Review Exercises

1. Following the procedure used in Theorem 10.1, prove that in a group, $y = a^{-1} \circ b$ is the unique solution of $a \circ y = b$.

2. Construct a commutative group with five elements.

3. Let the operation $x \cdot y$ be defined as $x + (y - 3)$. Show that the set G of all integers forms a group with respect to this operation and that 3 is the identity element of this group. What is the inverse element for an integer x?

4. (a) List four different subgroups of the group of all integers under ordinary addition.

   (b) For each of these subgroups, state whether it is isomorphic to the original group. If so, prove it and if not, explain why not.

5. Determine all the possible isomorphisms between the group of the integers $\{0, 1, 2, 3\}$ with addition modulo 4 and the group of rotations of the square $\{I, R, R', R''\}$.

6. Let $A = \{a, b, c\}$. Find all the distinct one-to-one correspondences from A onto A. Construct composites of all pairs of these one-to-one correspondences and express your results in the form of a multiplication table. What sort of mathematical configuration is represented by this multiplication table?

7. Let A^* be the set of all strings formed from some finite set A. For all strings x and y in A^* x is said to be a *conjugate* of y iff there are strings u and v such that $x = uv$ and $y = vu$.

   (a) Show that conjugacy is an equivalence relation and describe the partition it induces on A^*. For a string x_n of length n, what are the maximum and minimum number of strings in the equivalence class containing x_n?

(b) Prove that if x is a conjugate of y there is a string z such that $xz = zy$.

(c) Let $T = \{T_1, T_2, T_3, T_4\}$ be the set of functions each of which maps a string in A of length 4 into one of its conjugates, e.g., T_1 maps $a_1a_2a_3a_4$ into $a_1a_2a_3a_4$ and T_2 maps $a_1a_2a_3a_4$ into $a_2a_3a_4a_1$. Show that the operational structure consisting of T and the operation of composition of functions is an Abelian group.

8. Each of the following is a system of the form $\mathbf{A} = \langle A, \oplus, \odot \rangle$ consisting of a set and two binary operations. For each system, answer the following questions:

(a) Which of the group axioms are satisfied by $\langle A, \oplus \rangle$?

(b) Which of the group axioms are satisfied by $\langle A, \odot \rangle$?

(c) Is $\mathbf{A}$ an integral domain? If not, which axioms are not satisfied?

(1) $A =$ the set of all integers
$\oplus =$ ordinary addition
$\odot =$ ordinary multiplication

(2) $A =$ the set of all non-negative integers
$\oplus =$ ordinary subtraction
$\odot =$ ordinary multiplication

(3) $A = \{0, 1, 2, \ldots, 24\}$
$\oplus =$ addition modulo 25
$\odot =$ multiplication modulo 25

(4) $A = \{1, 2, 3, \ldots, 9, 10\}$
$\oplus =$ addition modulo 11
$\odot =$ multiplication modulo 11

(5) $A = \{1, 5, 7, 11\}$
$\oplus =$ addition modulo 12
$\odot =$ multiplication modulo 12

(6) $A =$ the set of all rational numbers p/q with $0 < p/q < 1$
$\oplus =$ ordinary addition
$\odot =$ ordinary multiplication

(7) $A =$ the set of all integers
$\oplus =$ ordinary addition

$\odot$ defined by $a \odot b = 0$ for all $a, b \in A$

9. Prove the following laws for integral domains. (You may use the results of previous problems as well as the axioms and theorems from the text.)

(a) $(a + b) \cdot (c + d) = (ac + bc) + (ad + bd)$ for all a, b, c, d.

(b) $-0 = 0$ (Note: $-x$ stands for "additive inverse of x", x^{-1} stands for "multiplicative inverse of x").

(c) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

(d) $-(-a) = a$.

10. The following correspondences are many-one mappings of the multiplicative group of all non-zero real numbers on part of itself. Which are homomorphisms? For those which are not, show why not. (*For those which are, *prove* that all the requirements are satisfied.)

(a) $x \mapsto |x|$

(b) $x \mapsto -x$

(c) $x \mapsto 2x$

(d) $x \mapsto 1/x$

(e) $x \mapsto x^2$

11.* Which of the following relations R are equivalence relations? For those which are, describe the equivalence classes.

(a) G is a group, S is a subgroup of G, and R is the set of all ordered pairs (a, b) with a, b in G such that $a^{-1} \cdot b \in S$.

(b) J is the integral domain of all integers and R is the set of all ordered pairs (a, b) with a, b in J such that $a + (-b)$ is even.

(c) J is the integral domain of all integers and R is the set of all ordered pairs (a, b) with a, b in J such that $a + (-b)$ is odd.

12. **(a)** Construct a group of symmetries of an equilateral triangle analogous to the group of symmetries of the square. (There should be six elements altogether.)

(b) Find all subgroups of that group.

(c) Construct a homomorphism between the whole group and one of its proper subgroups.

13. Using the axioms of Boolean algebras, prove that the set of elements of a Boolean algebra cannot form a group under the union operation.

14. Prove in a relatively pseudo-complemented lattice

(a) $a = b$ iff $a \Rightarrow b = 1 = b \Rightarrow a$

(b) $1 \Rightarrow b = b$

(c) $a \leq b \Rightarrow (a \wedge b)$