alternatives to the underlying (continuous) transform pair.

A well-known example is the Hartley transform, which we here describe in both continuous and discrete forms. Related to the Fourier transform (as expected), a principal feature is that 'real' expressions are predominant. (In fast computation of the discrete form, there is some reduction in the number of multiplications required – albeit at the expense of increased additions.)

We begin by defining a continuous transform

$$H(f) = \int_{-\infty}^{\infty} x(t')[\cos(2\pi f t') + \sin(2\pi f t')]\,\mathrm{d}t' \tag{B.1}$$

which is the sum of a two-sided Fourier cosine transform and a two-sided Fourier sine transform. By writing $x(t')$ as the sum of its even and odd components,

$$x(t') = \tfrac{1}{2}[x(t') + x(-t')] + \tfrac{1}{2}[x(t') - x(-t')]$$

we could write (B.1) in terms of one-sided transforms and make use of the results of Sections 1.4 and 1.6 to establish that the inverse is

$$x(t) = \int_{-\infty}^{\infty} H(f)[\cos(2\pi f t) + \sin(2\pi f t)]\,\mathrm{d}f \tag{B.2}$$

An alternative is to show directly that these two equations satisfy the Fourier integral theorem. It was shown that this can be written in the form

$$x(t) = \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} x(t')\cos\left[w(t-t')\right]\mathrm{d}t'\right\}\mathrm{d}w$$

which appeared previously as equation (1.9). Putting $w = 2\pi f$, $\mathrm{d}w = 2\pi \mathrm{d}f$ and noting that $\cos[2\pi f(t-t')]$ is an even function of f, this could be re-expressed as

$$x(t) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x(t')\cos\left[2\pi f(t-t')\right]\mathrm{d}t'\right\}\mathrm{d}f \qquad \text{(B.3)}$$

Now suppose that $H(f)$, as defined in (B.1), is substituted into the right-hand side of (B.2), and consider the expression

$$\begin{aligned} EX &= \lim_{l\to\infty}\int_{-l}^{l}\left\{\int_{-l}^{l} x(t')\left[\cos(2\pi f t') + \sin(2\pi f t')\right]\mathrm{d}t'\right\} \\ &\quad \cdot\left[\cos(2\pi f t) + \sin(2\pi f t)\right]\mathrm{d}f \\ &= \lim_{l\to\infty}\int_{-l}^{l} x(t')\left\{\int_{-l}^{l} \cos\left[2\pi f(t-t')\right] + \sin\left[2\pi f(t+t')\right]\mathrm{d}f\right\}\mathrm{d}t' \end{aligned}$$

on the changing the order of integration. Since $\sin[2\pi f(t+t')]$ is an odd function of f this reduces to

$$\begin{aligned} EX &= \lim_{l\to\infty}\int_{-l}^{l} x(t')\left\{\int_{-l}^{l} \cos\left[2\pi f(t-t')\right]\mathrm{d}f\right\}\mathrm{d}t' \\ &= \int_{-\infty}^{\infty} x(t')\left\{\int_{-\infty}^{\infty} \cos\left[2\pi f(t-t')\right]\mathrm{d}f\right\}\mathrm{d}t' \end{aligned}$$

The order of integration can be changed again, and reference to (B.3) shows that $EX = x(t)$. Equations (B.1) and (B.2) therefore constitute a transform pair.

If $H(f)$ is written in terms of its even and odd components

$$H(f) = \tfrac{1}{2}\left[H(f) + H(-f)\right] + \tfrac{1}{2}\left[H(f) - H(-f)\right] = H_{\mathrm{e}}(f) + H_{\mathrm{o}}(f),$$

then from equation (B.1) we see that

$$H_{\mathrm{e}}(f) = \int_{-\infty}^{\infty} x(t')\cos(2\pi f t')\,\mathrm{d}t'$$

and

$$H_o(f) = \int_{-\infty}^{\infty} x(t') \sin(2\pi f t')\,dt' \tag{B.4}$$

The Fourier transform can be obtained from the equation

$$X(f) = H_e(f) - jH_o(f) \tag{B.5}$$

because $e^{-j\theta} = \cos\theta - j\sin\theta$. This result holds, irrespective of whether $x(t)$ is real or complex.

If $x(t)$ is real, then $H(f)$ is real and also

$$H(f) = \mathrm{Re}\{X(f)\} - \mathrm{Im}\{X(f)\} \tag{B.6}$$

Equation (B.6) does *not* apply if $x(t)$ is a complex signal.

The Hartley transform is more symmetrical than the Fourier transform because the transform pair, (B.1) and (B.2), have an identical structure.

The discrete form of the Hartley transform is defined by the equation

$$H(n) = \sum_{k=0}^{N-1} x_k \left[\cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right)\right]$$

or, with an obvious notation,

$$H(n) = \sum_{k=0}^{N-1} x_k \operatorname{cas}\left(\frac{2\pi nk}{N}\right) \tag{B.7}$$

and the inversion is provided by

$$x(k) = \frac{1}{N} \sum_{n=0}^{N-1} H(n) \operatorname{cas}\left(\frac{2\pi nk}{N}\right) \tag{B.8}$$

(In some texts, the multiplier $1/N$ appears in the definition of $H(n)$. Here it has been transferred to the inversion sum, (B.8), to facilitate comparison with the DFT as defined in (5.5).)

These equations again show complete symmetry, and no modification (such as conjugation) is needed to use any particular algorithm for both transformation and inversion. To verify that we have a transform pair we begin as we did (in the case of the DFT) in Section 5.2. In (B.7) we replace the summation integer k by m and substitute into the right-hand side of (B.8), in which k is some fixed integer, to obtain

$$EX = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x_m \operatorname{cas}\left(\frac{2\pi nm}{N}\right) \right\} \operatorname{cas}\left(\frac{2\pi nk}{N}\right)$$

If the order of summation is changed,

$$EX = \frac{1}{N} \sum_{m=0}^{N-1} x_m \left\{ \sum_{n=0}^{N-1} \operatorname{cas}\left(\frac{2\pi nm}{N}\right) \operatorname{cas}\left(\frac{2\pi nk}{N}\right) \right\} \tag{B.9}$$

The summation over n can be written

$$\sum(m) = \sum_{n=0}^{N-1} \left\{ \cos\left[\frac{2\pi n(k-m)}{N}\right] + \sin\left[\frac{2\pi n(k+m)}{N}\right] \right\}$$

if the terms are expanded and simplified using trigonometric formulae. Considering the first term in $\sum(m)$, we can write

$$\sum_{n=0}^{N-1} \cos\left[\frac{2\pi n(k-m)}{N}\right] = \operatorname{Re}\left\{ \sum_{n=0}^{N-1} e^{j2\pi n(k-m)/N} \right\}$$

This sum appeared in (5.7) and, by considering the sum of a geometric progression, was shown to be zero if $k \neq m$ and N if $k = m$. Similarly, the second term in $\sum(m)$ can be written

$$\sum_{n=0}^{N-1} \sin\left[\frac{2\pi n(k+m)}{N}\right] = \operatorname{Im}\left\{ \sum_{n=0}^{N-1} e^{j2\pi n(k+m)/N} \right\}$$

in which the exponentials form another geometric progression, whose sum is

$$S_N = \frac{1 - e^{j2\pi(k+m)}}{1 - e^{j2\pi n(k+m)/N}}$$

(unless this is an indeterminate ratio). The numerator is always zero. The denominator is zero only if $(k + m) = N$, and k and m can assume values in the range $0, 1, 2, \ldots, N-1$ only. In that case,

$$\sum_{n=0}^{N-1} e^{j2\pi n(k+m)/N} = \sum_{n=0}^{N-1} e^{j2\pi n} = \sum_{n=0}^{N-1} (-1)^n$$

which is zero as N is an even number. Hence $\sum(m)$ is zero if $m \neq k$ and has value N if $m = k$. Equation (B.9) reduces to $EX = x_k$, and that (B.7) and (B.8) are a transform pair is confirmed.

(The orthogonality relation,

$$\sum_{n=0}^{N-1} \operatorname{cas}\left(\frac{2\pi nm}{N}\right) \operatorname{cas}\left(\frac{2\pi nk}{N}\right) = N\delta_{mk}$$

is analogous to the orthogonality relations used when finding expressions for the coefficients $\{a_n\}$ and $\{b_n\}$ in a trigonometric Fourier series.)

APPENDIX C

Further reading

In the preface it was explained that there is an extensive literature on the subject of discrete signals, systems and transforms but that the emphasis on particular applications and the level of assumed knowledge (mathematical or otherwise) are highly variable. What follows is a short selection of titles intended to reflect this diversity.

Bateman, A. and Yates, W. (1988) *Digital Signal Processing Design*, Pitman.

Bracewell, R.N. (1986) *The Fourier Transform and its Applications* (2nd edn), McGraw-Hill.

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Strum, R. and Kirk, D. (1988) *First Principles of Discrete Systems and Digital Signal Processing*, Addison Wesley.

Ziemer, R.W., Tranter, W.H. and Fannin, D.R. (1983) *Signals and Systems, Continuous and Discrete*, Macmillan.

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