Part III
Appendices

This third part of the book contains the Appendices, which put together mathematical materials used for the analytical work presented. The materials range from the equations of change for mass, momentum and energy of a fluid continuum, and some important vector analytical basics, to some useful materials about special functions of mathematical physics. The materials are selected in a way that other handbooks are not needed for verifying the text and solving other problems of the kinds discussed here.

Appendix A
The Equations of Change in Transport Processes

In this first part of the Appendix we put together the equations of change of continuum fluid mechanics. We simply list the equations as an encyclopedic table. In the versions of the equations specialised for *Newton*ian fluids we restrict the presentation to the incompressible case. The presentation is oriented at [2].

A.1 The Equation of Continuity

The equation of continuity in its differential formulation states that, at every place of a flow field, a change of fluid density with time is caused by the divergence of the mass flux vector. Deviations from this form of the equation are due to sources in the flow field, which can only exist in multi-phase or multi-component systems, where, e.g., liquid mass can "disappear" by evaporation into the gas phase, or a mixture component can condense from the gas phase and add to the liquid system. Other potential sources for mass of a mixture component are chemical reactions.

The *equation of continuity* in Cartesian, cylindrical, and spherical coordinate systems reads

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho u_\phi) = 0$$

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A.2 The Momentum Equation

The momentum equation represents *Newton*'s second law for a flowing fluid, which states that a change of the state of motion of the fluid is caused by the forces acting on it.

The *equations of motion* in Cartesian, cylindrical, and spherical coordinates read as follows.

The equations of motion in Cartesian coordinates (*x*, *y*, *z*)

x-component
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+ +\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho f_x^B$$

y-component
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+ +\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho f_y^B$$

z-component
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+ +\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho f_z^B$$

for a *Newton*ian fluid with constant ρ and μ

x-component
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+ +\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho f_x^B$$

$$\textit{y-component} \quad \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+ \\ +\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho f_y^B$$

$$\textit{z-component} \quad \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+ \\ +\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho f_z^B$$

The equations of motion in cylindrical coordinates

(r,θ,z); $x=r\cos\theta,\ y=r\sin\theta,\ z=z;\ r=\sqrt{x^2+y^2},\ \theta=\arctan(y/x),\ z=z$

$$\textit{r-component} \quad \rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+ \\ +\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r$$

$$\textit{\theta-component} \quad \rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+ \\ +\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}\right)+\rho g_\theta$$

$$\textit{z-component} \quad \rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+ \\ +\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

for a *Newton*ian fluid with constant ρ and μ

$$\textit{r-component} \quad \rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+ \\ +\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

θ-component
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+\rho g_\theta$$

z-component
$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+$$
$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho g_z$$

The equations of motion in spherical coordinates

$(r,\theta,\phi);\ x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta;$
$r=\sqrt{x^2+y^2+z^2},\ \theta=\arctan(\sqrt{x^2+y^2}/z),\ \phi=\arctan(y/x)$

r-component
$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\theta^2+u_\phi^2}{r}\right)=$$
$$=-\frac{\partial p}{\partial r}+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta)+\right.$$
$$\left.+\frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right)+\rho f_r^B$$

θ-component
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\cot\theta}{r}\right)=$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta)+\right.$$
$$\left.+\frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi}+\frac{\tau_{r\theta}}{r}-\frac{\cot\theta}{r}\tau_{\phi\phi}\right)+\rho f_\theta^B$$

φ-component
$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\phi u_r}{r}+\frac{u_\theta u_\phi}{r}\cot\theta\right)=$$
$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi})+\frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi}+\right.$$
$$\left.+\frac{\tau_{r\phi}}{r}+\frac{2\cot\theta}{r}\tau_{\theta\phi}\right)+\rho f_\phi^B$$

for a *Newton*ian fluid with constant ρ and μ

r-component
$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\theta^2+u_\phi^2}{r}\right)=$$
$$=-\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2u_r)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \phi^2}\right)+\rho f_r^B$$

θ-component
$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta}{r}-\frac{u_\phi^2\cot\theta}{r}\right)=$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(u_\theta\sin\theta)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2}+\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right)+\rho f_\theta^B$$

ϕ-component
$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\phi u_r}{r}+\frac{u_\theta u_\phi}{r}\cot\theta\right)=$$
$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(u_\phi\sin\theta)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2}+\right.$$
$$\left.+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi}\right)+\rho f_\phi^B$$

The components of the stress tensor for *Newton*ian fluids in Cartesian coordinates (x, y, z)

$$\tau_{xx}=\mu\left[2\frac{\partial u}{\partial x}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$
$$\tau_{yy}=\mu\left[2\frac{\partial v}{\partial y}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$
$$\tau_{zz}=\mu\left[2\frac{\partial w}{\partial z}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$
$$\tau_{xy}=\tau_{yx}=\mu\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]$$
$$\tau_{yz}=\tau_{zy}=\mu\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]$$
$$\tau_{zx}=\tau_{xz}=\mu\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]$$

The components of the stress tensor for *Newton*ian fluids in cylindrical coordinates (r, θ, z)

$$\tau_{rr}=\mu\left[2\frac{\partial u_r}{\partial r}-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$
$$\tau_{\theta\theta}=\mu\left[2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)-\frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{zz} = \mu\left[2\frac{\partial u_z}{\partial z} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right]$$

The components of the stress tensor for *Newton*ian fluids in spherical coordinates (r, θ, ϕ)

$$\tau_{rr} = \mu\left[2\frac{\partial u_r}{\partial r} - \frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{\theta\theta} = \mu\left[2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) - \frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{\phi\phi} = \mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right) - \frac{2}{3}(\nabla\cdot\mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right]$$

A.3 The Equation of Mechanical Energy

The equation of mechanical energy is obtained by multiplying the momentum equation with the velocity vector. The equation states that the change of mechanical energy of the fluid system is due to the powers of forces acting on it. The dissipation function represents the loss of mechanical energy into the form of thermal energy.

The *equation of mechanical energy* reads as follows.

$$\rho\left[\frac{\partial(\frac{1}{2}|\mathbf{v}|^2)}{\partial t}+u\frac{\partial(\frac{1}{2}|\mathbf{v}|^2)}{\partial x}+v\frac{\partial(\frac{1}{2}|\mathbf{v}|^2)}{\partial y}+w\frac{\partial(\frac{1}{2}|\mathbf{v}|^2)}{\partial z}\right]=\rho\frac{d}{dt}\left[\frac{1}{2}|\mathbf{v}|^2\right]=$$

$$=-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)+u\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+$$

$$+v\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+w\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)+\rho\left(uf_x^B+vf_y^B+wf_z^B\right)\ .$$

We may write this in a more compact form as

$$\rho\frac{d}{dt}\left[\frac{1}{2}|\mathbf{v}|^2\right]=-(\mathbf{v}\cdot\nabla p)+(\mathbf{v}\cdot[\nabla\cdot\tau])+\rho\left(\mathbf{v}\cdot\mathbf{f}^B\right)\ ,$$

which we re-write further, applying the product rule of differential calculus, into the form

$$\rho\frac{d}{dt}\left[\frac{1}{2}|\mathbf{v}|^2\right]=p\,(\nabla\cdot\mathbf{v})-(\nabla\cdot p\mathbf{v})+(\nabla\cdot[\tau\cdot\mathbf{v}])-(\tau:\nabla\mathbf{v})+\rho\left(\mathbf{v}\cdot\mathbf{f}^B\right)\ .$$

This is the equation of mechanical energy which we use in the following section to extract the thermal from the total energy equation.

A.4 The Equation of Thermal Energy

The equation of thermal energy is obtained by subtracting the mechanical energy equation from the equation for the total energy. The equation states that the change of thermal energy of the fluid system is due to the power of pressure forces acting on a compressible system, the rate of heat transported by conduction, viscous dissipation and sources of heat in the system. The dissipation function represents the heating of the system by the viscous losses of mechanical energy.

The *equation of the mechanical and thermal energies* reads as follows.

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{1}{2}|\mathbf{v}|^2\right)\right]+\left[\nabla\cdot\rho\mathbf{v}\left(e+\frac{1}{2}|\mathbf{v}|^2\right)\right]=\rho\left(\mathbf{v}\cdot\mathbf{f}^B\right)-$$
$$-(\nabla\cdot p\mathbf{v})+(\nabla\cdot[\tau\cdot\mathbf{v}])-(\nabla\cdot\mathbf{q})+\dot{q}_Q\ .$$

Making use of the continuity equation for re-writing the left-hand side of the equation, we may write this in a more compact form as

$$\rho\frac{d}{dt}\left(e+\frac{1}{2}|\mathbf{v}|^2\right)=\rho\left(\mathbf{v}\cdot\mathbf{f}^B\right)-(\nabla\cdot p\mathbf{v})+(\nabla\cdot[\tau\cdot\mathbf{v}])-(\nabla\cdot\mathbf{q})+\dot{q}_Q\ .$$

Subtracting the mechanical from the total energy equation, we obtain the thermal energy equation in the formulation with the internal energy as

$$\rho\left(\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}+w\frac{\partial e}{\partial z}\right)=-p\left(\nabla\cdot\mathbf{v}\right)+\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+\tau_{zx}\frac{\partial u}{\partial z}+$$

$$+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zy}\frac{\partial v}{\partial z}+$$

$$+\tau_{xz}\frac{\partial w}{\partial x}+\tau_{yz}\frac{\partial w}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+\dot{q}_Q\,.$$

In a more compact form we may rewrite this as

$$\rho\frac{de}{dt}=-p\left(\nabla\cdot\mathbf{v}\right)+\left(\tau:\nabla\mathbf{v}\right)-\left(\nabla\cdot\mathbf{q}\right)+\dot{q}_Q\,.$$

We may re-write this by introducing the enthalpy $h=e+p/\rho$. The result is

$$\rho\frac{dh}{dt}=\frac{dp}{dt}+\left(\tau:\nabla\mathbf{v}\right)-\left(\nabla\cdot\mathbf{q}\right)+\dot{q}_Q\,.$$

In the thermal energy equation, the dissipation function occurs, which we represent for *Newton*ian fluids as follows.

The dissipation function $[\tau:\nabla u]=\mu\Phi_\mu$ for *Newton*ian fluids

Cartesian

$$\Phi_\mu=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+$$

$$+\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]^2+\left[\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right]^2+\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]^2-$$

$$-\frac{2}{3}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]^2$$

Cylindrical

$$\Phi_\mu=2\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2\right]+$$

$$+\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2+\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right]^2+\left[\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right]^2-$$

$$-\frac{2}{3}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right]^2$$

Spherical

$$\Phi_\mu = 2\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)^2 + \right.$$

$$\left. + \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r}\right)^2\right] +$$

$$+ \left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]^2 +$$

$$+ \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right]^2 +$$

$$+ \left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right]^2 -$$

$$- \frac{2}{3}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]^2$$

Appendix B
Basic Vector Analytical Operations

In this second part of the Appendix we put together important differential operations from vector analysis with relevance to the transport equations governing fluid mechanics as well as heat and mass transfer discussed in the present book. We simply list the equations as an encyclopedic table.

B.1 The Gradient of a Scalar Field

The gradient of a scalar field is vectorial quantity determining the variability of the scalar function in space. We denote the scalar field S, which may be interpreted as the velocity potential of a flow.

The gradient of the potential in Cartesian, cylindrical, and spherical coordinates reads

Cartesian coordinates (x, y, z):

$$\nabla S = \frac{\partial S}{\partial x}\mathbf{e_x} + \frac{\partial S}{\partial y}\mathbf{e_y} + \frac{\partial S}{\partial z}\mathbf{e_z}$$

Cylindrical coordinates (r, θ, z):

$$\nabla_\mathbf{c} S = \frac{\partial S}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_{\boldsymbol{\theta}} + \frac{\partial S}{\partial z}\mathbf{e_z}$$

Spherical coordinates (r, θ, ϕ):

$$\nabla_\mathbf{s} S = \frac{\partial S}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial S}{\partial \theta}\mathbf{e}_{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\mathbf{e}_{\boldsymbol{\phi}}$$

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B.2 The Divergence of a Vector Field

The divergence of a vector field is formally the projection of the vector on the nabla (gradient) vector. As the vector field, here we take the velocity field of the flow.

The divergence of the velocity vector in Cartesian, cylindrical, and spherical coordinates reads

Cartesian coordinates (x, y, z), $\mathbf{v} = (u, v, w)$*:*

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Cylindrical coordinates (r, θ, z), $\mathbf{v} = (u_r,\ u_\theta,\ u_z)$*:*

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

Spherical coordinates (r, θ, ϕ), $\mathbf{v} = (u_r,\ u_\theta,\ u_\phi)$*:*

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

B.3 The Curl of a Vector Field

The curl of a vector field represents the rotation of the vector field as given by the vector (inner) product of the nabla operator with the vector. As the vector field, here we take the velocity field of the flow.

The curl of the velocity vector in Cartesian, cylindrical, and spherical coordinates reads

Cartesian coordinates (x, y, z), $\mathbf{v} = (u, v, w)$*:*

$$\nabla \times \mathbf{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e_x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e_y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e_z}$$

Cylindrical coordinates (r, θ, z), $\mathbf{v} = (u_r,\ u_\theta,\ u_z)$*:*

$$\nabla_\mathbf{c} \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e_r} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e_\theta} + \left(\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\mathbf{e_z}$$

Spherical coordinates (r, θ, φ), $\mathbf{v} = (u_r,\ u_\theta,\ u_\phi)$*:*

$$\nabla_\mathbf{s} \times \mathbf{v} = \left(\frac{1}{r \sin\theta} \frac{\partial(u_\phi \sin\theta)}{\partial\theta} - \frac{1}{r \sin\theta} \frac{\partial u_\theta}{\partial\phi} \right) \mathbf{e_r} + \left(\frac{1}{r \sin\theta} \frac{\partial u_r}{\partial\phi} - \frac{1}{r} \frac{\partial(r u_\phi)}{\partial r} \right) \mathbf{e_\theta} +$$

$$+ \left(\frac{1}{r} \frac{\partial(r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial\theta} \right) \mathbf{e_\phi}$$

B.4 The Laplacian of a Scalar Field

The Laplacian of a scalar field represents the divergence of the gradient field of the scalar. The scalar as a function of three spatial coordinates is denoted S. Satisfying the *Laplace* equation, it may be interpreted as a velocity potential of a solenoidal, curl-free vector field [2].

The Laplacian of the scalar S in Cartesian, cylindrical, and spherical coordinates reads

Cartesian coordinates (x, y, z):

$$\Delta S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}$$

Cylindrical coordinates (r, θ, z):

$$\Delta_c S = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial S}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 S}{\partial\theta^2} + \frac{\partial^2 S}{\partial z^2}$$

Spherical coordinates (r, θ, φ):

$$\Delta_s S = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial S}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial S}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 S}{\partial\phi^2}$$

B.5 The Laplacian of a Vector Field

The Laplacian of a vector field represents the divergence of the gradient field of the vector and is, therefore, itself a vectorial quantity. It differs from the Laplacian of a scalar field since, in general, the argument of the operation is a tensor representing the dependency of all the vector components on all the three coordinate directions. This includes the dependency of the unit vectors in the coordinate directions on the respective point in space. As the vector field, here we write the velocity field **v** as a function of three spatial coordinates, which is the most relevant case in the present context [2].

The Laplacian of the velocity vector in Cartesian, cylindrical, and spherical coordinates reads

Cartesian coordinates (x, y, z), $\mathbf{v} = (u,\ v,\ w)$:

$$[\Delta \mathbf{v}]_x = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$[\Delta \mathbf{v}]_y = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

$$[\Delta \mathbf{v}]_z = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

Cylindrical coordinates (r, θ, z), $\mathbf{v} = (u_r,\ u_\theta,\ u_z)$:

$$[\Delta_c \mathbf{v}]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}$$

$$[\Delta_c \mathbf{v}]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}$$

$$[\Delta_c \mathbf{v}]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}$$

Spherical coordinates (r, θ, ϕ), $\mathbf{v} = (u_r,\ u_\theta,\ u_\phi)$:

$$[\Delta_s \mathbf{v}]_r = \Delta u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{2u_\theta \cot\theta}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$[\Delta_s \mathbf{v}]_\theta = \Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial u_\phi}{\partial \phi}$$

$$[\Delta_s \mathbf{v}]_\phi = \Delta u_\phi - \frac{u_\phi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial u_\theta}{\partial \phi}$$

Appendix C
Special Functions of Mathematical Physics

In this third part of the Appendix we put together useful information about special functions of mathematical physics with relevance to transport processes in the coordinate systems represented here. An important source of the full detail of information about special functions is the book [1]. In the present context we restrict our presentation to the differential equations occurring in transport problems when seeking solutions as products of eigenfunctions after separation of variables. The special functions are the solutions of these differential equations. We present them together with graphs and, if applicable, recurrence relations needed in the derivation of the solutions.

In this presentation we follow the lines of the book [1]. We denote the unknown function $w(z)$, with the independent variable z, which may in general be complex. We denote derivatives of the function w with respect to its argument by primes.

C.1 Bessel Functions of Integer Order

Bessel's differential equation belongs to the hypergeometric type of ODEs and reads for the function $w = w(z)$

$$z^2 w'' + z w' + \left(z^2 - \nu^2\right) w = 0 \,.$$

The general solution of the equation is

$$w(z) = A J_\nu(z) + B Y_\nu(z) \,,$$

where $J_\nu(z)$ and $Y_\nu(z)$ are *Bessel* functions of the first and second kinds, respectively. The *Bessel* functions of the third kind (also called *Hankel* functions) may be expressed in terms of these two functions. *Hankel* functions are not discussed here, since they are not needed for representing the transport processes of this book.

G. Brenn, *Analytical Solutions for Transport Processes*,
Mathematical Engineering, DOI 10.1007/978-3-662-51423-8

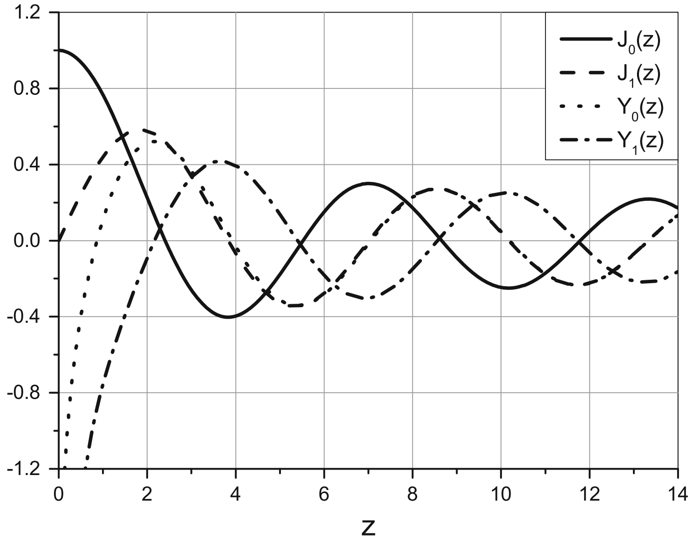

Fig. C.1 Graphs of the *Bessel* functions $J_0(z)$, $J_1(z)$, $Y_0(z)$ and $Y_1(z)$

Graphs of the *Bessel* functions of the first and second kinds for the degrees 0 and 1 are depicted in Fig. C.1. The graphs show that the functions $Y_\nu(z)$ diverge for zero value of the argument. They are therefore to be discarded from solutions for fields containing the origin of the coordinate system.

Recurrence relations and differentiation rules are

$$\begin{aligned} \mathscr{C}_{\nu-1}(z) + \mathscr{C}_{\nu+1}(z) &= \frac{2\nu}{z}\mathscr{C}_\nu(z) \\ \mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z) &= 2\mathscr{C}'_\nu(z) \\ \mathscr{C}_{\nu-1}(z) - \frac{\nu}{z}\mathscr{C}_\nu(z) &= \mathscr{C}'_\nu(z) \\ -\mathscr{C}_{\nu+1}(z) + \frac{\nu}{z}\mathscr{C}_\nu(z) &= \mathscr{C}'_\nu(z) \end{aligned}$$

where the function $\mathscr{C}$ represents any one of the functions J or Y or any linear combination of them. From the last relation follows in particular

$$J'_0(z) = -J_1(z)\ , \quad Y'_0(z) = -Y_1(z)\ .$$

The functions are represented by series expansions as

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{i=0}^{\infty} \frac{\left(-z^2/4\right)^i}{i!(\nu+i+1)!}$$

$$Y_\nu(z) = -\frac{1}{\pi}\left(\frac{1}{2}z\right)^{-\nu} \sum_{i=0}^{\nu-1} \frac{(\nu-i-1)!}{i!}\left(\frac{1}{4}z^2\right)^i + \frac{2}{\pi}\ln\left(\frac{1}{2}z\right) J_\nu(z) -$$
$$-\frac{1}{\pi}\left(\frac{1}{2}z\right)^\nu \sum_{i=0}^{\infty} [\psi(i+1)+\psi(\nu+i+1)] \frac{\left(-\frac{1}{4}z^2\right)^i}{i!(\nu+i)!}$$

where ψ is the digamma function [1]. The function of the second kind is as well represented as

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} ,$$

where the right hand side is replaced by its limiting value of ν is an integer or zero. Furthermore, the *Bessel* function of negative order $J_{-\nu}(z)$ is given as

$$J_{-\nu}(z) = (-1)^\nu J_\nu(z) .$$

The same relation holds for $Y_{-\nu}(z)$ as well.

Limiting forms for small arguments are

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \frac{1}{\Gamma(\nu+1)} \quad \text{for small arguments } z$$
$$Y_\nu(z) = -\frac{1}{\pi}\left(\frac{1}{2}z\right)^{-\nu} \Gamma(\nu) \quad \text{for small arguments } z$$

Limiting forms for large arguments are

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \quad \text{for large arguments } z$$
$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \quad \text{for large arguments } z$$

Approximations of the functions J_0, J_1, Y_0 and Y_1 by power series are given by *Newman* [4].

The *Bessel* functions of integer order for negative argument are given as

$$J_\nu(-z) = e^{i\nu\pi} J_\nu(z)$$
$$Y_\nu(-z) = e^{-i\nu\pi} Y_\nu(z) + i2\cos(\nu\pi) J_\nu(z)$$

In the case of integer order of the functions, then denoted n, these relations reduce to

$$J_n(-z) = (-1)^n J_n(z)$$
$$Y_n(-z) = (-1)^n Y_n(z) + 2i(-1)^n J_n(z)$$

C.2 Modified Bessel Functions

One version of *Bessel*'s differential equation differs from the latter by a minus in front of the last term on the left-hand side. This difference leads to imaginary arguments of the *Bessel* functions. For this type of argument of the *Bessel* functions, corresponding modified *Bessel* functions are defined. Their arguments are related to those of the *Bessel* functions by a function of the imaginary unit i.

This version of *Bessel*'s differential equation reads

$$z^2 w'' + z w' - \left(z^2 + \nu^2\right) w = 0\,.$$

The general solution of the equation is

$$w(z) = A I_\nu(z) + B K_\nu(z)\,,$$

where $I_\nu(z)$ and $K_\nu(z)$ are the modified *Bessel* functions of the first and second kinds, respectively. These functions are related to the *Bessel* functions by

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(z e^{i\pi/2}) \quad \text{for} -\pi < \arg z \leq \pi/2 \tag{C.1}$$
$$I_\nu(z) = e^{i3\nu\pi/2} J_\nu(z e^{-i3\pi/2}) \quad \text{for}\ \pi/2 < \arg z \leq \pi \tag{C.2}$$
$$K_\nu(z) = \frac{\pi}{2} e^{i(\nu+1)\pi/2} J_\nu(z e^{i\pi/2}) - \frac{\pi}{2} e^{i\nu\pi/2} Y_\nu(z e^{i\pi/2}) \quad \text{for} -\pi < \arg z \leq \pi/2$$

Graphs of the modified *Bessel* functions of the first and second kinds for the degrees 0 and 1 are depicted in Fig. C.2. The graphs show that the functions $K_\nu(x)$ diverge for zero value of the argument. They are therefore to be discarded from solutions for fields containing the origin of the coordinate system. The complement applies for the functions $I_\nu(x)$, which are discarded from solutions "outside" an interface.

Recurrence relations and differentiation rules are

$$\mathscr{Z}_{\nu-1}(z) - \mathscr{Z}_{\nu+1}(z) = \frac{2\nu}{z}\mathscr{Z}_\nu(z)$$
$$\mathscr{Z}_{\nu-1}(z) + \mathscr{Z}_{\nu+1}(z) = 2\mathscr{Z}'_\nu(z)$$
$$\mathscr{Z}_{\nu-1}(z) - \frac{\nu}{z}\mathscr{Z}_\nu(z) = \mathscr{Z}'_\nu(z)$$
$$\mathscr{Z}_{\nu+1}(z) + \frac{\nu}{z}\mathscr{Z}_\nu(z) = \mathscr{Z}'_\nu(z)$$

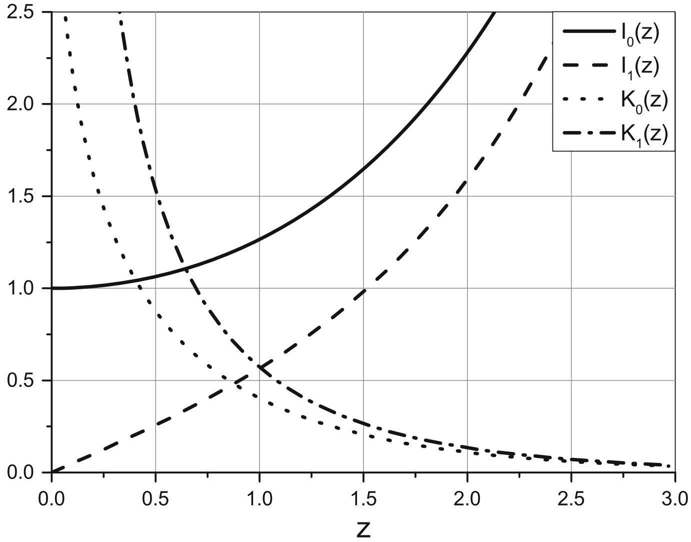

Fig. C.2 Graphs of the modified *Bessel* functions $I_0(x)$, $I_1(x)$, $K_0(x)$ and $K_1(x)$

where the function $\mathscr{Z}$ represents any one of the functions I_ν or $e^{i\nu\pi} K_\nu$ or any linear combination of them. From the last relation follows in particular

$$I_0'(z) = I_1(z) \ , \quad K_0'(z) = -K_1(z) \ .$$

Due to the mentioned difference in a sign in the underlying *Bessel*-type differential equations, there exist relations between the *Bessel* functions and their modified counterparts. The relation reads

$$\begin{aligned}
I_\nu(z) &= e^{-i\nu\pi/2} J_\nu(z e^{i\pi/2}) \quad \text{for} -\pi < \arg z \le \pi/2 \\
I_\nu(z) &= e^{-i3\nu\pi/2} J_\nu(z e^{i3\pi/2}) \quad \text{for}\ \pi/2 < \arg z \le \pi \\
K_\nu(z) &= i\,(\pi/2)\, e^{i\nu\pi/2} H_\nu^{(1)}(z e^{i\pi/2}) \quad \text{for} -\pi < \arg z \le \pi/2 \\
K_\nu(z) &= -i\,(\pi/2)\, e^{-i\nu\pi/2} H_\nu^{(2)}(z e^{-i\pi/2}) \quad \text{for}\ \pi/2 < \arg z \le \pi \ ,
\end{aligned}$$

where the functions $H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)$ and $H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$ are *Hankel* functions [1].

The functions are represented by series expansions as

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{i=0}^{\infty} \frac{\left(z^2/4\right)^i}{i!(\nu+i+1)!}$$

$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^{-\nu} \sum_{i=0}^{\nu-1} \frac{(\nu-i-1)!}{i!}\left(-\frac{1}{4}z^2\right)^i + (-1)^{\nu+1}\ln\left(\frac{1}{2}z\right) I_\nu(z) +$$
$$+ (-1)^\nu \frac{1}{2}\left(\frac{1}{2}z\right)^\nu \sum_{i=0}^{\infty} [\psi(i+1)+\psi(\nu+i+1)] \frac{\left(\frac{1}{4}z^2\right)^i}{i!(\nu+i)!}$$

where ψ is the digamma function [1]. The function of the second kind may as well be be determined as

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} ,$$

where the right hand side is replaced by its limiting value of ν is an integer or zero. Furthermore, the modified *Bessel* function of negative order $I_{-\nu}(z)$ is given as

$$I_{-\nu}(z) = I_\nu(z) .$$

The same relation holds for $K_{-\nu}(z)$ as well.

Limiting forms for small arguments are

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \frac{1}{\Gamma(\nu+1)} \quad \text{for small arguments } z$$
$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^{-\nu} \Gamma(\nu) \quad \text{for small arguments } z$$

Limiting forms for large arguments are

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} - \right.$$
$$\left. - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \ldots\right] \quad \text{for large arguments } z$$
$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left[1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \right.$$
$$\left. + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \ldots\right] \quad \text{for large arguments } z$$

where $\mu = 4\nu^2$.

Approximations of the modified *Bessel* functions may be derived from the power series approximations of the functions J_0, J_1, Y_0 and Y_1 given by *Newman* [4]. Here we make use of the relations (C.1) and (C.2).

The modified *Bessel* functions for negative argument are given as

$$I_\nu(-z) = e^{i\nu\pi} I_\nu(z)$$
$$K_\nu(-z) = e^{-i\nu\pi} K_\nu(z) - i\pi I_\nu(z)$$

In the case of integer order of the functions, then denoted n, these relations reduce to

$$I_n(-z) = (-1)^n I_n(z)$$
$$K_n(-z) = (-1)^n K_n(z) - i\pi I_n(z)$$

C.3 Spherical Bessel Functions

Another version of *Bessel*'s differential equation differs from the latter by a particular numerical structure of one parameter in front of the term with the unknown function. This difference leads to *Bessel* functions of fractional order, called the spherical *Bessel* functions.

$$z^2 w'' + 2zw' + \left[z^2 - n(n+1)\right] w = 0 \, .$$

The general solution of the equation is

$$w(z) = A j_n(z) + B y_n(z) \, ,$$

where $j_n(z) = \sqrt{\pi/2z} J_{n+1/2}(z)$ and $y_n(z) = \sqrt{\pi/2z} Y_{n+1/2}(z)$ are spherical *Bessel* functions of the first and second kinds, respectively. The spherical *Bessel* functions of the third kind may be expressed als complex combinations of these two functions. They are not discussed here since they are not needed for representing the transport processes of this book.

Graphs of the spherical *Bessel* functions of the first and second kinds for the degrees 0 and 1 are depicted in Fig. C.3. The graphs show that the functions $y_\nu(x)$ diverge for zero value of the argument. They are therefore to be discarded from solutions for fields containing the origin of the coordinate system.

Recurrence relations and differentiation rules are

$$\mathscr{F}_{n-1}(z) + \mathscr{F}_{n+1}(z) = \frac{2n+1}{z} \mathscr{F}_n(z)$$
$$n\mathscr{F}_{n-1}(z) - (n+1)\mathscr{F}_{n+1}(z) = 2(n+1)\mathscr{F}_n'(z)$$
$$\mathscr{F}_{n-1}(z) - \frac{n+1}{z} \mathscr{F}_n(z) = \mathscr{F}_n'(z)$$

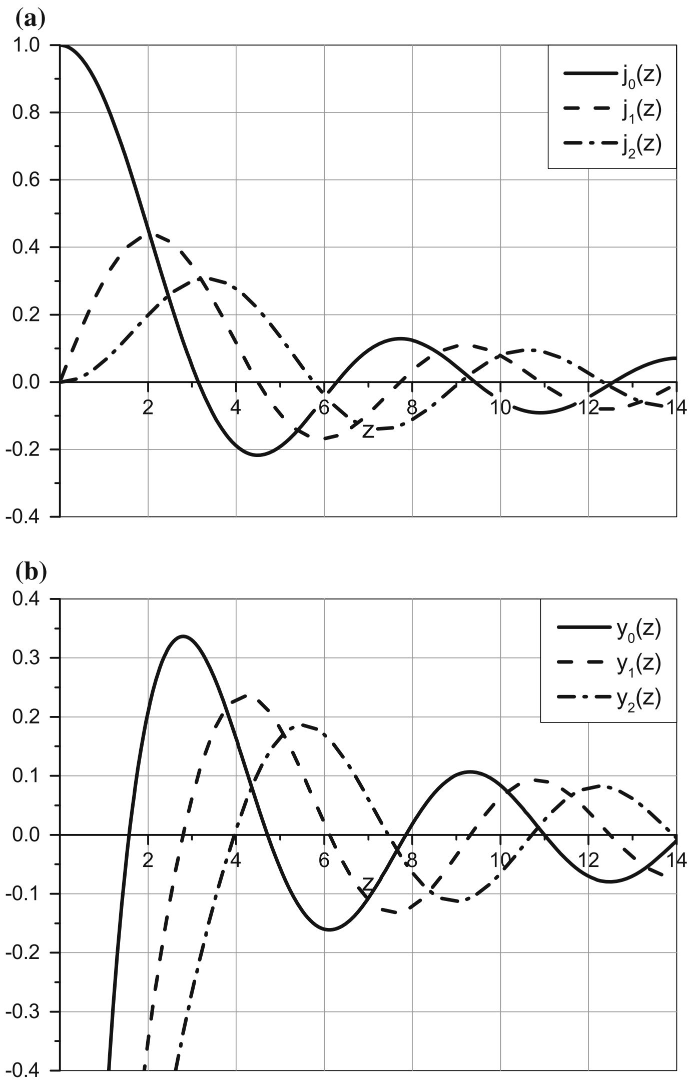

Fig. C.3 Graphs of the spherical *Bessel* functions **a** $j_0(z)$, $j_1(z)$ and $j_2(z)$, and **b** $y_0(z)$, $y_1(z)$ and $y_2(z)$

$$-\mathscr{F}_{n+1}(z) + \frac{n}{z}\mathscr{F}_n(z) = \mathscr{F}_n'(z)$$

where the function $\mathscr{F}$ represents any one of the functions j_n or y_n or any linear combination of them. From the last relation follows in particular

$$j_0'(z) = -j_1(z)\ , \quad y_0'(z) = -y_1(z)\ .$$

The spherical *Bessel* functions may be represented by sine and cosine functions as per

$$j_0(z) = \frac{\sin z}{z}$$
$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$
$$j_2(z) = -\left(z^2 - 3\right)\frac{\sin z}{z^3} - 3\frac{\cos z}{z^2}$$
$$y_0(z) = -\frac{\cos z}{z}$$
$$y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$
$$y_2(z) = \left(z^2 - 3\right)\frac{\cos z}{z^3} - 3\frac{\sin z}{z^2}$$

The solution of the so-called *Riccati-Bessel* differential equation

$$z^2 w'' + \left[z^2 - n\,(n+1)\right] = 0$$

may be written in terms of the spherical *Bessel* functions as

$$w(z) = Azj_n(z) + Bzy_n(z)$$

or with the functions $j_n(z)$ and $y_n(z)$ replaced by the spherical *Hankel* functions $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$ [1].

C.4 Integrals of Bessel Functions

The present section puts together some useful relations for integrals of *Bessel* functions, which include orthogonality relations needed for series expansions in *Bessel* functions. In the representation of these equations we follow the lines of [1].

$$\int_0^z t^n J_{n-1}(t)dt = z^n J_n(z) \tag{C.3}$$

$$\int_0^z t^{-n} J_{n+1}(t)dt = \frac{1}{2^n \Gamma(n+1)} - z^{-n} J_n(z) \tag{C.4}$$

The orthogonality relations involve integrals over products of *Bessel* functions.

$$\int_a^b t\mathscr{F}_n(\lambda_i t)\mathscr{F}_n(\lambda_j t)dt = 0 \quad \text{if i} \neq \text{j}$$

$$= \left\{\frac{1}{2}t^2\left[\left(1-\frac{n^2}{\lambda_i^2 t^2}\right)\mathscr{F}_n^2(\lambda_i t) + \mathscr{F}_i'^2(\lambda_i t)\right]\right\}_a^b \quad \text{if i = j} \tag{C.5}$$

where $0 < a < b$. This relation is valid if λ_i is a real root of $h_1\mathscr{F}_{n+1}(\lambda b) - h_2\mathscr{F}(\lambda b) = 0$ and if there exist two numbers k_1 and k_2 (both not zero) such that for all i the equation $k_1\lambda_i\mathscr{F}_{n+1}(\lambda_i a) - k_2\mathscr{F}_n(\lambda_i a) = 0$ holds.

An extensive list of integrals of *Bessel* functions and their products, and of their products with elementary functions, is found in [5].

C.5 Legendre Functions

Legendre's differential equation governs the dependency of a function on the polar angle θ in some spherical problems. The independent variable of the function, here denoted z, is interpreted as the cosine of the polar angle θ in the spherical coordinate system. The equation for the function $w(z)$ reads

$$\left(1-z^2\right)w'' - 2zw' + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]w = 0\,.$$

The fundamental system of the differential equation is given by the *Legendre* functions of the first and second kinds. The general solution of the equation is therefore given as

$$w(z) = AP_\nu^\mu(z) + BQ_\nu^\mu(z)\,,$$

where $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are the *Legendre* functions of the first and second kinds, respectively. In many applications, the parameter $\mu = 0$. This simplifies the differential equation and makes the *Legendre* functions of the first kind $P_\nu^\mu(z)$ reduce to the *Legendre* polynomials $P_n(z)$ $(\nu = n)$. The latter are given by *Rodrigues*' formula as [1]

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}\left(z^2-1\right)^n\,.$$

Graphs of the *Legendre* functions of the first and second kinds, $P_m^n(\cos\theta)$ and $Q_m^n(\cos\theta)$, for the special case that $n = 0$, are depicted in Fig. C.4.

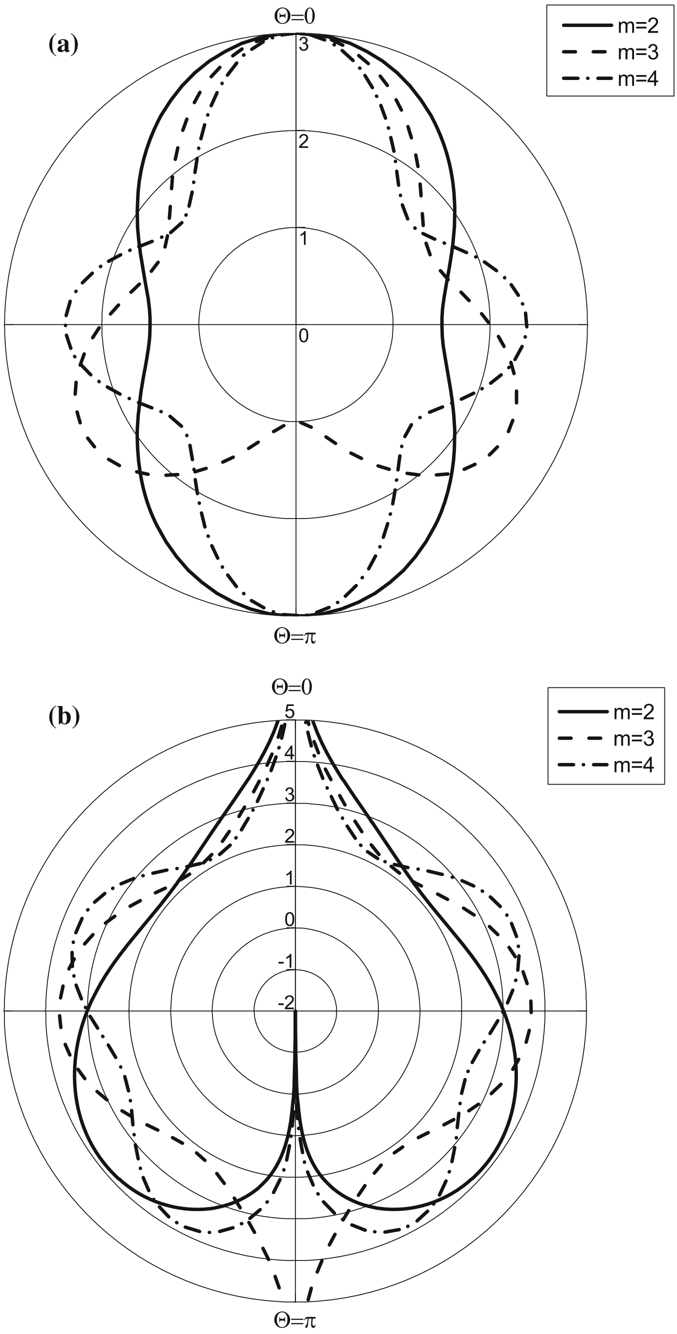

Fig. C.4 Graphs of the *Legendre* functions **a** $P_m^n(\cos\theta)$ of the first and **b** $Q_m^n(\cos\theta)$ of the second kinds for the special case that $n = 0$

A special type of *Legendre*'s differential equation may also occur in the analysis of spherical flow problems. This differential equation reads

$$\left(1 - z^2\right) w'' + m\,(m+1)\,w = 0\,.$$

The solution is [3]

$$w(z) = \left[A\,P_m^1(z) + B\,Q_m^1(z)\right]\left(1 - z^2\right)^{1/2}\,, \tag{C.6}$$

where A and B are constants. Due to the relationships

$$P_m'(z) = -\left(1 - z^2\right)^{-1/2} P_m^1(z)\,, \quad Q_m'(z) = -\left(1 - z^2\right)^{-1/2} Q_m^1(z)$$

between the functions $P_m^1(z)$ and $Q_m^1(z)$ and the derivatives $P_m'(z)$ and $Q_m'(z)$ of the functions of zero order [1], the solution (C.6) may be re-written into the form

$$w(z) = \left[\bar{A}\,P_m'(z) + \bar{B}\,Q_m'(z)\right]\left(1 - z^2\right)\,.$$

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