Appendix A
Essential Elements of Probability Theory and Random Processes

Probability theory is a mathematical construction to describe phenomena whose outcomes occur non-deterministically. Depending on the context, the probability of the outcome in a random phenomenon can be interpreted either as its long-run frequency of occurrence or as a measure of its subjective uncertainty. These two points of view correspond to the frequentist and Bayesian interpretations of probabilities. This distinction is not necessary here, and we restrict the presentation to materials used in this monograph. In particular, we only introduce notions of real-valued random variables. Further elements of the probability theory can be found in standard references; see for instance [78, 104, 137, 181].

This appendix follows the exposition in Chaps. 1 and 2 of the book *Stochastic calculus: applications in science and engineering* by M. Grigoriu [94]. The discussion below distills essential elements from this reference.

A.1 Probability Theory

A.1.1 Measurable Space

We consider a random phenomenon or experiment. We denote Θ the set of all possible outcomes of the experiment. The set Θ is called the **sample space**, while θ denotes an element of Θ. The sample space may have a finite or infinite (countable or uncountable) number of elements.

Let Σ be an non-empty collection of subsets of Θ. The collection Σ is called a **σ-field on** Θ if

- $\emptyset \in \Sigma$
- $A \in \Sigma \Rightarrow \overline{A} \in \Sigma$
- $A_{i \in I} \in \Sigma \Rightarrow \cup_{i \in I} A_i \in \Sigma$

where $\overline{A}$ is the complement of A in Θ and I is a countable set. An element $A \in \Sigma$ is called an event or Σ-measurable subset of Θ; it is a collection of outcomes. The pair

O.P. Le Maître, O.M. Knio, *Spectral Methods for Uncertainty Quantification,* Scientific Computation,
DOI 10.1007/978-90-481-3520-2,

(Θ, Σ) is called a **measurable space**. A **Borel** σ-field is generated by the collection of all open sets of a topological set. The members of a Borel σ-field are called Borel sets. An important example of Borel σ-field is $\mathcal{B}(\mathbb{R})$, the collection of all open intervals of $\mathbb{R}$. The definition of $\mathcal{B}(\mathbb{R})$ extends to intervals; for instance $\mathcal{B}([a, b])$ denotes the Borel σ-field on the interval $[a, b]$. Denoting $\sigma(\mathcal{A})$ the smaller σ-field generated by a collection $\mathcal{A}$ of subsets of Θ, we have

$$\mathcal{B}(\mathbb{R}) = \sigma\left((a, b), -\infty \leq a \leq b \leq +\infty\right).$$

A.1.2 Probability Measure

A set function $\mu : \Sigma \mapsto [0, \infty]$ is a **measure** on Σ if it is countably additive:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i), \quad A_i \in \Sigma, A_i \cap A_{i \neq j} = \emptyset.$$

The triple (Θ, Σ, μ) is called a **measure space**. A **finite measure** is a measure such that $\mu(\Theta) < \infty$. A finite measure P such that

- $P : \Sigma \mapsto [0, 1]$
- $P(\Theta) = 1$

is called a **probability measure** or simply a probability.

A.1.3 Probability Space

The triple (Θ, Σ, P) is called a **probability space**. A probability space (Θ, Σ, P) such that for $A, B \in \Sigma$ with $A \subset B$ and $P(B) = 0 \Rightarrow P(A) = 0$ is said complete. A probability space can always be completed so we shall implicitly assume complete probability spaces. We then have the following properties of a probability space (Θ, Σ, P) for $A, B \in \Sigma$:

- $P(A) \leq P(B)$ for $A \subseteq B$,
- $P(\overline{A}) = 1 - P(A)$,
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Consider the probability spaces $(\Theta_k, \Sigma_k, P_k)$ for $k = 1, \ldots, n$. From this collection, we define the **product probability space** (Θ, Σ, P) as

$$\Theta = \Theta_1 \times \cdots \times \Theta_n,$$
$$\Sigma = \Sigma_1 \times \cdots \times \Sigma_n,$$
$$P = P_1 \times \cdots \times P_n.$$

This definition also holds for $n = \infty$. Whenever, the spaces $(\Theta_k, \Sigma_k, P_k)$ are all identical, the product sample space, σ-field and probability measure will be denoted Θ^n, Σ^n and P^n respectively.

Let (Θ, Σ, P) be a probability space and let $B \in \Sigma$ be a given event such that $P(B) > 0$. We define a new probability measure, called the **probability conditional on** B as

$$P(A \in \Sigma | B) \equiv \frac{P(A \cap B)}{P(B)}.$$

From this definition, if the set of events $A_i \in \Sigma$ is a partition of the sample space (*i.e.* if $A_i \cap A_{j \neq i} = \emptyset$ and $\bigcup_i A_i = \Theta$), then

$$P(B) = \sum_i P(B \cap A_i) = \sum_i P(B|A_i)P(A_i),$$

(**law of total probability**) and

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{P(B)} = \frac{P(A_j)P(B|A_j)}{\sum_i P(A_i)P(B|A_i)}, \quad P(B) > 0$$

(**Bayes formula**).

A.2 Measurable Functions

A.2.1 Induced Probability

Consider two measurable spaces (Θ, Σ_Θ) and $(\Theta', \Sigma_{\Theta'})$ and a function $h : \Theta \mapsto \Theta'$ with domain Θ and range Θ'. The function h is said **measurable from** (Θ, Σ_Θ) **to** $(\Theta', \Sigma_{\Theta'})$ if for any event B in $\Sigma_{\Theta'}$

$$h^{-1}(B) \equiv \{\theta : h(\theta) \in B\} \in \Sigma_\Theta.$$

If h is measurable from (Θ, Σ_Θ) to $(\Theta', \Sigma_{\Theta'})$ and $(\Theta, \Sigma_\Theta, P)$ is a probability then $Q : \Sigma_{\Theta'} \mapsto [0, 1]$ defined by

$$Q(B \in \Sigma_{\Theta'}) \equiv P(h^{-1}(B))$$

is a probability measure on $(\Theta', \Sigma_{\Theta'})$ called the **probability induced** by h, or simply the distribution of h.

A.2.2 Random Variables

Consider a probability space (Θ, Σ, P) and a function $X : \Theta \mapsto \mathbb{R}$ measurable from (Θ, Σ) to $(\mathbb{R}, \mathcal{B})$. Then, X is a **$\mathbb{R}$-valued random variable**, sometimes denoted

$X(\theta)$. The **distribution** of X is the probability measure induced by the mapping $X : \Theta \mapsto \mathbb{R}$ on the measurable space $(\mathbb{R}, \mathcal{B})$, defined by

$$Q(B) = P(X^{-1}(B)), \quad B \in \mathcal{B}.$$

This definition implies that X is a random variable if and only if $X^{-1}((-\infty, x]) \in \Sigma$, $\forall x \in \mathbb{R}$. In other words, a $\mathbb{R}$-valued random variable is a function which maps the sample space Θ to $\mathbb{R}$.

The previous definition of a random variable can be extended to $\mathbb{R}^d$-valued functions $\mathbf{X}$ measurable from (Θ, Σ) to $(\mathbb{R}^d, \mathcal{B}^d)$. If all the coordinates of $\mathbf{X}$ are random variables, then $\mathbf{X}$ is called a **random vector**.

A.2.3 Measurable Transformations

This monograph is concerned with numerical techniques for the characterization of the output of physical models involving random input. We are then dealing with transformations of random variables. Denoting $\mathbf{X}$ the $\mathbb{R}^d$-valued random vector representing the model input, we are for instance interested in a model output $Y = g(\mathbf{X})$ where $g : \mathbb{R}^d \mapsto \mathbb{R}$. The model output in then defined by the mapping

$$(\Theta, \Sigma) \overset{\mathbf{X}}{\mapsto} (\mathbb{R}^d, \mathcal{B}^d) \overset{g}{\mapsto} (\mathbb{R}, \mathcal{B}).$$

It can be shown that if $\mathbf{X}$ and g are measurable functions from (Θ, Σ) to $(\mathbb{R}^d, \mathcal{B}^d)$ and from $(\mathbb{R}^d, \mathcal{B}^d)$ to $(\mathbb{R}, \mathcal{B})$, respectively, then the composed mapping $g \circ \mathbf{X} : (\Theta, \Sigma) \mapsto (\mathbb{R}, \mathcal{B})$ is measurable. As a result, the model output is a $\mathbb{R}$-valued random variable.

A.3 Integration and Expectation Operators

A.3.1 Integrability

For $X : (\Theta, \Sigma) \mapsto (\mathbb{R}, \mathcal{B})$, the integral of X with respect to P over the event $A \in \Sigma$ is

$$\int_\Theta I_A(\theta) X(\theta) \mathrm{d}P(\theta) = \int_A X(\theta) \mathrm{d}P(\theta),$$

where I_A is the indicator function of A. If this integral exists and is finite, X is said to be **P-integrable over $\mathbf{A}$**. A random vector $\mathbf{X}$ is P-integrable over A, if all of its components are individually P-integrable over A.

Let X and Y be two random variables defined on a probability space (Θ, Σ, P). We have the following properties:

- For $A \in \Sigma$,

$$\int_A |X| \, \mathrm{d}P < \infty \Rightarrow \int_A X \, \mathrm{d}P \quad \text{is finite.}$$

- If X, Y are P-integrable over $A \in \Sigma$, the random variable $aX + bY$ is P-integrable over A and (linearity):

$$\int_A (aX + bY) \, \mathrm{d}P = a \int_A X \, \mathrm{d}P + b \int_A Y \, \mathrm{d}P.$$

- If X is P-integrable over Θ and $\{A_i \in \Sigma\}$ is a partition of Θ, then

$$\int_{\cup_i A_i} X \, \mathrm{d}P = \sum_i \int_{A_i} X \, \mathrm{d}P.$$

- If X is P-integrable on Θ, then X is finite a.s. (almost surely), *i.e.* $B = \{\theta : X(\theta) = \pm\infty\} \in \Sigma$ is such that $P(B) = 0$.
- If $X \geq 0$ a.s., then

$$\int_A X \, \mathrm{d}P \geq 0.$$

- If $Y \leq X$ a.s. then

$$\int_A Y \, \mathrm{d}P \leq \int_A X \, \mathrm{d}P.$$

A.3.2 Expectation

The particular case of $A = \Theta$ for the integration domain corresponds to the expectation operator which will be denoted $\mathrm{E}[\cdot]$:

$$\mathrm{E}[X] \equiv \int_\Theta X(\theta) \, \mathrm{d}P(\theta).$$

Provided it exists and is finite, the expectation has the following properties. Let X and Y be two $\mathbb{R}$-valued random variables defined on a probability space (Θ, Σ, P) and P-integrable over Θ. Then

- Linearity of the expectation:

$$\mathrm{E}[aX + bY] = a\mathrm{E}[X] + b\mathrm{E}[Y].$$

- $X \geq 0$ a.s. $\Rightarrow \mathrm{E}[X] \geq 0$.
- $Y \leq X$ a.s. $\Rightarrow \mathrm{E}[Y] \leq \mathrm{E}[X]$.
- $|\mathrm{E}[X]| \leq \mathrm{E}[|X|]$.

A.3.3 L_2 Space

Consider a probability space (Θ, Σ, P). For $q \geq 1$, we denote $L_q(\Theta, \Sigma, P)$ the collection of $\mathbb{R}$-valued random variables X defined on (Θ, Σ, P) such that

$$\mathrm{E}\left[|X|^q\right] < \infty.$$

The case $q = 2$, corresponding to the L_2-space is of particular importance in this book as it possesses some essential properties.

First, L_2 **is a vector space**. Indeed, for $X \in L_2, \lambda \in \mathbb{R} \Rightarrow \lambda X \in L_2$, since $\mathrm{E}[(\lambda X)^2] = \lambda^2 \mathrm{E}[X^2] < \infty$. It remains to show that for $X, Y \in L_2$, $X + Y \in L_2$, so $\mathrm{E}[(X+Y)^2] = \mathrm{E}[X^2] + \mathrm{E}[Y^2] + 2\mathrm{E}[XY] < \infty$. To prove the last inequality, we have to show that $\mathrm{E}[XY]$ is finite; to this end consider $X \neq Y$ such that

$$p(\lambda) = \mathrm{E}\left[(X + \lambda Y)^2\right] = \mathrm{E}\left[X^2\right] + \lambda^2 \mathrm{E}\left[Y^2\right] + \lambda 2\mathrm{E}[XY],$$

has no real root; this leads to $\mathrm{E}[XY]^2 - \mathrm{E}[X^2]\mathrm{E}[Y^2] \leq 0$ or equivalently

$$|\mathrm{E}[XY]| \leq \mathrm{E}\left[X^2\right]^{1/2} \mathrm{E}\left[Y^2\right]^{1/2} \quad \text{for } X, Y \in L_2,$$

which is known as the **Cauchy-Schwarz** inequality.

Second, the expectation $\mathrm{E}[XY]$ defines an **inner product** on L_2, denoted $\langle X, Y \rangle$ with the associated L_2 **norm** $\|X\|_{L_2} = \mathrm{E}[X^2]^{1/2}$. Indeed, $\langle \cdot, \cdot \rangle$ is an inner product since

$$\begin{aligned}
&\langle 0, X \rangle = 0, \quad X \in L_2, \\
&\langle X, X \rangle > 0, \quad X \in L_2, X \neq 0, \\
&\langle X, Y \rangle = \langle Y, X \rangle, \quad X, Y \in L_2, \\
&\langle X + Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle, \quad X, Y, Z \in L_2, \\
&\langle \lambda X, Y \rangle = \lambda \langle X, Y \rangle, \quad X, Y \in L_2, \lambda \in \mathbb{R},
\end{aligned}$$

and $d : L_2 \times L_2 \mapsto [0, +\infty)$ defined by $d(X, Y) = \|X - Y\|_{L_2}$ is a metric on L_2 since

$$\begin{aligned}
&d(X, Y) = 0 \quad \text{iif } X = Y \text{ a.s.}, \\
&d(X, Y) = d(Y, X), \quad X, Y \in L_2, \\
&d(X, Y) \leq d(X, Z) + d(Z, Y), \quad \forall X, Y, Z \in L_2.
\end{aligned}$$

Finally, L_2 equipped with the inner product $\langle \cdot, \cdot \rangle$ and the L_2-norm is a **Hilbert space**. As a result, if $X \in L_2(\Theta, \Sigma, P)$ and Σ' is a sub-σ-field of Σ, then there is a unique random variable $X' \in L_2(\Theta, \Sigma', P)$ such that

$$\|X - X'\|_{L_2} = \min\left\{\|X - Z\|_{L_2} : Z \in L_2(\Theta, \Sigma', P)\right\},$$

and

$$\langle X - X', Z\rangle = 0, \quad \forall Z \in L_2(\Theta, \Sigma', P).$$

The random variable X' is the **orthogonal projection** of X on $L_2(\Theta, \Sigma', P)$, or best mean square estimator of X, and has the smallest mean square error of all members of $L_2(\Theta, \Sigma', P)$.

A.4 Random Variables

We recall (see A.2.2) that a $\mathbb{R}$-valued random variable defined on a probability space (Θ, Σ, P) is a measurable function from (Θ, Σ) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

A.4.1 Distribution Function of a Random Variable

The **cumulative distribution function**, or simply distribution function, of a random variable X defined on a probability space (Θ, Σ, P) is defined by

$$F_X(x) = P\left(X^{-1}((-\infty, x])\right) = P\left(\{\theta : X(\theta) \leq x\}\right) = P(X \leq x).$$

The distribution function is right-continuous, increasing, with range [0, 1]. In addition,

$$\begin{aligned}
&\lim_{x\to+\infty} F_X(x) = 1, \qquad \lim_{x\to-\infty} F_X(x) = 0,\\
&P(a < x \leq b) = F_X(b) - F_X(a) \geq 0,\\
&P(a \leq x < b) = F_X(b) - F_X(a) + P(x = a) - P(x = b).
\end{aligned}$$

The distribution F_X of a random variable X has only a countable number of jump discontinuities and is continuous at $x \in \mathbb{R}$ if and only if $P(X = x) = 0$.

A.4.2 Density Function of a Random Variable

If F_X is absolutely continuous in $\mathbb{R}$, there is an integrable function f called the probability density function, or density function, such that

$$F_X(b) - F_X(a) = \int_a^b f_X(x)\,\mathrm{d}x, \quad a \leq b.$$

A density function has the essential properties:

$$f_X(x) = F'_X(x) \quad \text{so} \quad \int_{-\infty}^{x} f_X(y)\,\mathrm{d}y = F_X(x),$$

$$f_X(x) \geq 0,$$

$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1.$$

Let X be a random variable defined on a probability space (Θ, Σ, P), and consider a measurable function $g : (\mathbb{R}, \mathcal{B}) \mapsto (\mathbb{R}, \mathcal{B})$. Then, $Y = g \circ X$ is a random variable defined on (Θ, Σ, P) and

$$\begin{aligned} \mathrm{E}[Y] &= \int_{\Theta} Y(\theta)\,\mathrm{d}P(\theta) = \int_{\Theta} g\,(X(\theta))\,\mathrm{d}P(\theta) \\ &= \int_{\mathbb{R}} g(x)\,\mathrm{d}Q(x) = \int_{\mathbb{R}} g(x)\,\mathrm{d}F_X(x) = \int_{\mathbb{R}} g(x) f_X(x)\,\mathrm{d}x. \end{aligned}$$

This relation provides an expression of the expectation of Y involving the probability density function of X. In this book, we heavily rely on such expression of expectation operator in terms of density functions.

A.4.3 Moments of a Random Variable

Let X be a $\mathbb{R}$-valued random variable defined on a probability space (Θ, Σ, P), and $Y = g(X) = X^r$ for $r \geq 1$. Since $g(x)$ is continuous, it is Borel-measurable and therefore $Y = X^r$ is a random variable. The expectation of Y is called the **moment of order** r of X, and is denoted $m_r(X)$:

$$m_r(X) = \mathrm{E}\left[X^r\right] = \int_{-\infty}^{+\infty} x^r\,\mathrm{d}F_X(x) = \int_{-\infty}^{+\infty} x^r f_X(x)\,\mathrm{d}x.$$

If $X \in L_r(\Theta, \Sigma, P)$, then $m_r(X)$ exists and is finite. If instead we consider $Y = g(X) = (X - m_1(X))^r$, then the expectation of Y is the **central moment of order** r.

A.4.4 Convergence of Random Variables

Let X be a $\mathbb{R}$-valued random variable and $X_{n\geq 1}$ be a sequence of random variables defined on a probability space (Θ, Σ, P). The convergence of the sequence X_n to X depends on the way we measure $X - X_n$. Alternatives are:

- **Almost sure** convergence,

$$X_n \xrightarrow{\text{a.s.}} X \quad \text{if } \lim_{n\to\infty} X_n(\theta) = X(\theta),\ \forall\theta \in \Theta \setminus N,\ P(N) = 0.$$

- Convergence in **probability**,

$$X_n \xrightarrow{\text{pr}} X \quad \text{if } \lim_{n\to\infty} P(|X_n(\theta) - X(\theta)|) > \epsilon) = 0, \forall\epsilon > 0.$$

- Convergence in **distribution**,

$$X_n \xrightarrow{\text{d}} X \quad \text{if } \lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x \in \mathbb{R}.$$

- Convergence in L_p,

$$X_n \xrightarrow{\text{m.p.}} X \quad \text{if } \lim_{n\to\infty} \mathrm{E}\left[|X_n - X|^p\right] = 0.$$

We will mostly use the mean square convergence, *i.e.* convergence in L_2, and denote it as $X_n \xrightarrow{\text{m.s.}} X$.

A.5 Random Vectors

Consider a probability space (Θ, Σ, P); the measurable function

$$\mathbf{X} : (\Theta, \Sigma) \mapsto (\mathbb{R}^d, \mathcal{B}^d), \quad d > 1,$$

defines a random vector in $\mathbb{R}^d$ or $\mathbb{R}^d$-valued random variable, with the induced probability measure

$$Q(B) = P(\mathbf{X} \in B) = P(\mathbf{X}^{-1}(B)).$$

A.5.1 Joint Distribution and Density Functions

The **joint distribution function** of $\mathbf{X}$ is the direct extension of the definition of the random variable distribution function:

$$F_{\mathbf{X}}(\mathbf{x}) = P\left(\bigcap_{i=1}^{d}\{X_i \le x_i\}\right), \quad \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

The joint distribution function of a random vector $\mathbf{X}$ has the following properties:

$$\lim_{x_k\to-\infty} F_{\mathbf{X}}(\mathbf{x}) = 0, \quad k = 1, \ldots, d,$$

$$x_k \mapsto F_{\mathbf{X}}(\mathbf{x}) \quad \text{is increasing, } k = 1, \ldots, d,$$
$$x_k \mapsto F_{\mathbf{X}}(\mathbf{x}) \quad \text{is right-continuous, } k = 1, \ldots, d.$$

In addition,

$$\lim_{x_k \to +\infty} F_{\mathbf{X}}(\mathbf{x}) = F_{\mathbf{X}_{|k}}(\mathbf{x}_{|k}), \qquad 1 \leq k \leq d,$$

is the joint distribution of the $\mathbb{R}^{d-1}$-valued random vector

$$\mathbf{X}_{|k} = (X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_d)\,.$$

The **joint density function** of the random vector $\mathbf{X}$, if it exists, is given by:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^d F_{\mathbf{X}}(\mathbf{x})}{\partial x_1 \cdots \partial x_d}.$$

The probability that the random vector takes values within the domain $(x_1, x_1 + dx_1] \times (x_d, x_d + dx_d]$, for given infinitesimal vector $\mathbf{d}x = (dx_1, \ldots, dx_d)$ is related to the joint density by

$$P\left(\bigcap_{i=1}^{d} \{X_i \in (x_i, x_i + dx_i]\}\right) \approx f_{\mathbf{X}}(\mathbf{x}) \|\mathbf{d}x\|.$$

From the joint distribution and density, one can derive expressions for the joint density or distribution of a subset of coordinates of $\mathbf{X}$. Such reduction is known as **marginalization**. For instance, the joint distribution of $\mathbf{X}_{|k}$ is

$$F_{\mathbf{X}_{|k}}(\mathbf{x}_{|k}) = F_{\mathbf{X}}(x_1, \ldots, x_{k-1}, \infty, x_{k+1}, \ldots, x_d),$$

whereas

$$f_{\mathbf{X}_{|k}}(\mathbf{x}_{|k}) = \int_{-\infty}^{+\infty} f_{\mathbf{X}}(x_1, \ldots, x_d)\, \mathrm{d}x_k.$$

In particular, $F_{X_i}(x_i) = F_{\mathbf{X}}(\infty, \ldots, \infty, x_i, \infty, \ldots, \infty)$ is the **marginal distribution** of X_i and

$$f_{X_i}(x_i) = \int_{\mathbb{R}^{d-1}} f_{\mathbf{X}}(\mathbf{x})\, \mathrm{d}\mathbf{x}_{|i}$$

is the **marginal density** of X_i.

Another useful expression concerns the **conditional density** of a subset of coordinates of $\mathbf{X}$. Consider for instance that the coordinates of $\mathbf{X}$ have been split into

two subsets corresponding to two vectors $\mathbf{X}_1 \in \mathbb{R}^{0<d_1<d}$ and $\mathbf{X}_2 \in \mathbb{R}^{d-d_1}$. Then the joint density of $\mathbf{X}_1$ conditioned by $\mathbf{X}_2 = \mathbf{x}_2$ is

$$f_{\mathbf{X}_1|\mathbf{X}_2}(\mathbf{x}_1, \mathbf{x}_2) = \frac{f_{\mathbf{X}}(\mathbf{x}_1, \mathbf{x}_2)}{f_{\mathbf{X}_2}(\mathbf{x}_2)}.$$

The joint density of transformed random vectors can also be derived. Consider a one to one mapping $\mathbf{h} : \mathbf{x} \in \mathbb{R}^d \mapsto \mathbf{y} = \mathbf{h}(\mathbf{x}) \in \mathbb{R}^d$, and define the random vector $Y = \mathbf{h}(\mathbf{X})$. The joint densities $f_{\mathbf{Y}}$ and $f_{\mathbf{X}}$ are related by

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\left|\nabla_{\mathbf{y}}\mathbf{h}^{-1}\right|,$$

where $\mathbf{h}^{-1}$ is the inverse mapping and $\nabla_{\mathbf{y}}\mathbf{h}^{-1}$ its Jacobian matrix. This relation can be intuitively derived from the principle of probability conservation. Indeed, denoting $D_{\mathbf{y}}$ a neighborhood of $\mathbf{y}$ and $D_{\mathbf{x}}$ its image by the inverse transformation $\mathbf{h}^{-1}$, the equality $P(\mathbf{Y} \in D_{\mathbf{y}}) = P(\mathbf{X} \in D_{\mathbf{x}})$ leads to

$$\begin{aligned} P(\mathbf{X} \in D_{\mathbf{x}}) &= \int_{D_{\mathbf{x}}} f_{\mathbf{X}}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = \int_{D_{\mathbf{y}}} f_{\mathbf{X}}(\mathbf{h}^{-1}(\boldsymbol{\xi}))\left|\nabla_{\boldsymbol{\xi}}\mathbf{h}^{-1}\right|\,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{D_{\mathbf{y}}} f_{\mathbf{Y}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = P(\mathbf{Y} \in D_{\mathbf{y}}). \end{aligned}$$

A.5.2 Independence of Random Variables

Consider a set of random variables X_i, $i \in I$, defined on a probability space (Θ, Σ, P). These random variables are **independent** if and only if

$$P(X_i \leq x_i, i \in J) = \prod_{i \in J} P(X_i \leq x_i), \quad \forall J \subset I, x_i \in \mathbb{R}.$$

For finite index set $I = \{1, \ldots, d\}$, the collection of random variables can be seen as a random vector $\mathbf{X}$ of $\mathbb{R}^d$, and the independence implies a factorized structure for the joint distribution $F_{\mathbf{X}}$,

$$F_{\mathbf{X}}(x_1, \ldots, x_d) = \prod_{i=1}^{d} F_{X_i}(x_i),$$

and joint density $f_{\mathbf{X}}$,

$$f_{\mathbf{X}}(x_1, \ldots, x_d) = \prod_{i=1}^{d} f_{X_i}(x_i).$$

A.5.3 Moments of a Random Vector

Consider a random vector $\mathbf{X}^d$ and an integer multi-index $\mathbf{l} = (l_1, \ldots, l_d) \in \mathbb{N}^d$. We define $|\mathbf{l}| = \sum_i l_i$, and the function

$$g_{\mathbf{l}} : \mathbf{x} \in \mathbb{R}^d \mapsto g(\mathbf{x}) = \prod_{i=1}^{i=d} x_i^{l_i}. \tag{A.1}$$

The function $g_{\mathbf{l}}(\mathbf{X})$ is a $\mathbb{R}$-valued random variable. If $\mathbf{X} \in L_{|\mathbf{l}|}$, *i.e.* for each of the vector coordinate $X_i \in L_{|\mathbf{l}|}$, then the moments of order $|\mathbf{l}|$ of $\mathbf{X}$ are given by

$$m_{\mathbf{l}}(\mathbf{X}) = \mathrm{E}\big[g_{\mathbf{l}}(\mathbf{X})\big]. \tag{A.2}$$

If the coordinates of $\mathbf{X}$ are mutually independent, the moments $m_{\mathbf{l}}$ can be expressed as:

$$m_{\mathbf{l}}(\mathbf{X}) = \mathrm{E}\left[\prod_{i=1}^{i=d} X_i^{l_i}\right] = \prod_{i=1}^{i=d} \mathrm{E}\left[X_i^{l_i}\right] = \prod_{i=1}^{i=d} m_{l_i}(X_i). \tag{A.3}$$

Of particular importance are the mean μ_i of X_i which corresponds to the first-order moment with $l_j = \delta_{ij}$, and the correlation r_{ij} of (X_i, X_j) which is the second-order moment with all $l_k = 0$ except $l_i = l_j = 1$. Combining these first and second moments, we obtain the covariance c_{ij} of (X_i, X_j),

$$c_{ij} = \mathrm{E}\big[(X_i - \mu_i)(X_j - \mu_j)\big] = r_{ij} - \mu_i \mu_j, \tag{A.4}$$

and the variance σ_i^2 of X_i:

$$\sigma_i^2 = \mathrm{E}\Big[(X_i - \mu_i)^2\Big] = r_{ii} - \mu_i^2 \geq 0. \tag{A.5}$$

If $c_{ij} = 0$ for $i \neq j$, then X_i and X_j are said to be **uncorrelated**. If $r_{ij} = 0$ for $i \neq j$, X_i and X_j are said to be **orthogonal**. The first and second-order moments of a random vector can be summarized using the mean vector $\mathbf{m}$ and correlation $\mathbf{r}$ or covariance $\mathbf{c}$ matrices:

$$\begin{aligned} \mathbf{m} &= \mathrm{E}[\mathbf{X}], \\ \mathbf{r} &= [r_{ij}] = \mathrm{E}\big[\mathbf{X}\mathbf{X}^t\big], \\ \mathbf{c} &= [c_{ij}] = \mathrm{E}\big[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^t\big]. \end{aligned} \tag{A.6}$$

Note that if X_i and X_j are independent, we have

$$c_{ij} = \mathrm{E}\big[(X_i - \mu_i)(X_j - \mu_j)\big] = \mathrm{E}[X_i - \mu_i]\mathrm{E}\big[X_j - \mu_j\big] = 0. \tag{A.7}$$

In other words, independent random variables are uncorrelated. However, the converse is usually not true: uncorrelated random variables are not independent in general. Also, random vectors with mutually uncorrelated coordinates have diagonal correlation matrices.

A.5.4 Gaussian Vector

The case of Gaussian vector is of particular importance in probability theory and in Polynomial Chaos expansions on Hermite bases. A Gaussian vector of $\mathbb{R}^d$ with mean $\mathbf{m}$ and covariance matrix $\mathbf{c}$ has for joint density function:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}\sqrt{|\mathbf{c}|}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^t \mathbf{c}^{-1}(\mathbf{x}-\mathbf{m})\right], \tag{A.8}$$

where $|\mathbf{c}|$ is the determinant of the correlation matrix. A Gaussian vector is then completely specified by its second-order properties and we shall write

$$\mathbf{X} \sim \mathrm{N}(\mathbf{m}, \mathbf{c}). \tag{A.9}$$

An important property of Gaussian vector is that if $\mathbf{c}$ is diagonal (its coordinate are uncorrelated) then the X_i are mutually independent. Indeed,

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \frac{1}{(2\pi)^{d/2}\sqrt{|\mathbf{c}|}} \exp\left[-\frac{1}{2}\sum_{i=1}^{d} \sigma_i^{-1}(x_i-\mu_i)^2\right] \\ &= \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{1}{2\sigma_i^2}(x_i-\mu_i)^2\right], \end{aligned} \tag{A.10}$$

because $c_{ii} = \sigma_i^2$ and $|\mathbf{c}| = \prod c_{ii}$ for diagonal matrices. Also, if $\mathbf{X}$ and $\mathbf{Y}$ are two $\mathbb{R}^d$-valued Gaussian vectors, then $(\alpha\mathbf{X} + \beta\mathbf{Y})$ is a Gaussian random vector.

A.6 Stochastic Processes

A.6.1 Motivation and Basic Definitions

In this monograph, we are interested in the computation of the stochastic solutions of equations, essentially ODEs or PDEs. Therefore these solutions are functions of time and/or space. Then, the notion of random variable and random vector need be extended to incorporate a dependence on time or/and space coordinates.

Consider a function $\mathbf{X} : T \times \Theta \mapsto \mathbb{R}^d$ with arguments –time– $t \in T$ (a real interval) and –event– $\theta \in \Theta$. If $\mathbf{X}(t)$ is a $\mathbb{R}^d$-valued random vector on the probability space (Θ, Σ, P) for all $t \in T$, then $\mathbf{X}$ is called a $\mathbb{R}^d$-valued **stochastic process** or vector stochastic process. The function $\mathbf{X}(\cdot, \theta) : t \in T \mapsto \mathbb{R}^d$ for given $\theta \in \Theta$ is called a **sample path** or simply a **realization** of $\mathbf{X}$. Conversely, the definition shows that $\mathbf{X}(t, \cdot) : \theta \in \Theta \mapsto \mathbb{R}^d$ is a random vector on (Θ, Σ, P).

The definition can be extended to random vectors indexed by a spatial coordinate. For instance, consider $\mathbf{y} \in \mathcal{D} \subset \mathbb{R}^n$, then $\mathbf{X} : \mathcal{D} \times \Theta \mapsto \mathbf{X} \in \mathbb{R}^d$ is called a $\mathbb{R}^d$-valued

stochastic field if $\mathbf{X}(\mathbf{y}, \cdot)$ is a random vector for each $\mathbf{y} \in \mathcal{D}$. The function $\mathbf{X}(\cdot, \theta)$ is called a realization of the stochastic field.

The two definitions can be combined and results in **space-time stochastic processes**:

$$\mathbf{X} : T \times \mathcal{D} \times \Theta \mapsto \mathbb{R}^d, \quad \mathbf{X}(t, \mathbf{y}, \cdot) \in (\Theta, \Sigma, P) \; \forall t \in T, \mathbf{y} \in \mathcal{D}. \tag{A.11}$$

There are noticeable differences between stochastic processes and stochastic fields, arising from oriented nature of time. However, these differences will not appear in the present monograph, and we shall call stochastic processes, or random process, any type of random variable or vector indexed by a deterministic set of coordinates in $\mathcal{D}$. We shall often rely on the loose terminology of random process. Consequently, we shall denote $\mathcal{D}$ the domain of indexation of the stochastic process, making no distinction between space and time coordinates in $\mathbf{y} \in \mathbb{R}^s$.

In the following, we restrict ourself to stochastic processes $\mathbf{X} : \mathcal{D} \times \Theta \mapsto \mathbb{R}^d$ defined on a probability space (Θ, Σ, P), with $\mathcal{D} \subset \mathbb{R}$, *i.e.* $n = 1$. The definitions and concepts can be extended to stochastic fields $n \geq 1$, subject to appropriate resolution of some technical difficulties and notation issues.

A.6.2 Properties of Stochastic Processes

If $\mathbf{X}$ is such that

$$\lim_{s \to t} P\left(\|\mathbf{X}(s) - \mathbf{X}(t)\| \geq \epsilon\right) \quad \forall \epsilon > 0, \tag{A.12}$$

and any $t \in \mathcal{D}$, the process is said **continuous in probability**. If for given integer $p \geq 1$,

$$\lim_{s \to t} \mathrm{E}\left[\|\mathbf{X}(s) - \mathbf{X}(t)\|^p\right] = 0, \quad \forall t \in \mathcal{D}, \tag{A.13}$$

the process is said **continuous in the p-th mean**, and **continuous in the mean square sense** in the case of $p = 2$. If

$$P\left(\left\{\theta : \lim_{s \to t} \|\mathbf{X}(s, \theta) - \mathbf{X}(t, \theta)\| \neq 0\right\}\right) = 0, \quad \forall t \in \mathcal{D}, \tag{A.14}$$

the process is said **almost surely continuous**.

In the previous definitions of continuity, the norm $\|\cdot\|$ is the classical Euclidean norm for vectors. The continuity can also be restricted to particular $t \in \mathcal{D}$ or subintervals.

A.6.2.1 Finite Dimensional Distributions and Densities

Let $m \geq 1$ be an integer, and a set of m distinct points t_i in $\mathcal{D}$. The **finite dimensional distribution (fdd) of order** m of X is defined by:

$$F_m\left(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}; t_1, \ldots, t_m\right) = P\left(\cap_{i=1}^{m}\{\mathbf{X}(t_i) \in \times_{k=1}^{d}(-\infty, x_k^{(i)}]\}\right), \quad \text{(A.15)}$$

where $\mathbf{x}^{(i)} = (x_1^{(i)}, \ldots, x_d^{(i)}) \in \mathbb{R}^d$. Thus, the finite dimensional distributions, F_m, are probability measures induced by $\mathbf{X}$ on the measurable space $(\mathbb{R}^{m\times d}, \mathcal{B}^{m\times d})$. The first-order fdd of $\mathbf{X}$ at t is called the marginal distribution. It is in fact the distribution of the random vector $\mathbf{X}(t, \cdot)$. For $d = 1$, the m-th order fdd's are

$$F_m(x^{(1)}, \ldots, x^{(m)}; t_1, \ldots, t_m) = P\left(X(t_1) \leq x^{(1)}, \ldots, X(t_m) \leq x^{(m)}\right), \quad \text{(A.16)}$$

from which the **finite dimensional densities** of X can be obtained by differentiation:

$$f_m\left(x^{(1)}, \ldots, x^{(m)}, t_1, \ldots, t_m\right) = \frac{\partial^m}{\partial x^{(1)} \cdots \partial x^{(m)}} F_m(x^{(1)}, \ldots, x^{(m)}; t_1, \ldots, t_m). \quad \text{(A.17)}$$

The marginal density of X at time t corresponds to the first-order fdd $f_1(x; t)$.

The $\mathbb{R}^d$-valued stochastic process is said **stationary in the strict sense** if all its fdd's are invariant under arbitrary shift τ of all the $t_i \to t_i + \tau$. If $\mathbf{X}$ is stationary, the marginal distribution of $\mathbf{X}$ is t-invariant.

A stochastic process $\mathbf{X}$ is a **Gaussian process** if all its fdd's are Gaussian.

A.6.3 Second Moment Properties

A **second-order stochastic process** $\mathbf{X}$ has all its components $X_i \in L_2(\Theta, \Sigma, P)$ for all $t \in \mathcal{D}$. For a second-order random process, we define

- the mean function: $\overline{\mathbf{x}}(t) = \mathrm{E}[\mathbf{X}]$
- the correlation function $\mathbf{r}(t, s) = \mathrm{E}[\mathbf{X}(t, \cdot)\mathbf{X}(s, \cdot)^t]$
- the covariance function $\mathbf{c}(t, s) = \mathrm{E}[(\mathbf{X}(t, \cdot) - \overline{\mathbf{x}}(t))(\mathbf{X}(s, \cdot) - \overline{\mathbf{x}}(s))^t]$

The **second moment properties** of a stochastic process are given by the couple $(\overline{\mathbf{x}}, \mathbf{r})$ or $(\overline{\mathbf{x}}, \mathbf{c})$. For a stationary process, the mean function $\overline{\mathbf{x}}$ is constant and the correlation and covariance functions depend only on the lag $\tau = t - s$. Conversely, a process with constant mean function and correlation and covariance functions of the form $\mathbf{r}(t, s) = \mathbf{r}(t - s)$ and $\mathbf{c}(t, s) = \mathbf{c}(t - s)$, is said **weakly stationary**. A weakly stationary process is non-necessarily strictly stationary.

The correlation function has the following remarkable properties:

- $r_{ii}(t, s) = r_{ii}(s, t)$. If the process is weakly stationary $r_{ii}(\tau) = r_{ii}(-\tau)$ where $\tau = t - s$.

- $\|r_{ii}(t,s)\|^2 \leq r_{ii}(t,t)r_{ii}(s,s)$. If the process is weakly stationary, $|r_{ii}(\tau)| \leq r_{ii}(0)$.
- $r_{ii}(t,s)$ is positive definite.
- $\mathbf{X}$ is mean square continuous at t' if and only if $r_{ii}(t,s)$ is continuous at $t = s = t'$ for $1 \leq i \leq d$.
- For $i \neq j$, $r_{ij}(t,s) = r_{ji}(s,t)$. If the process is weakly stationary, $r_{ij}(\tau) = r_{ji}(-\tau)$.
- $|r_{ij}(s,t)|^2 \leq r_{ii}(t,t)r_{jj}(s,s)$. If the process is weakly stationary, $|r_{ij}(\tau)|^2 \leq r_{ii}(0)r_{jj}(0)$.

Appendix B
Orthogonal Polynomials

By a system of orthogonal polynomials, we denote a set $\{P_n(x)\}_{n=1}^{\infty}$, where $P_n(x)$ is a polynomial of degree n. The polynomials are deemed orthogonal in the sense that the inner product

$$\langle P_n P_m \rangle$$

vanishes whenever $n \neq m$.

The variable x may be either continuous or discrete. In the former case, the inner product is defined as:

$$\langle P_n P_m \rangle = \int_a^b P_n(x) P_m(x) w(x) \, \mathrm{d}x \tag{B.1}$$

whereas in the latter it is given by:

$$\langle P_n P_m \rangle = \sum_{i=1}^{M} P_n(x_i) P_m(x_i) w(x_i). \tag{B.2}$$

In both cases, $w(x)$ is a positive weight function. Note that the limits of integration a and b in (B.1), and the limit M in (B.2) may be either finite or infinite.

We are specifically interested in orthogonal families of polynomials that form a basis of the function space $L_w^2 = \{f : \langle ff \rangle < \infty\}$. Within this general setting, we further focus our attention on families for which expansions of the form:

$$f(x) = \sum_{n=1}^{N} a_n P_n(x)$$

result, for suitable functions f, in spectra that decay exponentially as $N \to \infty$.

In this appendix, we provide a brief outline of families of orthogonal polynomials that are frequently used in PC expansions.

O.P. Le Maître, O.M. Knio, *Spectral Methods for Uncertainty Quantification,* Scientific Computation,
DOI 10.1007/978-90-481-3520-2, © Springer Science+Business Media B.V. 2010

B.1 Classical Families of Continuous Orthogonal Polynomials

In this section, we shall focus on the case of "continuous" polynomials defined for on the interval $a \leq x \leq b$. We shall denote by h_n the square of the L^2 norm of P_n, i.e.

$$h_n = \langle P_n P_n \rangle = \alpha \int_a^b P_n^2(x) w(x) \, \mathrm{d}x. \tag{B.3}$$

Note that for applications to PC expansions, it is convenient to adopt the convention that the space L^2_w has measure one, or alternatively that the norm of the indicator function of the entire space is equal to one, i.e.

$$\int_a^b w(x) \, \mathrm{d}x = 1. \tag{B.4}$$

This amounts to a constant scaling of weights functions that are normally utilized in the definition of orthogonal polynomials.

We also recall [1] that orthogonal polynomials: (a) satisfy the differential equation:

$$g_2(x) P_n'' + g_1(x) P' + a_n P_n = 0 \tag{B.5}$$

where g_1 and g_2 are independent of n and the a_n's are constants that depend on n only; and (b) can be generated using Rodrigues' formula:

$$P_n(x) = \frac{1}{e_n w(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left\{ w(x) \left[g(x) \right]^n \right\} \tag{B.6}$$

where g is a polynomial in x that is independent of n and the e_n's are arbitrary normalization factors that depend on n only. Note that Rodrigues' formula immediately shows that a constant multiplicative scaling of w has no impact on the corresponding representation or normalization.

B.1.1 Legendre Polynomials

The Legendre polynomials, $\{Le_n(x), n = 0, 1, \ldots\}$, are an orthogonal basis of $L^2_w[-1, 1]$ with respect to the weight function $w(x) = 1/2$ for all $x \in [-1, 1]$. The are typically normalized so that $Le_n(1) = 1$, in which case they are given by [26]:

$$Le_n(x) = \frac{1}{2^n} \sum_{l=0}^{[n/2]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} x^{n-2l}. \tag{B.7}$$

Here $[n/2]$ denotes the integral part of $n/2$. The polynomials are even when n is even and odd when n is odd.

The Legendre polynomials satisfy the recurrence relation:

$$Le_{n+1}(x) = \frac{2n+1}{n+1} x Le_n(x) - \frac{n}{n+1} Le_{n-1}(x) \tag{B.8}$$

with $Le_0(x) = 1$ and $Le_1(x) = x$. The first seven Legendre polynomials are given by [1, 26]:

$$\begin{aligned}
Le_0(x) &= 1, \\
Le_1(x) &= x, \\
Le_2(x) &= \frac{1}{2}\left(3x^2 - 1\right), \\
Le_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right), \\
Le_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \\
Le_5(x) &= \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right), \\
Le_6(x) &= \frac{1}{16}\left(2313x^6 - 315x^4 + 105x^2 5\right).
\end{aligned}$$

With the definition of w and the normalization convention specified above, the square norms of the Legendre polynomials are given by [1]:

$$h_n = \langle Le_n, Le_n \rangle = \int_1^1 Le_n^2(x) w(x)\, \mathrm{d}x = \frac{1}{2n+1}. \tag{B.9}$$

They satisfy (B.5) with [1]:

$$g_2(x) = 1 - x^2, \quad g_1(x) = -2x, \quad \text{and} \quad a_n = n(n+1), \tag{B.10}$$

and (B.6) with:

$$g(x) = 1 - x^2, \quad \text{and} \quad e_n = (-1)^n 2^n n!. \tag{B.11}$$

B.1.2 Hermite Polynomials

There are two widespread definitions of the Hermite polynomials, according to whether the weight functions are given by:

$$w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{B.12}$$

or

$$w(x) = \frac{1}{\sqrt{\pi}} \exp\left(-x^2\right). \tag{B.13}$$

In both cases, x is defined over the real line, i.e. $a = -\infty$ and $b = \infty$.

To distinguish between the two families, we shall use the conventional notation $He_n(x)$ to indicate that (B.12) is used, and $H_n(x)$ to indicate that (B.13) is adopted. Note that in PC expansions, we rely exclusively on He_n, but for completeness and computational convenience (see Appendix C) we address both cases.

For both definitions of w, the Hermite polynomials are conventionally normalized so that the factors appearing in Rodrigues' formula (B.6) are given by [1]:

$$e_n = (-1)^n. \tag{B.14}$$

The functions $g(x)$ appearing in (B.6) are also identical, $g(x) = 1$. It follows that the Hermite families are defined according to [111]:

$$He_n(x) = \frac{1}{(-1)^n \exp\left(-x^2/2\right)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\exp\left(-x^2/2\right)\right], \tag{B.15}$$

$$He(x) = \frac{1}{(-1)^n \exp\left(-x^2\right)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\exp\left(-x^2\right)\right]. \tag{B.16}$$

They have the following explicit representations:

$$He_n(x) = n! \sum_{m=0}^{[n/2]} (-1)^m \frac{1}{m! 2^m (n-2m)!} x^{n-2m}, \tag{B.17}$$

$$H_n(x) = n! \sum_{m=0}^{[n/2]} (-1)^m \frac{1}{m!(n-2m)!} (2x)^{n-2m}. \tag{B.18}$$

The polynomials $He_n(x)$ obey (B.5) with:

$$g_2(x) = 1, \quad g_1(x) = -x, \quad \text{and} \quad a_n = n, \tag{B.19}$$

and the recursion relation:

$$He_{n+1}(x) = x He_n(x) - n He_{n-1}(x). \tag{B.20}$$

With the standardization above, the square norms of the Hermite polynomials He_n are given by:

$$h_n = \langle He_n, He_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [He_n(x)]^2 \exp\left(-x^2/2\right) \mathrm{d}x = n!. \tag{B.21}$$

The first seven polynomials $He_n(x)$ are given by:

$$He_0(x) = 1, \tag{B.22}$$

$$He_1(x) = x, \tag{B.23}$$

$$He_2(x) = x^2 - 1, \tag{B.24}$$

$$He_3(x) = x^3 - 3x, \tag{B.25}$$

$$He_4(x) = x^4 - 6x^2 + 3, \tag{B.26}$$

$$He_5(x) = x^5 - 10x^3 + 15x, \tag{B.27}$$

$$He_6(x) = x^6 - 15x^4 + 45x^2 - 15. \tag{B.28}$$

The polynomials $H_n(x)$ obey (B.5) with:

$$g_2(x) = 1, \quad g_1(x) = -2x, \quad \text{and} \quad a_n = 2n, \tag{B.29}$$

and the recursion relation:

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x). \tag{B.30}$$

With the standardization above, the square norms of the Hermite polynomials H_n are given by:

$$h_n = \langle H_n, H_n \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [H_n(x)]^2 \exp\left(-x^2\right) \mathrm{d}x = 2^n n!. \tag{B.31}$$

The first seven polynomials $He_n(x)$ are given by:

$$H_0(x) = 1, \tag{B.32}$$

$$H_1(x) = 2x, \tag{B.33}$$

$$H_2(x) = 4x^2 - 2, \tag{B.34}$$

$$H_3(x) = 8x^3 - 12x, \tag{B.35}$$

$$H_4(x) = 16x^4 - 48x^2 + 12, \tag{B.36}$$

$$H_5(x) = 32x^5 - 160x^3 + 120x, \tag{B.37}$$

$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120. \tag{B.38}$$

B.1.3 Laguerre Polynomials

The Laguerre polynomials, $\{L_n(x), n = 0, 1, \ldots\}$, are an orthogonal basis of $L^2_w[0, \infty]$ with respect to the weight function $w(x) = \exp(-x)$. They are conventionally normalized so that the factors appearing in Rodrigues' formula (B.6) are given by [1]:

$$e_n = n!. \tag{B.39}$$

The function g appearing in (B.6) is $g(x) = x$. Accordingly, the polynomials La are defined by:

$$La(x) = \frac{1}{n! \exp(-x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\exp(-x) x^n \right), \tag{B.40}$$

and admit the following explicit representation:

$$La_(x) = \sum_{m=o}^{n} (-1)^m \frac{n!}{(n-m)!(m!)^2} x^m. \tag{B.41}$$

The polynomials $La_n(x)$ obey (B.5) with:

$$g_2(x) = x, \quad g_1(x) = 1 - x, \quad \text{and} \quad a_n = n, \tag{B.42}$$

and the recursion relation:

$$La_{n+1}(x) = \frac{1}{n+1} \left[(2n + 1 - x) La_n(x) - n La_{n-1}(x) \right]. \tag{B.43}$$

With the standardization above, the Laguerre polynomials La_n form an orthonormal basis of $L^2_w[0, \infty)$, i.e.

$$\langle La_n, La_m \rangle = \int_0^\infty La_n(x) La_m(x) \exp(-x) \, \mathrm{d}x = \delta_{nm}. \tag{B.44}$$

In particular, the square norm $h_n = 1$. The first six Laguerre polynomials are:

$$La_0(x) = 1, \tag{B.45}$$

$$La_1(x) = -x + 1, \tag{B.46}$$

$$La_2(x) = \frac{1}{2}(x^2 - 4x + 2), \tag{B.47}$$

$$La_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6), \tag{B.48}$$

$$La_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24), \tag{B.49}$$

$$La_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120). \tag{B.50}$$

B.2 Gauss Quadrature

In the application of PC methods, one needs in particular to evaluate moments of PC expansions. As further discussed in Appendix C, this task can be generally boiled down to evaluating inner products of one-dimensional polynomials. This section discusses evaluation of these inner products by means of Gauss quadratures which,

due to their high order accuracy, provide efficient means of computing the corresponding integrals.

We focus our attention to the Legendre, Hermite and Laguerre families discussed in the previous section. Accordingly, we are interested in evaluating integrals of polynomial functions defined over the intervals $[-1, 1]$, $(-\infty, \infty)$, and $[0, \infty)$.

B.2.1 Gauss-Legendre Quadrature

The Gauss-Legendre quadrature is based on the following formula:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{nq} w_i f(x_i) + R_{nq}, \tag{B.51}$$

where

$$R_{nq} = \frac{2^{2nq+1}(nq!)^4}{(2nq+1)[(2nq)!]^3} f^{(2nq)}(\xi), \quad -1 < \xi < 1. \tag{B.52}$$

Here, nq is the number of collocation points, x_i denote the coordinate of the i-th collocation point, and w_i the corresponding weight. R_n denotes the remainder of the quadrature; it indicates that the formula is exact if the order of the polynomials No $< 2nq$.

The coordinates x_i are the zeros of $Le_{nq}(x)$, and the weights are given by:

$$w_i = \frac{2}{(1-x_i^2)[Le'_{nq}(x_i)]^2}. \tag{B.53}$$

Explicit formulas for the Gauss-Legendre quadrature nodes are not known, and so they must be evaluated numerically.

The GAUSSQ routine available from netlib (http://netlib.org) can be used to determine the weights and abscissas appearing in (B.51). Table B.1 provides results using this routine for selected values of nq.

B.2.2 Gauss-Hermite Quadratures

Gauss-Hermite quadratures can be based on the following formula (see [1] p. 890):

$$\int_{-\infty}^{\infty} f(x)\exp(-x^2)\,\mathrm{d}x = \sum_{i=1}^{nq} w_i f(x_i) + R_{nq} \tag{B.54}$$

where nq is the number of collocation points, x_i denote the quadrature nodes and w_i are the corresponding weights. x_i is the i-th zero of the polynomial H_{nq}, and the

Table B.1 Nodes and weights for Gauss-Legendre quadrature for different values of nq

$nq = 5$		$nq = 10$		$nq = 15$	
x_i	w_i	x_i	w_i	x_i	w_i
−0.906180	0.236927E+00	−0.973907	0.666713E−01	−0.987993	0.307532E−01
−0.538469	0.478629E+00	−0.865063	0.149451E+00	−0.937273	0.703660E−01
0.000000	0.568889E+00	−0.679410	0.219086E+00	−0.848207	0.107159E+00
0.538469	0.478629E+00	−0.433395	0.269267E+00	−0.724418	0.139571E+00
0.906180	0.236927E+00	−0.148874	0.295524E+00	−0.570972	0.166269E+00
–	–	0.148874	0.295524E+00	−0.394151	0.186161E+00
–	–	0.433395	0.269267E+00	−0.201194	0.198431E+00
–	–	0.679410	0.219086E+00	0.000000	0.202578E+00
–	–	0.865063	0.149451E+00	0.201194	0.198431E+00
–	–	0.973907	0.666713E−01	0.394151	0.186161E+00
–	–	–	–	0.570972	0.166269E+00
–	–	–	–	0.724418	0.139571E+00
–	–	–	–	0.848207	0.107159E+00
–	–	–	–	0.937273	0.703660E−01
–	–	–	–	0.987993	0.307532E−01

weight w_i is given by:

$$w_i = \frac{2^{nq-1} nq! \sqrt{\pi}}{nq^2 [H_{nq-1}(x_i)]^2}. \tag{B.55}$$

The remainder in the quadrature (B.54) is given by:

$$R_n q = \frac{nq! \sqrt{\pi}}{2^n q (2nq)!} f^{(2nq)}(\zeta) \tag{B.56}$$

for some finite ζ. Thus, the formula (B.54) is exact if the polynomial $f(x)$ has degree smaller than $2nq$.

The GAUSSQ routine available from netlib (http://netlib.org) can be used to determine the weights and nodes appearing in (B.54). Table B.2 provides results using this routine for selected values of nq.

Note, however, the application of PC methods typically rely on the Hermite family He_n, and so the quadrature in (B.54) does not coincide with to the measure in the corresponding Hilbert space, which can be expressed as:

$$\langle f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-x^2/2) \, \mathrm{d}x. \tag{B.57}$$

Table B.2 Gauss nodes and weights for the quadrature in (B.54) for different values of nq

$nq = 5$		$nq = 10$		$nq = 15$	
x_i	w_i	x_i	w_i	x_i	w_i
−2.020183	0.199532E−01	−3.436159	0.764043E−05	−4.499991	0.152248E−08
−0.958572	0.393619E+00	−2.532732	0.134365E−02	−3.669950	0.105912E−05
0.000000	0.945309E+00	−1.756684	0.338744E−01	−2.967167	0.100004E−03
0.958572	0.393619E+00	−1.036611	0.240139E+00	−2.325732	0.277807E−02
2.020183	0.199532E−01	−0.342901	0.610863E+00	−1.719993	0.307800E−01
–	–	0.342901	0.610863E+00	−1.136116	0.158489E+00
–	–	1.036611	0.240139E+00	−0.565070	0.412029E+00
–	–	1.756684	0.338744E−01	0.000000	0.564100E+00
–	–	2.532732	0.134365E−02	0.565070	0.412029E+00
–	–	3.436159	0.764043E−05	1.136116	0.158489E+00
–	–	–	–	1.719993	0.307800E−01
–	–	–	–	2.325732	0.277807E−02
–	–	–	–	2.967167	0.100004E−03
–	–	–	–	3.669950	0.105912E−05
–	–	–	–	4.499991	0.152248E−08

In order to make use of the results above, one can apply the following change of variables:

$$\langle f \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} g(x) \exp(-x^2)\, \mathrm{d}x \tag{B.58}$$

where $g(x) = f(\sqrt{2}x)$. Consequently, application of the previous quadrature to the above integral immediately results in:

$$\langle f \rangle \simeq \frac{1}{\sqrt{\pi}} \sum_{i=1}^{nq} w_i\, g(x_i) \tag{B.59}$$

or alternatively,

$$\langle f \rangle \simeq \sum_{i=1}^{nq} w'_i\, f(x'_i) \tag{B.60}$$

where $w'_i \equiv w_i/\sqrt{\pi}$ and $x'_i \equiv \sqrt{2}x_i$. Equation (B.60) can now be readily used to evaluate moments and products of Hermite polynomials, He_n.

Table B.3 provides values of x'_i and w'_i for selected values of nq.

Note that our earlier observation that the quadrature in (B.54) is exact for polynomials of degree less than $2nq$ still applies to the quadrature in (B.60). In particular,

Table B.3 Gauss nodes and weights for the quadrature in (B.60) for different values of nq

$nq = 5$		$nq = 10$		$nq = 15$	
x_i'	w_i'	x_i'	w_i'	x_i'	w_i'
−2.856970	0.112574E−01	−4.859463	0.431065E−05	−6.363948	0.858965E−09
−1.355626	0.222076E+00	−3.581823	0.758071E−03	−5.190094	0.597542E−06
0.000000	0.533333E+00	−2.484326	0.191116E−01	−4.196208	0.564215E−04
1.355626	0.222076E+00	−1.465989	0.135484E+00	−3.289082	0.156736E−02
2.856970	0.112574E−01	−0.484936	0.344642E+00	−2.432437	0.173658E−01
–	–	0.484936	0.344642E+00	−1.606710	0.894178E−01
–	–	1.465989	0.135484E+00	−0.799129	0.232462E+00
–	–	2.484326	0.191116E−01	0.000000	0.318260E+00
–	–	3.581823	0.758071E−03	0.799129	0.232462E+00
–	–	4.859463	0.431065E−05	1.606710	0.894178E−01
–	–	–	–	2.432437	0.173658E−01
–	–	–	–	3.289082	0.156736E−02
–	–	–	–	4.196208	0.564215E−04
–	–	–	–	5.190094	0.597542E−06
–	–	–	–	6.363948	0.858965E−09

when sufficient quadrature points are provided, (B.60) reproduces the correct norm:

$$\left\langle He_m^2 \right\rangle = m!. \tag{B.61}$$

B.2.3 Gauss-Laguerre Quadrature

The Gauss-Laguerre quadrature is based on the following formula:

$$\int_0^\infty f(x) \exp(-x)\, \mathrm{d}x = \sum_{i=1}^{nq} w_i f(x_i) + R_{nq}, \tag{B.62}$$

where

$$R_{nq} = \frac{(nq!)^2}{(2nq)!} f^{(2nq)}(\xi), \quad 0 < \xi < \infty. \tag{B.63}$$

As before, nq is the number of collocation points, while x_i and w_i respectively denote the nodes and weights. From (B.63), it follows immediately that the quadrature (B.62) is exact when the order of the polynomial, f, is smaller than $2nq$.

Table B.4 Nodes and weights for Gauss-Laguerre quadrature for different values of nq

$nq = 5$		$nq = 10$		$nq = 15$	
x_i	w_i	x_i	w_i	x_i	w_i
0.263560	0.521756E+00	0.137793	0.308441E+00	0.093308	0.218235E+00
1.413403	0.398667E+00	0.729455	0.401120E+00	0.492692	0.342210E+00
3.596426	0.759424E−01	1.808343	0.218068E+00	1.215595	0.263028E+00
7.085810	0.361176E−02	3.401434	0.620875E−01	2.269950	0.126426E+00
12.640801	0.233700E−04	5.552496	0.950152E−02	3.667623	0.402069E−01
–	–	8.330153	0.753008E−03	5.425337	0.856388E−02
–	–	11.843786	0.282592E−04	7.565916	0.121244E−02
–	–	16.279258	0.424931E−06	10.120229	0.111674E−03
–	–	21.996586	0.183956E−08	13.130282	0.645993E−05
–	–	29.920697	0.991183E−12	16.654408	0.222632E−06
–	–	–	–	20.776479	0.422743E−08
–	–	–	–	25.623894	0.392190E−10
–	–	–	–	31.407519	0.145652E−12
–	–	–	–	38.530683	0.148303E−15
–	–	–	–	48.026086	0.160059E−19

The coordinates x_i are the zeros of $La_{nq}(x)$, and the weights are given by:

$$w_i = \frac{(nq!)^2 x_i}{(nq+1)^2 [La_{nq+1}(x_i)]^2}. \tag{B.64}$$

The GAUSSQ routine available from netlib (http://netlib.org) can be used to determine the weights and abscissas appearing in (B.62). Table B.4 provides results using this routine for selected values of nq.

B.3 Askey Scheme

As discussed by Xiu and Karniadakis [249], the above families of orthogonal polynomials are members of the so-called Askey scheme of polynomials [6]. The schemes classifies continuous and discrete hypergeometric orthogonal polynomials that obey certain difference and difference equations, and identifies limit relationship between them [117, 203]. The interesting aspect of the Askey family is that the weighting functions associated with some its members corresponds to frequently used probability distributions.

Specifically, in addition to the orthogonal polynomial systems discussed above, the Askey family includes the Jacobi polynomials, which in turn include the Legendre family, as well as the Charlier, Meixner, Krawtchouk and Hahn families of discrete orthogonal polynomials. The weighting functions associated with the Jacobi

polynomials correspond to the beta distribution, whereas the weighting functions of the Charlier, Meixner, Krawtchouk and Hahn families respectively correspond to the Poisson, negative binormal, binormal, and hypergeometric distributions.

For brevity, and since these additional families are not used in the examples provided in the present volume, we restrict ourselves to providing definitions of corresponding polynomials, and outlining the orthogonality relationships that they obey. Implementation of these families into the corresponding Wiener chaos can be readily performed once the orthogonality relationships are exploited to define a suitable inner product, specifically in a similar fashion to the development above.

B.3.1 Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $n = 0, 1, \ldots$, are an orthogonal family of polynomials over $[-1, 1]$ with respect to the weight function

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

The are typically normalized so that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

in which case they are given by [1]:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m. \tag{B.65}$$

The Jacobi polynomials satisfy (B.5) with [1]:

$$g_2(x) = 1-x^2, \quad g_1(x) = \beta-\alpha-(\alpha+\beta+2)x, \quad \text{and} \quad a_n = n(n+\alpha+\beta+1), \tag{B.66}$$

the Rodrigues formula (B.6) with:

$$g(x) = 1-x^2, \quad \text{and} \quad e_n = (-1)^n 2^n n!, \tag{B.67}$$

the orthogonality condition:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w(x)\,\mathrm{d}x = \frac{2^{\alpha}+\beta+1}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\delta_{nm}, \tag{B.68}$$

and the recurrence relation:

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x)$$

$$= \left[(2n+\alpha+\beta+1)(\alpha^2-\beta^2) + (2n+\alpha+\beta)_3 x \right] P_n^{(\alpha,\beta)}(x)$$
$$- 2(n+\alpha)(n+\beta)(2n+\alpha+beta+n) P_{n-1}^{(\alpha,\beta)}(x) \tag{B.69}$$

where $(m)_n = m(m+1)\cdots(m+n-1)$ denotes the Pochhammer symbol.

Note that the orthogonality condition (B.68) may be readily used to rescale $w(x)$ so that the resulting space $L_w^2[-1,1]$ has measure 1, and that the GAUSSQ routine also provides nodes and weights for Gauss-Jacobi quadratures.

B.3.2 Discrete Polynomials

The Charlier, Meixner, Krawtchouk, and Hahn polynomials are respectively denoted by $C_n(x,a)$, $M_n(x;\beta,c)$, $K_n(x;p,N)$, and $Q_n(x;\alpha,\beta,N)$. In all cases, x is discrete variable. For the Hahn and Krawtchouk polynomials, x is defined over the set $\{0,1,\ldots,N\}$, whereas for the Charlier and Meixner polynomials x is defined over the integers $0,1,\ldots$.

As discussed in [117], the Charlier, Meixner, Krawtchouk, and Hahn polynomials may be defined in terms of the hypergeometric series:

$${}_\tau F_s\left(\begin{matrix} a_1,\ldots,a_\tau \\ b1,\ldots,b_s \end{matrix} \middle| z \right) \equiv \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_\tau)_k}{(b_1)_k \ldots (b_s)_k} \frac{z^k}{k!}. \tag{B.70}$$

Relevant definitions are summarized Table B.5

The first two polynomials are:

$$C_0(x,a) = 1, \tag{B.71}$$
$$C_1(x,a) = 1 - \frac{x}{a}, \tag{B.72}$$
$$M_0(x;\beta,c) = 1, \tag{B.73}$$

Table B.5 Definition of Charlier, Meixner, Krawtchouk, and Hahn polynomials in terms of hypergeometric series

Polynomial	Definition
Charlier	${}_2F_0\left(\begin{matrix} -n,-x \\ -- \end{matrix} \middle| -\frac{1}{a} \right)$
Meixner	${}_2F_1\left(\begin{matrix} -n,-x \\ \beta \end{matrix} \middle| 1-\frac{1}{c} \right)$
Krawtchouk	${}_2F_1\left(\begin{matrix} -n,-x \\ -N \end{matrix} \middle| \frac{1}{p} \right)$
Hahn	${}_3F_2\left(\begin{matrix} -n,n+\alpha+\beta+1,-x \\ \alpha+1,-N \end{matrix} \middle| 1 \right)$

Table B.6 Weights and square norms of the Charlier, Meixner, Krawtchouk, and Hahn polynomials

Polynomial	$w(x)$	h_n
Charlier	$\frac{a^x}{x!}$	$a^{-n}\exp(-a)n!$
Meixner	$\frac{(\beta)_x c^x}{x!}$	$\frac{c^{-n}n!}{(\beta)_n(1-c)^\beta}$
Krawtchouk	$\binom{N}{x}p^x(1-p)^{N-x}$	$\frac{(1)^{-n}n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n$
Hahn	$\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$	$\frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}$

$$M_1(x;\beta,c) = 1 + \frac{x}{\beta}\left(1 - \frac{1}{c}\right), \tag{B.74}$$

$$K_0(x;p,N) = 1, \tag{B.75}$$

$$K_1(x;p,N) = 1 - \frac{x}{Np}, \tag{B.76}$$

$$Q_0(x;\alpha,\beta,N) = 1, \tag{B.77}$$

$$Q_1(x;\alpha,\beta,N) = 1 - \frac{(\alpha+\beta+2)x}{(\alpha+1)N}. \tag{B.78}$$

With the definitions above, these families of discrete polynomials satisfy the discrete orthogonality relation:

$$\sum_i f_n(x_i) f_m(x_i) w(x_i) = h_n \delta_{nm} \tag{B.79}$$

where the index i ranges over the entire domain of the discrete variable x. Relevant formulas for the weights, w, and square norms, h_n, are given in Table B.6.

The discrete polynomials satisfy the Rodrigues-type formula:

$$w(x) f_n(x) = r_n \nabla^n \left[g(x)\right] \tag{B.80}$$

where $w(x)$ is the weight function, r_n is a normalizing function independent of x, g is a polynomial in x, and ∇ denotes the backward difference operator $\nabla f(x) = f(x) - f(x-1)$. Relevant formulas for the factors r and g are given in Table B.7.

For computational purposes, polynomial evaluations may be conveniently performed on the basis of recursion relations. Relevant formulas for the Charlier, Meixner, Krawtchouk and Hahn polynomials are given by:

$$aC_{n+1} = (n + a - x)C_n - nC_n - 1, \tag{B.81}$$

$$c(n+\beta)M_{n+1} = \left[n + (n+\beta)c - (c-1)x\right]M_n - nM_{n-1}, \tag{B.82}$$

Table B.7 Factors r and g in (B.80) for the Charlier, Meixner, Krawtchouk, and Hahn polynomials

Polynomial	r_n	$g(x)$
Charlier	1	$\frac{a^x}{x!}$
Meixner	1	$\frac{(\beta+n)_x c^x}{x!}$
Krawtchouk	$(1-p)^{-N}$	$\binom{N-n}{x}\left(\frac{p}{1-p}\right)^x$
Hahn	$\frac{(-1)^n(\beta+1)_n}{(-N)_n}$	$\binom{\alpha+n+x}{x}\binom{\beta+N-x}{N-n-x}$

$$p(N-n)K_{n+1} = \big[p(N-n)+n(1-p)-x\big]K_n - n(1-p)K_{n-1}, \quad \text{(B.83)}$$

$$A_n Q_{n+1}(x) = (A_n + D_n - x)Q_n(x) - D_n Q_{n-1}(x) \quad \text{(B.84)}$$

where $C_n \equiv C_n(x;a)$, $M_n \equiv M_n(x;\beta,c)$, $K_n \equiv K_n(x;p,N)$, $Q_n(x) \equiv Q_n(x : \alpha,\beta,N)$, and

$$A_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

$$D_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

We finally note that, with the standardizations above, the weighting functions of Charlier, Meixner, Krawtchouk, and Hahn polynomials correspond, up to a scaling factor independent of x, to the probability density functions of the Poisson, negative binormal, binormal, and hypergeometric distributions. The scaling factors can be readily obtained from the orthogonality expressions, and can be used to rescale the standard forms of w to define a suitable inner product, namely one for which the measure of the corresponding ℓ_2 space is unity.

Appendix C
Implementation of Product and Moment Formulas

In the development of computer codes implementing PC expansions, one is faced with the problem of evaluating moments of classical orthogonal polynomials, or more often of multidimensional generalizations of these polynomials. This Appendix outlines the main ingredient of a general approach that has proven to be quite useful in the construction of utility libraries that are capable of performing fundamental operations and nonlinear transformations of PC representations of random variables.

C.1 One-Dimensional Polynomials

For most situations, it is quite advantageous for users to compute and store polynomials appearing in PC expansions. As mentioned in Chapter 2, these expansions involve orthogonal polynomials that are typically defined in terms of series expansions. For computational purposes, however, it is more convenient to evaluate the polynomials recursion relations. As discussed in Appendix B, in the Hermite case, one has

$$\begin{aligned} \psi_0(x) &= 1, \\ \psi_1(x) &= x, \\ \psi_n(x) &= x\psi_{n-1}(x) - (n-1)\psi_{n-2}(x), \quad n = 2, 3, \ldots \end{aligned} \tag{C.1}$$

and similar recursion relations exist for orthogonal polynomials that are member of the Askey family. The exploitation of these relationships is recommended whenever possible.

O.P. Le Maître, O.M. Knio, *Spectral Methods for Uncertainty Quantification,* Scientific Computation,
DOI 10.1007/978-90-481-3520-2, © Springer Science+Business Media B.V. 2010

C.1.1 Moments of One-Dimensional Polynomials

We are interested in evaluating second-order moments of the form

$$\langle f \rangle \quad \text{where } f(x) \equiv \psi_i(x)\psi_j(x), \tag{C.2}$$

and more generally in higher-order moments which may be expressed as:

$$\langle f \rangle \quad \text{where } f(x) \equiv \prod_{m=1}^{M} \psi_{i_m}(x). \tag{C.3}$$

Such moments may be evaluated on the basis of analytical expressions. Alternatively as discussed in Appendix B, Gauss quadratures may be readily applied, and these yield exact values so long as a sufficient number of collocation points is provided, namely to ensure a vanishing remainder.

C.2 Multidimensional PC Basis

As discussed in Chap. 2, in the multidimensional case the PC basis functions are defined from an incomplete tensorization of the corresponding 1D basis functions. As a result, each member of the multidimensional PC basis can written as a product of 1D polynomials in the appropriate random variable. In order to specify (and hence immediately evaluate) the elements of the multidimensional basis, it is advantageous to associate with each basis function, Ψ_i a multi-index $\boldsymbol{\alpha}^i = (\alpha_1^i, \alpha_2^i, \ldots, \alpha_{\mathrm{N}}^i)$, where α_k^i denotes the order of the 1D polynomial in ξ_k. Once these multi-indices are available, evaluation of the multidimensional basis function is straightforward. We illustrate this for the case of the N-dimensional Hermite polynomials. We denote the latter by Ψ_k, in order to distinguish them from, ψ_j, their 1D counterparts.

By definition of the multi-indices, the N-dimensional polynomial chaoses can be obtained from:

$$\Psi_k(\xi_1, \xi_2, \ldots, \xi_{\mathrm{N}}) = \prod_{i=1}^{\mathrm{N}} \psi_{\alpha_i^k}(\xi_i). \tag{C.4}$$

Thus, knowledge of the multi-indices, together with the 1D basis functions, provides a very efficient and convenient means of evaluating the N-dimensional PCs.

C.2.1 Multi-Index Construction

To complete this construction, one needs to define the multi-indices $\boldsymbol{\alpha}^i, i = 0, \ldots, \mathrm{P}$. While multiple definitions are possible, we have adopted the same indexing scheme used in [90]. Using this indexing scheme, the multi-indices are defined recursively for arbitrary order, No, and dimension, N, as described in the following pseudo-code:

1. For the 0-th order polynomial, set:

$$\alpha_i^0 = 0, \quad i = 1, \ldots, \mathrm{N}.$$

   Note that if $\mathrm{No} = 0$, then we set $\mathrm{P} = 0$ and the process is interrupted.
2. For the N first-order polynomials, set:

$$\alpha_j^i = \delta_{ij} \quad \text{for } 1 \leq i, j \leq \mathrm{N}.$$

   Note that if $\mathrm{No} = 1$, then we set $\mathrm{P} = \mathrm{N}$ and the process is interrupted.
3. Set $\mathrm{P} = \mathrm{N}$
4. Set $p_i(1) = 1, i = 1, \ldots, \mathrm{N}$
5. For $k = 2, \ldots, \mathrm{No}$:
   - Set $L = \mathrm{P}$
   - Set $p_i(k) = \sum_{m=i}^{\mathrm{N}} p_m(k-1), i = 1, \ldots, \mathrm{N}$
   - For $j = 1, \ldots, \mathrm{N}$:
     + For $m = L - p_j(k) + 1, \ldots, L$:
       · Set $\mathrm{P} = \mathrm{P} + 1$
       · Set $\boldsymbol{\alpha}_i^P = \boldsymbol{\alpha}_i^m, i = 1, \ldots, \mathrm{N}$
       · Set $\boldsymbol{\alpha}_j^P = \boldsymbol{\alpha}_j^P + 1$

In this algorithm, for a specific k, L denotes the number of terms that have been constructed so far with order less than k. The terms of order k are then constructed from the terms of order $(k-1)$ by increasing the order of the factors in those terms belonging to a specific dimension, one at the time. For example, for the three-dimensional stochastic case, there are 6 terms of order 2. The first 6 terms of order 3 are constructed by taking all 6 terms of order 2, and increasing the multi-index corresponding to the order of the first stochastic dimension by 1. The next 3 terms of order 3 are constructed by taking the last 3 terms of order 2, and increasing the order of the contribution of the second stochastic dimension. The last term is obtained by increasing the multi-index corresponding to the third dimension with 1 in the last term of order 2. Investigation of the process shows that the terms at the current order, in which the order of the contribution of dimension j will be increased to get terms at the next order, are the terms that were generated at the current order from the previous order by increasing the order of dimension j. This bookkeeping is done by the $p_i(k)$.

Note that the procedure above also provides the number, P, of N-dimensional Polynomial Chaoses of order $\leq$ No.

C.2.2 Moments of Multidimensional Polynomials

We are now interested in generalizing the results of the previous section to the N-dimensional case. To this end, we exploit the tensor product construction of

the multidimensional inner product, and again rely on the multi-indices $\boldsymbol{\alpha}^i$ to express the moments of N-dimensional Polynomial Chaoses in terms of products of 1-dimensional moments. For instance, the second-order moment $\langle \Psi_i \Psi_j \rangle$ can be expressed as:

$$\left\langle \Psi_i \Psi_j \right\rangle_{\mathrm{N}} = \prod_{k=1}^{\mathrm{N}} \left\langle \psi_{\alpha_k^i} \psi_{\alpha_k^j} \right\rangle_1 \tag{C.5}$$

where we have used subscripts to distinguish between the inner products in 1D and N-dimensional space. Similarly, for a moment of order M we have

$$\left\langle \prod_{m=1}^{M} \Psi_{i_m} \right\rangle_{\mathrm{N}} = \prod_{k=1}^{\mathrm{N}} \left\langle \prod_{m=1}^{M} \psi_{\alpha_k^{i_m}} \right\rangle_1 . \tag{C.6}$$

The advantage of this product factorization is that in practice only a few one-dimensional moments need to be actually computed, and this results in order-of-magnitude reduction in the effort required to form the required N-dimensional moments. It also greatly facilitates the computation of moments involving mixed bases, namely those involving tensor products of different families of orthogonal polynomials.

C.2.3 Implementation Details

We conclude with a brief outline of an attractive approach for the implementation of PC computations, especially in the multidimensional case. This approach is based on the establishment of a code infrastructure in which the multi-indices defined in Sect. C.2.1 are computed in a pre-processing step, together with the necessary multidimensional moments, namely those appearing in the definition of multiplication tensors used to evaluate products and other nonlinear transformations (see Sect. 4.5). As can be anticipated, it is generally advantageous to compute and store these tensors in a pre-processing step. One should particularly seek to take advantage of the sparse nature of these tensors, namely by storing (and using) only the non-vanishing entries of these tensors. Once properly defined in this compact format, these "transformation" tensors can be used as a basis for the construction of utility libraries that implement various transformations of PC representations.

The approach outlined above in fact forms the basis of the UQ toolkit,[1] which constitutes one incarnation of a PC-utility-library software infrastructure.

[1] B.J. Debusschere *et al.*, http://www.sandia.gov/UQToolkit/.

References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. Dover, New York (1970)
2. Agarwal, R., Wong, P.: Error Inequalities in Polynomial Interpolation and Their Applications vol. 262. Kluwer, Dordrecht (1993)
3. Alpert, B.: A class of bases in L_2 for the sparse representation of integral operators. J. Math. Anal. **24**, 246–262 (1993)
4. Alpert, B., Beylkin, G., Gines, D., Vozovoi, L.: Adaptive solution of partial differential equations in multiwavelet bases. J. Comput. Phys. **182**, 149–190 (2002)
5. Andreev, V., Pliss, N., Righetti, P.: Computer simulation of affinity capillary electrophoresis. Electrophoresis **23**(6), 889–895 (2002)
6. Askey, R., Wilson, J.: Some basic hypergeometric polynomials that generalize Jacobi polynomials. Mem. Am. Math. Soc. **54**, 1–55 (1985)
7. Babuška, I., Nobile, F., Tempone, R.: A stochastic collocation method for elliptic partial differential equations with random input data. SIAM J. Numer. Anal. **45**(3), 1005–1034 (2007)
8. Babuška, I., Tempone, R., Zouraris, G.E.: Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation. Comput. Methods Appl. Mech. Eng. **194**, 1251–1294 (2005)
9. Babuska, I., Chatzipantelidis, P.: On solving elliptic stochastic partial differential equations. Comput. Methods Appl. Mech. Eng. **191**, 4093–4122 (2002)
10. Babuska, I., Liu, K.M.: On solving stochastic initial-value differential equations. Technical report, TICAM Report 02–17, The University of Texas at Austin (2002)
11. Babuška, I., Miller, A.: A feedback finite element method with a posteriori error estimation. Comput. Methods Appl. Mech. Eng. **61**, 1–40 (1987)
12. Babuška, I., Rheinboldt, W.: A posteriori error estimates for the finite element method. Int. J. Numer. Methods Eng. **12**, 1597–1615 (1978)
13. Bachman, G., Narici, L.: Functional Analysis. Dover, New York (2000)
14. Batchelor, G.: An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge (1985)
15. Beale, J.: On the accuracy of vortex methods at large times. In: Computational Fluid Dynamics and Reacting Gas Flows. Springer, Berlin (1988)
16. Beaudouin, A., Huberson, S., Rivoalen, E.: Simulation of anisotropic diffusion by means of a diffusion velocity method. J. Comput. Phys. **186**, 122–135 (2003)
17. Becker, R., Rannacher, R.: An optimal control approach to a posteriori error estimation in finite element methods. Acta Numer. **10**, 1–102 (2001)
18. Benth, F.E., Gjerde, J.: Convergence rates for finite element approximations of stochastic partial differential equations. Stoch. Stoch. Rep. **63**(3–4), 313–326 (1998)
19. Beran, P., Pettit, C., Millman, D.: Uncertainty quantification of limit-cycle oscillations. J. Comput. Phys. **217**, 217–247 (2006)

O.P. Le Maître, O.M. Knio, *Spectral Methods for Uncertainty Quantification,* Scientific Computation,
DOI 10.1007/978-90-481-3520-2, © Springer Science+Business Media B.V. 2010

20. Berveiller, M., Sudret, B., Lemaire, M.: Stochastic finite element: a non intrusive approach by regression. Eur. J. Comput. Mech. **15**, 81–92 (2006)
21. Bielewicz, E., Górski, J.: Shells with random geometric imperfections simulation-based approach. Int. J. Non-Linear Mech. **37**, 777–784 (2002)
22. Bier, M., Palusinski, O., Mosher, R., Saville, D.: Electrophoresis: Mathematical modeling and computer simulation. Science **219**(4590), 1281–1287 (1983)
23. Blatman, G., Sudret, B.: Sparse polynomial chaos expansions and adaptive stochastic finite elements using a regression approach. C. R. Méc. **336**, 518–523 (2008)
24. Boyd, J.: Chebishev and Fourier Spectral Methods. Dover, New York (2001)
25. Cameron, R., Martin, W.: The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals. Ann. Math. **48**, 385–392 (1947)
26. Canuto, C., Hussaini, M., Quarteroni, A., Zang, T.: Spectral Methods in Fluids Dynamics. Springer Series in Computational Physics. Springer, Berlin (1990)
27. Chenoweth, D., Paolucci, S.: Natural convection in an enclosed vertical layer with large horizontal temperature differences. J. Fluid Mech. **169**, 173–210 (1986)
28. Choquin, J., Lucquin-Desreux, B.: Accuracy of a deterministic particle method for Navier-Stokes equations. J. Numer. Methods Fluids **8**, 1439–1458 (1988)
29. Chorin, A.: A numerical method for solving incompressible viscous flow problems. J. Comput. Phys. **2**, 12–26 (1967)
30. Chorin, A.: Numerical study of slightly viscous flow. J. Fluid Mech. **57**, 785–796 (1973)
31. Chorin, A.: Gaussian fields and random flow. J. Fluid Mech. **63**, 21–32 (1974)
32. Christiansen, I.: Numerical simulation of hydrodynamics by the method of point vortex. J. Comput. Phys. **13**(3), 363–379 (1973)
33. Christon, M., Gresho, P., Sutton, S.: Computational predictability of natural convection flows in enclosure. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics. Proceedings of First MIT Conference on Computational Fluid and Solid Mechanics, pp. 1465–1468. Elsevier, Amsterdam (2001)
34. Christopher, D.: Numerical prediction of natural convection flows in a tall enclosure. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1469–1471. Elsevier, Amsterdam (2001)
35. Clément, P.: Approximation by finite element functions using local regularization. Rev. Fr. Autom. Inf. Rech. Opér., Sér. Rouge Anal. Numér. **9**(R-2), 77–84 (1975)
36. Clenshaw, C., Curtis, A.: A method for numerical integration on an automatic computer. Numer. Math. **2**, 197–205 (1960)
37. Comini, G., Manzan, M., Nonino, C., Saro, O.: Finite element solutions for natural convection in a tall rectangular cavity. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1472–1476. Elsevier, Amsterdam (2001)
38. Cools, R., Maerten, B.: A hybrid subdivision strategy for adaptive integration routines. J. Univ. Comput. Sci. **4**(5), 486–500 (1998)
39. Cormick, S.M.: Multigrids Methods. SIAM Books, Philadelphia (1987)
40. Cottet, G.: Large time behavior of deterministic particle approximations to the Navier-Stokes equations. Math. Comput. **56**, 45–59 (1991)
41. Cottet, G., Koumoutsakos, P.: Vortex Methods Theory and Practice. Cambridge University Press, Cambridge (2000)
42. Cottet, G., Mas-Gallic, S.: A particle method to solve the Navier-Stokes system. Numer. Math. **57**, 805–827 (1990)
43. Courant, D., Hilbert, C.: Method of Mathematical Physics. Interscience, New York (1943)
44. Crow, S., Canavan, G.: Relationship between a Wiener-Hermite expansion and an energy cascade. J. Fluid Mech. **41**, 387–403 (1970)
45. Das, S., Ghanem, R., Spall, J.: Asymptotic sampling distribution for Polynomial Chaos representation, from data: a maximum entropy and Fisher information approach. SIAM J. Sci. Comput. **30**(5), 2207–2234 (2007)
46. Daube, O.: An influence matrix technique for the resolution of 2d Navier-Stokes equation in velocity-vorticity formulation. In: Int. Conf. Num. Meth. in Laminar and Turbulent Flows (1986)

47. Daubechies, I.: Ten Lectures on Wavelets. SIAM Books, Philadelphia (1992)
48. De Vahl Davis, G., Jones, I.: Natural convection in a square cavity: a comparison exercise. Int. J. Numer. Methods Fluids **3**, 227–248 (1983)
49. De Valh Davis, G., Jones, I.: Natural convection in a square cavity: a comparison exercise. Int. J. Numer. Methods Fluids **3**, 227–248 (1983)
50. Deb, M., Babuska, I., Oden, J.: Solution of stochastic partial differential equations using Galerkin finite element techniques. Comput. Methods Appl. Mech. Eng. **190**, 6359–6372 (2001)
51. Debusschere, B., Najm, H., Matta, A., Knio, O., Ghanem, R., Le Maître, O.: Protein labeling reactions in electrochemical microchannel flow: numerical prediction and uncertainty propagation. Phys. Fluids **15**(8), 2238–2250 (2003)
52. Debusschere, B., Najm, H., Pébray, P., Knio, O., Ghanem, R., Le Maître, O.: Numerical challenges in the use of polynomial chaos representations for stochastic processes. J. Sci. Comput. **26**(2), 698–719 (2004)
53. Degond, P., Mas-Gallic, S.: The weighted particle method for convection diffusion equations, part 1: the case of an isotropic viscosity. Math. Comput. **33**(53), 485–507 (1989)
54. Degond, P., Mustieles, F.J.: A deterministic approximation of diffusion using particles. J. Sci. Stat. Comput. **1**(2), 293–310 (1990)
55. Desceliers, C., Ghanem, R., Soize, C.: Maximum likelihood estimation of stochastic chaos representations from experimental data. Int. J. Numer. Methods Eng. **66**(6), 978–1001 (2006)
56. Desceliers, C., Soize, C., Ghanem, R.: Identification of chaos representations of elastic properties of random media using experimental vibration tests. Comput. Mech. **39**(6), 831–838 (2007)
57. Ditlevsen, O., Tarp-Johansen, N.: Choice of input fields in stochastic finite-elements. Probab. Eng. Mech. **14**, 63–72 (1999)
58. Doostan, A., Ghanem, R., Red-Horse, J.: Stochastic model reduction for chaos representations. Comput. Methods Appl. Mech. Eng. **196**, 3951–3966 (2007)
59. Drazin, P., Reid, W.: Hydrodynamic Stability. Cambridge University Press, Cambridge (1982)
60. Edwards, W., Tuckerman, L., Friesner, R., Sorensen, D.: Krylov methods for the incompressible Navier-Stokes equations. J. Comput. Phys. **110**(1), 82–102 (1994)
61. Efron, B., Tibshirani, R.: An Introduction to the Bootstrap. Monographs on Statistics and Applied Probability, vol. 57. Chapman & Hall/CRC Press, London (1994)
62. Eldred, M.: Optimization strategies for complex engineering applications. Sandia Technical Report SAND98-0340, Sandia National Laboratories (1998)
63. Eldred, M., Giunta, A., Wojkiewicz, S., van Bloemen Waanders, B., Hart, W., Alleva, M.: Dakota, a multilevel parallel object-oriented framework for design optimization, parameter estimation, sensitivity analysis, and uncertainty quantification. version 3.0 reference manual. Sandia Technical Report SAND02-XXXX (in preparation), Sandia National Laboratories (2002). http://endo.sandia.gov/DAKOTA/papers/Dakota_hardcopy.pdf
64. Eldredge, J.: Numerical simulation of the fluid dynamics of 2D rigid body motion with the vortex particle method. J. Comput. Phys. **221**, 626–648 (2007)
65. Eldredge, J., Leonard, A.: A general deterministic treatment of derivatives in particles methods. J. Comput. Phys. **180**, 686–709 (2002)
66. Ermakov, S., Jacobson, S., Ramsey, J.: Computer simulations of electrokinetic transport in microfabricated channel structures. Anal. Chem. **70**, 4494–4504 (1998)
67. Ermakov, S., Jacobson, S., Ramsey, J.: Computer simulations of electrokinetic injection techniques in microfluidic devices. Anal. Chem. **72**, 3512–3517 (2000)
68. Ermakov, S., Righetti, P.: Computer simulation for capillary zone electrophoresis, a quantitative approach. J. Chromatogr. A **667**, 257–270 (1994)
69. Ermakov, S., Zhukov, M., Capelli, L., Righetti, P.: Artifactual peak splitting in capillary electrophoresis, 2: defocussing phenomena for ampholytes. Anal. Chem. **67**, 2957–2965 (1995)
70. Ern, A., Guermond, J.L.: Theory and Practice of Finite Elements vol. 159. Springer, Berlin (2004)

71. Ernst, O., Powell, C., Silvester, D., Ullmann, E.: Efficient solvers for a linear stochastic Galerkin mixed formulation of diffusion problems with random data. SIAM J. Sci. Comput. **31**(2), 1424–1447 (2009)
72. Nobile, F.R.T., Webster, C.: An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. SIAM J. Numer. Anal. **46**(5), 2411–2442 (2008)
73. Nobile, F.R.T., Webster, C.: A sparse grid stochastic collocation method for partial differential equations with random input data. SIAM J. Numer. Anal. **46**(5), 2309–2345 (2008)
74. Fletcher, C.: Computational Techniques for Fluid Dynamics, vol. 1. Springer, Berlin (1988)
75. Frauenfelder, P., Schwab, C., Todor, R.A.: Finite elements for elliptic problems with stochastic coefficients. Comput. Methods Appl. Mech. Eng. **194**(2–5), 205–228 (2005)
76. Furnival, D., Elman, H., Powell, C.: H(div) preconditioning for a mixte finite element formulation of the stochastic diffusion problem. Technical Report 2008-66, MIMS EPrint (2008)
77. Ganapathysubramanian, B., Zabaras, N.: Sparse grid collocation schemes for stochastic natural convection problems. J. Comput. Phys. **225**, 652–685 (2007)
78. Gardiner, C.: Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences. Springer series in Synergetics. Springer, Berlin (2004)
79. Ge, L., Cheung, K., Kobayashi, M.: Stochastic solution for uncertainty propagation in nonlinear shallow-water equations. J. Hydraul. Eng. **134**, 1732–1743 (2008)
80. Gebhart, B., Jaluria, Y., Mahajan, R., Sammakia, B.: Buoyancy-induced Flows and Transport. Hemisphere, Washington (1988)
81. Gentleman, W.: Implementing the Clenshaw-Curtis quadrature, I: methodology and experience. Commun. ACM **15**(5), 337–342 (1972)
82. Gentleman, W.: Implementing the Clenshaw-Curtis quadrature, II: computing the cosine transform. Commun. ACM **15**(5), 343–346 (1972)
83. Genz, A., Keister, B.: Fully symmetric interpolatory rules for multiple integrals over infinite regions with Gaussian weight. J. Comput. Appl. Math. **71**, 299–309 (1996)
84. Gerstner, T., Griebel, M.: Numerical integration using sparse grids. Numer. Algorithms **18**, 209–232 (1998)
85. Ghanem, R.: Probabilistic characterization of transport in heterogeneous media. Comput. Methods Appl. Mech. Eng. **158**, 199–220 (1998)
86. Ghanem, R.: The nonlinear Gaussian spectrum of lognormal stochastic processes and variables. ASME J. Appl. Mech. **66**, 964–973 (1999)
87. Ghanem, R., Dham, S.: Stochastic finite element analysis for multiphase flow in heterogeneous porous media. Transp. Porous Media **32**, 239–262 (1998)
88. Ghanem, R., Kruger, R.: Numerical solution of spectral stochastic finite element systems. Comput. Methods Appl. Mech. Eng. **129**, 289–303 (1996)
89. Ghanem, R., Spanos, P.: A spectral stochastic finite element formulation for reliability analysis. J. Eng. Mech. ASCE **117**, 2351–2372 (1991)
90. Ghanem, R., Spanos, P.: Stochastic Finite Elements: A Spectral Approach, 2nd edn. Dover, New York (2002)
91. Ghosh, D., Ghanem, R.: Stochastic convergence acceleration through basis enrichment of polynomial chaos expansions. Int. J. Numer. Methods Eng. **73**, 162–184 (2008)
92. Goluh, G., Van Loan, C.: Matrix Computations, 3rd edn. Johns Hopkins University Press, Baltimore (1996)
93. Gresho, P., Sutton, S.: 8:1 thermal cavity problem. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1482–1485. Elsevier, Amsterdam (2001)
94. Grigoriu, M.: Stochastic Calculus, Applications in Sciences and Engineering. Birkhäuser, Basel (2002)
95. Groce, G., Favero, M.: Simulation of natural convection flow in enclosures by an unstaggered grid finite volume algorithm. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1477–1481. Elsevier, Amsterdam (2001)
96. Grudmann, H., Waubke, H.: Non-linear stochastic dynamics of systems with random properties, a spectral approach combined with statistical linearization. Int. J. Non-Linear Mech. **31**(5), 619–630 (1997)

97. Hairer, E., Nørsett, S., Wanner, G.: Solving Ordinary Differential Equations I, Nonstiff Problems, 2nd edn. Springer, Berlin (1992)
98. Halton, J.: On the efficiency of certain quasi-random sequences of points in evaluating multidimensional integrals. Numer. Math. **2**(1), 84–90 (1960)
99. Hardin, R., Sloane, N.: A new approach to the construction of optimal designs. J. Stat. Plann. Inference **37**, 339–369 (1993)
100. Heuveline, V., Rannacher, R.: Duality-base adaptivity in the *hp*-finite element method. J. Numer. Math. **12**(2), 95–113 (2003)
101. Hien, T., Kleiber, M.: Stochastic finite element modeling in linear transient heat transfer. Comput. Methods Appl. Mech. Eng. **144**, 111–124 (1997)
102. Holden, H., Oksendal, B., Uboe, J., Zhang, T.: Stochastic Partial Differential Equations. Birkhäuser, Basel (1996)
103. Hortmann, M., Peric, M., Scheuerer, G.: Finite volume multigrid prediction of laminar natural convection: Benchmark solutions. Int. J. Numer. Methods Fluids **11**, 189–207 (1990)
104. Jacod, J., Protter, P.: Probability Essentials. Universitext/Springer, Berlin (1991)
105. Johnston, H., Krasny, R.: Computational predictability of natural convection flows in enclosures: A benchmark problem. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1486–1489. Elsevier, Amsterdam (2001)
106. Kac, M., Siegert, A.: An explicit representation of a stationary Gaussian process. Ann. Math. Stat. **18**, 438–442 (1947)
107. Kaminski, M., Hien, T.: Stochastic finite element modeling of transient heat transfer in layered composites. Int. Commun. Heat Mass Transfer **26**(6), 801–810 (1999)
108. Karhunen, K.: Uber lineare Methoden in der Wahrscheinlichkeitsrechnung. Am. Acad. Sci., Fennicade, Ser. A, I **37**, 3–79 (1947)
109. Keese, A.: Numerical solution of systems with stochastic uncertainties: a general purpose framework for stochastic finite elements. PhD thesis, Tech. Univ. Braunschweigh (2004)
110. Keese, A., Matthies, H.: Numerical methods and Smolyak quadrature for nonlinear stochastic partial differential equations. Technical report, Institute of Scientific Computing TU Braunschweig Brunswick (2003)
111. Keryszig, E.: Advanced Engineering Mathematics. Wiley, New York (1979)
112. Kim, J., Moin, P.: Application of a fractional-step method to incompressible Navier-Stokes equations. J. Comput. Phys. **59**, 308–323 (1985)
113. Kim, S.E., Choudhury, D.: Numerical investigation of laminar natural convection flow inside a tall cavity using a finite volume based Navier-Stokes solver. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1490–1492. Elsevier, Amsterdam (2001)
114. Knio, O., Ghanem, R.: Polynomial chaos product and moment formulas: A user utility. Technical report, Johns Hopkins University (2001)
115. Knio, O., Le Maître, O.: Uncertainty propagation in CFD using polynomial chaos decomposition. Fluid Dyn. Res. **38**, 616–640 (2006)
116. Knio, O., Najm, H., Wyckoff, P.: A semi-implicit numerical scheme for reacting flow, II: stiff, operator-split formulation. J. Comput. Phys. **154**, 428–467 (1999)
117. Koekoek, R., Swarttouw, R.: The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Technical report 98-17, Department of Technical Mathematics and Informatics, Delft University of Technology (1998). http://fa.its.tudelft.nl/~koekoek/askey
118. Koumoutsakos, P., Leonard, A., Pépin, P.: Boundary conditions for viscous vortex methods. J. Comput. Phys. **113**, 52–61 (1994)
119. Le Maître, O.: A Newton method for the resolution of steady stochastic Navier-Stokes equations. Comput. Fluids (2007, submitted)
120. Le Maître, O.: Polynomial chaos expansion of a Lagrangian model for the flow around an airfoil. C. R. Acad. Sci. **334**(11), 693–698 (2006)
121. Le Maître, O., Knio, O.: A stochastic particle-mesh scheme for uncertainty propagation in vortical flows. J. Comput. Phys. **226**, 645–671 (2007)
122. Le Maître, O., Knio, O., Debusschere, B., Najm, H., Ghanem, R.: A multigrid solver for two-dimensional stochastic diffusion equations. Comput. Methods Appl. Mech. Eng. **192**(41–42), 4723–4744 (2003)

123. Le Maître, O., Knio, O., Najm, H., Ghanem, R.: A stochastic projection method for fluid flow. I. Basic formulation. J. Comput. Phys. **173**, 481–511 (2001)
124. Le Maître, O., Knio, O., Najm, H., Ghanem, R.: Uncertainty propagation using Wiener-Haar expansions. J. Comput. Phys. **197**(1), 28–57 (2004)
125. Le Maître, O., Najm, H., Ghanem, R., Knio, O.: Multi-resolution analysis of Wiener-type uncertainty propagation schemes. J. Comput. Phys. **197**(2), 502–531 (2004)
126. Le Maître, O., Najm, H., Pébay, P., Ghanem, R., Knio, O.: Multi-resolution-analysis scheme for uncertainty quantification in chemical systems. SIAM J. Sci. Comput. **29**(2), 864–889 (2007)
127. Le Maître, O., Reagan, M., Debusschere, B., Najm, H., Ghanem, R., Knio, O.: Natural convection in a closed cavity under stochastic, non-Boussinesq conditions. SIAM J. Sci. Comput. **26**(2), 375–394 (2004)
128. Le Maître, O., Reagan, M., Najm, H., Ghanem, R., Knio, O.: A stochastic projection method for fluid flow. II. Random process. J. Comput. Phys. **181**, 9–44 (2002)
129. Le Maître, O., Scanlan, R., Knio, O.: Numerical estimation of the flutter derivatives of a NACA airfoil by means of Navier-Stokes simulation. J. Fluids Struct. **17**, 1–28 (2003)
130. Le Quéré, P.: Accurate solution to the square thermally driven cavity at high Rayleigh number. Comput. Fluids **20**(1), 29–41 (1991)
131. Le Quéré, P., de Roquefort, T.: Computation of natural convection in two-dimensional cavities with Chebyshev polynomials. J. Comput. Phys. **57**, 210–228 (1985)
132. Le Quéré, P., Masson, R., Perrot, P.: A Chebyshev collocation algorithm for 2d non-Boussinesq convection. J. Comput. Phys. **103**, 320–334 (1992)
133. Leonard, A.: Vortex methods for flow simulations. J. Comput. Phys. **37**, 289–335 (1980)
134. Liu, J.: Monte Carlo Strategies in Scientific Computing. Springer, Berlin (2001)
135. Liu, N., Hu, B., Yu, Z.W.: Stochastic finite element method for random temperature in concrete structures. Int. J. Solids Struct. **38**, 6965–6983 (2001)
136. Loève, M.: Fonctions aléatoires du second ordre. In: Lévy, P. (ed.) Processus Stochastique et Mouvement Brownien. Gauthier Villars, Paris (1948)
137. Loève, M.: Probability Theory. Springer, Berlin (1977)
138. Lumley, J.: Stochastic Tools in Turbulence. Academic Press, New York (1970)
139. Ma, X., Zabaras, N.: An adaptive hierarchical sparse grid collocation algorithm for the solution of stochastic differential equations. J. Comput. Phys. (2009, in press)
140. Majda, A., Sethian, J.: The derivation and numerical solution of the equations for zero Mach number combustion. Comb. Sci. Technol. **42**, 185–205 (1985)
141. Makeev, A., Maroudas, D., Kevrekidis, I.: Coarse stability and bifurcation analysis using stochastic simulators: kinetic Monte Carlo examples. J. Chem. Phys. **116**, 10083–10091 (2002)
142. Mao, Q., Pawliszyn, J., Thormann, W.: Dynamics of capillary isoelectric focusing in the absence of fluid flow: high-resolution computer simulation and experimental validation with whole column optical imaging. Anal. Chem. **72**, 5493–5502 (2000)
143. Marzouk, Y., Najm, H.: Dimensionality reduction and polynomial chaos acceleration of Bayesian inference in inverse problems. J. Comput. Phys. **228**, 1862–1902 (2009)
144. Marzouk, Y., Najm, H., Rahn, L.: Stochastic spectral methods for efficient Bayesian solution of inverse problems. J. Comput. Phys. **224**, 560–586 (2007)
145. Mathelin, L., Hussaini, M.: A stochastic collocation algorithm for uncertainty analysis. Technical Report NASA/CR-2003-212153, NASA Langley Research Center (2003)
146. Mathelin, L., Hussaini, M., Zang, T.: Stochastic approaches to uncertainty quantification in CFD simulations. Numer. Algorithms **38**(1), 209–236 (2005)
147. Mathelin, L., Le Maître, O.: Dual based a posteriori error estimation for stochastic finite element methods. Commun. Appl. Math. Comput. Sci. **2**(1), 83–115 (2007)
148. Mathelin, L., Zang, T., Bataille, F.: Uncertainty propagation for turbulent, compressible nozzle flow using stochastic methods. AIAA J. **42**(8), 1669–1676 (2004)
149. Matthies, H., Brenner, C., Bucher, C., Soares, C.: Uncertainties in probabilistic numerical analysis of structures and solids—stochastic finite-elements. Struct. Saf. **19**(3), 283–336 (1997)

150. Matthies, H., Bucher, C.: Finite element for stochastic media problems. Comput. Methods Appl. Mech. Eng. **168**, 3–17 (1999)
151. Matthies, H.G., Keese, A.: Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. Comput. Methods Appl. Mech. Eng. **194**(12–16), 1295–1331 (2005)
152. McGuffin, V., Tavares, M.: Computer assisted optimization of separations in capillary zone electrophoresis. Anal. Chem. **69**, 152–164 (1997)
153. McKay, M., Conover, W., Beckman, R.: A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. Technometrics **21**, 239–245 (1979)
154. Meecham, W., Jeng, D.: Use of the Wiener-Hermite expansion for nearly normal turbulence. J. Fluid Mech. **32**, 225 (1968)
155. Micheletti, S., Perotto, S., Picasso, M.: Stabilized finite elements on anisotropic meshes: a priori error estimates for the advection-diffusion and the stokes problems. SIAM J. Numer. Anal. **41**(3), 1131–1162 (2003)
156. de Montigny, P., Stobaugh, J., Givens, R., Carlson, R., Srinivasachar, K., Sternson, L., Higuchi, T.: Naphtalene-2,3-dicarboxaldehyde/cyanide ion: a rationally designed fluorogenic reagent for primary amines. Anal. Chem. **59**, 1096–1101 (1987)
157. Moré, J.J., Garbow, B.S., Hillstrom, K.E.: User guide for MINPACK-1. Technical Report ANL-80-74, Argone National Lab (1980)
158. Morokoff, W., Caflisch, R.: Quasi-Monte Carlo integration. J. Comput. Phys. **122**(2), 218–230 (1995)
159. Mosher, R., Dewey, D., Thormann, W., Saville, D., Bier, M.: Computer simulation and experimental validation of the electrophoretic behavior of proteins. Anal. Chem. **61**, 362–366 (1989)
160. Mosher, R., Gebauer, P., Thormann, W.: Computer simulation and experimental validation of the electrophoretic behavior of proteins. III. use of titration data predicted by the protein's amino acid composition. J. Chromatogr. **638**, 155–164 (1993)
161. Nair, P., Keane, A.: Stochastic reduced basis methods. AIAA J. **40**, 1653–1664 (2002)
162. Najm, H.: A conservative low Mach number projection method for reacting flow modeling. In: Chan, S. (ed.) Transport Phenomena in Combustion, vol. 2, pp. 921–932. Taylor and Francis, Washington (1996)
163. Najm, H.: Uncertainty quantification and polynomial chaos techniques in computational fluid dynamics. Annu. Rev. Fluid Mech. **41**, 35–52 (2009)
164. Najm, H., Wyckoff, P., Knio, O.: A semi-implicit numerical scheme for reacting flow, I: stiff chemistry. J. Comput. Phys. **143**(2), 381–402 (1998)
165. Nicola, B., Verlinden, B., Beuselinck, A., Jancsok, P., Quenon, V., Scheerlinck, N., Verbosen, P., de Baerdemaeker, J.: Propagation of stochastic temperature fluctuations in refrigerated fruits. Int. J. Refrig. **222**, 81–90 (1999)
166. Nobile, F., Tempone, R.: Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients. Technical Report 22/2008, MOX (2008)
167. Nouy, A.: Generalized spectral decomposition method for solving stochastic finite element equations: invariant subspace problem and dedicated algorithms. Comput. Methods Appl. Mech. Eng. (2007, submitted)
168. Nouy, A.: A generalized spectral decomposition technique to solve a class of linear stochastic partial differential equations. Comput. Methods Appl. Methods Eng. **196**(45–48), 4521–4537 (2007)
169. Nouy, A.: Méthode de construction de bases spectrales généralisées pour l'approximation de problèmes stochastiques. Méc. Ind. **8**(3), 283–288 (2007)
170. Nouy, A., Clément, A., Schoefs, F., Moës, N.: An extended stochastic finite element method for solving stochastic partial differential equations on random domains. Comput. Methods Appl. Mech. Eng. (2007, submitted)
171. Nouy, A., Le Maître, O.: Generalized spectral decomposition for stochastic nonlinear problems. J. Comput. Phys. **228**, 202–235 (2009)

172. Nouy, A., Schoefs, F., Moës, N.: X-SFEM, a computational technique based on X-FEM to deal with random shapes. Eur. J. Comput. Mech. **16**, 277–293 (2007)
173. Novak, E., Ritter, K.: Simple cubature formulas with high polynomial exactness. Constr. Approx. **15**, 499–522 (1999)
174. Ogden, R.T.: Essentials Wavelets for Statistical Applications and Data Analysis. Birkhäuser, Basel (1997)
175. Oksendal, B.: Stochastic Differential Equations. An Introduction with Applications, 5th edn. Springer, Berlin (1998)
176. Paillere, H., Le Quéré, P.: Modelling and simulation of natural convection flows with large temperature differences: a benchmark problem for low Mach number solvers. In: 12th Seminar "Computational Fluid Dynamics" CEA/Nuclear Reactor Division (2000)
177. Paillere, H., Viozat, C., Kumbaro, A., Toumi, I.: Comparison of low Mach number models for natural convection problems. Heat Mass Transfer **36**, 567–573 (2000)
178. Palusinski, O., Graham, A., Mosher, R., Bier, M., Saville, D.: Theory of electrophoretic separations, part II: construction of a numerical simulation scheme and its applications. AIChE J. **32**(2), 215–223 (1986)
179. Pan, T.W., Glowinski, R.: A projection/wave-like equation method for natural convection flows in enclosures. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1493–1496. Elsevier, Amsterdam (2001)
180. Panton, R.: Incompressible Flow. Wiley, New York (1984)
181. Papoulis, A., Pillai, S.: Probability, Random Variables and Stochastic Processes. McGraw-Hill, New York (2002)
182. Patankar, N., Hu, H.: Numerical simulation of electroosmotic flow. Anal. Chem. **70**, 1870–1881 (1998)
183. Pellissetti, M., Ghanem, R.: Iterative solution of systems of linear equations arising in the context of stochastic finite elements. Adv. Eng. Softw. **31**, 607–616 (2000)
184. Peraire, J., Peiró, J., Morgan, K.: Adaptive remeshing for three-dimensional compressible flow computations. J. Comput. Phys. **103**(2), 269–285 (1992)
185. Pernice, M., Walker, H.: NITSOL: A Newton iterative solver for nonlinear systems. SIAM J. Sci. Comput. **19**(1), 302–318 (1998)
186. Petras, K.: On the Smolyak cubature error for analytic functions. Adv. Comput. Math. **12**, 71–93 (2000)
187. Petras, K.: Fast calculation in the Smolyak algorithm. Numer. Algorithms **26**, 93–109 (2001)
188. Peyret, R.: Spectral Methods for Incompressible Viscous Flow. Springer, Berlin (2002)
189. Poëtte, G., Després, B., Lucor, D.: Uncertainty quantification for systems of conservation laws. J. Comput. Phys. **228**, 2443–2467 (2009)
190. Powell, C.: Robust preconditioning for second-order elliptic PDEs with random field coefficients. Technical Report 2006.419, MIMS Eprint (2006)
191. Powell, C., Elman, H.: Block diagonal preconditioning for spectral stochastic finite element systems. IMA J. Numer. Anal. **29**(2), 350–375 (2009)
192. Probstein, R.: Physicochemical Hydrodynamics: An Introduction, 2nd edn. Wiley, New York (1994)
193. Pulkelsheim, F.: Optimal Design of Experiments. Classics in Applied Mathematics, vol. 50. SIAM, Philadelphia (2006)
194. Raviart, P.: An Analysis of Particles Methods. Lecture Notes in Math., vol. 1127, pp. 243–324. Springer, Berlin (1985)
195. Reagan, M., Najm, H., Debusschere, B., Maître, O.L., Knio, O., Ghanem, R.: Spectral stochastic uncertainty quantification in chemical systems. Combust. Theory Model. **8**, 607–632 (2004)
196. Rivoalen, E., Huberson, S.: Numerical simulation of axisymmetric flows by means of a particle method. J. Comput. Phys. **152**, 1–31 (1999)
197. Roberts, G., Rhodes, P., Snyder, R.: Dispersion effects in capillary zone electrophoresis. J. Chromatogr. **480**, 35–67 (1989)
198. Rosenhead, L.: The formation of vortices from a surface of discontinuity. Proc. R. Soc. **134** (1931)

199. Saad, Y.: Iterative Methods for Sparse Linear Systems, 2nd edn. SIAM, Philadelphia (2003)
200. Saad, Y., Schultz, M.: GMRES: a generalized minimal residual algorithm for solving non-symmetric linear system. J. Sci. Stat. Comput. **7**, 856–869 (1986)
201. Salinger, A., Lehoucq, R., Pawlowski, R., Shadid, J.: Understanding the 8:1 cavity problem via scalable stability analysis algorithms. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1497–1500. Elsevier, Amsterdam (2001)
202. Scales, P., Grieser, F., Healy, T., White, L., Chan, D.: Electrokinetics of the silica-solution interface: a flat plate streaming potential study. Langmuir **8**, 965–974 (1992)
203. Schoutens, W.: Stochastic Processes and Orthogonal Polynomials. Springer, Berlin (2000)
204. Schuëller, G.: Computational stochastic mechanics—recent advances. Comput. Struct. **79**, 2225–2234 (2001)
205. Shafer-Nielsen, C.: A computer model for time-based simulation of electrophoresis systems with freely defined initial and boundary conditions. Electrophoresis **16**, 1369–1376 (1995)
206. Shen, W.Z., Loc, T.P.: Simulation of 2d external viscous flow by means of a domain decomposition method using an influence matrix technique. Int. J. Numer. Methods Fluids **220**, 1111 (1995)
207. Sleijpen, G., Fokkema, D.: BICGSTAB(L) for linear equations involving unsymmetric matrices with complex spectrum. Electon. Trans. Numer. Anal. **1**, 11–32 (1993)
208. Slepian, D., Pollak, H.: Prolate spheroidal wave functions; Fourier analysis and uncertainty, I. Bell Syst. Tech. J., 3–63 (1961)
209. Slepian, D., Pollak, H.: Prolate spheroidal wave functions; Fourier analysis and uncertainty, IV: extension to many dimensions; generalizes prolate spheroidal functions. Bell Syst. Tech. J., 009–3057 (1964)
210. Sluzalec, A.: Random heat flow with phase change. Int. J. Heat Mass Transfer **43**, 2303–2312 (2000)
211. Sluzalec, A.: Stochastic finite element analysis of two-dimensional eddy current problems. Appl. Math. Model. **24**, 401–406 (2000)
212. Smolyak, S.: Quadrature and interpolation formulas for tensor products of certain classes of functions. Dokl. Akad. Nauk SSSR **4**, 240–243 (1963)
213. Sobol, I.: Uniformly distributed sequences with an additional uniform property. USSR Comput. Math. Math. Phys. **16**, 236–242 (1976)
214. Soize, C.: The Fokker-Planck Equation for Stochastic Dynamical Systems ans Its Explicit Steady Solutions. World Scientific, Singapore (1994)
215. Soize, C.: A comprehensive overview of a non-parametric probabilistic approach of model uncertainty for predictive models in structural dynamics. J. Sound Vib. **288**, 623–652 (2005)
216. Soize, C.: Random matrix theory for modeling uncertainty in computational mechanics. Comput. Methods Appl. Mech. Eng. **194**(12–16), 1333–1366 (2005)
217. Soize, C.: Construction of probability distributions in high dimension using the maximum entropy principle: Applications to stochastic processes, random fields and random matrices. Int. J. Numer. Methods Eng. **76**(10), 1583–1611 (2008)
218. Soize, C., Ghanem, R.: Physical systems with random uncertainties: chaos representations with arbitrary probability measure. J. Sci. Comput. **26**(2), 395–410 (2004)
219. Soize, C., Ghanem, R.: Reduced chaos decomposition with random coefficients of vector-valued random variables and random fields. Comput. Methods Appl. Mech. Eng. **198**(21–26), 1926–1934 (2009)
220. Stoer, J., Bulirsch, R.: Introduction to Numerical Analysis, 2nd edn. Texts in Applied Mathematics, vol. 12. Springer, Berlin (1993)
221. Strang, G.: Introduction to Applied Mathematics. Wellesley-Cambridge Press, Wellesley (1986)
222. Stroud, A., Secrest, D.: Gaussian Quadrature Formula. Prentice-Hall, New York (1966)
223. Suslov, S., Paolucci, S.: A Petrov-Galerkin method for flows in cavities. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1501–1504. Elsevier, Amsterdam (2001)
224. Theting, T.: Solving Wick-stochastic boundary value problems using a finite element method. Stoch. Stoch. Rep. **70**, 241–270 (2000)

225. Thormann, W., Mosher, R.: Theory of electrophoretic transport and separations: the study of electrophoretic boundaries and fundamental separation principles by computer simulation. In: Chrambach, A., Dunn, M., Radola, B. (eds.) Advances in Electrophoresis, vol. 2, pp. 45–108. VCH, Weinheim (1988)
226. Thormann, W., Zhang, C.X., Caslavska, J., Gebauer, P., Mosher, R.: Modeling of the impact of ionic strength on the electroosmotic flow in capillary electrophoresis with uniform and discontinuous buffer systems. Anal. Chem. **70**, 549–562 (1998)
227. Trefethen, L.: Is Gauss quadrature better than Clenshaw-Curtis? SIAM Rev. **50**(1), 67–87 (2008)
228. Trottenberg, U., Oosterlee, C., Schüller, A.: Multigrid. Academic Press, New York (2001)
229. Tryoen, J.: Méthode intrusive de modélisation d'incertitudes dans les systèmes hyperboliques de lois de conservation. MS thesis, Université Paris VI (2008)
230. Tuckerman, L.: Steady-state solving via stokes preconditioning; recursion relations of elliptic operators. In: 11th Int. Conf. on Numer. Meth. in Fluid Dynamics. Springer, Berlin (1989)
231. Tuckerman, L., Huepe, C., Brachet, M.: Numerical methods for bifurcation problems. In: Descalzi, O., Martinez, J., Rica, S. (eds.) Instabilities and Non-equilibrium Structures IX, pp. 75–86. Kluwer Academic, Dordrecht (2004)
232. Vanel, J., Peyret, R., Bontoux, P.: A pseudo-spectral solution of vorticity-stream-function equations using the influence matrix technique. Numer. Methods Fluid Dyn. **II**, 463–475 (1986)
233. Waldvogel, J.: Fast construction of the Fejér and Clenshaw-Curtis quadrature rules. BIT Numer. Math. **43**(1), 001–018 (2003)
234. Walnut, D.F.: An Introduction to Wavelets Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser, Basel (2002)
235. Walter, G.: Wavelets and Other Orthogonal Systems with Applications. CRC Press, Boca Raton (1994)
236. Walters, R., Huyse, L.: Uncertainty analysis for fluid mechanics with applications. Technical report, NASA CR-2002-211449 (2002)
237. Wan, Q.H.: Effect of electroosmotic flow on the electrical conductivity of packed capillary columns. J. Phys. Chem. B **101**(24), 4860–4862 (1997)
238. Wan, X., Karniadakis, G.: An adaptive multi-element generalized polynomial chaos method for stochastic differential equations. J. Comput. Phys. **209**, 617–642 (2005)
239. Webster, C.: Sparse grid collocation techniques for the numerical solution of partial differential equations with random input data. PhD thesis, Florida State University (2007)
240. Westerberg, K.: Thermally driven flow in a cavity using the Galerkin finite element method. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1505–1508. Elsevier, Amsterdam (2001)
241. Wiener, S.: The homogeneous chaos. Am. J. Math. **60**, 897–936 (1938)
242. Wojtkiewicz, S., Eldred, M., Field, R., Urbina, A., Red-Horse, J.: A toolkit for uncertainty quantification in large computational engineering models. In: Proceedings of the 42nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, AIAA Paper 2001-1455 (2001)
243. Xin, S., Le Quéré, P.: An extended Chebyshev pseudo-spectral contribution to cpncfe benchmark. In: Bathe, K. (ed.) Computational Fluid and Solid Mechanics, pp. 1509–1513. Elsevier, Amsterdam (2001)
244. Xiu, D., Hesthaven, J.: High-order collocation methods for differential equations with random inputs. SIAM J. Sci. Comput. **27**(3), 1118–1139 (2005)
245. Xiu, D., Karniadakis, G.: Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos. Comput. Methods Appl. Mech. Eng. **191**, 4927–4948 (2002)
246. Xiu, D., Karniadakis, G.: The Wiener-Askey polynomial chaos for stochastic differential equations. SIAM J. Sci. Comput. **24**, 619–644 (2002)
247. Xiu, D., Karniadakis, G.: Modeling uncertainty in flow simulations via generalized polynomial chaos. J. Comput. Phys. **187**, 137–167 (2003)
248. Xiu, D., Karniadakis, G.: Supersensitivity due to uncertain boundary conditions. Int. J. Numer. Methods Eng. **61**(12), 2114–2138 (2004)

249. Xiu, D., Lucor, D., Su, C., Karniadakis, G.: Stochastic modeling of flow structure interactions using generalized polynomial chaos. J. Fluids Eng. **124**, 51–59 (2002)
250. Xiu, D., Shen, J.: Efficient stochastic Galerkin methods for random diffusion equations. J. Comput. Phys. **228**, 266–281 (2009)
251. Youla, D.: The solution of the homogeneous Wiener-Hopf integral equation occurring in the expansion of second-order stationary random functions. IRE Tans. Inf. Theory, 18–193 (1957)

Index

O.P. Le Maître, O.M. Knio, *Spectral Methods for Uncertainty Quantification,* Scientific Computation,
DOI 10.1007/978-90-481-3520-2, © Springer Science+Business Media B.V. 2010