Appendix A
Bounded Operators and Classes of Compact Operators on Hilbert Spaces

Most of the results on bounded or compact operators we use can be found in standard texts on bounded Hilbert space operators such as [RS1, AG, GK, We].

Let $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ be complex Hilbert spaces with scalar products and norms denoted by $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ and $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_2$, respectively.

A linear operator $T:\mathcal{H}_1\to\mathcal{H}_2$ is called *bounded* if there is a constant $C>0$ such that $\|Tx\|_2\le C\|x\|_1$ for all $x\in\mathcal{D}(T)$; the *norm* of T is then defined by

$$\|T\|=\sup\bigl\{\|Tx\|_2:\|x\|_1\le 1,\ x\in\mathcal{D}(T)\bigr\}.$$

Let $\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ denote the set of bounded linear operators $T:\mathcal{H}_1\to\mathcal{H}_2$ such that $\mathcal{D}(T)=\mathcal{H}_1$. Then $(\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2),\|\cdot\|)$ is a Banach space. For each $T\in\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$, there exists a unique *adjoint operator* $T^*\in\mathbf{B}(\mathcal{H}_2,\mathcal{H}_1)$ such that

$$\langle Tx,y\rangle_2=\bigl\langle x,T^*y\bigr\rangle_1\quad\text{for all } x\in\mathcal{H}_1,\ y\in\mathcal{H}_2.$$

The set $\mathbf{B}(\mathcal{H}):=\mathbf{B}(\mathcal{H},\mathcal{H})$ is a $*$-algebra with composition of operators as product, adjoint operation as involution, and the identity operator $I=I_{\mathcal{H}}$ on $\mathcal{H}$ as unit.

An operator $T\in\mathbf{B}(\mathcal{H})$ is said to be *normal* if $T^*T=TT^*$.

An operator $T\in\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ is called an *isometry* if $T^*T=I_{\mathcal{H}_1}$ and *unitary* if $T^*T=I_{\mathcal{H}_1}$ and $TT^*=I_{\mathcal{H}_2}$. Clearly, $T\in\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ is an isometry if and only if $\langle Tx,Ty\rangle_2=\langle x,y\rangle_1$ for all $x,y\in\mathcal{H}_1$. Further, $T\in\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ is called a *finite-rank operator* if $\dim\mathcal{R}(T)<\infty$. If $T\in\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ is a finite-rank operator, so is T^*.

A sequence $(T_n)_{n\in\mathbb{N}}$ from $\mathbf{B}(\mathcal{H})$ is said to *converge strongly* to $T\in\mathbf{B}(\mathcal{H})$ if $\lim_{n\to\infty}T_nx=Tx$ for all $x\in\mathcal{H}$; in this case we write $T=\text{s-}\lim_{n\to\infty}T_n$.

A sequence $(x_n)_{n\in\mathbb{N}}$ of vectors $x_n\in\mathcal{H}$ *converges weakly* to a vector $x\in\mathcal{H}$ if $\lim_{n\to\infty}\langle x_n,y\rangle=\langle x,y\rangle$ for all $y\in\mathcal{H}$; then we write $x=\text{w-}\lim_{n\to\infty}x_n$.

Now we collect some facts on *projections*. Let $\mathcal{K}$ be a closed linear subspace of $\mathcal{H}$. The operator $P_{\mathcal{K}}\in\mathbf{B}(\mathcal{H})$ defined by $P_{\mathcal{K}}(x+y)=x$, where $x\in\mathcal{K}$ and $y\in\mathcal{K}^{\perp}$, is called the (orthogonal) *projection* of $\mathcal{H}$ on $\mathcal{K}$. An operator $P\in\mathbf{B}(\mathcal{H})$ is a projection if and only if $P=P^2$ and $P=P^*$; in this case, P projects on $P\mathcal{H}$.

Let P_1 and P_2 be projections of $\mathcal{H}$ onto subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. The sum $Q:=P_1+P_2$ is a projection if and only if $P_1P_2=0$, or equivalently, $\mathcal{K}_1\perp\mathcal{K}_2$; then Q projects on $\mathcal{K}_1\oplus\mathcal{K}_2$. The product $P:=P_1P_2$ is a projection if

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and only if $P_1P_2 = P_2P_1$; in this case, P projects on $\mathcal{K}_1 \cap \mathcal{K}_2$. Further, $P_1 \le P_2$, or equivalently $\mathcal{K}_1 \subseteq \mathcal{K}_2$, holds if and only if $P_1P_2 = P_1$, or equivalently $P_2P_1 = P_1$; if this is fulfilled, then $P_2 - P_1$ is the projection on $\mathcal{K}_2 \ominus \mathcal{K}_1$.

Let $P_n, n \in \mathbb{N}$, be pairwise orthogonal projections (that is, $P_kP_n = 0$ for $k \ne n$) on $\mathcal{H}$. Then the infinite sum $\sum_{n=1}^{\infty} P_n$ converges strongly to a projection P, and P projects on the closed subspace $\bigoplus_{n=1}^{\infty} P_n\mathcal{H}$ of $\mathcal{H}$.

If $T \in \mathbf{B}(\mathcal{H})$ and P is a projection of $\mathcal{H}$ on $\mathcal{K}$, then the operator $PT\lceil\mathcal{K}$ is in $\mathbf{B}(\mathcal{K})$; it is called the *compression* of T to $\mathcal{K}$.

In the remaining part of this appendix we are dealing with *compact operators*.

Definition A.1 A linear operator $T : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(T) = \mathcal{H}$ is called *compact* if the image $T(M)$ of each bounded subset M is relatively compact in $\mathcal{H}$, or equivalently, if for each bounded sequence $(x_n)_{n\in\mathbb{N}}$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathcal{H}$.

The compact operators on $\mathcal{H}$ are denoted by $\mathbf{B}_\infty(\mathcal{H})$. Each compact operator is bounded. Various characterizations of compact operators are given in the following:

Proposition A.1 *For any $T \in \mathbf{B}(\mathcal{H})$, the following statements are equivalent*:

(i) *T is compact.*
(ii) *$|T| := (T^*T)^{1/2}$ is compact.*
(iii) *T^*T is compact.*
(iv) *T maps each weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ into a convergent sequence in $\mathcal{H}$, that is, if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{H}$, then $Tx_n \to Tx$ in $\mathcal{H}$.*
(v) *There exists a sequence $(T_n)_{n\in\mathbb{N}}$ of finite-rank operators $T_n \in \mathbf{B}(\mathcal{H})$ such that* $\lim_{n\to\infty} \|T - T_n\| = 0$.

Proof [RS1, Theorems VI.11 and VI.13] or [We, Theorems 6.3, 6.4, 6.5]. □

Proposition A.2 *Let $T, T_n, S_1, S_2 \in \mathbf{B}(\mathcal{H})$ for $n \in \mathbb{N}$. If $\lim_{n\to\infty} \|T - T_n\| = 0$ and each operator T_n is compact, so is T. If T is compact, so are S_1TS_2 and T^*.*

Proof [RS1, Theorem VI.12] or [We, Theorem 6.4]. □

Basic results about the spectrum of compact operators are collected in the following two theorems.

Theorem A.3 *Let $T \in \mathbf{B}_\infty(\mathcal{H})$. The spectrum $\sigma(T)$ is an at most countable set which has no nonzero accumulation point. If $\mathcal{H}$ is infinite-dimensional, then $0 \in \sigma(T)$. If λ is a nonzero number of $\sigma(T)$, then λ is an eigenvalue of T of finite multiplicity, and $\overline{\lambda}$ is an eigenvalue of T^* which has the same multiplicity as λ.*

Proof [RS1, Theorem VI.15] or [We, Theorems 6.7, 6.8] or [AG, Nr. 57]. □

From now on we assume that $\mathcal{H}$ is an *infinite-dimensional separable* Hilbert space.

Theorem A.4 *Suppose that* $T \in \mathbf{B}_\infty(\mathcal{H})$ *is normal. Then there exist a complex sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *and an orthonormal basis* $\{e_n : n \in \mathbb{N}\}$ *of* $\mathcal{H}$ *such that*

$$\lim_{n\to\infty} \lambda_n = 0 \quad \textit{and} \quad Te_n = \lambda_n e_n, \qquad T^* e_n = \overline{\lambda_n} e_n \quad \textit{for } n \in \mathbb{N}.$$

Proof [AG, Nr. 62] or [We, Theorem 7.1], see, e.g., [RS1, Theorem VI.16]. □

Conversely, let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$, and $(\lambda_n)_{n\in\mathbb{N}}$ a bounded complex sequence. Then the normal operator $T \in \mathbf{B}(\mathcal{H})$ defined by $Te_n = \lambda_n e_n$, $n \in \mathbb{N}$, is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.

Next we turn to *Hilbert–Schmidt operators*.

Definition A.2 An operator $T \in \mathbf{B}(\mathcal{H})$ is called a *Hilbert–Schmidt operator* if there exists an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$ such that

$$\|T\|_2 := \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2} < \infty. \tag{A.1}$$

The set of Hilbert–Schmidt operators on $\mathcal{H}$ is denoted by $\mathbf{B}_2(\mathcal{H})$. The number $\|T\|_2$ in (A.1) does not depend on the particular orthonormal basis, and $\mathbf{B}_2(\mathcal{H})$ is a Banach space equipped with the norm $\|\cdot\|_2$. Each Hilbert–Schmidt operator is compact.

An immediate consequence of Theorem A.4 and the preceding definition is the following:

Corollary A.5 *If* $T \in \mathbf{B}_2(\mathcal{H})$ *is normal, then the sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *of eigenvalues of* T *counted with multiplicities belongs to* $l^2(\mathbb{N})$.

Theorem A.6 *Let* M *be an open subset of* $\mathbb{R}^d$, *and* $K \in L^2(M \times M)$. *Then the integral operator* T_K *defined by*

$$(T_K f)(x) = \int_M K(x, y) f(y)\, dy, \qquad f \in L^2(M),$$

is a Hilbert–Schmidt operator on $\mathcal{H} = L^2(M)$, *and* $\|T_K\|_2 = \|K\|_{L^2(M\times M)}$. *In particular,* T_K *is a compact operator on* $L^2(M)$.

Proof [RS1, Theorem VI.23] or [AG, Nr. 32] or [We, Theorem 6.11]. □

Finally, we review some basics on *trace class operators*.

Let $T \in \mathbf{B}_\infty(\mathcal{H})$. Then $|T| = (T^*T)^{1/2}$ is a positive self-adjoint compact operator on $\mathcal{H}$. Let $(s_n(T))_{n\in\mathbb{N}}$ denote the sequence of eigenvalues of $|T|$ counted with multiplicities and arranged in decreasing order. The numbers $s_n(T)$ are called *singular numbers* of the operator T. Note that $s_1(T) = \|T\|$.

Definition A.3 An operator $T \in \mathbf{B}_\infty(\mathcal{H})$ is said to be of *trace class* if

$$\|T\|_1 := \sum_{n=1}^{\infty} s_n(T) < \infty.$$

The set $\mathbf{B}_1(\mathcal{H})$ of all trace class operators on $\mathcal{H}$ is a Banach space equipped with the trace norm $\|\cdot\|_1$. Trace class operators are Hilbert–Schmidt operators.

If $T \in \mathbf{B}_1(\mathcal{H})$ and $S_1, S_2 \in \mathbf{B}(\mathcal{H})$, then $T^* \in \mathbf{B}_1(\mathcal{H})$, $S_1 T S_2 \in \mathbf{B}_1(\mathcal{H})$, and

$$\|T\| \leq \|T\|_1 = \left\|T^*\right\|_1 \quad \text{and} \quad \|S_1 T S_2\|_1 \leq \|T\|_1 \|S_1\| \|S_2\|. \tag{A.2}$$

Proposition A.7 *The set of finite-rank operators of* $\mathbf{B}(\mathcal{H})$ *is a dense subset of the Banach space* $(\mathbf{B}_1(\mathcal{H}), \|\cdot\|_1)$.

Proof [RS1, Corollary, p. 209]. □

Theorem A.8 (*Trace of trace class operators*) *Let* $T \in \mathbf{B}_1(\mathcal{H})$. *If* $\{x_n : n \in \mathbb{N}\}$ *is an orthonormal basis of* $\mathcal{H}$, *then*

$$\operatorname{Tr} T := \sum_{n=1}^{\infty} \langle T x_n, x_n \rangle \tag{A.3}$$

is finite and independent of the orthonormal basis. It is called the trace *of* T.

Proof [RS1, Theorem VI.24] or [GK, Chap. III, Theorem 8.1] or [AG, Nr. 66]. □

Combining Theorems A.4 and A.8, we easily obtain the following:

Corollary A.9 *For* $T = T^* \in \mathbf{B}_1(\mathcal{H})$, *there are a real sequence* $(\alpha_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$ *and an orthonormal basis* $\{x_n : n \in \mathbb{N}\}$ *of* $\mathcal{H}$ *such that*

$$T = \sum_{n=1}^{\infty} \alpha_n \langle \cdot, x_n \rangle x_n, \qquad \|T\|_1 = \sum_{n=1}^{\infty} |\alpha_n|, \qquad \operatorname{Tr} T = \sum_{n=1}^{\infty} \alpha_n. \tag{A.4}$$

(A.4) implies that $T \to \operatorname{Tr} T$ is a continuous linear functional on $(\mathbf{B}_1(\mathcal{H}), \|\cdot\|_1)$.

Theorem A.10 (*Lidskii's theorem*) *Let* $T \in \mathbf{B}_1(\mathcal{H})$. *Let* $\lambda_n(T)$, $n \in \mathbb{N}$, *denote the eigenvalues of* T *counted with multiplicities*; *if* T *has only finitely many eigenvalues* $\lambda_1(T), \dots, \lambda_k(T)$, *set* $\lambda_n(T) = 0$ *for* $n > k$; *if* T *has no eigenvalue, put* $\lambda_n(T) = 0$ *for all* $n \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{\infty} \left|\lambda_n(T)\right| \leq \|T\|_1 \quad \text{and} \quad \operatorname{Tr} T = \sum_{n=1}^{\infty} \lambda_n(T). \tag{A.5}$$

Proof [Ld] or [RS4, Corollary, p. 328] or [GK, Chap. III, Theorem 8.4]. □

Appendix B
Measure Theory

The material collected in this appendix can be found in advanced standard texts on measure theory such as [Ru2] or [Cn].

Let Ω be a nonempty set, and let $\mathfrak{A}$ be a σ-algebra on Ω. A *positive measure* (briefly, a *measure* on $(\Omega, \mathfrak{A})$) is a mapping $\mu : \mathfrak{A} \to [0, +\infty]$ which is σ-additive, that is,

$$\mu\left(\bigcup_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} \mu(M_n) \tag{B.1}$$

for any sequence $(M_n)_{n\in\mathbb{N}}$ of disjoints sets $M_n \in \mathfrak{A}$. A positive measure μ is said to be *finite* if $\mu(\Omega) < \infty$ and σ*-finite* if Ω is a countable union of sets $M_n \in \mathfrak{A}$ with $\mu(M_n) < \infty$. A *measure space* is a triple $(\Omega, \mathfrak{A}, \mu)$ of a set Ω, a σ-algebra $\mathfrak{A}$ on Ω, and a positive measure μ on $\mathfrak{A}$. A property is said to hold μ*-a.e.* on Ω if it holds except for a μ-null set. A function f on Ω with values in $(-\infty, +\infty]$ is called $\mathfrak{A}$-*measurable* if for all $a \in \mathbb{R}$, the set $\{t \in \Omega : f(t) \le a\}$ is in $\mathfrak{A}$. A function f with values in $\mathbb{C} \cup \{+\infty\}$ is said to be μ*-integrable* if the integral $\int_\Omega f\,d\mu$ exists and is finite.

A *complex measure* on $(\Omega, \mathfrak{A})$ is a σ-additive map $\mu : \mathfrak{A} \to \mathbb{C}$. Note that complex measures have only finite values, while positive measures may take the value $+\infty$.

Let μ be a complex measure. Then μ can written as $\mu = \mu_1 - \mu_2 + \mathrm{i}(\mu_3 - \mu_4)$ where μ_1, μ_2, μ_3, μ_4 are finite positive measures. We say that μ is *supported* by a set $K \in \mathfrak{A}$ if $\mu(M) = \mu(M \cap K)$ for all $M \in \mathfrak{A}$. Given $M \in \mathfrak{A}$, let

$$|\mu|(M) = \sup \sum_{n=1}^{k} \mu(M_n), \tag{B.2}$$

where the supremum is taken over all finite partitions $M = \bigcup_{n=1}^{k} M_n$ of disjoint sets $M_n \in \mathfrak{A}$. Then $|\mu|$ is a positive measure on $(\Omega, \mathfrak{A})$ called the *total variation* of μ.

There are three basic limit theorems. Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space.

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Theorem B.1 (*Lebesgue's dominated convergence theorem*) *Let f_n, $n \in \mathbb{N}$, and f be $\mathfrak{A}$-measurable complex functions on Ω, and let $g : \Omega \to [0, +\infty]$ be a μ-integrable function. Suppose that*

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \textit{and} \quad \left|f_n(t)\right| \le g(t), \quad n \in \mathbb{N},\ \mu\textit{-a.e. on } \Omega. \tag{B.3}$$

Then the functions f and f_n are μ-integrable,

$$\lim_{n\to\infty} \int_\Omega |f_n - f|\, d\mu = 0, \quad \textit{and} \quad \lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu.$$

Proof [Ru2, Theorem 1.34]. □

Theorem B.2 (*Lebesgue's monotone convergence theorem*) *Let f_n, $n \in \mathbb{N}$, and f be $[0, +\infty]$-valued $\mathfrak{A}$-measurable functions on Ω such that*

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \textit{and} \quad f_n(t) \le f_{n+1}(t), \quad n \in \mathbb{N},\ \mu\textit{-a.e. on } \Omega. \tag{B.4}$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu.$$

Proof [Ru2, Theorem 1.26]. □

Theorem B.3 (*Fatou's lemma*) *If f_n, $n \in \mathbb{N}$, are $[0, +\infty]$-valued $\mathfrak{A}$-measurable functions on Ω, then*

$$\int_\Omega \Big(\liminf_{n\to\infty} f_n\Big)\, d\mu \le \liminf_{n\to\infty} \int_\Omega f_n\, d\mu.$$

Proof [Ru2, Lemma 1.28]. □

Let $p, q, r \in (1, \infty)$ and $p^{-1} + q^{-1} = r^{-1}$. If $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$, then $fg \in L^r(\Omega, \mu)$, and the *Hölder inequality* holds:

$$\|fg\|_{L^r(\Omega,\mu)} \le \|f\|_{L^p(\Omega,\mu)} \|g\|_{L^q(\Omega,\mu)}. \tag{B.5}$$

Let φ be a mapping of Ω onto another set Ω_0. Then the family $\mathfrak{A}_0$ of all subsets M of Ω_0 such that $\varphi^{-1}(M) \in \mathfrak{A}$ is a σ-algebra on Ω_0 and $\mu_0(M) := \mu(\varphi^{-1}(M))$ defines an measure on $\mathfrak{A}_0$. Thus, $(\Omega_0, \mathfrak{A}_0, \mu_0)$ is also a measure space.

Proposition B.4 (*Transformations of measures*) *If f is a μ_0-a.e. finite $\mathfrak{A}_0$-measurable function on Ω_0, then $f \circ \varphi$ is a μ-a.e. finite $\mathfrak{A}$-measurable function on Ω, and*

$$\int_{\Omega_0} f(s)\, d\mu_0(s) = \int_\Omega f\big(\varphi(t)\big)\, d\mu(t), \tag{B.6}$$

where if either integral exists, so does the other.

Proof [Ha, §39, Theorem C]. □

Let $(\Omega, \mathfrak{A}, \mu)$ and $(\Phi, \mathfrak{B}, \nu)$ be σ-finite measure spaces. Let $\mathfrak{A} \times \mathfrak{B}$ denote the σ-algebra on $\Omega \times \Phi$ generated by the sets $M \times N$, where $M \in \mathfrak{A}$ and $N \in \mathfrak{B}$. Then there exists a unique measure $\mu \times \nu$, called the *product* of μ and ν, on $\mathfrak{A} \times \mathfrak{B}$ such that $(\mu \times \nu)(M \times N) = \mu(M)\nu(N)$ for all $M \in \mathfrak{A}$ and $N \in \mathfrak{B}$.

Theorem B.5 (*Fubini's theorem*) *Let f be an $\mathfrak{A} \times \mathfrak{B}$-measurable complex function on $\Omega \times \Phi$. Then*

$$\int_\Omega \left(\int_\Phi |f(x,y)|\, d\nu(y) \right) d\mu(x) = \int_\Phi \left(\int_\Omega |f(x,y)|\, d\mu(x) \right) d\nu(y). \tag{B.7}$$

If the double integral in (B.7) is finite, then we have

$$\begin{aligned} \int_\Omega \left(\int_\Phi f(x,y)\, d\nu(y) \right) d\mu(x) &= \int_\Omega \left(\int_\Phi f(x,y)\, d\mu(x) \right) d\nu(y) \\ &= \int_{\Omega \times \Phi} f(x,y) d(\mu \times \nu)(x,y). \end{aligned}$$

Proof [Ru2, Theorem 8.8]. □

Thus, if f is $[0, +\infty]$-valued, one can always interchange the order of integrations. If f is a complex function, it suffices to check that the integral in (B.7) is finite.

Let μ and ν be complex or positive measures on $\mathfrak{A}$. We say that μ and ν are *mutually singular* and write $\mu \perp \nu$ if μ and ν are supported by disjoint sets.

Suppose that ν is positive. We say that μ is *absolutely continuous* with respect to ν and write $\mu \ll \nu$ if $\nu(N) = 0$ for $N \in \mathfrak{A}$ implies that $\mu(N) = 0$.

For any $f \in L^1(\Omega, \nu)$, there is a complex measure μ defined by

$$\mu(M) = \int_M f\, d\nu, \quad M \in \mathfrak{A}.$$

Obviously, $\mu \ll \nu$. We then write $d\mu = f\, d\nu$.

Theorem B.6 (*Lebesgue–Radon–Nikodym theorem*) *Let ν be a σ-finite positive measure, and μ a complex measure on $(\Omega, \mathfrak{A})$.*

(i) *There is a unique pair of complex measures μ_a and μ_s such that*

$$\mu = \mu_a + \mu_s, \quad \mu_s \perp \nu, \; \mu_a \ll \nu. \tag{B.8}$$

(ii) *There exists a unique function $f \in L^1(\Omega, \nu)$ such that $d\mu_a = f\, d\nu$.*

Proof [Ru2, Theorem 6.10]. □

Assertion (ii) is the *Radon–Nikodym theorem*. It says that each complex measure μ which is absolutely continuous w.r.t. ν is of the form $d\mu = f d\nu$ with

$f \in L^1(\Omega, \nu)$ uniquely determined by ν and μ. The function f is called the *Radon–Nikodym derivative* of ν w.r.t. μ and denoted by $\frac{d\nu}{d\mu}$.

Assertion (i) is the *Lebesgue decomposition theorem*. We refer to the relation $d\mu = f\,d\nu + d\mu_s$ as the *Lebesgue decomposition* of μ relative to ν.

Let μ be a σ-finite positive measure. Then there is also a decomposition (B.8) into a unique pair of positive measures μ_a and μ_s, and there exists a $[0, +\infty]$-valued $\mathfrak{A}$-measurable function f, uniquely determined ν-a.e., such that $d\mu_a = f\,d\nu$.

Next we turn to Borel measures. Let Ω be a locally compact Hausdorff space that has a countable basis. The *Borel algebra* $\mathfrak{B}(\Omega)$ is the σ-algebra on Ω generated by the open subsets of Ω. A $\mathfrak{B}(\Omega)$-measurable function on Ω is called a *Borel function*. A *regular Borel measure* is a measure μ on $\mathfrak{B}(\Omega)$ such that for all M in $\mathfrak{B}(\Omega)$,

$$\mu(M) = \sup\bigl\{\mu(K) : K \subseteq M, K \text{ compact}\bigr\} = \inf\bigl\{\mu(U) : M \subseteq U,\ U \text{ open}\bigr\}.$$

The *support* of μ is the smallest closed subset M of Ω such that $\mu(\Omega \backslash M) = 0$; it is denoted by $\operatorname{supp} \mu$. A *regular complex Borel measure* is a complex measure of the form $\mu = \mu_1 - \mu_2 + \mathrm{i}(\mu_3 - \mu_4)$, where μ_1, μ_2, μ_3, μ_4 are finite regular positive Borel measures.

Theorem B.7 (*Riesz representation theorem*) *Let $C(\Omega)$ be the Banach space of continuous functions on a compact Hausdorff space Ω equipped with the supremum norm. For each continuous linear functional F on $C(\Omega)$, there exists a unique regular complex Borel measure μ on Ω such that*

$$F(f) = \int_\Omega f\,d\mu, \quad f \in C(\Omega).$$

Moreover, $\|F\| = |\mu|(\Omega)$.

Proof [Ru2, Theorem 6.19]. □

The Lebesgue measure on $\mathbb{R}^d$ is denoted by m. We write simply dx instead of $dm(x)$ and *a.e.* when we mean m-a.e. If Ω is an open or closed subset of $\mathbb{R}^d$, then $L^p(\Omega)$ always denotes the space $L^p(\Omega, m)$ with respect to the Lebesgue measure.

In the case where ν is the Lebesgue measure on $\Omega = \mathbb{R}$ we specify our terminology concerning Theorem B.6(i). In this case we write μ_{ac} for μ_a and μ_{sing} for μ_s and call μ_{ac} and μ_{sing} the *absolutely continuous part* and *singular part* of μ, respectively.

Let μ be a positive regular Borel measure on $\mathbb{R}$. A point $t \in \mathbb{R}$ is called an *atom* of μ if $\mu(\{t\}) \neq 0$. The set P of atoms of μ is countable, since μ is finite for compact sets. There is a measure μ_{p} on $\mathfrak{B}(\mathbb{R})$ defined by $\mu_{\mathrm{p}}(M) = \mu(M \cap P)$. Then μ_{p} is a *pure point measure*, that is, $\mu_{\mathrm{p}}(M) = \sum_{t \in M} \mu(\{t\})$ for $M \in \mathfrak{B}(\mathbb{R})$. Clearly, $\mu_{\mathrm{sc}} := \mu_{\mathrm{sign}} - \mu_{\mathrm{p}}$ is a positive measure. Further, μ_{sc} and m are mutually singular, and μ_{sc} is a *continuous measure*, that is, μ_{sc} has no atoms. Such a measure is called *singularly continuous*. We restate these considerations.

Proposition B.8 *For any regular positive Borel measure μ on $\mathbb{R}$, there is a unique decomposition $\mu = \mu_{\mathrm{p}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{ac}}$ into a pure point part μ_{p}, a singularly continuous part μ_{sc}, and an absolutely continuous part μ_{ac}. These parts are mutually singular.*

Let μ be a complex Borel measure on $\mathbb{R}$. The *symmetric derivative $D\mu$* of μ with respect to the Lebesgue measure is defined by

$$(D\mu)(t) = \lim_{\varepsilon\to+0}(2\varepsilon)^{-1}\mu\big([t-\varepsilon, t+\varepsilon]\big), \quad t \in \mathbb{R}. \tag{B.9}$$

The following theorem relates singular and absolutely continuous parts of a measure and its symmetric derivative. The description of the singular part in (ii) is the classical *de la Vallée Poussin theorem.*

Theorem B.9 *Suppose that μ is a complex regular Borel measure on $\mathbb{R}$. Let $d\mu = f\,dm + d\mu_{\mathrm{sign}}$ be its Lebesgue decomposition with respect to the Lebesgue measure.*

(i) *$(D\mu)(t)$ exists a.e. on $\mathbb{R}$, and $D\mu = f$ a.e. on $\mathbb{R}$, that is, $d\mu_{\mathrm{ac}}(t) = (D\mu)(t)\,dt$.*
(ii) *Suppose that the measure μ is positive. Then its singular part μ_{sign} is supported by the set $S(\mu) = \{t \in \mathbb{R} : (D\mu)(t) = +\infty\}$, and its absolutely continuous part μ_{ac} is supported by $L(\mu) = \{t \in \mathbb{R} : 0 < (D\mu)(t) < \infty\}$.*

Proof (i): [Ru2, Theorem 7.8].

(ii): The result about μ_{sign} is proved in [Ru2, Theorem 7.15].

The assertion concerning μ_{ac} is derived from (i). Set $M_0 := \{t : (D\mu)(t) = 0\}$ and $M_\infty := \{t : (D\mu)(t) = +\infty\}$. Because μ is a positive measure, $(D\mu)(t) \ge 0$ a.e. on $\mathbb{R}$. Since $d\mu_{\mathrm{ac}} = (D\mu)\,dt$ by (i) and $D\mu = f \in L^1(\mathbb{R})$, we have $\mu_{\mathrm{ac}}(M_0) = 0$ and $m(M_\infty) = 0$, so $\mu_{\mathrm{ac}}(M_\infty) = 0$. Hence, μ_{ac} is supported by $L(\mu)$. □

Finally, we discuss the relations between complex regular Borel measures and functions of bounded variation. Let f be a function on $\mathbb{R}$. The *variation* of f is

$$V_f := \sup \sum_{k=1}^{n} \big| f(t_k) - f(t_{k-1}) \big|,$$

the supremum being taken over all collections of numbers $t_0 < \dots < t_n$ and $n \in \mathbb{N}$. We say that f is of *bounded variation* if $V_f < \infty$. Clearly, each bounded nondecreasing real function is of bounded variation. Let *NBV* denote the set of all right-continuous functions f of bounded variation satisfying $\lim_{t\to-\infty} f(t) = 0$.

Theorem B.10 (i) *There is a one-to-one correspondence between complex regular Borel measures μ on $\mathbb{R}$ and functions f in NBV. It is given by*

$$f_\mu(t) = \mu\big((-\infty, t]\big), \quad t \in \mathbb{R}. \tag{B.10}$$

(ii) *Each function $f \in NBV$ is differentiable a.e. on $\mathbb{R}$, and $f' \in L^1(\mathbb{R})$. Further, if $f = f_\mu$ is given by (B.10), then $f'(t) = (D\mu)(t)$ a.e. on $\mathbb{R}$.*

Proof (i): [Cn, Proposition 4.4.3] or [Ru1, Theorem 8.14].
(ii): [Ru1, Theorem 8.18]. □

Obviously, if μ is a positive measure, then the function f_μ is nondecreasing.

By a slight abuse of notation one usually writes simply $\mu(t)$ for the function $f_\mu(t)$ defined by (B.10).

Appendix C
The Fourier Transform

In this appendix we collect basic definitions and results on the Fourier transform. Proofs and more details can be found, e.g., in [RS2, Chap. IX], [GW], and [Vl].

Definition C.1 The *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ is the set of all functions $\varphi \in C^\infty(\mathbb{R}^d)$ satisfying

$$p_{\alpha,\beta}(\varphi) := \sup\bigl\{\bigl|\bigl(x^\alpha D^\beta \varphi\bigr)(x)\bigr| : x \in \mathbb{R}^d\bigr\} < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^d.$$

Equipped with the countable family of seminorms $\{p_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^d\}$, $\mathcal{S}(\mathbb{R}^d)$ is a locally convex Fréchet space. Its dual $\mathcal{S}'(\mathbb{R}^d)$ is the vector space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. The elements of $\mathcal{S}'(\mathbb{R}^d)$ are called *tempered distributions*.

Each function $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$, gives rise to an element $F_f \in \mathcal{S}'(\mathbb{R}^d)$ by defining $F_f(\varphi) = \int f \varphi \, dx$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We shall consider $L^p(\mathbb{R}^d)$ as a linear subspace of $\mathcal{S}'(\mathbb{R}^d)$ by identifying f with F_f.

For $f \in L^1(\mathbb{R}^d)$, we define two functions $\hat{f}, \check{f} \in L^\infty(\mathbb{R}^d)$ by

$$\hat{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-\mathrm{i}x\cdot y} f(y)\, dy, \tag{C.1}$$

$$\check{f}(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot y} f(x)\, dx, \tag{C.2}$$

where $x \cdot y := x_1 y_1 + \cdots + x_d y_d$ denotes the scalar product of $x, y \in \mathbb{R}^d$.

Lemma C.1 (*Riemann–Lebesgue lemma*) *If* $f \in L^1(\mathbb{R}^d)$, *then* $\lim_{\|x\|\to\infty} \hat{f}(x) = 0$.

Proof [RS2, Theorem IX.7] or [GW, Theorem 17.1.3]. □

Theorem C.2 *The map* $\mathcal{F} : f \to \mathcal{F}(f) := \hat{f}$ *is a continuous bijection of* $\mathcal{S}(\mathbb{R}^d)$ *onto* $\mathcal{S}(\mathbb{R}^d)$ *with inverse* $\mathcal{F}^{-1}(f) = \check{f}$, $f \in \mathcal{S}(\mathbb{R}^d)$. *For* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* $\alpha \in \mathbb{N}_0^d$,

$$D^\alpha f = \mathcal{F}^{-1} x^\alpha \mathcal{F} f. \tag{C.3}$$

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Proof [RS2, Formula (IX.1) and Theorem IX.1] or [GW, Theorem 19.3.1]. □

For $F \in \mathcal{S}'(\mathbb{R}^d)$, define $\mathcal{F}(F)(\varphi) := F(\hat{\varphi}) \equiv F(\mathcal{F}(\varphi))$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. By Theorem C.2, we have $\mathcal{F}(F) \in \mathcal{S}'(\mathbb{R}^d)$. The tempered distribution $\mathcal{F}(F)$ is called the *Fourier transform* of F. An immediate consequence (see, e.g., [RS2, Theorem IX.2]) of Theorem C.2 is the following:

Corollary C.3 *$\mathcal{F}$ is a bijection of $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$. The inverse of $\mathcal{F}$ is given by $\mathcal{F}^{-1}(F)(\varphi) = F(\check{\varphi}) \equiv F(\mathcal{F}^{-1}(\varphi))$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.*

Since $L^p(\mathbb{R}^d)$ is contained in $\mathcal{S}'(\mathbb{R}^d)$, we know from Corollary C.3 that $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ are well-defined elements of $\mathcal{S}'(\mathbb{R}^d)$ for any $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$. The following two fundamental theorems describe some of these elements.

Theorem C.4 (*Plancherel theorem*) *$\mathcal{F}$ is a unitary linear map of $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$, the corresponding functions $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ of $L^2(\mathbb{R}^d)$ are given by*

$$\hat{f}(x) := \mathcal{F}(f)(x) = \lim_{R\to\infty} (2\pi)^{-d/2} \int_{\|y\|\le R} e^{-\mathrm{i}x\cdot y} f(y)\,dy, \tag{C.4}$$

$$\check{f}(y) := \mathcal{F}^{-1}(f)(y) = \lim_{R\to\infty} (2\pi)^{-d/2} \int_{\|x\|\le R} e^{\mathrm{i}x\cdot y} f(x)\,dx. \tag{C.5}$$

Proof [RS2, Theorem IX.6] or [GW, Theorem 22.1.4]. □

Theorem C.5 (*Hausdorff–Young theorem*) *Suppose that $q^{-1} + p^{-1} = 1$ and $1 \le p \le 2$. Then the Fourier transform $\mathcal{F}$ maps $L^p(\mathbb{R}^d)$ continuously into $L^q(\mathbb{R}^d)$.*

Proof [RS2, Theorem IX.8] or [Tl, Theorem 1.18.8] or [LL, Theorem 5.7]. □

Definition C.2 Let $f, g \in L^2(\mathbb{R}^d)$. The *convolution* of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy, \quad x \in \mathbb{R}^d.$$

Note that the function $y \to f(x-y)g(y)$ is in $L^1(\mathbb{R}^d)$, because $f, g \in L^2(\mathbb{R}^d)$.

Theorem C.6 *$\mathcal{F}^{-1}(\mathcal{F}(f)\cdot\mathcal{F}(g)) = (2\pi)^{-d/2} f * g$ for $f, g \in L^2(\mathbb{R}^d)$.*

Proof [GW, Proposition 33.3.1] or [LL, Theorem 5.8]; see also [RS2, Theorem IX.3]. □

Appendix D
Distributions and Sobolev Spaces

Let Ω be an open subset of $\mathbb{R}^d$. In this appendix we collect basic notions and results about distributions on Ω and the Sobolev spaces $H^n(\Omega)$ and $H^n_0(\Omega)$.

Distributions are treated in [GW, Gr, LL, RS2], and [Vl]. Sobolev spaces are developed in many books such as [At, Br, EE, Gr, LL, LM], and [Tl].

We denote by $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ the set of all $f \in C^\infty(\Omega)$ for which its support

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R}^d : f(x) \neq 0\}}$$

is a compact subset of Ω. Further, recall the notations $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_d})^{\alpha_d}$, $\partial_k := \frac{\partial}{\partial x_k}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $k = 1, \dots, d$.

Definition D.1 A linear functional F on $\mathcal{D}(\Omega)$ is called a *distribution* on Ω if for every compact subset K of Ω, there exist numbers $n_K \in \mathbb{N}_0$ and $C_K > 0$ such that

$$\bigl|F(\varphi)\bigr| \leq C_K \sup_{x \in K,\, |\alpha| \leq n_K} \bigl|\partial^\alpha \varphi(x)\bigr| \quad \text{for all } \varphi \in \mathcal{D}(\Omega), \operatorname{supp} \varphi \subseteq K.$$

The vector space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$.

Let $L^1_{\text{loc}}(\Omega)$ denote the vector space of measurable functions f on Ω such that $\int_K |f|\, dx < \infty$ for each compact set $K \subseteq \Omega$. Each $f \in L^1_{\text{loc}}(\Omega)$ yields a distribution $F_f \in \mathcal{D}'(\Omega)$ defined by $F_f(\varphi) = \int f\varphi\, dx$, $\varphi \in \mathcal{D}(\Omega)$. By some abuse of notation, we identify the function $f \in L^1_{\text{loc}}(\Omega)$ with the distribution $F_f \in \mathcal{D}'(\Omega)$.

Suppose that $F \in \mathcal{D}'(\Omega)$. If $a \in C^\infty(\Omega)$, then $aF(\cdot) := F(a\cdot)$ defines a distribution $aF \in \mathcal{D}'(\Omega)$. For $\alpha \in \mathbb{N}_0^d$, there is a distribution $\partial^\alpha F$ defined by

$$\partial^\alpha F(\varphi) := (-1)^{|\alpha|} F\bigl(\partial^\alpha \varphi\bigr), \quad \varphi \in \mathcal{D}(\Omega).$$

Let $u \in L^1_{\text{loc}}(\Omega)$. A function $v \in L^1_{\text{loc}}(\Omega)$ is called the αth *weak derivative* of u if the distributional derivative $\partial^\alpha u \in \mathcal{D}'(\Omega)$ is given by the function v, that is, if

$$(-1)^{|\alpha|} \int_\Omega u \partial^\alpha(\varphi)\, dx = \int_\Omega v\varphi\, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

In this case, v is uniquely determined a.e. on Ω by u and denoted also by $\partial^\alpha u$.

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If $u \in C^n(\Omega)$, integration by parts implies that the αth weak derivative $\partial^\alpha u$ exists when $|\alpha| \leq n$ and is given by the "usual" derivative $\partial^\alpha u(x)$ on Ω.

Next, we introduce the Sobolev spaces $H^n(\Omega)$ and $H_0^n(\Omega)$, $n \in \mathbb{N}_0$.

Definition D.2 $H^n(\Omega)$ is the vector space of functions $f \in L^2(\Omega)$ for which the weak derivative $\partial^\alpha f$ exists and belongs to $L^2(\Omega)$ for all $\alpha \in N_0^d$, $|\alpha| \leq n$, endowed with the scalar product

$$\langle f, g\rangle_{H^n(\Omega)} := \sum_{|\alpha|\leq n} \left\langle \partial^\alpha f, \partial^\alpha g\right\rangle_{L^2(\Omega)},$$

and $H_0^n(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $(H^n(\Omega), \|\cdot\|_{H^n(\Omega)})$.

Then $H^n(\Omega)$ and $H_0^n(\Omega)$ are Hilbert spaces.

The two compact embedding Theorems D.1 and D.4 are due to *F. Rellich*.

Theorem D.1 *If Ω is bounded, the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact.*

Proof [At, A 6.1] or [EE, Theorem 3.6]. □

Theorem D.2 (*Poincaré inequality*) *If Ω is bounded, there is a constant $c_\Omega > 0$ such that*

$$c_\Omega \|u\|^2_{L^2(\Omega)} \leq \sum_{k=1}^{d} \int_\Omega |\partial_k u|^2\, dx \equiv \|\nabla u\|^2_{L^2(\Omega)} \quad \textit{for all } u \in H_0^1(\Omega). \tag{D.1}$$

Proof [At, 4.7] or [Br, Corollary 9.19]. □

The Poincaré inequality implies that $\|\nabla\cdot\|_{L^2(\Omega)}$ defines an equivalent norm on the Hilbert space $(H_0^1(\Omega), \|\cdot\|_{H_0^1(\Omega)})$.

For $\Omega = \mathbb{R}^d$, the Sobolev spaces can be described by the Fourier transform.

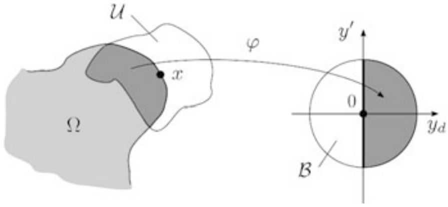
Theorem D.3 $H^n(\mathbb{R}^d) = H_0^n(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \|x\|^{n/2}\hat{f}(x) \in L^2(\mathbb{R}^d)\}$. *Further, if $k \in \mathbb{N}_0$ and $n > k + d/2$, then $H^n(\mathbb{R}^d) \subseteq C^k(\mathbb{R}^d)$.*

Proof [LM, p. 30] or [RS2, p. 50 and Theorem IX.24]. □

The next results require some "smoothness" assumption on the boundary $\partial\Omega$. Let $m \in \mathbb{N} \cup \{\infty\}$. An open set $\Omega \subseteq \mathbb{R}^d$ is said to be of *class* C^m if for each point $x \in \partial\Omega$, there are an open neighborhood $\mathcal{U}$ of x in $\mathbb{R}^d$ and a bijection φ of $\mathcal{U}$ on some open ball $\mathcal{B} = \{y \in \mathbb{R}^d : \|y\| < r\}$ such that φ and φ^{-1} are C^m-mappings,

$$\varphi(\mathcal{U} \cap \Omega) = \left\{\left(y', y_d\right) \in \mathcal{B} : y_d > 0\right\}, \quad \text{and} \quad \varphi(\mathcal{U} \cap \partial\Omega) = \left\{\left(y', 0\right) \in \mathcal{B}\right\}.$$

Here $y \in \mathbb{R}^d$ was written as $y = (y', y_d)$, where $y' \in \mathbb{R}^{d-1}$ and $y_d \in \mathbb{R}$.



It should be noted that, according to this definition, $\Omega \cap \mathcal{U}$ lies "at one side of $\partial\Omega$." It may happen that $\partial\Omega$ is a C^m-manifold, but Ω is not of class C^m; for instance, $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 \neq 0\}$ is not of class C^1. But $\{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0\}$ is C^∞.

From now on we assume that Ω is a *bounded open set of class* C^1. We denote by $\nu(x) = (\nu_1, \dots, \nu_d)$ the *outward unit normal vector* at $x \in \partial\Omega$ and by $d\sigma$ the *surface measure* of $\partial\Omega$.

Theorem D.4 *The embedding map* $H^1(\Omega) \to L^2(\Omega)$ *is compact.*

Proof [At, A 6.4] or [Br, Theorem 9.19] or [LM, Theorem 16.1]. □

Theorem D.5 $C^\infty(\mathbb{R}^d)\lceil\Omega$ *is a dense subset of* $H^n(\Omega)$ *for* $n = 1, 2$.

Proof [At, A 6.7] or [EE, Chap. V, Theorem 4.7]. □

The following two results are called *trace theorems*. Theorem D.6 gives the possibility to assign "boundary values" $\gamma_0(f)$ along $\partial\Omega$ to any function $f \in H^1(\Omega)$.

Theorem D.6 *There exists a uniquely determined continuous linear mapping* $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega, d\sigma)$ *such that* $\gamma_0(f) = f\lceil\partial\Omega$ *for* $f \in C^\infty(\mathbb{R}^d)$. *Moreover,* $H_0^1(\Omega) = \{f \in H^1(\Omega) : \gamma_0(f) = 0\}$.

Proof [At, A 6.6] or [LM, Theorem 8.3]. □

Theorem D.7 *Let* Ω *be of class* C^2 *and bounded. There is a continuous linear map* $(\gamma_0, \gamma_1) : H^2(\Omega) \to L^2(\partial\Omega, d\sigma) \oplus L^2(\partial\Omega, d\sigma)$ *such that* $\gamma_0(g) = g\lceil\partial\Omega$ *and* $\gamma_1(g) = \frac{\partial g}{\partial \nu}\lceil\partial\Omega$ *for* $g \in C^\infty(\mathbb{R}^d)$. *The kernel of this map is* $H_0^2(\Omega)$.

Proof [LM, Theorem 8.3]. □

The maps γ_0 and (γ_0, γ_1) are called *trace maps*. Taking fractional Sobolev spaces on the boundary $\partial\Omega$ for granted, their ranges can be nicely described (see [LM, p. 39]) by

$$\gamma_0\big(H^1(\Omega)\big) = H^{1/2}(\partial\Omega), \; (\gamma_0, \gamma_1)\big(H^2(\Omega)\big) = H^{3/2}(\partial\Omega) \oplus H^{1/2}(\partial\Omega). \tag{D.2}$$

For $f \in H^1(\Omega)$ and $g \in H^2(\Omega)$, we consider $\gamma_0(f)$ and $\gamma_1(g)$ as *boundary values*, and we also write $f \lceil \partial\Omega$ for $\gamma_0(f)$ and $\frac{\partial g}{\partial \nu} \lceil \partial\Omega$ for $\gamma_1(g)$.

Theorem D.8 (*Gauss' formula*) *For $f, g \in H^1(\Omega)$ and $k = 1, \dots, d$, we have*

$$\int_\Omega \partial_k f \, dx = \int_{\partial\Omega} f \nu_k \, d\sigma, \tag{D.3}$$

$$\int_\Omega (\partial_k f \overline{g} + f \overline{\partial_k g}) \, dx = \int_{\partial\Omega} f \overline{g} \nu_k \, d\sigma. \tag{D.4}$$

Proof [At, A 6.8] or [Br, p. 316]. □

Theorem D.9 (*Green's formulas*) *Let Ω be of class C^2. For all $h \in H^1(\Omega)$ and $f, g \in H^2(\Omega)$, we have*

$$\int_\Omega (-\Delta f) \overline{h} \, dx = \sum_{k=1}^d \int_\Omega \partial_k f \overline{\partial_k h} \, dx - \int_{\partial\Omega} \frac{\partial f}{\partial \nu} \overline{h} \, d\sigma, \tag{D.5}$$

$$\int_\Omega (-\Delta f) \overline{g} \, dx - \int_\Omega f \overline{(-\Delta g)} \, dx = \int_{\partial\Omega} \left(f \frac{\overline{\partial g}}{\partial \nu} - \frac{\partial f}{\partial \nu} \overline{g} \right) d\sigma. \tag{D.6}$$

Proof [Br, p. 316]. □

Strictly speaking, the corresponding traces $\gamma_0(f)$, $\gamma_0(\overline{g})$, $\gamma_1(f)$, $\gamma_0(\overline{h})$, $\gamma_1(f)$, $\gamma_1(\overline{g})$ are meant in the integrals along the boundary $\partial\Omega$ in (D.3)–(D.6). Clearly, (D.3) implies (D.4), while (D.5) implies (D.6).

In the literature these formulas are often stated only for functions from $C^\infty(\overline{\Omega})$. The general case is easily derived by a limit procedure. Let us verify (D.5). By Theorem D.5, $h \in H^1(\Omega)$ and $f \in H^2(\Omega)$ are limits of sequences (h_n) and (f_n) from $C^\infty(\mathbb{R}^d) \lceil \Omega$ in $H^1(\Omega)$ resp. $H^2(\Omega)$. Then (D.5) holds for h_n and f_n. Passing to the limit by using the continuity of the trace maps, we obtain (D.5) for h and f.

The next theorem deals with the *regularity of weak solutions* of the Dirichlet and Neumann problems. To formulate these results, we use the following equation:

$$\int_\Omega \nabla f \cdot \nabla h \, dx + \int_\Omega f h \, dx = \int_\Omega g h \, dx. \tag{D.7}$$

Theorem D.10 *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set of class C^2, and $g \in L^2(\Omega)$.*

(i) (*Regularity for the Dirichlet problem*)
Let $f \in H_0^1(\Omega)$. If (D.7) holds for all $h \in H_0^1(\Omega)$, then $f \in H^2(\Omega)$.

(ii) (*Regularity for the Neumann problem*)
Let $f \in H^1(\Omega)$. If (D.7) holds for all $h \in H^1(\Omega)$, then $f \in H^2(\Omega)$.

Proof [Br, Theorems 9.5 and 9.6]. □

Appendix E
Absolutely Continuous Functions

Basics on absolutely continuous functions can be found, e.g., in [Ru2, Cn], or [HS].

Throughout this appendix we suppose that $a, b \in \mathbb{R}$, $a < b$.

Definition E.1 A function f on the interval $[a, b]$ is called *absolutely continuous* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{k=1}^{n} \bigl| f(b_k) - f(a_k) \bigr| < \varepsilon$$

for every finite family of pairwise disjoint subintervals (a_k, b_k) of $[a, b]$ satisfying

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

The set of all absolutely continuous functions on $[a, b]$ is denoted by $AC[a, b]$. Each $f \in AC[a, b]$ is continuous on $[a, b]$ and a function of bounded variation.

Theorem E.1 *A function f on $[a, b]$ is absolutely continuous if and only if there is a function $h \in L^1(a, b)$ such that*

$$f(x) - f(a) = \int_a^x h(t)\, dt \quad \text{for } x \in [a, b]. \tag{E.1}$$

In this case, f is differentiable a.e., and we have $f'(x) = h(x)$ a.e. on $[a, b]$. The function $h \in L^1(a, b)$ is uniquely determined by f.

Theorem E.2 *For $f, g \in AC[a, b]$, the integration-by-parts formula holds*:

$$\int_a^b f'(t) g(t)\, dt + \int_a^b f(t) g'(t)\, dt = f(b)g(b) - f(a)g(a). \tag{E.2}$$

Theorems E.1 and E.2 are proved, e.g., in [Cn, Corollaries 6.3.7 and 6.3.8].

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The Sobolev spaces $H^n(a,b)$ and $H^n_0(a,b)$, $n \in \mathbb{N}$, can be expressed in terms of absolutely continuous functions. That is,

$$\begin{aligned} H^1(a,b) &= \big\{ f \in AC[a,b] : f' \in L^2(a,b) \big\}, \\ H^n(a,b) &= \big\{ f \in C^{n-1}\big([a,b]\big) : f^{(n-1)} \in H^1(a,b) \big\}, \\ H^n_0(a,b) &= \big\{ f \in H^n(a,b) : f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0, \\ &\qquad f(b) = f'(b) = \cdots = f^{(n-1)}(b) = 0 \big\}. \end{aligned}$$

In particular, $C^n([a,b]) \subseteq H^n(a,b) \subseteq C^{n-1}([a,b])$ for $n \in \mathbb{N}$.

Now let $\mathcal{J}$ be an unbounded open interval of $\mathbb{R}$, and let $\overline{\mathcal{J}}$ denote the closure of $\mathcal{J}$ in $\mathbb{R}$, that is, $\overline{\mathcal{J}} = [a, +\infty)$ if $\mathcal{J} = (a, +\infty)$ and $\overline{\mathcal{J}} = \mathbb{R}$ if $\mathcal{J} = \mathbb{R}$. Then

$$\begin{aligned} H^1(\mathcal{J}) &= \big\{ f \in L^2(\mathcal{J}) : f \in AC[a,b] \text{ for } [a,b] \subseteq \overline{\mathcal{J}} \text{ and } f' \in L^2(\mathcal{J}) \big\}, \\ H^n(\mathcal{J}) &= \big\{ f \in L^2(\mathcal{J}) : f \in C^{n-1}(\overline{\mathcal{J}}) \text{ and } f^{(n-1)} \in H^1(\mathcal{J}) \big\}, \\ H^n_0(a,\infty) &= \big\{ f \in H^n(a,\infty) : f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0 \big\}, \\ H^n_0(\mathbb{R}) &= H^n(\mathbb{R}). \end{aligned}$$

For $f \in H^n(\mathcal{J})$, we have $f \in L^2(\mathcal{J})$ and $f^{(n)} \in L^2(\mathcal{J})$ by the preceding formula. It can be shown that $f \in H^n(\mathcal{J})$ implies that $f^{(j)} \in L^2(\mathcal{J})$ for *all* $j = 1, \dots, n$.

The preceding formulas are proved in [Br, Chap. 8] and [Gr, Sect. 4.3], where Sobolev spaces on intervals are treated in detail.

Appendix F
Nevanlinna Functions and Stieltjes Transforms

In this appendix we collect and discuss basic results on Nevanlinna functions and on Stieltjes transforms and their boundary values. For some results, it is difficult to localize easily accessible proofs in the literature, and we have included complete proofs of those results.

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the upper half-plane.

Definition F.1 A holomorphic complex function f on $\mathbb{C}_+$ is called a *Nevanlinna function* (or likewise, a *Pick function* or a *Herglotz function*) if

$$\operatorname{Im} f(z) \geq 0 \quad \text{for all } z \in \mathbb{C}_+.$$

We denote the set of all Nevanlinna functions by $\mathfrak{N}$.

If $f \in \mathfrak{N}$ and $f(z_0)$ is real for some point $z_0 \in \mathbb{C}_+$, then f is not open and hence a constant. That is, all nonconstant Nevanlinna functions map $\mathbb{C}_+$ into $\mathbb{C}_+$.

Each Nevanlinna function f can be extended to a holomorphic function on $\mathbb{C}\backslash\mathbb{R}$ by setting $f(\overline{z}) := \overline{f(z)}$ for $z \in \mathbb{C}_+$.

Theorem F.1 (*Canonical integral representation of Nevanlinna functions*) *For each Nevanlinna function f, there exist numbers $a, b \in \mathbb{R}$, $b \geq 0$, and a finite positive regular Borel measure μ on the real line such that*

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1+zt}{t-z}\, d\mu(t), \quad z \in \mathbb{C}/\mathbb{R}, \tag{F.1}$$

where the numbers a, b and the measure μ are uniquely determined by f.

Conversely, any function f of this form is a Nevanlinna function.

Proof [AG, Nr. 69, Theorem 2] or [Dn1, p. 20, Theorem 1]. □

Often the canonical representation (F.1) is written in the form

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\nu(t), \quad z \in \mathbb{C}/\mathbb{R}, \tag{F.2}$$

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where ν is a regular positive Borel measure on $\mathbb{R}$ satisfying $\int (1+t^2)^{-1}\,d\nu(t) < \infty$. The two forms (F.1) and (F.2) are related by the formula $d\mu(t) = (1+t^2)^{-1}\,d\nu(t)$.

Let now μ be a *regular complex* Borel measure on $\mathbb{R}$.

Definition F.2 The *Stieltjes transform* of μ is defined by

$$I_\mu(z) = \int_{\mathbb{R}} \frac{1}{t-z}\,d\mu(t), \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{F.3}$$

Stieltjes transforms are also called *Cauchy transforms* or *Borel transforms*.

Theorem F.2 (*Stieltjes–Perron inversion formula*) *The measure μ is uniquely determined by the values of its Stieltjes transform on $\mathbb{C}\backslash\mathbb{R}$. In fact, for $a, b \in \mathbb{R}$, $a < b$, we have*

$$\mu\big((a,b)\big) + \frac{1}{2}\mu\big(\{a\}\big) + \frac{1}{2}\mu\big(\{b\}\big) = \lim_{\varepsilon\to+0} \frac{1}{2\pi\mathrm{i}} \int_a^b \big[I_\mu(t+\mathrm{i}\varepsilon) - I_\mu(t-\mathrm{i}\varepsilon)\big]\,dt, \tag{F.4}$$

$$\mu\big(\{a\}\big) = \lim_{\varepsilon\to+0} -\mathrm{i}\varepsilon I_\mu(a+\mathrm{i}\varepsilon). \tag{F.5}$$

Proof Since μ is a linear combination of finite positive Borel measures, one can assume without loss of generality that μ is positive. In this case proofs are given in [AG, Nr. 69] and [We, Appendix B].

(F.4) can be proved by similar arguments as used in the proof of Proposition 5.14. Since $\frac{-\mathrm{i}\varepsilon}{t-(a+\mathrm{i}\varepsilon)} \to \chi_{\{a\}}(t)$ as $\varepsilon \to +0$, (F.5) follows from (F.3) by interchanging limit and integration using the dominated convergence Theorem B.1. □

From formula (F.4) it follows at once that $I_\mu \equiv 0$ on $\mathbb{C}\backslash\mathbb{R}$ implies that $\mu = 0$. However, there are complex measures $\mu \neq 0$ for which $I_\mu = 0$ on $\mathbb{C}_+$.

The following fundamental theorem, due to I.I. Privalov, is about *boundary values* of Stieltjes transforms. Again, we assume that μ is a regular complex Borel measure on $\mathbb{R}$. Recall that the function $\mu \equiv f_\mu$ defined by (B.10) is of bounded variation and differentiable a.e. on $\mathbb{R}$ by Theorem B.10.

Theorem F.3 (*Sokhotski–Plemelj formula*) *The limits* $I_\mu(t \pm \mathrm{i}0) := \lim_{\varepsilon\to\pm 0} I_\mu(t+\mathrm{i}\varepsilon)$ *exist, are finite, and*

$$I_\mu(t \pm \mathrm{i}0) = \pm\mathrm{i}\pi \frac{d\mu}{dt}(t) + (\mathrm{PV})\int_{\mathbb{R}} \frac{1}{s-t}\,d\mu(s) \quad \textit{a.e. on } \mathbb{R}, \tag{F.6}$$

where (PV) $\int$ *denotes the principal value of the integral.*

Proof [Pv] or [Mi]. □

From now on we assume that μ is a *finite positive* regular Borel measure on $\mathbb{R}$. Then $\overline{I_\mu(z)} = I_\mu(\overline{z})$ on $\mathbb{C}\setminus\mathbb{R}$. Hence, formulas (F.4) and (F.5) can be written as

$$\mu\big((a,b)\big)+\frac{1}{2}\mu\big(\{a\}\big)+\frac{1}{2}\mu\big(\{b\}\big)=\lim_{\varepsilon\to+0}\pi^{-1}\int_a^b \operatorname{Im} I_\mu(t+\mathrm{i}\varepsilon)\,dt, \qquad (F.7)$$

$$\mu\big(\{a\}\big)=\lim_{\varepsilon\to+0}\varepsilon \operatorname{Im} I_\mu(a+\mathrm{i}\varepsilon). \qquad (F.8)$$

Thus, the *positive* measure μ is uniquely determined by the values of I_μ on $\mathbb{C}_+$.

The Stieltjes transform I_μ of the finite *positive* (!) Borel measure μ is obviously a Nevanlinna function, since

$$\operatorname{Im} I_\mu(z)=\operatorname{Im} z\int_{\mathbb{R}}\frac{1}{|t-z|^2}\,d\mu(t), \quad z\in\mathbb{C}\backslash\mathbb{R}.$$

The next result characterizes these Stieltjes transforms among Nevanlinna functions.

Theorem F.4 *A Nevanlinna function f is the Stieltjes transform I_μ of a finite positive Borel measure μ on $\mathbb{R}$ if and only if*

$$\sup\big\{\big|yf(\mathrm{i}y)\big| : y\in\mathbb{R},\ y\geq 1\big\}<\infty. \qquad (F.9)$$

Proof [AG, Nr. 69, Theorem 3]. □

The necessity of condition (F.9) is easily seen:

Since $\mu(\mathbb{R})<\infty$, Lebesgue's convergence Theorem B.1 applies and implies that $\mu(\mathbb{R})=\lim_{y\in\mathbb{R},y\to\infty}-\mathrm{i}yI_\mu(\mathrm{i}y)$, which yields (F.9). □

Theorem F.5 *Let K be a closed subset of $\mathbb{R}$. The Stieltjes transform $I_\mu(z)$ has a holomorphic extension to $\mathbb{C}\backslash K$ if and only if* $\operatorname{supp}\mu\subseteq K$.

Proof [Dn1, Lemma 2, p. 26]. □

Let us sketch the proof of Theorem F.5:

If $\operatorname{supp}\mu\subseteq K$, then (F.3) with $z\in\mathbb{C}\backslash K$ defines a holomorphic extension of I_μ to $\mathbb{C}\backslash K$. Conversely, suppose that I_μ has a holomorphic extension, say f, to $\mathbb{C}\backslash K$. Then $\lim_{\varepsilon\to+0} I_\mu(t\pm\mathrm{i}\varepsilon)=f(t)$, and hence $\lim_{\varepsilon\to+0}\operatorname{Im} I_\mu(t+\mathrm{i}\varepsilon)=0$ for $t\in\mathbb{R}\backslash K$. Therefore, by (F.7) and (F.8), μ has no mass in $\mathbb{R}\backslash K$, that is, $\mu(\mathbb{R}\backslash K)=0$. □

Since μ is a positive measure, the integral $(PV)\int$ in (F.6) is real, and we obtain

$$\frac{d\mu}{dt}(t)=\pi^{-1}\operatorname{Im} I_\mu(t+\mathrm{i}0) \quad \text{a.e. on } \mathbb{R}. \qquad (F.10)$$

Obviously, if for some $t\in\mathbb{R}$, the boundary value $I_\mu(t+\mathrm{i}0)$ of I_μ exists, so does the boundary value $(\operatorname{Im} I_\mu)(t+\mathrm{i}0)$ of $\operatorname{Im} I_\mu$ and $\operatorname{Im}(I_\mu(t+\mathrm{i}0))=(\operatorname{Im} I_\mu)(t+\mathrm{i}0)$. However, $(\operatorname{Im} I_\mu)(t+\mathrm{i}0)$ may exist, but $I_\mu(t+\mathrm{i}0)$ does not. But we always have $\operatorname{Im}(I_\mu(t+\mathrm{i}0))=(\operatorname{Im} I_\mu)(t+\mathrm{i}0)$ a.e. on $\mathbb{R}$.

The next result describes the parts of the Lebesgue decomposition (B.8) of the measure μ with respect to the Lebesgue measure $\nu=m$ in terms of boundary values of the imaginary part of its Stieltjes transform I_μ.

Theorem F.6 (i) *The singular part* μ_{sign} *of* μ *is supported by the set*

$$S_\mu := \{t \in \mathbb{R} : (\operatorname{Im} I_\mu)(t + \mathrm{i}0) = +\infty\}.$$

(ii) *The absolutely continuous part* μ_{ac} *is given by* $d\mu_{\mathrm{ac}}(t) = \pi^{-1}(\operatorname{Im} I_\mu)(t + \mathrm{i}0)\, dt$ *and supported by the set*

$$L_\mu := \{t \in \mathbb{R} : 0 < (\operatorname{Im} I_\mu)(t + \mathrm{i}0) < +\infty\}.$$

Proof (i): The de la Vallée Poussin theorem (Theorem B.9(ii)) states that μ_{sign} is supported by the set $S(\mu) = \{t \in \mathbb{R} : (D\mu)(t) = +\infty\}$. From the inequalities

$$\begin{aligned}(\operatorname{Im} I_\mu)(t + \mathrm{i}\varepsilon) &= \int_{\mathbb{R}} \frac{\varepsilon}{(s-t)^2 + \varepsilon^2}\, d\mu(s) \geq \varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \frac{\varepsilon^2}{(s-t)^2 + \varepsilon^2}\, d\mu(s) \\ &\geq \varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \frac{\varepsilon^2}{\varepsilon^2 + \varepsilon^2}\, d\mu(s) = (2\varepsilon)^{-1} \mu\big([t+\varepsilon, t-\varepsilon]\big)\end{aligned}$$

we conclude that $(D\mu)(t) = +\infty$ implies $(\operatorname{Im} I_\mu)(t + \mathrm{i}0) = +\infty$, so $S(\mu) \subseteq S_\mu$. This proves (i).

(ii): From formula (F.10) it follows that

$$(D\mu)(t) = \frac{d\mu}{dt}(t) = \pi^{-1} \operatorname{Im} I_\mu(t + \mathrm{i}0) = \pi^{-1}(\operatorname{Im} I_\mu)(t + \mathrm{i}0) \quad \text{a.e. on } \mathbb{R}, \tag{F.11}$$

so that $d\mu_{\mathrm{ac}}(t) = \pi^{-1}(\operatorname{Im} I_\mu)(t + \mathrm{i}0)\, dt$ by Theorem B.9(i). Hence, μ_{ac} is supported by the set $\tilde{L}_\mu := \{t : 0 < (\operatorname{Im} I_\mu)(t + \mathrm{i}0)\}$. By (F.11) and Theorem B.9(i) we have $\pi^{-1} \operatorname{Im} I_\mu(\cdot + \mathrm{i}0) = D\mu = f \in L^1(\mathbb{R})$. Therefore, $\mu_{\mathrm{ac}}(S_\mu) = 0$. (This follows also from (i).) Hence, μ_{ac} is also supported by the set $L_\mu = \tilde{L}_\mu \backslash S_\mu$. □

Theorem F.7 *Suppose that* f *is a Nevanlinna function such that* (F.9) *holds and*

$$\sup\{|\operatorname{Im} f(z)| : z \in \mathbb{C}_+\} < \infty. \tag{F.12}$$

Then the function $\operatorname{Im} f(t + \mathrm{i}0)$ *is in* $L^1(\mathbb{R})$, *and we have*

$$f(z) = \pi^{-1} \int_{\mathbb{R}} \frac{\operatorname{Im} f(t + \mathrm{i}0)}{t - z}\, dt, \quad z \in \mathbb{C}_+. \tag{F.13}$$

Proof Since we assumed that condition (F.9) holds, Theorem F.4 applies, and hence $f = I_\mu$ for some finite positive Borel measure μ.

We prove that (F.12) implies that μ is absolutely continuous and (F.13) holds. From (F.8) and (F.12) it follows that $\mu(\{c\}) = 0$ for all $c \in \mathbb{R}$. Therefore, by (F.7),

$$\mu\big((a,b)\big) = \lim_{\varepsilon \to +0} \pi^{-1} \int_a^b \operatorname{Im} I_\mu(t + \mathrm{i}\varepsilon)\, dt = \pi^{-1} \int_a^b \operatorname{Im} I_\mu(t + \mathrm{i}0)\, dt, \tag{F.14}$$

where assumption (F.12) ensures that limit and integration can be interchanged by Lebesgue's convergence Theorem B.1. From (F.14) we easily conclude that $d\mu(t) = \pi^{-1} \operatorname{Im} f(t + \mathrm{i}0)\, dt$. Inserting this into (F.3), we obtain (F.13). □

We close this appendix by stating another related but finer result. It was not used in this text. Let ν be a positive regular Borel measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\nu(t) < \infty.$$

Then the *Poisson–Stieltjes transform* P_ν is defined by

$$P_\nu(x,y) = \int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}\, d\nu(t), \quad x \in \mathbb{R},\ y > 0.$$

Note that $P_\nu(x,y)$ is a nonnegative harmonic function on $\mathbb{C}_+$. If the measure ν is finite, the Stieltjes transform I_ν is defined, and obviously $P_\nu(x,y) = \operatorname{Im} I_\nu(x+iy)$. If ν is the Lebesgue measure, one computes that $P_\nu(x,y) = \pi$ for $x + \mathrm{i}y \in \mathbb{C}_+$.

Then there is the following *Fatou-type theorem*; a proof is given in [Dn3].

Theorem F.8 *Let $t \in \mathbb{R}$.*

(i) *The limit $P_\nu(t,+0) := \lim_{\varepsilon\to+0} P_\nu(t,\varepsilon)$ exists and is finite if and only if the symmetric derivative $(D\nu)(t)$ (defined by (B.9)) does. In this case we have $P_\nu(t,+0) = \pi(D\nu)(t)$.*

(ii) *If $(D\nu)(t) = +\infty$, then $P_\nu(t,+0) \equiv \lim_{\varepsilon\to+0} P_\nu(t,\varepsilon) = +\infty$; the converse is not true in general.*

Suppose that the positive measure ν is finite. Since $P_\nu(x,y) = \operatorname{Im} I_\nu(x+\mathrm{i}y)$, it is obvious that $P_\nu(t,+0) = (\operatorname{Im} I_\nu)(t+\mathrm{i}0)$. Hence, the assertion of Theorem F.6(ii) can be derived from Theorem F.8(i), thereby avoiding the use of the Sokhotski–Plemelj formula (F.10). Indeed, since $\pi(D\nu)(t) = (\operatorname{Im} I_\nu)(t+\mathrm{i}0)$ by Theorem F.8(i), we have $d\nu_{ac}(t) = \pi^{-1}(\operatorname{Im} I_\nu)(t+\mathrm{i}0)\,dt$ by Theorem B.9(i). This implies the assertion of Theorem F.6(ii).

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