APPENDIX

INTEGRAL TRANSFORMS (*)

1. Fourier Transforms

We define the operators $\mathfrak{F}$, $\mathfrak{F}_c$ and $\mathfrak{F}_s$ by the equations

$$\mathfrak{F}[f(x);\xi] = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(x)\exp(ix\xi)dx$$

$$\mathfrak{F}_c[f(x);\xi] = \frac{2}{\pi}\int_{0}^{\infty} f(x)\cos(x\xi)dx$$

$$\mathfrak{F}_s[f(x);\xi] = \frac{2}{\pi}\int_{0}^{\infty} f(x)\sin(x\xi)dx$$

respectively. If we introduce the operator $\mathfrak{F}^*$ through the equation

$$\mathfrak{F}^*[f(x);\xi] = \mathfrak{F}[f(x);-\xi]$$

the *Fourier inversion theorem* may be written in the form

$$\hat{f}(\xi) = \mathfrak{F}[f(x);\xi] \implies f(x) = \mathfrak{F}^*[f(x);\xi]$$

or alternatively as

$$\mathfrak{F}^{-1} = \mathfrak{F}^*.$$

(*) A useful general reference is Sneddon, 1972.

In a similar way we can write the *Fourier cosine inversion theorem* as

$$\mathfrak{F}_c^{-1} = \mathfrak{F}_c$$

and the *Fourier sine inversion theorem* as

$$\mathfrak{F}_s^{-1} = \mathfrak{F}_s .$$

If $f(x)$ is continuous for all real values of x we have

$$\mathfrak{F}[f'(x);\xi] = -i\xi \hat{f}(\xi) \tag{A.1}$$

but if $f(x)$ is piecewise continuous with n points of (finite) discontinuity at $a_1,\ldots,a_n$ this formula should be replaced by

$$\mathfrak{F}[f'(x);\xi] = -i\xi\hat{f}(\xi) - (2\pi)^{-1/2}\sum_{r=0}^{n}[f]_{a_r}\exp(i\xi a_r) \tag{A.2}$$

where

$$[f]_{a_r} = f(a_r+) - f(a_r-) . \tag{A.3}$$

If $f(x)$ and its first m derivatives are continuous for all real values of x , equation (A.1) is generalized to

$$\mathfrak{F}[f^{(m)}(x);\xi] = (-i\xi)^m\hat{f}(\xi) . \tag{A.4}$$

Similarly, if we write

$$\hat{f}_c(\xi) = \mathfrak{F}_c[f(x);\xi], \quad \hat{f}_s(\xi) = \mathfrak{F}_s[f(x);\xi],$$

we have the formulae

$$\mathcal{F}_c[f'(t);\xi] = \xi\hat{f}_s(\xi) - (2/\pi)^{1/2} f(0) \tag{A.5}$$

$$\mathcal{F}_s[f'(t);\xi] = -\xi\hat{f}_c(\xi) \tag{A.6}$$

from which we easily deduce that

$$\mathcal{F}_c[f''(t);\xi] = -\xi^2\hat{f}_s(\xi) - (2/\pi)^{1/2} f'(0) \tag{A.7}$$

$$\mathcal{F}_s[f''(t);\xi] = -\xi^2\hat{f}_s(\xi) + (2/\pi)^{1/2}\xi f(0). \tag{A.8}$$

Similarly if we write

$$\hat{f}_c(\xi,y) = \mathcal{F}_c[f(x,y);x\to\xi],$$

$$\hat{f}_s(\xi,y) = \mathcal{F}_s[f(x,y);x\to\xi], \quad f_r(y) = \sqrt{\frac{2}{\pi}}\left[\frac{\partial^r f}{\partial x^r}\right]_{x=0}$$

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad D = \frac{\partial}{\partial y},$$

we have the formulae

$$\mathcal{F}_c[\Delta_2 f(x,y);x\to\xi] = (D^2-\xi^2)\hat{f}_c(\xi,y) - f_1(y) \tag{A.9}$$

$$\mathcal{F}_s[\Delta_2 f(x,y);x\to\xi] = (D^2-\xi^2)\hat{f}_s(\xi,y) + \xi f_0(y) \tag{A.10}$$

$$\mathcal{F}_c[\Delta_2^2 f(x,y);x\to\xi] = (D^2-\xi^2)^2\hat{f}_c(\xi,y) + (\xi^2-2D^2)f_1(y) - f_3(y), \tag{A.11}$$

$$\mathcal{F}_s[\Delta_2^2 f(x,y);x\to\xi] = (D^2-\xi^2)^2\hat{f}_s(\xi,y) - \xi(\xi^2+2D^2)f_0(y) + \xi f_2(y). \tag{A.12}$$

If we define the *convolution* of two functions f and g by

$$f \cdot g = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x-u) g(u) du$$

we have the result

$$\mathfrak{F}[f \cdot g; \xi] = \hat{f}(\xi) \hat{g}(\xi) \tag{A.13}$$

of which a special case is

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(t)|^2 dt . \tag{A.14}$$

Similarly we have the formulae

$$\int_{0}^{\infty} |\hat{f}_c(\xi)|^2 d\xi = \int_{0}^{\infty} |f(t)|^2 dt \tag{A.15}$$

$$\int_{0}^{\infty} |\hat{f}_s(\xi)|^2 d\xi = \int_{0}^{\infty} |f(t)|^2 dt . \tag{A.16}$$

Equations (A.14), (A.15) and (A.16) are called *Parseval's relations for Fourier transforms*.

If we have a function of n independent variables $x_1, x_2, \ldots, x_n$ we define the Fourier transform $f_{(n)}(\xi_1, \xi_2, \ldots, \xi_n)$ by the equation

$$\hat{f}_{(n)}(\vec{\xi}) = \mathfrak{F}_{(n)}[f(\vec{x}); \vec{x} \rightarrow \vec{\xi}] = (2\pi)^{-1/2n} \int_{E_n} f(\vec{x}) \exp\{i(\vec{\xi} \cdot \vec{x})\} d\vec{x} \tag{A.17}$$

where E_n denotes the n-dimensional (real) Euclidean space and

$$(\vec{x} \cdot \vec{\xi}) = x_1 \xi_1 + x_2 \xi_2 + \ldots + x_n \xi_n .$$

Defining the convolution by

$$(f \cdot g)(\vec{x}) = (2\pi)^{-1/2n} \int_{E_n} f(\vec{x} - \vec{u}) g(\vec{u}) d\vec{u}$$

we have the convolution theorem

$$\mathfrak{F}_{(n)}[(f \cdot g)(\vec{x}); \vec{\xi}] = \hat{f}_{(n)}(\vec{\xi}) \hat{g}_{(n)}(\vec{\xi}) . \qquad (A.18)$$

Corresponding to equations (A.11) and (A.12) we have

$$\mathfrak{F}_{(n)}[\Delta_n^r f(x); \xi] = (-1)^r \xi^{2r} \hat{f}_n(\xi) \qquad (A.19)$$

where Δ_n^r is defined by the equations

$$\Delta_n = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

and

$$\Delta_n^r = \Delta_n \Delta_n^{r-1}$$

and ξ^2 by the equation

$$\xi^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 .$$

In applications we often have to find the Fourier transform of a function $f(r)$ which is a function of

$$r = |\vec{x}| = \sqrt{(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

only. It turns out that

$$\mathfrak{F}_{(n)}[f(r); \xi] = \xi^{-\nu} \mathfrak{H}_\nu[r^\nu f(r); \xi], \quad \left(\nu = \frac{1}{2}n - 1\right), \qquad (A.20)$$

where the operator $\mathcal{H}_\nu$ is defined by the equation

$$\mathcal{H}_\nu[f(x);\xi] = \int_0^\infty x f(x) J_\nu(\xi x) dx$$

and $\mathcal{H}_\nu f$ is called the *Hankel transform of order* ν of the function f . In particular, if $\varrho^2 = x_1^2 + x_2^2$, $\lambda^2 = \xi_1^2 + \xi_2^2$,

$$\mathcal{F}_{(2)}[f(\varrho);(\xi_1,\xi_2)] = \mathcal{H}_0[f(\varrho);\lambda] , \qquad \text{(A.21)}$$

and if $r^2 = x_1^2 + x_2^2 + x_3^2$, $\mu^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$,

$$\mathcal{F}_{(3)}[f(r);(\xi_1,\xi_2,\xi_3)] = \mu^{-1}\mathcal{F}_s[rf(r);\mu] . \qquad \text{(A.22)}$$

2. Laplace Transforms

We define the *Laplace transform* of a function $f(x)$ on the positive real line by the equation

$$\bar{f}(p) = \mathcal{L}[f(x),p] = \int_0^\infty f(x)e^{-px}dx . \qquad \text{(A.23)}$$

If $\bar{f}(p)$ is an analytic function of the complex variable p and is of order $O(p^{-k})$ in some half-plane $\operatorname{Re} p > \gamma$, where γ and k are real constants ($k > 1$) , then as $\omega \to \infty$

$$\frac{1}{2\pi i}\int_{c+i\omega}^{c+i\omega} e^{px}\bar{f}(p)dp , \quad (c > \gamma)$$

converges to a function $f(c)$ which is independent of c and whose Laplace transform is $\bar{f}(p)$. We shall refer to this result as the *Laplace inversion theorem.*

We also write (A.23) in the inverse form

$$f(x) = \mathcal{L}^{-1}[\bar{f}(p);x] .$$

Corresponding to equation (A.2) we have the relation

$$\mathcal{L}[f'(x);p] = p\bar{f}(p) - f(0) - \sum_{r=1}^{\infty} [f]_{a_r} \exp(-p a_r) \tag{A.24}$$

which, in the case in which f is continuous reduces to

$$\mathcal{L}[f'(x);p] = p\bar{f}(p) - f(0) . \tag{A.25}$$

Similarly if $f \in C^{n-1}(R^+)$, we have

$$\mathcal{L}[f^{(n)}(x);p] = p^n \bar{f}(p) - \sum_{r=0}^{n-1} p^{n-r-1} f^{(r)}(0) . \tag{A.26}$$

The convolution integral appropriate to the Laplace transform is

$$(f^* g)(x) = \int_0^x f(x-u)g(u)du$$

which has the property

$$\mathcal{L}[(f^* g)(x);p] = \bar{f}(p)\bar{g}(p) . \tag{A.27}$$

A straightforward application of the Laplace transform shows that the integral equation

$$\int_0^x \frac{f(t)dt}{(x^2-t^2)^a} = g(x), \quad x > 0 , \quad 0 < a < 1$$

has solution

$$f(t) = \frac{2\sin(\pi a)}{\pi}\frac{d}{dt}\int_0^t \frac{xg(x)dx}{(t^2-x^2)^{1-a}} , \quad x>0 ,$$

and that the integral equation

$$\int_x^\infty \frac{f(t)dt}{(t^2-x^2)^a} = g(x), \quad x>0, \quad 0<a<1$$

has solution

$$f(t) = -\frac{2\sin(\pi a)}{\pi}\frac{d}{dt}\int_t^\infty \frac{xg(x)dx}{(x^2-t^2)^{1-a}} .$$

We may write these results in the case $a=\frac{1}{2}$ in the form of inversion theorems for the *Abel transforms* $\mathcal{A}_1, \mathcal{A}_2$ defined by the equations

$$\hat{f}_1(x) \equiv \mathcal{A}_1[f(t);x] = \sqrt{\frac{2}{\pi}}\int_0^x \frac{f(t)dt}{\sqrt{(x^2-t^2)}} , \quad x>0 \tag{A.28}$$

$$\hat{f}_2(x) \equiv \mathcal{A}_2[f(t);x] = \sqrt{\frac{2}{\pi}}\int_x^\infty \frac{f(t)dt}{\sqrt{(t^2-x^2)}} , \quad x>0 . \tag{A.29}$$

The appropriate inversion formulae are

$$f(t) = \mathcal{A}_1^{-1}[\hat{f}_1(x);t] \equiv D_t \mathcal{A}_1[x\hat{f}_1(x);t] \tag{A.30}$$

$$f(t) = \mathcal{A}_2^{-1}[\hat{f}_2(x);t] \equiv -D_t \mathcal{A}_2[x\hat{f}_2(x);t] \tag{A.31}$$

with $D_t = d/dt$.

3. Hankel Transforms

The Hankel transform of order ν, $\mathcal{H}_\nu f$, of a function f was defined in the previous section.

From the recurrence formulae for the Bessel functions of the first kind we deduce readily that

$$\mathcal{H}_\nu\left[\rho^{\nu-1}\frac{\partial}{\partial\rho}\{\rho^{1-\nu}f(\rho)\};\xi\right] = -\xi\mathcal{H}_{\nu-1}[f(\rho);\xi] \qquad (A.32)$$

$$\mathcal{H}_\nu\left[\rho^{-\nu-1}\frac{\partial}{\partial\rho}\{\rho^{\nu}f(\rho)\};\xi\right] = \xi\mathcal{H}_{\nu+1}[f(\rho);\xi] \qquad (A.33)$$

and from these deduce that

$$\mathcal{H}_\nu[\mathcal{B}_\nu f(\rho);\xi] = -\xi^2\mathcal{H}_\nu[f(\rho);\xi] \qquad (A.34)$$

where $\mathcal{B}_\nu$ denotes the differential operator

$$\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{\nu^2}{\rho^2}.$$

If Δ_3 is expressed in terms of cylindrical coordinates

$$\mathcal{H}_\nu[\Delta_3 f(\rho,z)e^{i\nu\phi};\rho\to\xi] = (D^2-\xi^2)\bar{f}_\nu(\xi,z)e^{i\nu\phi} \qquad (A.35)$$

where $\bar{f}_\nu(\xi,z) = \mathcal{H}_\nu[f(\rho,z);\rho\to\xi]$ and $D = \partial/\partial z$.

In problems in which there is axial symmetry, we may, by choosing the axis of symmetry to be the z-axis, take the Laplacian operator to be

$$\Delta_a = \mathcal{B}_0 + D^2$$

in which case the result corresponding to (A.35) is

$$\mathcal{H}_0[\Delta_a f(\rho, z); \rho \to \xi] = (D^2 - \xi^2)\bar{f}_0(\xi, z) \qquad \text{(A.36)}$$

Putting $\nu = 1$ in (A.32) and $\nu = 0$ in (A.33) we obtain the important special cases

$$\mathcal{H}_1\left[\frac{\partial f}{\partial \rho}; \rho \to \xi\right] = -\xi \bar{f}_0(\xi) \qquad \text{(A.37)}$$

$$\mathcal{H}_0\left[\frac{1}{\rho}\cdot\frac{\partial f}{\partial \rho}; \rho \to \xi\right] = \xi \bar{f}_1(\xi) . \qquad \text{(A.38)}$$

There are close connections between Hankel and Fourier transforms. For instance, we have the equations

$$\mathcal{H}_0[\hat{f}_c(\xi); \rho] = -\mathcal{A}_2[f'(t); \rho] \qquad \text{(A.39)}$$

$$\mathcal{H}_0[\xi^{-1}\hat{f}_c(\xi); \rho] = \mathcal{A}_1[f(t); \rho] \qquad \text{(A.40)}$$

$$\mathcal{H}_0[\hat{f}_s(\xi); \rho] = \mathcal{A}_1[f'(t); \rho] \qquad \text{(A.41)}$$

$$\mathcal{H}_0[\xi^{-1}\hat{f}_s(\xi); \rho] = \mathcal{A}_2[f(t); \rho] \qquad \text{(A.42)}$$

where $\hat{f}_c$ and $\hat{f}_s$ denotes respectively the Fourier cosine and sine transforms of f.

Similarly, if we denote the Hankel transform of order 0 of f by $\bar{f}_0$ we have the equations

$$\mathcal{F}_c[\bar{f}_0(\xi);x] = \mathcal{A}_2[\varrho f(\varrho);x]$$

$$\mathcal{F}_s[\bar{f}_0(\xi);x] = \mathcal{A}_1[\varrho f(\varrho);x] \, .$$

Two further relations which are useful in applications are

$$\mathcal{F}_c[\mathcal{H}_0\{t^{-1}g(t)H(a-t);\xi\};x] = (2/\pi)^{1/2} H(a-x)\int_x^a \frac{g(t)dt}{\sqrt{(t^2-x^2)}} \tag{A.45}$$

$$\mathcal{F}_s[\mathcal{H}_0\{t^{-1}g(t)H(a-t);\xi\};x] = (2/\pi)^{1/2} \int_0^{\min(x,a)} \frac{g(t)dt}{\sqrt{(t^2-x^2)}} \, . \tag{A.46}$$

4. Mellin Transforms

The Mellin transform $f^*(s)$ of a function $f(x);x\in R^+$, is defined by

$$f^*(s) = \mathfrak{M}[f(x);s] = \int_0^\infty x^{s-1} f(x)dx \tag{A.47}$$

and is easily shown to have the properties

$$\mathfrak{M}[f^{(n)}(x);s] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n) \tag{A.48}$$

$$\mathfrak{M}\left[\left(x\frac{d}{dx}\right)^n f(x);s\right] = (-s)^n f^*(s) \tag{A.49}$$

$$\mathfrak{M}[x^n f^{(n)}(x);s] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s) \, . \tag{A.50}$$

If we express the two-dimensional Laplacian operator Δ_2 in terms of plane polar coordinates ϱ and ϕ we find that

(A.51) $$\mathfrak{M}[\Delta_2 f(\varrho,\phi);\varrho\to s] = [D_\phi^2+(s-2)^2]f^*(s-2,\phi)$$

where $D_\phi = \partial/\partial\phi$ and

(A.52) $$f^*(s,\phi) = \mathfrak{M}[f(\varrho,\phi);\varrho\to s] .$$

Applying (A.51) twice, we obtain

(A.53) $$\mathfrak{M}[\Delta_2^2 f(\varrho,\phi);\varrho\to s] = [D^2+(s-2)^2][D^2+(s-4)^2]f^*(s-4,\phi) .$$

Use is frequently made of the Mellin transforms of certain integral expressions. For instance we have the relations

(A.54) $$\mathfrak{M}\left[x^\lambda \int_0^\infty u^\mu f(xu)g(u)du;s\right] = f^*(s+\lambda)g^*(\mu+1-\lambda-s) ,$$

(A.55) $$\mathfrak{M}\left[x^\lambda \int_0^\infty u^\mu f(x/u)g(u)du;s\right] = f^*(s+\lambda)g^*(\lambda+\mu+1+s) .$$

The case $\lambda = 0$, $\mu = -1$ of (A.55) written in inverse form is of value

(A.56) $$\mathfrak{M}^{-1}[f^*(s)g^*(s);x] = \int_0^\infty f(x/u)g(u)u^{-1}du .$$

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