Appendix A

Maximal Rings of Quotients

Every commutative domain can be embedded into a field, namely, into its field of fractions. A vast number of more general constructions are known in ring theory. Incidentally, not everything is so simple in the noncommutative context; for example, not every domain can be embedded into a division ring (see e.g., [133]). Thus, a simple minded attempt to take formal inverses of elements may not always work, and more sophisticated approaches are necessary.

It is not our intention to treat the general theory of rings of quotients. We shall confine ourselves to maximal rings of quotients, and even this only for the case when the original rings are semiprime. The main reason for considering these rings of quotients is that they are simply most suitable for our purposes (cf. Sections 5.2 and 5.3). In principle they do have one disadvantage, namely, as their name already suggests, they may be very "big", much bigger than the original rings, and so they do not always reflect well their structure. There are other well-known rings of quotients, which are smaller than the maximal ones (in the literature these rings are often called Martindale rings of quotients, while in [40] the terms symmetric and two-sided rings of quotients are used). However, dealing with any of them in the FI context would lead to serious technical problems. Versions of Corollary A.5 below do not hold for them, and this is basically what causes the main problem. Anyway, concerning concrete applications of FI's to prime (and semiprime) rings (at least those that are known so far), maximal rings of quotients are as good as any others would be. Namely, the unknown functions in FI's arriving from concrete problems as a rule turn out to be quasi-polynomials. But then only the center of the bigger ring matters. And it is a fact that all these rings of quotients have the same center, called the extended centroid of the original ring.

Maximal rings of quotients and extended centroids are studied in many books, for instance in [40, 133, 134, 197], to mention just a few. In our exposition we shall mostly follow [40]. We will present the results in a rigorous fashion, while their proofs will be mostly just outlined, pointing out the main ideas and neglecting technicalities.

Assume from now on that $\mathcal{A}$ *is a semiprime ring.* This assumption is not needed in everything that follows, but for simplicity we restrict out attention to the situation in which we are really interested. A left ideal $\mathcal{L}$ of $\mathcal{A}$ is said to be *dense* if given $a_1, a_2 \in \mathcal{A}$ with $a_1 \neq 0$, there exists $a \in \mathcal{A}$ such that $aa_1 \neq 0$ and $aa_2 \in \mathcal{L}$. If $\mathcal{I}$ is a two-sided ideal, then it is easy to see that $\mathcal{I}$ is dense (as a left ideal) if and only if $\mathcal{I}b \neq 0$ for every nonzero $b \in \mathcal{A}$, which is further equivalent to $b\mathcal{I} \neq 0$ for every nonzero $b \in \mathcal{A}$. Furthermore, such ideals are exactly the *essential* ideals, that is ideals having nonzero intersections with all nonzero ideals. If $\mathcal{A}$ is prime, then every nonzero ideal is essential.

Assume for a moment that $\mathcal{A}$ is a commutative domain and $\mathcal{Q}$ is its field of fractions. Pick $q \in \mathcal{Q}$. Then $q = ba^{-1}$ for some $a, b \in \mathcal{A}$ with $a \neq 0$. Let $\mathcal{L}$ be any nonzero ideal of $\mathcal{A}$ contained in $a\mathcal{A}$. Note that $f(x) = xq$ defines an $\mathcal{A}$-module homomorphism from $\mathcal{L}$ into $\mathcal{A}$. Conversely, every $\mathcal{A}$-module homomorphism g from a nonzero ideal $\mathcal{I}$ of $\mathcal{A}$ into $\mathcal{A}$ is of such a form. Indeed, for all $x, y \in \mathcal{I}$ we have $g(x)y = g(xy) = xg(y)$, and so fixing a nonzero y it follows that $g(x) = xr$ where $r = g(y)y^{-1} \in \mathcal{Q}$.

We now return to an arbitrary semiprime ring $\mathcal{A}$. The only aim of the previous paragraph was to help the reader to understand the ideas hidden behind the construction that follows. Let us now consider the set of all pairs $(f; \mathcal{L})$, where $\mathcal{L}$ is a dense left ideal and $f : \mathcal{L} \to \mathcal{A}$ is a left $\mathcal{A}$-module homomorphism. We define $(f; \mathcal{L}) \sim (g; \mathcal{M})$ if f and g coincide on some dense left ideal contained in $\mathcal{L} \cap \mathcal{M}$. It is easy to see that $\sim$ is an equivalence relation. By $[f; \mathcal{L}]$ we denote the equivalence class determined by $(f; \mathcal{L})$. We define the addition and multiplication of equivalence classes as follows:

$$[f; \mathcal{L}] + [g; \mathcal{M}] = [f + g; \mathcal{L} \cap \mathcal{M}],$$
$$[f; \mathcal{L}][g; \mathcal{M}] = [gf; f^{-1}(\mathcal{M})].$$

So basically the sum of equivalence classes corresponds to the sum of homomorphisms, and the product to their composition. One just has to take care about domains so that everything makes sense. Let us point out that $\mathcal{L} \cap \mathcal{M}$ and $f^{-1}(\mathcal{M})$ (the preimage of $\mathcal{M}$), are indeed dense left ideals, as can be easily checked. One can also check that both operations are well-defined, and that the set of all equivalence classes becomes a ring under these operations. All these require some work, but it is elementary and easy. One can embed $\mathcal{A}$ into this ring via $a \mapsto [R_a; \mathcal{A}]$ where R_a is the right multiplication by $a \in \mathcal{A}$, i.e., $R_a(x) = xa$. Identifying each a with $[R_a; \mathcal{A}]$ we thus have $a[f; \mathcal{L}] = f(a)$ for every $a \in \mathcal{L}$. Using this one can easily show that the ring that we constructed has the properties given in the next theorem.

Theorem A.1. *Let $\mathcal{A}$ be a semiprime ring. Then there exists a ring $\mathcal{Q}_{ml}(\mathcal{A})$ satisfying the following conditions:*

(i) *$\mathcal{A}$ is a subring of $\mathcal{Q}_{ml}(\mathcal{A})$;*

(ii) *For every $q \in \mathcal{Q}_{ml}(\mathcal{A})$ there exists a dense left ideal $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{L}q \subseteq \mathcal{A}$;*

(iii) *If* $0 \neq q \in \mathcal{Q}_{ml}(\mathcal{A})$*, then* $\mathcal{L}q \neq 0$ *for every dense left ideal* $\mathcal{L}$ *of* $\mathcal{A}$*;*

(iv) *If* $\mathcal{L}$ *is a dense left ideal of* $\mathcal{A}$ *and* $f : \mathcal{L} \to \mathcal{A}$ *is a left* $\mathcal{A}$*-module homomorphism, then there exists* $q \in \mathcal{Q}_{ml}(\mathcal{A})$ *such that* $f(x) = xq$ *for all* $x \in \mathcal{L}$*.*

Moreover, the properties (i)–(iv) *characterize* $\mathcal{Q}_{ml}(\mathcal{A})$ *up to an isomorphism.*

The last assertion can also be easily established. Indeed, let $\mathcal{Q}$ be a ring satisfying (i)–(iv). Given $q \in \mathcal{Q}$, by assumption there exists a dense left ideal $\mathcal{L}$ such that $\mathcal{L}q \subseteq \mathcal{A}$. One can check that the map $q \mapsto [R_q; \mathcal{L}]$ is a ring isomorphism from $\mathcal{Q}$ onto $\mathcal{Q}_{ml}(\mathcal{A})$.

The ring $\mathcal{Q}_{ml}(\mathcal{A})$ is called the *maximal left ring of quotients* of $\mathcal{A}$. These rings first appeared in the work by Utumi [193], and in the literature they are sometimes also called Utumi left rings of quotients.

One can similarly introduce and study maximal *right* rings of quotients. We have chosen to deal with the left ones by chance. After all, results on FI's are in principle left-right symmetric.

Let us mention just a couple of concrete examples, in order to give some evidence that the concept of $\mathcal{Q}_{ml}(\mathcal{A})$ is a natural one. If $\mathcal{A}$ is a semiprime left Goldie ring, then $\mathcal{Q}_{ml}(\mathcal{A})$ is just the classical left ring of quotients of $\mathcal{A}$. So, for instance, $\mathcal{Q}_{ml}(M_n(\mathbb{Z})) = M_n(\mathbb{Q})$. Next, let $\mathcal{A}$ be a primitive ring containing an idempotent $e \in \mathcal{A}$ such that $\mathcal{D} = e\mathcal{A}e$ is a division ring (more details about such rings can be found at the end of this appendix and in appendix D). Then $\mathcal{Q}_{ml}(\mathcal{A}) = \mathrm{End}_{\mathcal{D}}(e\mathcal{A})$. For more examples we refer to the aforementioned books; especially [133] has plenty of them.

As already mentioned, the intersection of two, and hence also of finitely many dense left ideals is again a dense left ideal. Therefore (ii) can be strengthened as follows.

Corollary A.2. *For any* $q_1, \ldots, q_n \in \mathcal{Q}_{ml}(\mathcal{A})$ *there exists a dense left ideal* $\mathcal{L}$ *of* $\mathcal{A}$ *such that* $\mathcal{L}q_i \subseteq \mathcal{A}$ *for every* i*.*

The next lemma is a very special case of the general theory (cf. [40, Section 6.4]). But as this lemma is all we need, we shall give a simple direct proof. Let us first mention that $\mathcal{Q}_{ml}(\mathcal{A})$ is again a semiprime ring, and moreover it is prime in case $\mathcal{A}$ is prime. This can be easily checked.

Lemma A.3. *Let* $a, b \in \mathcal{Q} = \mathcal{Q}_{ml}(\mathcal{A})$*, and let* $\mathcal{I}$ *be an essential ideal of* $\mathcal{A}$*. If* $a\mathcal{I}b = 0$*, then* $a\mathcal{Q}b = b\mathcal{Q}a = 0$*.*

Indeed, from $a\mathcal{I}b = 0$ it follows that $(\mathcal{I}b\mathcal{Q}a)^2 = 0$. Thus $\mathcal{J} = \mathcal{I}b\mathcal{Q}a \cap \mathcal{A}$ is a left ideal of $\mathcal{A}$ such that $\mathcal{J}^2 = 0$. Since $\mathcal{A}$ is semiprime, $\mathcal{J} = 0$. If $b\mathcal{Q}a \neq 0$ pick $q \in \mathcal{Q}$ such that $bqa \neq 0$. Then there exists $r \in \mathcal{A}$ such that $0 \neq rbqa \in \mathcal{A}$. Since $\mathcal{I}$ is essential in $\mathcal{A}$ we arrive at the contradiction $0 \neq \mathcal{I}rbqa \subseteq \mathcal{J}$. Thus $b\mathcal{Q}a = 0$. Accordingly, $(a\mathcal{Q}b)\mathcal{Q}(a\mathcal{Q}b) = 0$, forcing $a\mathcal{Q}b = 0$ since $\mathcal{Q}$ is semiprime.

Theorem A.4. *Let* $\mathcal{I}$ *be an essential ideal of* $\mathcal{A}$ *and let* $\mathcal{B}$ *be any ring such that* $\mathcal{I} \subseteq \mathcal{B} \subseteq \mathcal{Q}_{ml}(\mathcal{A})$*. Then* $\mathcal{Q}_{ml}(\mathcal{B}) = \mathcal{Q}_{ml}(\mathcal{A})$*.*

One should first note that $\mathcal{B}$ is semiprime. Now, to prove Theorem A.4 it is enough to show that $\mathcal{Q}_{ml}(\mathcal{A})$ satisfies the properties (i)–(iv) of Theorem A.1 (in which we take $\mathcal{B}$ to play the role of $\mathcal{A}$). Since $\mathcal{B}$ is a subring of $\mathcal{Q}_{ml}(\mathcal{A})$, we get (i) for free. Proving the other three properties is not so trivial, but still elementary. We omit details.

A particular case of Theorem A.4 is of special importance.

Corollary A.5. $\mathcal{Q}_{ml}(\mathcal{Q}_{ml}(\mathcal{A})) = \mathcal{Q}_{ml}(\mathcal{A})$.

The center of $\mathcal{Q}_{ml}(\mathcal{A})$ is called the *extended centroid* of $\mathcal{A}$. This term was introduced in the prime ring context by Martindale who also discovered the basic properties and the usefulness of the extended centroid in the study of Lie homomorphisms [151] and generalized polynomial identities [152]. Somewhat later Amitsur considered the extended centroid of semiprime rings [3].

The extended centroid of $\mathcal{A}$ will be denoted by $\mathcal{C}$. In terms of the construction of $\mathcal{Q}_{ml}(\mathcal{A})$ given earlier, it is easy to see that $\mathcal{C}$ is characterized as the set of all equivalence classes $\lambda = [f; \mathcal{L}]$ where $\mathcal{L}$ is an essential ideal of $\mathcal{A}$ and $f : \mathcal{L} \to \mathcal{A}$ is an $(\mathcal{A}, \mathcal{A})$-bimodule map (thus $\lambda\mathcal{L} \subseteq \mathcal{A}$). It can be shown that $\mathcal{C}$ is a von Neumann regular ring, i.e., for every $\lambda \in \mathcal{C}$ there exists $\mu \in \mathcal{C}$ such that $\lambda^2\mu = \lambda$. The *centroid* Ω of $\mathcal{A}$ is the subring of $\mathcal{C}$ consisting of all equivalence classes of the form $[f; \mathcal{A}]$. The center $\mathcal{Z}$ of $\mathcal{A}$ is embeddable in Ω, so we have $\mathcal{Z} \subseteq \Omega \subseteq \mathcal{C}$. In case $\mathcal{A}$ is unital, $\mathcal{Z}$ is isomorphic to Ω (in which case there is no need for the notion of the centroid). One can check that the centralizer of $\mathcal{A}$ in $\mathcal{Q}$ is just $\mathcal{C}$; moreover, the same is true for the centralizer of every essential ideal of $\mathcal{A}$ in $\mathcal{Q}$.

The $\mathcal{C}$-subalgebra of $\mathcal{Q}_{ml}(\mathcal{A})$ generated by $\mathcal{A}$ is called the *central closure* of $\mathcal{A}$. It will be denoted by $\mathcal{AC}$. Thus a typical element in $\mathcal{AC}$ is of the form $\sum_i \lambda_i a_i$ with $\lambda_i \in \mathcal{C}$ and $a_i \in \mathcal{A}$. We say that $\mathcal{A}$ is a *centrally closed ring* if it is equal to its own central closure. A centrally closed ring is not necessarily unital. For a unital ring, saying that it is centrally closed is the same as saying that its extended centroid coincides with its center. A centrally closed ring is clearly an algebra over the extended centroid. By a *centrally closed algebra over* $\mathcal{C}$ we shall mean an algebra over a commutative ring $\mathcal{C}$ such that its extended centroid is $\mathcal{C}$.

It is not difficult to show that the central closure is a centrally closed semiprime (and prime if $\mathcal{A}$ is prime) ring. Simple rings are always centrally closed. So, a unital simple ring is a centrally closed algebra over its center. However, one usually refers to these algebras as *central simple algebras* (recall that an algebra over a commutative ring $\mathcal{C}$ is said to be central if $\mathcal{C}$ is its center).

FI's have turned out to be useful in solving some problems in algebras that appear in functional analysis. But we did not consider these topics, in order to avoid making the book too diverse. Let us now make a short digression. If one takes, for example, a semiprime Banach algebra, then its extended centroid of course exists, but it may not have any reasonable topological properties and so it is just a"creature from another planet", apparently useless for the category of Banach algebras. If, however, we restrict ourselves to some special classes of algebras, then this is no longer the case. Let us mention just two nice examples:

primitive (complex) Banach algebras and prime C^*-algebras are centrally closed algebras over $\mathbb{C}$. Therefore, all results that involve prime rings and their extended centroids are directly applicable to these algebras.

The extended centroid plays a particularly important role in prime rings. Here is one of the main reasons:

Theorem A.6. *$\mathcal{C}$ is a field if and only if $\mathcal{A}$ is prime.*

Let us sketch the proof. First suppose $\mathcal{A}$ is prime. Let $0 \neq \lambda \in \mathcal{C}$ and let $\mathcal{I}$ be a nonzero ideal of $\mathcal{A}$ such that $\lambda\mathcal{I} \subseteq \mathcal{A}$. Then the inverse of λ is determined by the map $f : \lambda\mathcal{I} \to \mathcal{A}$ given by $f(\lambda x) = x$ (well-definedness follows from λ not being a zero divisor). Conversely, suppose $\mathcal{A}$ is not prime, and accordingly let $\mathcal{I} \neq 0$ be a non-essential ideal of $\mathcal{A}$. Then $\mathcal{J} = \{x \in \mathcal{A} \,|\, x\mathcal{I} = 0\}$ is a nonzero ideal of $\mathcal{A}$ and $\mathcal{K} = \mathcal{I} \oplus \mathcal{J}$ is an essential ideal of $\mathcal{A}$. Define $f, g : \mathcal{K} \to \mathcal{A}$ respectively by $f(x + y) = x$ and $g(x + y) = y$. Then $[f; \mathcal{K}]$ and $[g; \mathcal{K}]$ are nonzero orthogonal idempotents in $\mathcal{C}$, whence $\mathcal{C}$ cannot be a field.

The centroid Ω of a prime ring $\mathcal{A}$ is a commutative unital domain containing the center $\mathcal{Z}$ (note that $\mathcal{Z}$ could well be 0). In general $\mathcal{C}$ need not be the field of fractions of Ω (or of $\mathcal{Z}$), even if Ω (or $\mathcal{Z}$) should be a field itself (cf. Examples 5.29 and 6.10). However, in some cases $\mathcal{C}$ is the field of fractions of $\mathcal{Z}$; e.g., if $\mathcal{A}$ is a prime PI-ring then $\mathcal{Z} \neq 0$ and $\mathcal{C}$ is the field of fractions of $\mathcal{Z}$.

Until further notice $\mathcal{A}$ will be a prime ring. Suppose that $0 \neq a, b \in \mathcal{Q}_{ml}(\mathcal{A})$ are such that $axb = bxa$ for all $x \in \mathcal{A}$. We claim that then a and b are linearly dependent over $\mathcal{C}$, i.e., $b = \lambda a$ for some $\lambda \in \mathcal{C}$. Indeed, pick a dense left ideal $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{L}a \subseteq \mathcal{A}$. Then $\mathcal{I} = \mathcal{L}a\mathcal{A}$ is a nonzero (and hence automatically essential) ideal of $\mathcal{A}$. Define $f : \mathcal{I} \to \mathcal{A}$ by $f(\sum_i u_i a x_i) = \sum_i u_i b x_i$. To show that f is well-defined, assume that $\sum_i u_i a x_i = 0$. Then also $(\sum_i u_i a x_i) y b = 0$ for every $y \in \mathcal{A}$. However, according to our assumption we have $a x_i y b = b x_i y a$, and so it follows that $(\sum_i u_i b x_i) y a = 0$. Since $\mathcal{A}$ is prime and $a \neq 0$ this yields $\sum_i u_i b x_i = 0$, as desired. Since f is a left $\mathcal{A}$-module homomorphism we have that $f(y) = y\lambda$ for some $\lambda \in \mathcal{Q}_{ml}(\mathcal{A})$ and all $y \in \mathcal{I}$. But f is clearly also a right $\mathcal{A}$-module homomorphism, from which we easily infer that $\lambda \in \mathcal{C}$. Consequently, $b = \lambda a$.

What we just proved is a very special case of the following result.

Theorem A.7. *Let $\mathcal{A}$ be prime, and let $a_i, b_i, c_j, d_j \in \mathcal{Q}_{ml}(\mathcal{A})$ be such that*

$$\sum_{i=1}^{n} a_i x b_i = \sum_{j=1}^{m} c_j x d_j \quad \textit{for all } x \in \mathcal{A}.$$

If $a_1, \ldots, a_n$ are linearly independent over $\mathcal{C}$, then each b_i is a $\mathcal{C}$-linear combination of $d_1, \ldots, d_m$. Similarly, if $b_1, \ldots, b_n$ are linearly independent over $\mathcal{C}$, then each a_i is a $\mathcal{C}$-linear combination of $c_1, \ldots, c_m$.

The proof of Theorem A.7 can be quite easily reduced to the $axb = bxa$ case that we have just settled. In the first step we reduce the problem to the case where each $c_j = 0$. This is easy. Assume the linear independence of the a_i's, and choose a basis of the linear span of all a_i's and c_j's that contains all a_i's. Then write each c_j as a linear combination of elements from this basis, which gives $\sum_{i=1}^{n} a_i x b_i' - \sum_{j=1}^{k} e_j x f_j = 0$ where the set $\{a_1, \ldots, a_n, e_1, \ldots, e_k\}$ is independent and each b_i' is the sum of b_i and a linear combination of the d_j's. This shows that indeed we may assume that each $c_j = 0$. Now our goal is to prove that every $b_i = 0$. Let $\mathcal{L}$ be a dense left ideal of $\mathcal{A}$ such that $\mathcal{L}b_n \subseteq \mathcal{A}$. For all $u \in \mathcal{L}$ and $y \in \mathcal{A}$ we have

$$\sum_{i=1}^{n-1} a_i u(b_i y b_n - b_n y b_i) = \Big(\sum_{i=1}^{n} a_i u b_i y\Big) b_n - \sum_{i=1}^{n} a_i (u b_n y) b_i = 0.$$

This makes it possible for one to use induction on n. We already know that $b_i y b_n - b_n y b_i \neq 0$ for some $y \in \mathcal{A}$, unless b_n and b_i are linearly dependent. The rest of the proof is easy.

To prove the second assertion, i.e., the one concerning the case where the b_i's are linearly independent, one can follow the same pattern, although some care is needed since the concept of $\mathcal{Q}_{ml}(\mathcal{A})$ is not left-right symmetric.

The next result is reminiscent of the density theorems.

Theorem A.8. *Let $\mathcal{A}$ be prime, and let $a_1, a_2, \ldots, a_n \in \mathcal{Q}_{ml}(\mathcal{A})$ be such that a_1 does not lie in the $\mathcal{C}$-linear span of $a_2, \ldots, a_n$. Then there exists $\mathcal{E} \in \mathcal{M}(\mathcal{A})$, the multiplication ring of $\mathcal{A}$, such that*

$$\mathcal{E}(a_1) \neq 0 \quad \text{and} \quad \mathcal{E}(a_2) = \ldots = \mathcal{E}(a_n) = 0.$$

The proof given in [40] is based on the so-called weak density theorem, while the proof in the original paper [104] is more direct. We will give the proof only for the special case where $n = 2$ and $\mathcal{A}$ is unital, just to indicate why the result is not so surprising. The following simple argument is taken from [76] in which a generalization of Theorem A.8 is proved. Let $a_1, a_2 \in \mathcal{Q}_{ml}(\mathcal{A})$ be linearly independent. As shown above, there exists $x \in \mathcal{A}$ such that $a_1 x a_2 \neq a_2 x a_1$. Accordingly, $\mathcal{E} = {}_1M_{xa_2} - {}_{a_2x}M_1 \in \mathcal{M}(\mathcal{A})$ satisfies $\mathcal{E}(a_1) \neq 0$ and $\mathcal{E}(a_2) = 0$.

Our final result in this appendix is of great importance for the theory of (generalized) polynomial identities. It links the concept of the extended centroid with the structure theory of rings. Before stating it we first recall some elementary facts about minimal one-sided ideals.

A nonzero left (resp. right) ideal $\mathcal{I}$ of a ring $\mathcal{A}$ is said to be *minimal* if it does not properly contain a nonzero left (resp. right) ideal of $\mathcal{A}$. Minimal left and right ideals of semiprime rings are generated by idempotents. Indeed, let $\mathcal{I}$ be a minimal left ideal of a semiprime ring $\mathcal{A}$. Then $\mathcal{I}^2 \neq 0$. Picking $a \in \mathcal{I}$ such that $\mathcal{I}a \neq 0$ it follows that $\mathcal{I}a = \mathcal{I}$ by the minimality of $\mathcal{I}$. In particular, $ea = a$ for some $e \in \mathcal{I}$. The set $\mathcal{N} = \{x \in \mathcal{I} \mid xa = 0\}$ is a left ideal of $\mathcal{A}$ and a proper subset of $\mathcal{I}$ as $e \notin \mathcal{N}$. Therefore $\mathcal{N} = 0$. Noting that $e^2 - e \in \mathcal{N}$ it follows that e is an idempotent.

Again using the minimality of $\mathcal{I}$ we get that $\mathcal{I} = \mathcal{A}e$, i.e., $\mathcal{I}$ is generated by an idempotent. Of course, similarly we see that every minimal right ideal of $\mathcal{A}$ is of the form $f\mathcal{A}$ for some idempotent f. But actually the connection between minimal left and right ideals is even closer. For an idempotent e in a semiprime ring $\mathcal{A}$ the following three conditions are equivalent: (a) $\mathcal{A}e$ is a minimal left ideal, (b) $e\mathcal{A}$ is a minimal right ideal, and (c) $e\mathcal{A}e$ is a division ring. The proof is just an exercise. For example, let us show that (a) implies (c). If $\mathcal{A}e$ is a minimal left ideal and $b \in \mathcal{A}$ is such that $ebe \neq 0$, then we have $\mathcal{A}ebe = \mathcal{A}e$ by the minimality condition. Hence there is $c \in \mathcal{A}$ such that $cebe = e$, and hence $ecebe = e$. Thus every nonzero element in $e\mathcal{A}e$ has a left inverse, and so $e\mathcal{A}e$ is a division ring. An idempotent e in a semiprime ring $\mathcal{A}$ is called a *minimal idempotent* if it satisfies the (equivalent) conditions (a)–(c).

Theorem A.9. *Let $\mathcal{A}$ be a centrally closed prime ring. Suppose there exists a nonzero $\mathcal{E} \in \mathcal{M}(\mathcal{A})$ such that its range is finite dimensional over $\mathcal{C}$. Then $\mathcal{A}$ contains a minimal idempotent e such that* $\dim_{\mathcal{C}} e\mathcal{A}e < \infty$.

The first step of the proof is to show that there exist nonzero elements $b, c \in \mathcal{A}$ such that $\dim_{\mathcal{C}} b\mathcal{A}c < \infty$. Indeed, we may write $\mathcal{E}(x) = \sum_{i=1}^{n} a_i x b_i$, $n \geq 1$, with $a_1, \ldots, a_n$ $\mathcal{C}$-independent and $\mathcal{E}(x) \in \mathcal{V}$ where $\mathcal{V}$ is finite dimensional over $\mathcal{C}$. By Theorem A.8 there exists $\mathcal{F} \in \mathcal{M}(\mathcal{A})$, with $\mathcal{F}(x) = \sum_{j=1}^{m} s_j x t_j$, such that $\mathcal{F}(a_1) \neq 0$ and $\mathcal{F}(a_i) = 0$, $i \geq 2$. Therefore $\mathcal{F}(a_1)xb_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} s_j a_i t_j x b_i \in \sum_{j=1}^{m} s_j \mathcal{V}$, noting that $\sum_{j=1}^{m} s_j \mathcal{V}$ is finite dimensional over $\mathcal{C}$. So we may take $b = \mathcal{F}(a_1)$ and $c = b_1$. Thus $\mathcal{A}$ contains nonzero left ideals $\mathcal{L}$ and right ideals $\mathcal{R}$ such that $\dim_{\mathcal{C}} \mathcal{R}\mathcal{L} \leq \infty$. Pick a left ideal $\mathcal{L}_0$ and a right ideal $\mathcal{R}_0$ such that $\mathcal{R}_0\mathcal{L}_0$ has minimal (nonzero) dimension. Set $\mathcal{I} = \mathcal{A}\mathcal{R}_0\mathcal{L}_0$. Suppose that $\mathcal{I}'$ is a left ideal of $\mathcal{A}$ such that $0 \neq \mathcal{I}' \subseteq \mathcal{I}$. Then $\mathcal{I}' \subseteq \mathcal{L}_0$ and hence $\mathcal{R}_0\mathcal{I}' \subseteq \mathcal{R}_0\mathcal{L}_0$, which forces $\mathcal{R}_0\mathcal{I}' = \mathcal{R}_0\mathcal{L}_0$. Consequently, $\mathcal{I}' \supseteq \mathcal{A}\mathcal{R}_0\mathcal{I}' = \mathcal{A}\mathcal{R}_0\mathcal{L}_0 = \mathcal{I}$, so that $\mathcal{I}' = \mathcal{I}$. Thus $\mathcal{I}$ is a minimal left ideal of $\mathcal{A}$, and so there exists a minimal idempotent $e \in \mathcal{A}$ such that $\mathcal{I} = \mathcal{A}e$. Since $e \in \mathcal{I} \subseteq \mathcal{A}\mathcal{R}_0$ and $\mathcal{A}e = \mathcal{I} \subseteq \mathcal{L}_0$ it follows that $e\mathcal{A}e = e \cdot \mathcal{A}e$ is finite dimensional.

Appendix B

The Orthogonal Completion

The theory of orthogonal completions was created by Beidar and Mikhalev in a series of papers [15, 41, 42, 43, 169]. An account of it is given in the book [40]. The material in this appendix is drawn from various parts of [40, Chapters 2 and 3] and is designed to provide the necessary background material for proving d-freeness of semiprime rings (under appropriate conditions) in Section 5.3. We shall do this in a self-contained manner, in particular avoiding using the tools of mathematical logic. Unlike in the other three appendices, in this one we shall give complete proofs.

Throughout this appendix, $\mathcal{A}$ will be a semiprime ring with extended centroid $\mathcal{C}$ and maximal left quotient ring $\mathcal{Q} = \mathcal{Q}_{ml}(\mathcal{A})$. For sets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{Q}$ we let $\ell(\mathcal{T}; \mathcal{S})$ denote the left annihilator of $\mathcal{S}$ in $\mathcal{T}$. The set $\mathcal{B}$ of idempotents in $\mathcal{C}$ will play a key role in the theory we outline in this appendix. Its importance is immediately recognized in view of the following lemma.

Lemma B.1. *For every subset $\mathcal{S} \subseteq \mathcal{Q}$ there exists a unique element $E(\mathcal{S}) \in \mathcal{B}$ such that $\ell(\mathcal{Q}; \mathcal{QS}) = (1 - E(\mathcal{S}))\mathcal{Q}$ (and hence $E(\mathcal{S})t = t$ for all $t \in \mathcal{S}$). Further, for every $e \in \mathcal{B}$ we have $E(e\mathcal{S}) = eE(\mathcal{S})$.*

Proof. Let $\mathcal{I}$ be the ideal of $\mathcal{Q}$ generated by $\mathcal{S}$ and let $\mathcal{J} = \ell(\mathcal{Q}; \mathcal{I}) = \ell(\mathcal{Q}; \mathcal{QS})$. It is easy to see that $\mathcal{I} \oplus \mathcal{J}$ is an essential ideal of $\mathcal{Q}$. We define a map from $\mathcal{I} \oplus \mathcal{J}$ into $\mathcal{Q}$ via $x + y \mapsto x$ for $x \in \mathcal{I}$ and $y \in \mathcal{J}$. Note that, in view of Theorem A.1 (iv), this map determines an element $f = E(\mathcal{S})$ in $\mathcal{Q}_{ml}(\mathcal{Q}) = \mathcal{Q}$ (see Corollary A.5). One can check that $f^2 = f$ and that f commutes with every element in $\mathcal{Q}$, that is to say, $f \in \mathcal{B}$. Furthermore, $fx = x$ for $x \in \mathcal{I}$ and $fy = 0$ for $y \in \mathcal{J}$. It is then easily seen that $\ell(\mathcal{Q}; \mathcal{QS}) = \mathcal{J} = (1 - f)\mathcal{Q}$. Of course f is uniquely determined by this property.

Pick $e \in \mathcal{B}$, and let $q \in \ell(\mathcal{Q}; \mathcal{Q}e\mathcal{S}) = (1 - E(e\mathcal{S}))\mathcal{Q}$. Then $eq \in \ell(\mathcal{Q}; \mathcal{QS})$, and so $q = eq + (1 - e)q \in (1 - E(\mathcal{S}))\mathcal{Q} + (1 - e)\mathcal{Q} \subseteq \ell(\mathcal{Q}; \mathcal{Q}e\mathcal{S})$. Thus $\ell(\mathcal{Q}; \mathcal{Q}e\mathcal{S}) = (1 - E(\mathcal{S}))\mathcal{Q} + (1 - e)\mathcal{Q}$. Using the fact that for any $e_1, e_2 \in \mathcal{B}$ we have $e_1\mathcal{Q} + e_2\mathcal{Q} = (e_1 - e_1e_2)\mathcal{Q} \oplus e_2\mathcal{Q} = (e_1 + e_2 - e_1e_2)\mathcal{Q}$, we see that

$\ell(\mathcal{Q};\mathcal{Q}e\mathcal{S}) = (1-E(\mathcal{S})+1-e-(1-E(\mathcal{S}))(1-e))\mathcal{Q} = (1-eE(\mathcal{S}))\mathcal{Q}$. Thus $(1-E(e\mathcal{S}))\mathcal{Q} = (1-eE(\mathcal{S}))\mathcal{Q}$, and so $E(e\mathcal{S}) = eE(\mathcal{S})$. □

We shall write $E(s)$ for $E(\{s\})$. Further, for $e \in \mathcal{B}$ we set $\mathcal{L}_e = \{x \in \mathcal{A} \mid ex \in \mathcal{A}\}$.

Lemma B.2. *$\mathcal{L}_e$ is an essential ideal of $\mathcal{A}$ for every $e \in \mathcal{B}$. Moreover, $\mathcal{L}_e e$ is an ideal of $\mathcal{A}$ and $\mathcal{Q}_{ml}(\mathcal{L}_e e) = \mathcal{Q}e$.*

Proof. Clearly $\mathcal{L}_e$ is an ideal of $\mathcal{A}$. According to Theorem A.1 (ii) it contains a dense left ideal of $\mathcal{A}$, which implies that $\mathcal{L}_e$ is an essential ideal. It is also clear that $\mathcal{L}_e e$ is an ideal of $\mathcal{A}$. Using Theorem A.4 we have

$$\mathcal{Q} = \mathcal{Q}_{ml}(\mathcal{L}_e) = \mathcal{Q}_{ml}(\mathcal{L}_e e \oplus \mathcal{L}_e(1-e)) = \mathcal{Q}_{ml}(\mathcal{L}_e e) \oplus \mathcal{Q}_{ml}(\mathcal{L}_e(1-e)),$$

from which $\mathcal{Q}_{ml}(\mathcal{L}_e e) = \mathcal{Q}e$ easily follows. □

A subset $\mathcal{U} \subseteq \mathcal{B}$ is said to be *dense* if $\ell(\mathcal{Q};\mathcal{U}) = 0$, i.e., $E(\mathcal{U}) = 1$, and $\mathcal{U}$ is *orthogonal* if $uv = 0$ for all $u, v \in \mathcal{U}$ with $u \neq v$. For future reference we record two simple observations concerned with such subsets. The first one is immediate.

Lemma B.3. *If $\mathcal{U}$ and $\mathcal{V}$ are dense orthogonal subsets of $\mathcal{B}$, then $\mathcal{U}\mathcal{V} = \{uv \mid u \in \mathcal{U}, v \in \mathcal{V}\}$ is also a dense orthogonal subset of $\mathcal{B}$.*

Lemma B.4. *If $\mathcal{U}$ is a dense subset of $\mathcal{B}$, then $\mathcal{I} = \sum_{u\in\mathcal{U}} \mathcal{L}_u u$ is an essential ideal of $\mathcal{A}$.*

Proof. Lemma B.2 implies that $\mathcal{I}$ is an ideal of $\mathcal{A}$. Let $b \in \mathcal{A}$ be such that $b\mathcal{I} = 0$. Then $(bu)\mathcal{L}_u = b\mathcal{L}_u u = 0$ for every $u \in \mathcal{U}$. Since $\mathcal{L}_u$ is an essential ideal of $\mathcal{A}$ by Lemma B.2, it is easy to see (e.g., by using Theorem A.1) that $\ell(\mathcal{Q};\mathcal{L}_u) = 0$. Therefore $bu = 0$ for every $u \in \mathcal{U}$, and hence $b = 0$ since $\mathcal{U}$ is dense. Thus $\mathcal{I}$ is essential. □

We now make the key definition: A subset $\mathcal{T} \subseteq \mathcal{Q}$ is said to be *orthogonally complete* if for any orthogonal dense subset $\mathcal{U} \subseteq \mathcal{B}$ and any elements $t_u \in \mathcal{T}$, $u \in \mathcal{U}$, there exists $t \in \mathcal{T}$ such that $tu = t_u u$ for all $u \in \mathcal{U}$. We denote this element t, which is clearly unique, by the suggestive notation

$$t = \sum_{u\in\mathcal{U}}^{\perp} t_u u.$$

To show this is not just an empty concept we have the following

Lemma B.5. *$\mathcal{Q}$ is orthogonally complete.*

Proof. Let $\mathcal{U}$ be a dense orthogonal subset of $\mathcal{B}$ and let $\{q_u \mid u \in \mathcal{U}\} \subseteq \mathcal{Q}$. Note that $\mathcal{D} = \sum_{u\in\mathcal{U}} \mathcal{Q}u$ is an essential ideal of $\mathcal{Q}$. We define $f : \mathcal{D} \to \mathcal{Q}$ according to $f(\sum x_u u) = \sum x_u q_u u$. We note that f is a well-defined left $\mathcal{Q}$-module homomorphism, and so there exists $q \in \mathcal{Q}(= \mathcal{Q}_{ml}(\mathcal{Q}))$ such that, in particular, for each $u \in \mathcal{U}$ we have $qu = f(u) = q_u u$. □

Lemma B.5 shows that $\sum_{u\in\mathcal{U}}^{\perp} q_u u$ always exists in $\mathcal{Q}$ for any choice of an orthogonal dense subset $\mathcal{U} \subseteq \mathcal{B}$ and any elements $q_u \in \mathcal{Q}$. The above definition can now be rephrased as follows: $\mathcal{T}$ is orthogonally complete if $\sum_{u\in\mathcal{U}}^{\perp} t_u u$ lies in $\mathcal{T}$ whenever every $t_u \in \mathcal{T}$.

We can now define the *orthogonal completion* $\mathcal{O}(\mathcal{T})$ of any subset $\mathcal{T} \subseteq \mathcal{Q}$ to be the intersection of all orthogonally complete subsets containing $\mathcal{T}$. It is straightforward to show that $\mathcal{O}(\mathcal{T})$ is in fact orthogonally complete. This, of course, is not very enlightening as to the nature of $\mathcal{O}(\mathcal{T})$, but fortunately one has the much more tangible characterization of $\mathcal{O}(\mathcal{T})$ given by the following

Lemma B.6. *Let $\mathcal{T} \subseteq \mathcal{Q}$. Then $\mathcal{O}(\mathcal{T})$ consists of all elements of the form $\sum_{u\in\mathcal{U}}^{\perp} t_u u$ where $\mathcal{U}$ is a dense orthogonal subset of $\mathcal{B}$ and $t_u \in \mathcal{T}$ for every $u \in \mathcal{U}$.*

Proof. Let $\mathcal{H}$ denote the set of all elements of the form $\sum_{u\in\mathcal{U}}^{\perp} t_u u$ where $\mathcal{U}$ is a dense orthogonal subset of $\mathcal{B}$ and $t_u \in \mathcal{T}$. Clearly $\mathcal{H} \subseteq \mathcal{O}(\mathcal{T})$; our task is to show that $\mathcal{H}$ itself is orthogonally complete. To this end we let $\mathcal{W}$ be a dense orthogonal subset of $\mathcal{B}$ and for each $w \in \mathcal{W}$ let $h_w \in \mathcal{H}$. By Lemma B.5 we know that $q = \sum_{w\in\mathcal{W}}^{\perp} h_w w$ exists in $\mathcal{Q}$. We have to show that $q \in \mathcal{H}$. Each h_w can be written as $h_w = \sum_{u_w\in\mathcal{U}_w}^{\perp} t_{u_w} u_w$ where $t_{u_w} \in \mathcal{T}$ and $\mathcal{U}_w$ is a dense orthogonal subset of $\mathcal{B}$. Now $\mathcal{V} = \{wu_w \mid w \in \mathcal{W}, u_w \in \mathcal{U}_w\}$ is easily seen to be a dense orthogonal subset of $\mathcal{B}$. For $v = wu_w \in \mathcal{V}$ we define $t_v = t_{u_w}$, and set $p = \sum_{v\in\mathcal{V}}^{\perp} t_v v \in \mathcal{H}$. For $v = wu_w \in \mathcal{V}$ we shall show that $pv = qv$. Indeed, $pv = t_v v = t_{u_w} wu_w$, and on the other hand $qv = qwu_w = h_w wu_w = h_w u_w w = t_{u_w} u_w w = t_{u_w} wu_w$. Since $\mathcal{V}$ is dense it follows that $q = p \in \mathcal{H}$. □

We will need the following facts about orthogonally complete subsets.

Lemma B.7. *Let $\mathcal{T}$ be an orthogonally complete set such that $0 \in \mathcal{T}$. Then:*

(i) *$e\mathcal{T} \subseteq \mathcal{T}$ for all $e \in \mathcal{B}$.*

(ii) *There exists $t \in \mathcal{T}$ such that $E(t) = E(\mathcal{T})$.*

Proof. (i) Clearly $\{e, 1-e\}$ is a dense orthogonal subset of $\mathcal{B}$. Let $t \in \mathcal{T}$, and set $t_e = t,\ t_{1-e} = 0$. Since $\mathcal{T}$ is orthogonally complete there exists $s \in \mathcal{T}$ such that $se = te$ and $s(1-e) = 0$. Therefore $te = se = s \in \mathcal{T}$.

(ii) Let $\mathcal{W} = \{E(t) \mid t \in \mathcal{T}\}$. For $t \in \mathcal{T}$ and $e \in \mathcal{B}$ we know from (i) that $te \in \mathcal{T}$. Therefore by Lemma B.1 we have $eE(t) = E(et) \in \mathcal{W}$, i.e., $e\mathcal{W} \subseteq \mathcal{W}$. By Zorn's Lemma there exists a maximal orthogonal subset $\mathcal{V} \subseteq \mathcal{W}$. We set $\mathcal{U} = \mathcal{V} \cup \{1 - E(\mathcal{W})\}$. Clearly $\mathcal{U}$ is an orthogonal subset of $\mathcal{B}$. Suppose $E(\mathcal{U}) \neq 1$. Then $e = 1 - E(\mathcal{U}) \neq 0$ in particular satisfies $e(1 - E(\mathcal{W})) = 0$ and therefore $ew \neq 0$ for some $w \in \mathcal{W}$. But $ew \in \mathcal{W}$ by what we proved, and so we have that $\mathcal{V} \cup \{ew\}$ is an orthogonal subset of $\mathcal{V}$, in contradiction to the maximality of $\mathcal{V}$. Thus $\mathcal{U}$ is a dense orthogonal subset of $\mathcal{B}$. By definition of $\mathcal{W}$ for each $v \in \mathcal{V}$ there exists $t_v \in \mathcal{T}$ such that $E(t_v) = v$. We set $t_{1-E(\mathcal{W})} = 0$. Since $\mathcal{T}$ is orthogonally complete, $t = \sum_{u\in\mathcal{U}}^{\perp} t_u u$ belongs to $\mathcal{T}$. We claim that $E(\mathcal{T}) = E(t)$. Since $t \in \mathcal{T}$ it follows easily that $E(t) = E(t)E(\mathcal{T})$. If $E(\mathcal{T}) \neq E(t)$, then $e = E(\mathcal{T}) - E(t) \neq 0$,

$e \in \mathcal{B}$, $et = 0$ but $e\mathcal{T} \neq 0$. From $et = 0$ we conclude that $0 = E(etv) = E(et_v v)$, which by Lemma B.1 yields $0 = evE(t_v) = ev^2 = ev$ for each $v \in \mathcal{V}$, i.e., $e\mathcal{V} = 0$. On the other hand there exists $x \in \mathcal{T}$ such that $ex \neq 0$, whence $E(ex) \neq 0$. But $E(ex) \in \mathcal{W}$ since $ex \in \mathcal{T}$, and from $E(ex) = eE(x)$ we obtain $E(ex)\mathcal{V} = 0$. This is a contradiction to the maximality of $\mathcal{V}$. □

Given dense orthogonal subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{B}$, and elements $x = \sum_{u\in\mathcal{U}}^{\perp} x_u u$ and $y = \sum_{v\in\mathcal{V}}^{\perp} y_v v$, it is an easy exercise to show that

$$x \pm y = \sum_{uv\in\mathcal{U}\mathcal{V}}^{\perp} (x_u \pm y_v)uv, \quad xy = \sum_{uv\in\mathcal{U}\mathcal{V}}^{\perp} x_u y_v uv.$$

From these and Lemma B.6 we see that the orthogonal completion of a subring of $\mathcal{Q}$ is again a subring of $\mathcal{Q}$. The ring we are especially interested in is

$$\mathcal{O} = \mathcal{O}(\mathcal{A}),$$

the orthogonal completion of $\mathcal{A}$, so we now direct our attention to $\mathcal{O}$. First we mention an illustrative example. Let $\{\mathcal{A}_i \mid i \in I\}$ be a family of prime rings and let $\mathcal{A} = \oplus_{i\in I}\mathcal{A}_i$ be their direct sum. One can check that in this case $\mathcal{Q} = \prod_{i\in I} \mathcal{Q}_i$, where $\mathcal{Q}_i = \mathcal{Q}_{ml}(\mathcal{A}_i)$, and $\mathcal{O} = \prod_{i\in I} \mathcal{A}_i$.

Now we explore $\mathcal{B}$ in more detail. First, we note that $\mathcal{B}$ becomes a Boolean ring under a new addition $e \oplus f = e + f - 2ef$ but with the same multiplication. Further, $\mathcal{B}$ becomes a partially ordered set by defining $e \leq f$ if $e = ef$. We let $Spec(\mathcal{B})$ denote the collection of maximal ideals of the Boolean ring $\mathcal{B}$. We note that an ideal $\mathcal{M}$ of $\mathcal{B}$ is maximal if and only if for all $e \in \mathcal{B}$ either $e \in \mathcal{M}$ or $1 - e \in \mathcal{M}$ but not both. Corresponding to $\mathcal{M} \in Spec(\mathcal{B})$ is the ideal of $\mathcal{O}$

$$\mathcal{O}\mathcal{M} = \{\sum_i s_i e_i \mid s_i \in \mathcal{O} \text{ and } e_i \in \mathcal{M}\}.$$

An important observation for us is the following

Lemma B.8. *Let $a \in \mathcal{O}$ and let $\mathcal{M} \in Spec(\mathcal{B})$. Then $a \in \mathcal{O}\mathcal{M}$ if and only if $E(a) \in \mathcal{M}$.*

Proof. If $E(a) \in \mathcal{M}$, then $a = aE(a) \in \mathcal{O}\mathcal{M}$. Conversely, suppose $a = \sum_{i=1}^{n} s_i e_i \in \mathcal{O}\mathcal{M}$. For each i we have $1 - e_i \notin \mathcal{M}$ and so $e = \prod_{i=1}^{n}(1 - e_i) \notin \mathcal{M}$. But $ae = 0$, whence $0 = E(ae) = eE(a)$. Since $e \notin \mathcal{M}$ it follows that $E(a) \in \mathcal{M}$. □

One of the key results of this theory is

Theorem B.9. *For $\mathcal{M} \in Spec(\mathcal{B})$, $\mathcal{O}\mathcal{M}$ is a prime ideal of $\mathcal{O}$, i.e., $\mathcal{O}_{\mathcal{M}} = \mathcal{O}/\mathcal{O}\mathcal{M}$ is a prime ring.*

Proof. Suppose $a\mathcal{O}b \subseteq \mathcal{O}\mathcal{M}$ for some $a, b \in \mathcal{O}$, with $b \notin \mathcal{O}\mathcal{M}$. It is easy to see that $a\mathcal{O}b$ is orthogonally complete. By Lemma B.7 (ii) $E(a\mathcal{O}b) = E(t)$ for some $t \in a\mathcal{O}b$.

Consequently, since $E(t) \in \mathcal{M}$ in view of Lemma B.8, $e = 1 - E(a\mathcal{O}b) \notin \mathcal{M}$. From Lemma B.8 we also see that $E(b) \notin \mathcal{M}$. We have $ea\mathcal{O}b = 0$, and hence $a\mathcal{Q}(eb) = 0$ by Lemma A.3. Thus $a \in (1 - E(eb))\mathcal{Q}$, so that $aE(eb) = 0$. Lemma B.1 now shows that $aeE(b) = 0$, and note that this yields $E(a)eE(b) = 0$. Since $eE(b) \notin \mathcal{M}$ this forces $E(a) \in \mathcal{M}$, and so, by Lemma B.8, $a \in \mathcal{OM}$. □

The following lemma will prove useful in that it converts a seemingly "infinite" situation into a "finite" one.

Lemma B.10. *For every $\mathcal{M} \in Spec(\mathcal{B})$ let $w_{\mathcal{M}} \in \mathcal{B}\backslash\mathcal{M}$. Then there exist $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_q \in Spec(\mathcal{B})$ and orthogonal idempotents $e_1, e_2, \ldots, e_q \in \mathcal{B}$ whose sum is* 1 *such that $e_p \leq w_{\mathcal{M}_p}$ for $p = 1, 2, \ldots, q$.*

Proof. Let $\mathcal{W}$ be the ideal of the Boolean ring $\mathcal{B}$ generated by all $w_{\mathcal{M}}$. If $\mathcal{W} \neq \mathcal{B}$ then $\mathcal{W} \subseteq \mathcal{M}$ for some $\mathcal{M} \in Spec(\mathcal{B})$, whence the contradiction that $w_{\mathcal{M}} \notin \mathcal{M}$. Thus $\mathcal{W} = \mathcal{B}$ and in particular $1 = w_1b_1 \oplus w_2b_2 \oplus \ldots \oplus w_qb_q$ (Boolean sum) for some $w_p = w_{\mathcal{M}_p}$ and $b_p \in \mathcal{B}$. From the definition of the Boolean operations it is easy to see that $\mathcal{B} = w_1\mathcal{B} + w_2\mathcal{B} + \ldots + w_q\mathcal{B}$ (usual sum). Set $e_1 = w_1$ and $e_2 = w_2 - w_1w_2$. Clearly $e_1\mathcal{B} + e_2\mathcal{B} = w_1\mathcal{B} + w_2\mathcal{B}$, with $e_1e_2 = 0$ and $e_2 \leq w_2$. This is just the first step in the well-known process of replacing idempotents by orthogonal ones, and so we eventually have that $\mathcal{B} = e_1\mathcal{B} + e_2\mathcal{B} + \ldots + e_q\mathcal{B}$ with the e_i's orthogonal and $e_p \leq w_p$. From this it follows that $1 = e_1 + e_2 + \ldots + e_q$ and the lemma is proved. □

Appendix C

Polynomial Identities

The theory of rings with polynomial identities is well documented in several monographs, for instance in [120], [184] and [187]. We shall survey those elements of the theory that are important for understanding functional identities. We will omit rigorous proofs, but rather try to give some informal evidence for the truthfulness of the results that will be stated.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set, and let $\mathbb{Z}\langle X\rangle$ be the free algebra on X over $\mathbb{Z}$. Let $f = f(x_1, \ldots, x_n) \in \mathbb{Z}\langle X\rangle$ be a polynomial such that at least one of its monomials of highest degree has coefficient 1. Let $\mathcal{R}$ be a nonempty subset of a ring $\mathcal{A}$. We say that f is a *polynomial identity* on $\mathcal{R}$ if $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \mathcal{R}$, that is, if the polynomial function determined by f vanishes on $\mathcal{R}^n$. In this case we also say that $\mathcal{R}$ satisfies the polynomial identity f. In what follows we will only consider the case where $\mathcal{R} = \mathcal{A}$, i.e., we will treat rings satisfying polynomial identities. Such rings are called *PI-rings.*

The simplest examples of PI-rings are commutative rings. Indeed, saying that a ring $\mathcal{A}$ is commutative is the same as saying that $\mathcal{A}$ satisfies the polynomial identity $x_1x_2 - x_2x_1$. Similarly, a ring $\mathcal{A}$ is Boolean if and only if it satisfies $x_1^2 - x_1$, and $\mathcal{A}$ is a nilpotent ring if and only if it satisfies $x_1x_2 \ldots x_n$ for some positive integer n.

As we shall see, PI-rings are rather special. Incidentally, the polynomial function determined by the polynomial px_1 vanishes on every ring with characteristic p, but this does not mean that rings of finite characteristic are necessarily PI-rings. Note that we have required that one of the monomials of highest degree in a polynomial identity should have coefficient 1. We remark that in general PI theory takes place in the framework of algebras $\mathcal{A}$ over a commutative domain $\mathcal{C}$ (e.g., a field), but in this book we only have need of the theory when $\mathcal{C} = \mathbb{Z}$, i.e., $\mathcal{A}$ is just a ring.

A polynomial $f = f(x_1, \ldots, x_n) \in \mathbb{Z}\langle X\rangle$ is said to be multilinear if every x_i, $1 \le i \le n$, appears exactly once in each of the monomials of f. Thus f is of the form

$$f(x_1,\dots,x_n) = \sum_{\pi\in S_n} n_\pi x_{\pi(1)}x_{\pi(2)}\dots x_{\pi(n)},$$

where S_n is the symmetric group of order n and n_π are integers. If $\mathcal{A}$ satisfies a polynomial identity of degree n, then it also satisfies a multilinear polynomial identity of degree $\le n$. One can show this by using the standard linearization procedure. A trivial example: the polynomial identity $x_1^2 - x_1$, through which Boolean rings are defined, leads to the polynomial identity $x_1x_2 + x_2x_1$. This suggests that sometimes one can lose some important information when reducing general identities to multilinear ones. But for our purposes such a loss is of no significance. We are interested only in structural properties of a ring that satisfies a polynomial identity of a certain degree, and so we may immediately assume the multilinearity of this identity.

A polynomial of extreme importance in PI theory is

$$St_d = St_d(x_1,\dots,x_d) = \sum_{\pi\in S_d}(-1)^\pi x_{\pi(1)}x_{\pi(2)}\dots x_{\pi(d)},$$

which we call the *standard polynomial* of degree d. Here, $(-1)^\pi$ denotes the sign of the permutation π. For example, $St_2 = x_1x_2 - x_2x_1$. It is easy to check that

$$St_d(x_1,\dots,x_d) = \sum_{i=1}^{d}(-1)^{i+1}x_iSt_{d-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d).$$

Therefore, if a ring $\mathcal{A}$ satisfies St_d, then it satisfies St_m for every $m \ge d$. Another useful property of St_d is that it vanishes if any two of its arguments are equal, i.e.,

$$St_d(x_1,\dots x_i,\dots,x_i,\dots,x_d) = 0.$$

This has an important consequence: every n-dimensional algebra $\mathcal{A}$ over a field $\mathcal{K}$ satisfies St_{n+1}. So, for example, $M_n(\mathcal{K})$ satisfies St_{n^2+1}. But in fact a much sharper result is true: $M_n(\mathcal{K})$, where $\mathcal{K}$ can be any commutative ring, satisfies St_{2n}. This is the celebrated Amitsur–Levitzki theorem. Various proofs are known, some of them short, but all nontrivial. The following fact gives another light to the meaning of the Amitsur–Levitzki theorem: $M_n(\mathcal{K})$ does not satisfy a polynomial identity of degree $< 2n$. This is easy to prove. Just consider the matrix units $e_{11}, e_{12}, e_{22}, e_{23}, \dots, e_{n-1\,n}, e_{nn}$; there are $2n-1$ of them, their product in the given order is e_{1n}, and their product in any other order is 0. Therefore, if f is a multilinear polynomial of degree $2n-1$ such that its coefficient at $x_1x_2\dots x_{2n-1}$ is 1, then $f(e_{11}, e_{12}, e_{22}, \dots, e_{nn}) = e_{1n} \ne 0$.

We shall now consider prime PI-rings and begin with

Theorem C.1. *Let $\mathcal{A}$ be a prime ring. Then $\mathcal{A}$ is a PI-ring if and only if its central closure $\mathcal{AC}$ is a finite dimensional central simple algebra over the extended centroid $\mathcal{C}$ of $\mathcal{A}$.*

This is a partial statement of Posner's theorem [183], one of the cornerstones of PI theory. Actually much more can be said: the center $\mathcal{Z}$ of $\mathcal{A}$ is nonzero and has $\mathcal{C}$ as its field of fractions. Consequently, every element in $\mathcal{AC}$ is of the form $\frac{a}{\lambda}$ where $a \in \mathcal{A}$ and $\lambda \in \mathcal{Z}$. This is a highly nontrivial result whose proof is based on the existence of the so-called central polynomials in matrix algebras.

Let us recall that by the classical Wedderburn theorem, a finite dimensional central simple algebra is up to an isomorphism the same as $M_n(D)$ where D is a finite dimensional division algebra. This is also apparent from the proof of Theorem C.1 which we now sketch.

The "if" part is obvious. Namely, finite dimensional algebras are PI-rings, and subrings of PI-rings are trivially also PI-rings. To prove the converse, assume that $\mathcal{A}$ is a PI-ring. Since $\mathcal{A}$ and $\mathcal{AC}$ clearly satisfy the same multilinear polynomial identities, we may assume without loss of generality that $\mathcal{A}$ is centrally closed. We now invoke a result from the next appendix, namely, Theorem D.1, that considers a more general situation when $\mathcal{A}$ satisfies a generalized polynomial identity, and an outline of whose proof is given. One thereby concludes that $\mathcal{A}$ contains an idempotent e such that $\mathcal{A}e$ is a minimal left ideal of $\mathcal{A}$ and $e\mathcal{A}e$ is a division ring with $\dim_{\mathcal{C}} e\mathcal{A}e < \infty$. We may regard $\mathcal{A}e$ as a faithful simple left $\mathcal{A}$-module (thus $\mathcal{A}$ is a primitive ring). It is easy to see that $\mathrm{End}_{\mathcal{A}}(\mathcal{A}e)$ is antiisomorphic to $e\mathcal{A}e$. According to the well-known corollary to Jacobson's density theorem we have two possibilities: either $\mathcal{A}$ is isomorphic to $M_n(e\mathcal{A}e)$ for some positive integer n or for every $m \in \mathbb{N}$ there exist a subring $\mathcal{A}_m$ of $\mathcal{A}$ and an ideal $\mathcal{I}_m$ of $\mathcal{A}_m$ such that $\mathcal{A}_m/\mathcal{I}_m \cong M_m(e\mathcal{A}e)$. Clearly, if f is a polynomial identity of $\mathcal{A}$, then it is also a polynomial identity of $\mathcal{A}_m/\mathcal{I}_m$. Therefore, in the latter case f would be a polynomial identity of $M_m(e\mathcal{A}e)$ for every $m \in \mathbb{N}$, and hence also of its subring $M_m(\mathcal{C}e) \cong M_m(\mathcal{C})$. However, as noticed above, $M_m(\mathcal{C})$ does not satisfy polynomial identities of degree $2m-1$, so no polynomial exists that would be a polynomial identity of $M_m(\mathcal{C})$ for every m. Therefore the first possibility occurs, i.e., $\mathcal{A} \cong M_n(e\mathcal{A}e)$ for some $n \in \mathbb{N}$, and hence $\mathcal{A}$ is a finite dimensional central simple algebra over $\mathcal{C}$.

The following theorem gives more detailed information about prime PI-rings. By $\deg(\,.\,)$ we denote the degree of algebraicity over $\mathcal{C}$ (cf. Section 5.2).

Theorem C.2. *Let $\mathcal{A}$ be prime ring, and let $n \in \mathbb{N}$. The following conditions are equivalent:*

(i) *$\mathcal{AC}$ is a finite dimensional central simple algebra over $\mathcal{C}$ with $\dim_{\mathcal{C}} \mathcal{AC} = n^2$;*

(ii) *$\mathcal{A}$ satisfies St_{2n} and does not satisfy any polynomial identity of degree $< 2n$;*

(iii) *There exists a field $\mathbb{F}$ such that $\mathcal{A}$ can be embedded into the ring $M_n(\mathbb{F})$, and $M_n(\mathbb{F})$ satisfies the same multilinear polynomial identities as $\mathcal{A}$ (and hence $\mathcal{A}$ cannot be embedded in $M_{n-1}(\mathcal{K})$ for any commutative ring $\mathcal{K}$);*

(iv) $\deg(\mathcal{A}) = n$.

Moreover, in this case there exist traces of k-additive maps $\alpha_k : \mathcal{A} \to \mathcal{C}$, $k = 1, \ldots, n$, such that

$$x^n + \alpha_1(x)x^{n-1} + \ldots + \alpha_{n-1}(x)x + \alpha_n(x) = 0$$

for all $x \in \mathcal{A}$. Also, we have $\mathcal{Q}_{ml}(\mathcal{A}) = \mathcal{AC}$.

If $\mathcal{A}$ was the algebra of all square matrices over a field, then the equivalence of (i)–(iv) would be easy to establish. Most of the implications can be proved by reducing the general situation to this simple and tractable one. The idea is to consider the scalar extension of the $\mathcal{C}$-algebra $\mathcal{AC}$ by the algebraic closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ (incidentally, $\mathbb{F}$ in (iii) can be chosen to be $\overline{\mathcal{C}}$). Not everything is entirely obvious. In particular, showing that (iv) implies any of (i)–(iii) requires some more effort since the condition $\deg(\mathcal{A}) = n$ is not a multilinear one, and so it is more difficult to deal with scalar extensions. Anyhow, making use of certain standard tools of PI theory this problem can be handled as well. The last assertion concerning α_i's is based on the Cayley–Hamilton theorem (cf. the discussion following Example 1.2).

Note that Theorem C.2 in particular implies that for every prime PI-ring $\mathcal{A}$ there exists $n \in \mathbb{N}$ such that $\mathcal{A}$ satisfies St_{2n}, but does not satisfy any polynomial identity of degree $< 2n$. So the minimal degree of all polynomial identities of $\mathcal{A}$ is an even number.

If $\mathcal{A}$ is a simple unital ring, then $\mathcal{A}$ is centrally closed and moreover, the extended centroid is just the center $\mathcal{Z}$ of $\mathcal{A}$. Therefore parts of Theorem C.2 can be written in a simpler way. Let us record this.

Corollary C.3. *Let $\mathcal{A}$ be a simple unital ring. Then $\dim_{\mathcal{Z}} \mathcal{A} = n^2$ if and only if $\deg(\mathcal{A}) = n$. Moreover, in this case there exist traces of k-additive maps $\alpha_k : \mathcal{A} \to \mathcal{Z}$, $k = 1, \ldots, n$, such that*

$$x^n + \alpha_1(x)x^{n-1} + \ldots + \alpha_{n-1}(x)x + \alpha_n(x) = 0$$

for all $x \in \mathcal{A}$.

As mentioned earlier, the center of a nonzero prime PI-ring is nonzero; moreover, the same is true for semiprime rings. This is the result by Rowen [186].

Theorem C.4. *A nonzero semiprime PI-ring has a nonzero center.*

The results stated so far could be described as folklore. The next two lemmas are more special. They are taken from papers on FI's ([29, Lemmas 2.1 and 2.2] and [22, Lemma 2.1]), although their connection to FI's is only indirect. The proofs use standard PI theory in order to reduce the general case to the one where $\mathcal{A}$ is the algebra $M_n(\mathbb{F})$ of matrices over a field. Then the problem of course becomes very concrete; using the fact that the set of matrices with zero trace is the only proper noncentral Lie ideal of $M_n(\mathbb{F})$ (in the first lemma), and that the transpose and the symplectic involution are basically (here we are neglecting certain technicalities) the only involutions on $M_n(\mathbb{F})$ (in the second lemma), one then just has to find matrices with a "big" degree of algebraicity in appropriate sets.

Lemma C.5. *If $\mathcal{L}$ is a noncentral Lie ideal of a prime ring $\mathcal{A}$, then* $\deg(\mathcal{L}) = \deg(\mathcal{A})$.

Lemma C.6. *If $\mathcal{A}$ is a prime ring with involution and* $\operatorname{char}(\mathcal{A}) \neq 2$, *then* $\deg(\mathcal{S}(\mathcal{A}) \cup \mathcal{K}(\mathcal{A})) = \deg(\mathcal{A})$. *Moreover, if* $\deg(\mathcal{A}) \geq 5$, *then* $\deg(\mathcal{L}) = \deg(\mathcal{A})$ *for every noncentral Lie ideal $\mathcal{L}$ of $\mathcal{K}(\mathcal{A})$.*

Appendix D

Generalized Polynomial Identities

Let us introduce the concept of a generalized polynomial identity through a typical example. Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$, and let $\mathcal{A} = \mathrm{End}_{\mathbb{F}}(\mathcal{V})$. Let $e \in \mathcal{A}$ be an idempotent of rank 1; this means that there are $u \in \mathcal{V}$ and a linear functional f on $\mathcal{V}$ such that $f(u) = 1$ and $e : v \mapsto f(v)u$. Note that for every $x \in \mathcal{A}$, exe is a scalar multiple of e. Accordingly,

$$ex_1ex_2e = ex_2ex_1e$$

holds for all $x_1, x_2 \in \mathcal{A}$. This is a model of a generalized polynomial identity. Thus, $\mathcal{A} = \mathrm{End}_{\mathbb{F}}(\mathcal{V})$ is a GPI-ring, while it is a PI-ring only when $\mathcal{V}$ is finite dimensional (see appendix C).

As this example suggests, informally we consider a *generalized polynomial identity* on a ring $\mathcal{A}$ as an identical relation

$$\sum a_{i_0} x_{j_1} a_{i_1} x_{j_2} \ldots a_{i_{n-1}} x_{j_n} a_{i_n} = 0$$

for all $x_{j_k} \in \mathcal{A}$, where a_{i_k} are fixed elements in $\mathcal{A}$ (and when $a_{i_0} \in \mathbb{Z}$ and each each $a_{i_k} = 1$, $k \geq 1$, this reduces to a polynomial identity). Of course this is not sufficiently precise. For example, every central element c gives rise to the identity $cx - xc = 0$. In the exact definition one has to get rid of such trivial cases. We refer the reader to the book [40] for a full account of GPI theory, and in particular for all details concerning the rigorous definition of a generalized polynomial identity. In this appendix we shall keep the exposition at an intuitive level. Let us just mention that in case $\mathcal{A}$ is a prime ring, one defines a generalized polynomial identity of $\mathcal{A}$ as an element $f = f(x_1, \ldots, x_n)$ of the coproduct of $\mathcal{Q}_s(\mathcal{A})$, the symmetric ring of quotients of $\mathcal{A}$, and the free algebra $\mathcal{C}\langle X \rangle$ over the extended centroid $\mathcal{C}$, such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in \mathcal{A}$ (so one allows that the elements a_{i_k}

lie in $\mathcal{Q}_s(\mathcal{A})$). We say that $\mathcal{A}$ is a *GPI-ring* if it satisfies a nonzero generalized polynomial identity.

As in the case of polynomial identities, for most purposes it is enough to consider multilinear generalized polynomial identities (their definition should be self-explanatory). We have dealt with linear generalized polynomial identities in one variable (i.e., elements of the form $\sum_i a_i x_1 b_i$) already in Theorem A.7. In fact, this theorem implies that a prime ring cannot satisfy a nonzero linear generalized polynomial identity in one variable. The next case of multilinear identities in two variables is more interesting, as our initial example clearly suggests. So assume that a prime ring $\mathcal{A}$ satisfies a generalized polynomial identity

$$0 \neq f = f(x_1, x_2) = \sum_{i=1}^{p} a_i x_1 b_i x_2 c_i + \sum_{j=1}^{q} d_j x_2 e_j x_1 f_j.$$

This means that this expression equals 0 if we replace the indeterminates x_1 and x_2 by any two elements in $\mathcal{A}$. Assume for simplicity that all $a_i, b_i, c_i, d_j, e_j, f_j$ lie in $\mathcal{A}$, and also that $\mathcal{A}$ is centrally closed. Further, without loss of generality we may assume that the first summation of f, $\sum_{i=1}^{p} a_i x_1 b_i x_2 c_i$, is also nonzero, and that $\{a_1, \ldots, a_n\}$ is a maximal linearly independent subset of $\{a_1, \ldots, a_p\}$. Note that we can rewrite f as

$$f = \sum_{i=1}^{n} a_i x_1 \mathcal{E}_i(x_2) + \sum_{j=1}^{q} d_j x_2 e_j x_1 f_j,$$

where $\mathcal{E}_i$ lies in $\mathcal{M}(\mathcal{A})$, the multiplication ring of $\mathcal{A}$. If every $\mathcal{E}_i$ was zero (as an element of $\mathcal{M}(\mathcal{A})$), then, by the result on linear identities in one variable, $\sum_{i=1}^{n} a_i x_1 \mathcal{E}_i(x_2) = \sum_{i=1}^{p} a_i x_1 b_i x_2 c_i$ would be 0, contrary to our assumption. So we may assume that $\mathcal{E}_1 \neq 0$. As $f(x, y) = 0$ for all $x, y \in \mathcal{A}$, we are now in a position to apply Theorem A.7. Hence it follows that for every $y \in \mathcal{A}$, $\mathcal{E}_1(y)$ is a $\mathcal{C}$-linear combination of $f_1, \ldots, f_q$, meaning that the range of $\mathcal{E}_1$ is finite dimensional. Now we can use Theorem A.9. Thus $\mathcal{A}$ contains a minimal idempotent e such that $e\mathcal{A}e$ is a finite dimensional division algebra over $\mathcal{C}$.

If $\mathcal{A}$ is not centrally closed, then the above conclusion holds for its central closure $\mathcal{AC}$. Namely, obviously f is also a generalized polynomial identity of $\mathcal{AC}$, and $\mathcal{AC}$ is centrally closed.

If $\mathcal{A}$ satisfies a multilinear identity in three or more variables, the result is the same. At the first glance it may not appear entirely obvious how to extend the above argument concerning the two variables case. Anyway, it turns out that it is possible, and the following theorem, established in [152] by Martindale, holds.

Theorem D.1. *Let $\mathcal{A}$ be a prime ring. Then $\mathcal{A}$ is a GPI-ring if and only if $\mathcal{AC}$ contains a minimal idempotent e such that* $\dim_{\mathcal{C}} e\mathcal{AC}e < \infty$.

We have proved the "only if" part for the case when the GPI is of degree ≤ 2. The "if" part is easy. Namely, if $m = \dim_{\mathcal{C}} e\mathcal{AC}e$, then $\mathcal{A}$ satisfies the generalized polynomial identity $St_{m+1}(ex_1e, ex_2e, \ldots, ex_{m+1}e)$.

It should be pointed out that the existence of an idempotent e satisfying the conditions of Theorem D.1 tells a great deal about the structure of $\mathcal{B} = \mathcal{AC}$ (and hence of $\mathcal{A}$). Apparently this is just a local property, it concerns only one element. But it has global consequences. Already the fact that there exist minimal left ideals in $\mathcal{B}$ is decisive. First of all, $\mathcal{B}$ is then a primitive ring since a minimal left ideal of a prime ring can be considered as a faithful simple module. More importantly, the existence of one minimal left ideal $\mathcal{I}$ implies the existence of "many". Indeed, for every $b \in \mathcal{B}$ we have that either $\mathcal{I}b = 0$ or $\mathcal{I}b$ is again a minimal left ideal; namely, if $\mathcal{I}b \neq 0$, then $\mathcal{I}$ and $\mathcal{I}b$ are obviously isomorphic as left $\mathcal{A}$-modules. This implies that the sum of all minimal left ideals in $\mathcal{B}$ is a two-sided ideal of $\mathcal{B}$. It is called the *socle* of $\mathcal{B}$. One can similarly consider the sum of all right ideals of $\mathcal{B}$, but fortunately we get the same ideal (see remarks about minimal one-sided ideals in appendix A; the sum of all minimal left ideals coincides with the sum of all minimal right ideals as long as the ring in question is semiprime). So, $\mathcal{B}$ contains a nonzero ideal that has a very concrete form: its elements are of the form $a_1e_1 + \ldots + a_ne_n$ where $a_i \in \mathcal{A}$ and every e_i is a minimal idempotent. When dealing with a prime ring, it is often the case that if one controls a nonzero ideal, then one controls the entire ring. So the ring $\mathcal{B}$ is really tractable. The information that $\dim_{\mathcal{C}} e\mathcal{B}e < \infty$ is also important. One can show that if e and f are two minimal idempotents in $\mathcal{B}$, then the division algebras $e\mathcal{B}e$ and $f\mathcal{B}f$ are isomorphic. So, in particular, $\dim_{\mathcal{C}} e\mathcal{B}e = \dim_{\mathcal{C}} f\mathcal{B}f$.

Actually, even more can be said about prime (and hence primitive) rings with nonzero socle. They can be represented as rings of linear operators on a vector space (over a division ring) which contain "many" finite rank operators. In fact, the socle is equal to the set of all finite rank operators in this ring. See [40, section 4.3] for details.

There is just one technical result that we still have to record. Its statement is somewhat lengthy, but it is exactly what is needed in the proof of Theorem 5.36.

Lemma D.2. *Let $\mathcal{A}$ be a non-GPI prime ring.*

(i) *If $\{q_{i1}, q_{i2}, \ldots, q_{in_i}\} \subseteq \mathcal{Q}_{ml}(\mathcal{A})$, $i = 1, 2, \ldots, p$, is a collection of $\mathcal{C}$-independent sets, then there exists $x \in \mathcal{A}$ such that the set*

$$\{xq_{i1}, xq_{i2}, \ldots, xq_{in_i}\}$$

is $\mathcal{C}$-independent for every $i = 1, 2, \ldots, p$, and moreover each $xq_{ik} \in \mathcal{A}$.

(ii) *If $\{a_{i1}, a_{i2}, \ldots, a_{in_i}\} \subseteq \mathcal{A}$, $i = 1, 2, \ldots, r$, is a collection of $\mathcal{C}$-independent sets, and $0 \neq a \in \mathcal{A}$, then there exists $y \in \mathcal{A}$ such that the set*

$$\{a_{i1}, a_{i2}, \ldots, a_{in_i}, a_{i1}ya, a_{i2}ya, \ldots, a_{in_i}ya\}$$

is $\mathcal{C}$-independent for every $i = 1, 2, \ldots, r$.

(iii) *If $\{b_{j1}, b_{j2}, \ldots, b_{jn_j}\} \subseteq \mathcal{A}$, $j = 1, 2, \ldots, s$, is a collection of $\mathcal{C}$-independent sets, and $0 \neq b \in \mathcal{A}$, then there exists $z \in \mathcal{A}$ such that the set*

$$\{b_{j1}, b_{j2}, \ldots, b_{jn_j}, bzb_{j1}, bzb_{j2}, \ldots, bzb_{jn_j}\}$$

is $\mathcal{C}$-independent for every $j = 1, 2, \ldots, s$.

The proof is based on the fact that the linear dependence of elements can be expressed through a standard polynomial. For example, if the elements

$$a_1, \ldots, a_n, a_1ya, \ldots, a_nya$$

are $\mathcal{C}$-dependent for every $y \in \mathcal{A}$, then also

$$a_1x, \ldots, a_nx, a_1yax, \ldots, a_nyax$$

are $\mathcal{C}$-dependent for every $y \in \mathcal{A}$ and every $x \in \mathcal{A}$, so that

$$St_{2n}(a_1x, \ldots, a_nx, a_1yax, \ldots, a_nyax) = 0.$$

But this can be interpreted as a nonzero generalized polynomial identity. A more complicated situation involving more sets is just seemingly more difficult; the problem can be resolved by simply multiplying the adequate standard polynomials. We have thereby indicated the idea of the proofs of (ii) and (iii). A modification of this idea, together with Lemma A.2, works for (i) as well. To be honest, the fact that (i) involves $\mathcal{Q}_{ml}(\mathcal{A})$ creates some technical difficulties. However, using [40, Proposition 2.10, Corollary 6.1.7 and Theorem 6.4.4] they can be overcome.

As one could expect, all assertions of the lemma are just special cases of more general phenomena; see [40, Lemma 6.1.8]. We have chosen, however, to avoid stating a more abstract version of the lemma, and rather confine ourselves to what we really need.

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