Appendix B-I

Alternative Notations and Connectives

The logical language we have studied in this chapter is the best known classical system of logic of statements. There are various alternative symbolic notations for the connectives we introduced, the most common of which are listed below.

Alternative notation for connectives

	our symbol	alternative(s)
negation	$\sim$	$\neg$, $-$, $\bar{p}$
conjunction	&	$\wedge$, $\bullet$
conditional	$\rightarrow$	$\supset$ (called 'horseshoe')

Table B.I–1

Furthermore, there are two connectives which we have not introduced, as they are not part of the system most commonly used. One, called "Quine's dagger", $\downarrow$, is sufficiently powerful to be the only connective in a system equivalent to the one given. Its truth table is shown in (B.II.-2).

P	Q	$P \downarrow Q$
1	1	0
1	0	0
0	1	0
0	0	1

Table B.I–2

Its nearest English correspondent is *neither ... nor*. As an interesting exercise one can show that negation can be defined in terms of this connective, and then that disjunction can be defined in terms of negation and this connective. We have already proven (Chapter, Exercise 7) that the five-connective system can be reduced to one containing just $\vee$ and $\sim$. Therefore $\downarrow$ suffices alone for the five connectives.

Similarly, there is another connective, written as $|$, which is called the "Sheffer stroke", whose interpretation by the truth table is as follows:

P	Q	$P \mid Q$
1	1	0
0	1	1
1	0	1
0	0	1

Table B.I–3

There is no simple English connective which has this meaning, but it can be expressed as 'not both p and q' or the neologism 'nand'. Again, you may want to convince yourself that a system with the Sheffer stroke as its only connective is truth-functionally complete (first define negation and then disjunction, as above).

Appendix B-II

Kleene's Three-valued Logic

The logic of statements and the predicate logic are two-valued, since there are but two truth values and every formula is either true or false. This is based on the assumption that the semantic assignments used in an interpretation are *total* functions. In some linguistic applications and especially in computational contexts, that assumption seems much too strong, since it requires that there is a clear semantic procedure which decides for any given x whether $[\![\varphi(x)]\!]$ is true or false for any arbitrary φ. But if we allow a *partial* interpretation function of predicates, such a procedure may not always exist, since its value on $\varphi(x)$ may be undefined for some x. Kleene developed a semantics for predicate logic with such partial functions which yield values 'true' or 'false' when defined, but which may also be undefined. Since it has certain linguistically useful aspects, we discuss it here briefly.

In case a partial function is undefined for an argument it may be because we lack information, or we may take it to mean that we disregard its value as it is does not matter to our interpretation. The following truth tables represent the strong Kleene semantics for the connectives, where 1 is 'true', 0 is 'false', and * means the truth value is undefined.

p	$\sim p$	p	q	$p \& q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
1	0	1	1	1	1	1	1
0	1	1	0	0	1	0	0
*	*	1	*	*	1	*	*
		0	1	0	1	1	0
		0	0	0	0	1	1
		0	*	0	*	1	*
		*	1	*	1	1	*
		*	0	0	*	*	*
		*	*	*	*	*	*

From this table we can see that if sufficient information is available to verify or falsify a statement, the undefined part does not alter it. But if the value of a part must be known to determine the value of a complex statement, the latter remains undefined until we know the value of its parts. This means that the value of a complex statement may be determined even when we do not know the value of *all* of its parts. For instance, as soon as we know that the antecedent of a conditional is false, we know that the conditional is true, irrespective of what the consequent may be. Kleene's operations are monotonic in the sense that any valuation function preserves its initial assignments when the domain is extended and new objects are added to the interpretation of the partial predicates. If we understand the undefined cases as arguments of which we have not yet determined the value, it is an unnatural consequence of these truth tables that a classical tautology such as $p \vee \sim p$ remains undefined until we know the value of p. There are ways to escape such consequences and preserve the classical tautologies and contradictions in a partial truth definition, but we cannot go into such systems here.

Three-valued logics have primarily been applied in linguistics in semantic theories of *presuppositions*. There is a lively controversy surrounding the analysis of presupposition in philosophy and linguistics, which cannot be surveyed here. For illustration we define this notion in semantic terms:

> Any statement p is a *presupposition* of a statement q iff if p is not true (but false or undefined), q is undefined.

The truth of a presupposition p is in some sense an *assertability condition* for q. In our everyday use of natural language we rely uncommonly often on such presuppositions. We use names to refer to people under the assumption that they exist; we use definite NPs like *the students who passed their exams* with the common understanding that we mean to say something about a particular non-empty set of students. We also presuppose that it is raining when we say *John knows it is raining* or *Jane does not regret that it is raining.* Characteristic of presuppositions is that if p is a presupposition of q, p is also a presupposition of $\sim q$. This means that presuppositions are preserved under negation, and this is captured in the fact that * is preserved under negation according to the truth tables above. It is a major research question how the presuppositions of a complex sentence are to be characterized in terms of the presuppositions of its component sentences and this is called the *projection-problem* of presuppositions. Kleene's system goes a long way, since it recognizes that the presuppositions of a complex sentence may not be just the sum of the presuppositions of its parts. Presuppositions of sentences may for example be cancelled in sentences of which they are a constituent. Consider for instance the conditional *If there is a president of the U.S., the president is elected.* This sentence contains two sentences, (1) *There is a president of the U.S.* and (2) *The president is elected.* (1) is a presupposition of (2), since if (1) is false, (2) must be undefined, because it does not describe anyone in that case. The Kleene interpretation of the conditional captures this nicely, since $p \rightarrow q$ is true even when q is undefined and p is false. So the entire conditional sentence does not have (1) as presupposition. But Kleene's interpretation can be seen to lead to problems in the following sentence: (3) *If revolutions are unconstitutional, the president is elected.* Now (1) does seem to be a presupposition of (3), since if (1) is false, the consequent of (3) is undefined and hence (3) must be undefined. The antecedent may, however, very well be false when (1) is false, since they are independent. If the antecedent is false, (3) is true. But if there is an assignment making (3) true and (1) false, (1) cannot be a presupposition of (3). This seems wrong and not in accordance with our intuitions. Generally speaking, in a conditional with a contingent antecedent whose truth value is independent of the presuppositions of its consequent, the presuppositions of the consequent are incorrectly cancelled in the Kleene interpretation of the connectives. There are ways to mend this problem, but none has yet found general acceptance. The suggested further reading for this chapter contains some main references to the literature on presuppositions.

Review Exercises

1. Suppose that $P \longleftrightarrow Q$ is true, what is $P \vee \sim Q$?

2. Prove:

(a)
$$\begin{array}{l} p \\ q \\ \hline \therefore (p \,\&\, q) \vee r \end{array}$$

(b)
$$\begin{array}{l} p \leftrightarrow q \\ \hline \therefore \sim q \vee p \end{array}$$

(c)
$$\begin{array}{l} p \rightarrow (q \rightarrow \sim\sim r) \\ p \,\&\, \sim r \\ \hline \therefore \sim q \end{array}$$

(d)
$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \sim q \\ \hline \therefore r \end{array}$$

(e)
$$\begin{array}{l} p \,\&\, (q \rightarrow (r \vee \sim\sim s)) \\ q \\ \hline \therefore p \,\&\, (s \vee r) \end{array}$$

(f)
$$\begin{array}{l} \sim (p \vee \sim q) \\ r \vee p \\ \hline \therefore q \,\&\, r \end{array}$$

(g)
$$\begin{array}{l} p \rightarrow q \\ p \rightarrow (q \rightarrow r) \\ q \rightarrow (r \rightarrow s) \\ \hline \therefore p \rightarrow s \end{array}$$

(h)
$$\begin{array}{l} r \rightarrow (p \vee s) \\ q \rightarrow (s \vee t) \\ \sim s \\ \hline \therefore (\sim p \,\&\, \sim t) \rightarrow (\sim r \,\&\, \sim q) \end{array}$$

(i)
$$\begin{array}{l} p \rightarrow q \\ \sim q \,\&\, r \\ \hline \therefore \sim p \end{array}$$

(j)
$$\begin{array}{l} p \vee q \\ r \,\&\, \sim p \\ \hline \therefore q \end{array}$$

(k)
$$\begin{array}{l} p \leftrightarrow (\sim q \rightarrow r) \\ \sim r \,\&\, \sim (s \rightarrow q) \\ \hline \therefore \sim p \end{array}$$

3. Show that the following set of statements is inconsistent:

(a) $r \,\&\, (p \vee q)$

(b) $\sim (p \,\&\, r)$

(c) $\sim (q \,\&\, r)$

4. Does conjunction distribute over conditionals? I.e., is $(p \,\&\, q) \rightarrow (p \,\&\, r)$ equivalent to $p \,\&\, (q \rightarrow r)$?

5. Translate the following expressions to predicate logic.

(a) All horses are quadrupeds, but some quadrupeds are not horses.
(b) Distinct utterances must have distinct phonemic transcriptions.
(c) Not all trees are deciduous.
(d) Some politicians are honest men.
(e) No ducks are amphibious.
(f) Every cloud has a silver lining.
(g) Only Rosicrucians experience complete happiness.
(h) Everything I like is immoral, illegal or fattening.
(i) I like anything that is immoral, illegal or fattening.
(j) Everyone wants everyone to be rich.
(k) Everyone wants to be rich.

6. For each of the following formulas give an interpretation in a model which makes the formula false.

(a) $((\exists x)F(x) \,\&\, (\exists x)G(x)) \rightarrow (\exists x)(F(x) \,\&\, G(x))$
(b) $(\forall x)(\exists y)(\forall z)H(x,y,z) \rightarrow (\exists y)(\forall x)(\forall z)H(x,y,z)$

7. Formalize and prove with natural deduction:

(a) All linemen for the Green Bay Packers weigh at least 200 pounds. Mathilda weighs less than 200 pounds. Therefore, Mathilda is not a lineman for the Green Bay Packers.
(b) All cabdrivers and headwaiters are surly and churlish. Therefore, all cabdrivers are surly.

8. Construct Beth Tableaux for

(a) $[(\exists x)F(x) \rightarrow (\forall x)G(x)] \Longrightarrow (\forall x)(F(x) \rightarrow G(x))$
(b) $*(\forall x)(G(x) \,\&\, (\sim F(x) \vee H(x))) \Longrightarrow (\forall x)(G(x) \;\&\; (\exists x)(F(x) \rightarrow (\forall x)H(x))$