Topics and Questions for Midterm Examinations

1 Series and Integrals Depending on a Parameter

1. The Cauchy criterion for convergence of a series. The comparison theorem and the basic sufficient conditions for convergence (majorant, integral, Abel–Dirichlet). The series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.
2. Uniform convergence of families and series of functions. The Cauchy criterion and the basic sufficient conditions for uniform convergence of a series of functions (M-test, Abel–Dirichlet).
3. Sufficient conditions for two limiting passages to commute. Continuity, integration, and differentiation and passage to the limit.
4. The region of convergence and the nature of convergence of a power series. The Cauchy–Hadamard formula. Abel's (second) theorem. Taylor expansions of the basic elementary functions. Euler's formula. Differentiation and integration of a power series.
5. Improper integrals. The Cauchy criterion and the basic sufficient conditions for convergence (M-test, Abel–Dirichlet).
6. Uniform convergence of an improper integral depending on a parameter. The Cauchy criterion and the basic sufficient conditions for uniform convergence (majorant, Abel–Dirichlet).
7. Continuity, differentiation, and integration of a proper integral depending on a parameter.
8. Continuity, differentiation, and integration of an improper integral depending on a parameter. The Dirichlet integral.
9. The Eulerian integrals. Domains of definition, differential properties, reduction formulas, various representations, interconnections. The Poisson integral.
10. Approximate identities. The theorem on convergence of the convolution. The classical Weierstrass theorem on uniform approximation of a continuous function by an algebraic polynomial.

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2 Problems Recommended as Midterm Questions

Problem 1 P is a polynomial. Compute $(\mathrm{e}^{t\frac{\mathrm{d}}{\mathrm{d}x}})P(x)$.

Problem 2 Verify that the vector-valued function $\mathrm{e}^{tA}x_0$ is a solution of the Cauchy problem $\dot{x} = Ax$, $x(0) = x_0$. (Here $\dot{x} = Ax$ is a system of equations defined by the matrix A.)

Problem 3 Find up to order $o(1/n^3)$ the asymptotics of the positive roots $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ of the equation $\sin x + 1/x = 0$ as $n \to \infty$.

Problem 4 a) Show that $\ln 2 = 1 - 1/2 + 1/3 - \cdots$. How many terms of this series must be taken to determine $\ln 2$ within 10^{-3}?

b) Verify that $\frac{1}{2}\ln\frac{1+t}{1-t} = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots$. Using this expansion it becomes convenient to compute $\ln x$ by setting $x = \frac{1+t}{1-t}$.

c) Setting $t = 1/3$ in b), obtain the equality

$$\frac{1}{2}\ln 2 = \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots .$$

How many terms of this series must one take to find $\ln 2$ within 10^{-3}? Compare this with the result of a).

This is one of the methods of improving convergence.

Problem 5 Verify that in the sense of Abel summation

a) $1 - 1 + 1 \cdots = \frac{1}{2}$.

b) $\sum_{k=1}^{\infty} \sin k\varphi = \frac{1}{2} \cdot \frac{1}{2}\varphi$, $\varphi \neq 2\pi n$, $n \in \mathbb{Z}$.

c) $\frac{1}{2} + \sum_{k=1}^{\infty} \cos k\varphi = 0$, $\varphi \neq 2\pi n$, $n \in \mathbb{Z}$.

Problem 6 Prove Hadamard's lemma:

a) If $f \in C^{(1)}(U(x_0))$, then $f(x) = f(x_0) + \varphi(x)(x - x_0)$, where $\varphi \in C(U(x_0))$ and $\varphi(x_0) = f'(x_0)$.

b) If $f \in C^{(n)}(U(x_0))$, then

$$f(x) = f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \cdots + \\ + \frac{1}{(n-1)!}f^{(n-1)}(x_0)(x - x_0)^{n-1} + \varphi(x)(x - x_0)^n,$$

where $\varphi \in C(U(x_0))$ and $\varphi(x_0) = \frac{1}{n!}f^{(n)}(x_0)$.

c) What do these relations look like in coordinate form, when $x = (x^1, \ldots, x^n)$, that is, when f is a function of n variables?

Problem 7 a) Verify that the function

$$J_0(x) = \frac{1}{\pi} \int_0^1 \frac{\cos xt}{\sqrt{1-t^2}} \, \mathrm{d}t$$

satisfies Bessel's equation $y'' + \frac{1}{x}y' + y = 0$.

b) Try to solve this equation using power series.

c) Find the power-series expansion of the function $J_0(x)$.

Problem 8 Verify that the following asymptotic expansions hold

a) $\Gamma(\alpha, x) := \int_x^{+\infty} t^{\alpha-1}\mathrm{e}^{-t} \, \mathrm{d}t \simeq \mathrm{e}^{-\alpha} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha-k+1)} x^{\alpha-k}$,

b) $\mathrm{Erf}(x) := \int_x^{+\infty} \mathrm{e}^{-t^2} \, \mathrm{d}t \simeq \frac{1}{2}\sqrt{\pi}\mathrm{e}^{-x^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(3/2-k)x^{2k-1}}$

as $x \to +\infty$.

Problem 9 a) Following Euler, obtain the result that the series $1 - 1!x + 2!x^2 - 3!x^3 + \cdots$ is connected with the function

$$S(x) := \int_0^{+\infty} \frac{\mathrm{e}^{-t}}{1+xt} \, \mathrm{d}t.$$

b) Does this series converge?

c) Does it give the asymptotic expansion of $S(x)$ as $x \to 0$?

Problem 10 a) A linear device A whose characteristics are constant over time responds to a signal $\delta(t)$ in the form of a δ-function by giving out the signal (function) $E(t)$. What will the response of this device be to an input signal $f(t)$, $-\infty < t < +\infty$?

b) Can the input signal f always be recovered uniquely from the transformed signal $\widehat{f} := Af$?

3 Integral Calculus (Several Variables)

1. Riemann integral on an n-dimensional interval. Lebesgue criterion for existence of the integral.
2. Darboux criterion for existence of the integral of a real-valued function on an n-dimensional interval.
3. Integral over a set. Jordan measure (content) of a set and its geometric meaning. Lebesgue criterion for existence of the integral over a Jordan-measurable set. Linearity and additivity of the integral.
4. Estimates of the integral.
5. Reduction of a multiple integral to an iterated integral. Fubini's theorem and its most important corollaries.

6. Formula for change of variables in a multiple integral. Invariance of measure and the integral.
7. Improper multiple integrals: basic definitions, majorant criterion for convergence, canonical integrals. Computation of the Euler–Poisson integral.
8. Surfaces of dimension k in $\mathbb{R}^n$ and basic methods of defining them. Abstract k-dimensional manifolds. Boundary of a k-dimensional manifold as a $(k-1)$-dimensional manifold without boundary.
9. Orientable and nonorientable manifolds. Methods of defining the orientation of an abstract manifold and a (hyper)surface in $\mathbb{R}^n$.

Orientability of the boundary of an orientable manifold. Orientation induced on the boundary from the manifold.
10. Tangent vectors and the tangent space to a manifold at a point. Interpretation of a tangent vector as a differential operator.
11. Differential forms in a region $D \subset \mathbb{R}^n$. Examples: differential of a function, work form, flux form. Coordinate expression of a differential form. Exterior derivative operator.
12. Mapping of objects and the adjoint mapping of functions on these objects. Transformation of points and vectors of tangent spaces at these points under a smooth mapping. Transfer of functions and differential forms under a smooth mapping. A recipe for carrying out the transfer of forms in coordinate form.
13. Commutation of transfer of differential forms with exterior multiplication and differentiation. Differential forms on a manifold. Invariance (unambiguous nature) of operations on differential forms.
14. A scheme for computing work and flux. Integral of a k-form over a k-dimensional smooth oriented surface, taking account of orientation. Independence of the integral of the choice of parametrization. General definition of the integral of a differential k-form over a k-dimensional compact oriented manifold.
15. Green's formula on a square, its derivation, interpretation, and expression in the language of integrals of the corresponding differential forms. The general Stokes formula. Reduction to a k-dimensional interval and proof for a k-dimensional interval. The classical integral formulas of analysis as particular versions of the general Stokes formula.
16. The volume element on $\mathbb{R}^n$ and on a surface. Dependence of the volume element on orientation. The integral of first kind and its independence of orientation. Area and mass of a material surface as an integral of first kind. Expression of the volume element of a k-dimensional surface $S^k \subset \mathbb{R}^n$ in local parameters and the expression of the volume element of a hypersurface $S^{n-1} \subset \mathbb{R}^n$ in Cartesian coordinates of the ambient space.
17. Basic differential operators of field theory (grad, curl, div) and their connection with the exterior derivative operator d in oriented Euclidean space $\mathbb{R}^3$.
18. Expression of work and flux of a field as integrals of first kind. The basic integral formulas of field theory in $\mathbb{R}^3$ as the vector expression of the classical integral formulas of analysis.
19. A potential field and its potential. Exact and closed forms. A necessary differential condition for a form to be exact and for a vector field to be a potential field. Its

sufficiency in a simply connected domain. Integral criterion for exactness of 1-forms and vector fields.

20. Local exactness of a closed form (the Poincaré lemma). Global analysis. Homology and cohomology. De Rham's theorem (statement).

21. Examples of the application of the Stokes (Gauss–Ostrogradskii) formula: derivation of the basic equations of the mechanics of continuous media. Physical meaning of the gradient, curl, and divergence.

22. Hamilton's nabla operator and work with it. The gradient, curl, and divergence in triorthogonal curvilinear coordinates.

4 Problems Recommended for Studying the Midterm Topics

The numbers followed by closing parentheses below refer to the topics 1–22 just listed. The closing parentheses dashes are followed by section numbers (for example 13.4 means Sect. 4 of Chap. 13), which in turn are separated by a dash from the numbers of the problems from the section related to the topic from the list above.

1) 11.1—2,3; 2) 11.1—4; 3) 11.2—1,3,4; 4) 11.3—1,2,3,4; 5) 11.4—6,7 and 13.2—6; 6) 11.5—9 and 12.5—5,6; 7) 11.6—1,5,7; 8) 12.1—2,3 and 12.4—1,4; 9) 12.2—1,2,3,4 and 12.5—11; 10) 15.3—1,2; 11) 12.5—9 and 15.3—3; 12) 15.3—4; 13) 12.5—8,10; 14) 13.1—3,4,5,9; 15) 13.1—1,10,13,14; 16) 12.4—10 and 13.2—5; 17) 14.1—1,2; 18) 14.2—1,2,3,4,8; 19) 14.3—7,13,14; 20) 14.3—11,12; 21) 13.3—1 and 14.1—8; 22) 14.1—4,5,6.

Examination Topics

1 Series and Integrals Depending on a Parameter

1. Cauchy criterion for convergence of a series. Comparison theorem and the basic sufficient conditions for convergence (majorant, integral, Abel–Dirichlet). The series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.
2. Uniform convergence of families and series of functions. Cauchy criterion and the basic sufficient conditions for uniform convergence of a series of functions (M-test, Abel–Dirichlet).
3. Sufficient conditions for commutativity of two limiting passages. Continuity, integration, and differentiation and passage to the limit.
4. Region of convergence and the nature of convergence of a power series. Cauchy–Hadamard formula. Abel's (second) theorem. Taylor expansions of the basic elementary functions. Euler's formula. Differentiation and integration of a power series.
5. Improper integrals. Cauchy criterion and the basic sufficient conditions for convergence (majorant, Abel–Dirichlet).
6. Uniform convergence of an improper integral depending on a parameter. Cauchy criterion and the basic sufficient conditions for uniform convergence (M-test, Abel–Dirichlet).
7. Continuity, differentiation, and integration of a proper integral depending on a parameter.
8. Continuity, differentiation, and integration of an improper integral depending on a parameter. Dirichlet integral.
9. Eulerian integrals. Domains of definition, differential properties, reduction formulas, various representations, interconnections. Poisson integral.
10. Approximate identities. Theorem on convergence of the convolution. Classical Weierstrass theorem on uniform approximation of a continuous function by an algebraic polynomial.
11. Vector spaces with an inner product. Continuity of the inner product and algebraic properties connected with it. Orthogonal and orthonormal systems of vectors.

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Pythagorean theorem. Fourier coefficients and Fourier series. Examples of inner products and orthogonal systems in spaces of functions.
12. Orthogonal complement. Extremal property of Fourier coefficients. Bessel's inequality and convergence of the Fourier series. Conditions for completeness of an orthonormal system. Method of least squares.
13. Classical (trigonometric) Fourier series in real and complex form. Riemann–Lebesgue lemma. Localization principle and convergence of a Fourier series at a point. Example: expansion of $\cos(\alpha x)$ in a Fourier series and the expansion of $\sin(\pi x)/\pi x$ in an infinite product.
14. Smoothness of a function, rate of decrease of its Fourier coefficients, and rate of convergence of its Fourier series.
15. Completeness of the trigonometric system and mean convergence of a trigonometric Fourier series.
16. Fourier transform and the Fourier integral (the inversion formula). Example: computation of $\widehat{f}$ for $f(x) := \exp(-a^2x^2)$.
17. Fourier transform and the derivative operator. Smoothness of a function and the rate of decrease of its Fourier transform. Parseval's equality. The Fourier transform as an isometry of the space of rapidly decreasing functions.
18. Fourier transform and convolution. Solution of the one-dimensional heat equation.
19. Recovery of a transmitted signal from the spectral function of a device and the signal received. Sampling theorem (Kotel'nikov–Shannon formula).
20. Asymptotic sequences and asymptotic series. Example: asymptotic expansion of $\mathrm{Ei}(x)$. Difference between convergent and asymptotic series. Asymptotic Laplace integral (principal term). Stirling's formula.

2 Integral Calculus (Several Variables)

1. Riemann integral on an n-dimensional interval. Lebesgue criterion for existence of the integral.
2. Darboux criterion for the existence of the integral of a real-valued function on an n-dimensional interval.
3. Integral over a set. Jordan measure (content) of a set and its geometric meaning. Lebesgue criterion for existence of the integral over a Jordan-measurable set. Linearity and additivity of the integral.
4. Estimates of the integral.
5. Reduction of a multiple integral to an iterated integral. Fubini's theorem and its most important corollaries.
6. Formula for change of variables in a multiple integral. Invariance of measure and the integral.
7. Improper multiple integrals: basic definitions, the majorant criterion for convergence, canonical integrals. Computation of the Euler–Poisson integral.

8. Surfaces of dimension k in $\mathbb{R}^n$ and the basic methods of defining them. Abstract k-dimensional manifolds. Boundary of a k-dimensional manifold as a $(k-1)$-dimensional manifold without boundary.
9. Orientable and nonorientable manifolds. Methods of defining the orientation of an abstract manifold and a (hyper)surface in $\mathbb{R}^n$.

Orientability of the boundary of an orientable manifold. Orientation on the boundary induced from the manifold.
10. Tangent vectors and the tangent space to a manifold at a point. Interpretation of a tangent vector as a differential operator.
11. Differential forms in a region $D \subset \mathbb{R}^n$. Examples: differential of a function, work form, flux form. Coordinate expression of a differential form. Exterior derivative operator.
12. Mapping of objects and the adjoint mapping of functions on these objects. Transformation of points and vectors of tangent spaces at these points under a smooth mapping. Transfer of functions and differential forms under a smooth mapping. A recipe for carrying out the transfer of forms in coordinate form.
13. Commutation of the transfer of differential forms with exterior multiplication and differentiation. Differential forms on a manifold. Invariance (unambiguous nature) of operations on differential forms.
14. A scheme for computing work and flux. Integral of a k-form over a k-dimensional smooth oriented surface. Taking account of orientation. Independence of the integral of the choice of parametrization. General definition of the integral of a differential k-form over a k-dimensional compact oriented manifold.
15. Green's formula on a square, its derivation, interpretation, and expression in the language of integrals of the corresponding differential forms. General Stokes formula. Reduction to a k-dimensional interval and proof for a k-dimensional interval. Classical integral formulas of analysis as particular versions of the general Stokes formula.
16. Volume element on $\mathbb{R}^n$ and on a surface. Dependence of volume element on orientation. The integral of first kind and its independence of orientation. Area and mass of a material surface as an integral of first kind. Expression of volume element of a k-dimensional surface $S^k \subset \mathbb{R}^n$ in local parameters and expression of volume element of a hypersurface $S^{n-1} \subset \mathbb{R}^n$ in Cartesian coordinates of the ambient space.
17. Basic differential operators of field theory (grad, curl, div) and their connection with the exterior derivative operator d in oriented Euclidean space $\mathbb{R}^3$.
18. Expression of work and flux of a field as integrals of first kind. Basic integral formulas of field theory in $\mathbb{R}^3$ as the vector expression of the classical integral formulas of analysis.
19. A potential field and its potential. Exact and closed forms. A necessary differential condition for a form to be exact and for a vector field to be a potential field. Its sufficiency in a simply connected domain. Integral criterion for exactness of 1-forms and vector fields.
20. Local exactness of a closed form (Poincaré's lemma). Global analysis. Homology and cohomology. De Rham theorem (formulation).

21. Examples of the application of the Stokes (Gauss–Ostrogradskii) formula: derivation of the basic equations of the mechanics of continuous media. Physical meaning of the gradient, curl, and divergence.

22. Hamilton's Nabla operator, and computation of work with it. Gradient, curl and divergence in a 3-dimensional orthogonal system of curvilinear coordinates.

Examination Problems
(Series and Integrals Depending on a Parameter)

1. We shall consider a sequence of real-valued functions $\{f_n\}$ defined on the interval [0, 1], for example.

a) What types of convergence for a sequence of functions do you know?
b) Provide the definition of each of them.
c) What are the relations between them? (Prove the relation or give an explanatory example when there is no such relation.)

2. Let f be a periodic function with period 2π. Suppose it is identically zero on the interval $]-\pi, 0[$ and $f(x) = 2x$ on the interval $[0, \pi]$. Calculate the sum S of the standard trigonometric Fourier series of this function.

3. a) We know the expansion in power series of the function $(1+x)^{-1}$ (geometric progression). Obtain from it the expansion in a power series of the function $\ln(1+x)$ and justify your steps.

b) What is the radius of convergence of the obtained series?
c) Does this series converge at $x = 1$, and if so, is its sum equal to $\ln 2$? Why?

4. a) It is known that the spectral function (characteristic function) p of a linear device (operator) A is everywhere nonzero. How can we find the transmitted signal f if we know the function p and the received signal $g = Af$.

b) Let the function p be defined by $p(\omega) \equiv 1$ for $|\omega| \le 10$ and $p(\omega) \equiv 0$ for $|\omega| > 10$. Suppose that we know the spectrum $\widehat{g}$ (Fourier transform) of the received signal g and that it is exactly $\widehat{g}(\omega) \equiv 1$ for $|\omega| \le 1$ and $\widehat{g}(\omega) \equiv 0$ for $|\omega| > 1$. Finally, suppose that it is also known that the input signal f does not contain some other frequencies apart from the frequencies transmitted by the device A (i.e., beyond the frequencies $|\omega| \le 10$). Find the input signal f.

5. Using Euler's Γ function and Laplace's method, obtain the very useful asymptotic Stirling's formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.

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Intermediate Problems (Integral Calculus of Several Variables)

1. Compute the values of the following forms ω in $\mathbb{R}^n$ on the given set of vectors.

a) $\omega = x^2\,\mathrm{d}x^1$ applied to the vector $\xi = (1,2,3) \in T\mathbb{R}^3_{(1,2,3)}$;

b) $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^3 + x^1\,\mathrm{d}x^2 \wedge \mathrm{d}x^4$ applied to the ordered pair of vectors $(\xi_1, \xi_2) \in T\mathbb{R}^4_{(1,0,0,0)}$. (Set $\xi_1 = (\xi_1^1, \dots, \xi_1^4)$, $\xi_2 = (\xi_2^1, \dots, \xi_2^4)$.)

2. Let $f^1, \dots, f^n$ be smooth functions with argument $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. Express the form $\mathrm{d}f^1 \wedge \cdots \wedge \mathrm{d}f^n$ in terms of the forms $\mathrm{d}x^1, \dots, \mathrm{d}x^n$.

3. Let F be a vector field of a force acting on a domain $D \subset \mathbb{R}^3$. By the action of this vector field an object was transferred along a smooth path $\gamma \subset D$ from the point $a \in D$ to the point $b \in D$. Calculate the work done by the vector field in this process.

a) Write the formula for the calculation of this work as an integral of the first type and as an integral of the second type (i.e., in terms of $\mathrm{d}s$ and $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, respectively).

b) Prove that in the case of the gravitational vector field, this work does not depend on the path and that it is equal to …?

4. Consider the following problem about the flux of a vector field.

a) One has the vector field V (for instance, the vector field velocity of some current) on the domain $D \in \mathbb{R}^3$. Write a formula for the calculation of the flux of the vector field V through the oriented surface $S = S^2_+ \subset D$ as an integral of the first type and as an integral of the second type (i.e., in terms of $\mathrm{d}\sigma$ and $\mathrm{d}y \wedge \mathrm{d}z$, $\mathrm{d}z \wedge \mathrm{d}x$, $\mathrm{d}x \wedge \mathrm{d}y$ respectively).

b) Consider a convex polyhedral domain $D \subset \mathbb{R}^3$. On each of its faces is constructed a vector pointing toward the exterior normal direction with magnitude equal to the area of the corresponding side. Physics states that the sum of these vectors is equal to zero (otherwise, we could build a perpetual motion device). Mathematics agrees. Prove this fact.

c) Deduce Archimedes's law by a direct computation (calculate the buoyancy force acting on a submerged body in a bathtub completely filled with water, for example, as the resulting pressure on the surface of the body).

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Appendix A
Series as a Tool (Introductory Lecture)

When a geological deposit is discovered, it is explored and then exploited. In mathematics, it is also like that. Axiomatics and useful formalisms arise as the result of solving concrete questions and problems. They do not fall down from the sky, as it seems to inexperienced students when everything starts with axioms.

This course is largely dedicated to series, i.e., basically limits of sequences. We shall give at least an initial idea of how and where this tool works, in order to convince ourselves that the study of this remarkably effective machinery, namely the theory of series, does not reduce to the abstract study of the convergence of series (the existence of a limit).

A.1 Getting Ready

A.1.1 The Small Bug on the Rubber Rope

(Problem proposed by the academician L.B. Okun to the academician A.D. Sakharov.)[1]

Problem 1 You hold one end of a 1 km long rubber rope. A small bug crawls toward you from the other end, which is fixed, with a speed of 1 cm/s. As soon as it crawls one centimeter, you stretch the rubber rope another kilometer every time. Does the insect ever reach your hand? And if it does, how long will it take?

[1]Martin Gardner in his book *Time Travel and Other Mathematical Bewilderments* (New York: W. H. Freeman & Company, 1987, English, p. 295) writes, "This delightful problem, which has the flavor of a Zeno paradox, was devised by Denys Wilquin of New Caledonia. It appeared first in December 1972 in Pierre Berloquin's lively puzzle column in the French monthly *Science et Vie*."

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A.1.2 Integral and Estimation of Sums

After some thinking, it may occur to you that the following sum might be useful in finding the answer $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Problem 2 Recall the integral, and show that $S_n - 1 < \int_1^n \frac{1}{x}\, dx < S_{n-1}$.

A.1.3 From Monkeys to Doctors of Science Altogether in 10^6 Years

Littlewood in his famous book *Littlewood's Miscellany*, speaking about large numbers, wrote that 10^6 years is the time needed to convert monkeys into doctors of science.[2]

Problem 3 Would the little bug arrive in time for the thesis defense or at least before the end of the universe?

A.2 The Exponential Function

A.2.1 Power Series Expansion of the Functions exp, sin, cos

According to Taylor's formula with remainder in Lagrange's form, one has

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + r_n(x),$$

where $r_n(x) = \frac{1}{(n+1)!} e^{\xi} \cdot x^{n+1}$ and $|\xi| < |x|$;

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{2n!}x^{2n} + r_{2n}(x),$$

where $r_{2n}(x) = \frac{1}{(2n+1)!} \cos(\xi + \frac{\pi}{2}(2n+1))x^{2n+1}$ and $|\xi| < |x|$;

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + r_{2n+1}(x),$$

where $r_{2n+1}(x) = \frac{1}{(2n+2)!} \sin(\xi + \frac{\pi}{2}(2n+2))x^{2n+2}$ and $|\xi| < |x|$. Since for every fixed value $x \in \mathbb{R}$, the remainder in each of the above formulas clearly tends to zero

[2] John E. Littlewood, *Littlewood's Miscellany*. Cambridge: Cambridge University Press, 1986, English, p. 212.

as $n \to \infty$, we can write

$$\begin{aligned}
\mathrm{e}^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots + \frac{1}{n!}x^n + \cdots, \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{2n!}x^{2n} + \cdots, \\
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots.
\end{aligned}$$

A.2.2 Exit to the Complex Domain and Euler's Formula

We substitute x for the complex number ix in the right-hand side of the first of these equalities. Then, after some simple arithmetic manipulations, we obtain Euler's outstanding relationship

$$\mathrm{e}^{ix} = \cos x + i \sin x.$$

Setting $x = \pi$, we find that $\mathrm{e}^{i\pi} + 1 = 0$. This is the famous equation connecting the fundamental constants of mathematics: e from analysis, i from algebra, π from geometry, 1 from arithmetic, and 0 from logic.

We defined the function exp for purely imaginary values of the argument and obtained Euler's formula $\mathrm{e}^{ix} = \cos x + i \sin x$, from which, clearly, it also follows that

$$\cos x = \frac{1}{2}\left(\mathrm{e}^{ix} + \mathrm{e}^{-ix}\right) \quad \text{and} \quad \sin x = \frac{1}{2i}\left(\mathrm{e}^{ix} - \mathrm{e}^{-ix}\right).$$

A.2.3 The Exponential Function as a Limit

We know that $(1 + \frac{x}{n})^n \to \mathrm{e}^x$ as $n \to \infty$ for $x \in \mathbb{R}$. It is natural to assume that $\mathrm{e}^z = \lim_{n\to\infty}(1 + \frac{z}{n})^n$, where now $z = x + iy$ is an arbitrary complex number. A computation of this limit gives $\mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y)$.

Problem 4 Verify this and obtain a formula for $\cos z$ and $\sin z$.

A.2.4 Multiplication of Series and the Basic Property of the Exponential Function

The expression $\mathrm{e}^z = \mathrm{e}^x(\cos y + i \sin y)$ for e^{x+iy} can be naturally obtained from the relation $\mathrm{e}^{x+iy} = \mathrm{e}^x \mathrm{e}^{iy}$ if it is valid for complex values of the argument of the function exp.

We shall prove this by direct multiplication. Let u and v be complex numbers. Setting $e^u := \sum_{k=0}^{\infty} \frac{1}{k!} u^k$ and $e^v := \sum_{m=0}^{\infty} \frac{1}{m!} v^m$ we find that

$$e^u \cdot e^v = \left(\sum_{k=0}^{\infty} \frac{1}{k!} u^k \right) \cdot \left(\sum_{m=0}^{\infty} \frac{1}{m!} v^m \right) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!} \frac{1}{m!} u^k v^m =$$

$$= \sum_{n=0}^{\infty} \sum_{n=k+m} \frac{1}{k!} \frac{1}{m!} u^k v^m = \sum_{n=0}^{\infty} \frac{1}{n!} (u+v)^n = e^{u+v}.$$

We used here the fact that $\sum_{n=k+m} \frac{n!}{k!m!} u^k v^m = (u+v)^n$, provided that $uv = vu$.

A.2.5 *Exponential of a Matrix and the Role of Commutativity*

What happens if in the expression

$$e^A = 1 + \frac{1}{1!} A + \frac{1}{2!} A^2 + \cdots + \frac{1}{n!} A^n + \cdots ,$$

we consider A a square matrix, and 1 is the identity matrix of the same size? For example, if A is the identity matrix, then it is easy to check that e^A turns out to be a diagonal matrix, with elements e on the main diagonal.

Problem 5 a) Calculate $\exp A$ for the following matrices A:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) Let A_1 and A_2 be the last two matrices of order two. Find e^{A_1}, e^{A_2} and check that $e^{A_1} \cdot e^{A_2} \neq e^{A_1 + A_2}$. What is going on here?

c) Show that $e^{tA} = I + tA + o(t)$, for $t \to 0$.

d) Check that $\det(I + tA) = 1 + t \cdot (\operatorname{tr} A) + o(t)$, where $\operatorname{tr} A$ is the trace of the square matrix A.

e) Prove the important relationship $\det e^A = e^{\operatorname{tr} A}$.

A.2.6 *Exponential of Operators and Taylor's Formula*

Let $P(x)$ be a polynomial and $A = \frac{d}{dx}$ the differentiation operator. Then $(AP)(x) = \frac{dP}{dx}(x) = P'(x)$.

Problem 6 a) Check that the relation $\exp(t\frac{\mathrm{d}}{\mathrm{d}x})P(x) = P(x+t)$ is what you know as Taylor's formula.

b) By the way, how many terms of the series e^x do you have to consider in order to obtain a polynomial that allows you to calculate e^x on the interval $[-3, 5]$ with an accuracy up to 10^{-2}?

A.3 Newton's Binomial

A.3.1 Expansion in Power Series of the Function $(1+x)^\alpha$

Newton knew the validity, for every natural number α, of the formula for the binomial expansion

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{2!}x^2 + \cdots ,$$

and then he remarked that this formula remains valid for arbitrary α, but the number of terms in the sum might be infinite.

For instance, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$.

A.3.2 Integration of a Series and Expansion of $\ln(1+x)$

By integrating the last series over the interval $[0, x]$, we find that

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad \text{for } |x| < 1.$$

A.3.3 Expansion of the Functions $(1+x^2)^{-1}$ *and* $\arctan x$

Analogously, we write the expansion $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$, we integrate its terms over the interval $[0, x]$, and we obtain

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots .$$

If we set $x = 1$, this expansion seems to imply that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

Perhaps this is true (and certainly it is), but we have the feeling that we are already going beyond the limits of what is permitted. The following example will only reinforce our concerns.

A.3.4 Expansion of $(1+x)^{-1}$ and Computing Curiosities

For $x = 1$, the expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ leads to the equality $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$.

By grouping terms, we can obtain $\frac{1}{2} = (1-1) + (1-1) + \cdots = 0$ and we can obtain $\frac{1}{2} = 1 + (-1+1) + (-1+1) + \cdots = 1$.

After this, it is necessary to question almost everything that we have done so successfully and nonchalantly by multiplying the infinite sums (series), rearranging and grouping their terms, and integrating them. All this must obviously be clarified. We shall do it soon, but before that, we mention yet another area where series are commonly used.

A.4 Solution of Differential Equations

A.4.1 Method of Undetermined Coefficients

Consider the simplest equation $\ddot{x} + x = 0$ of harmonic oscillations. We shall look for the solution as a series $x(t) = a_0 + a_1 t + a_2 t^2 + \cdots$. Substituting the series into the equation, grouping the terms with equal powers of t, and equating the coefficients with the same powers in t on both sides of the equation, we obtain an infinite system of equations:

$$2a_2 + a_0 = 0, \qquad 2 \cdot 3a_3 + a_1 = 0, \qquad 3 \cdot 4a_4 + a_2 = 0, \quad \ldots.$$

If the initial conditions $x(0) = x_0$ and $x'(0) = v_0$ are given, then from the series $x(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, and $x'(t) = a_1 + 2a_2 t + \cdots$, we find that $a_0 = x_0$ and $a_1 = v_0$. If we know a_0 and a_1, we can find successively and uniquely the remaining coefficients of the expansion.

For example, if $x(0) = 0$ and $x'(0) = 1$, then

$$x(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \cdots = \sin t,$$

and if $x(0) = 1$ and $x'(0) = 0$, then

$$x(t) = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots = \cos t.$$

A.4.2 Use of the Exponential Function

What happens if the solution that we are looking for has the form $x(t) = \mathrm{e}^{\lambda t}$? Then $\ddot{x} + x = \mathrm{e}^{\lambda t}(\lambda^2 - 1) = 0$, and therefore $\lambda^2 + 1 = 0$, i.e., $\lambda = i$ or $\lambda = -i$. But what are these strange complex oscillations $x(t) = \mathrm{e}^{it}$, $x(t) = \mathrm{e}^{-it}$, and $x(t) = c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it}$?

Problem 7 Analyze the situation and solve the problem, for example, if $x(0) = 0$ and $x'(0) = 1$ or if $x(0) = 1$ and $x'(0) = 0$. Recall Euler's formula and compare your results with those obtained above.

A.5 The General Idea About Approximation and Expansion

A.5.1 The Meaning of a Positional Number System. Irrational Numbers

Recall the usual representation of the number $\pi = 3.1415926\ldots$ or in general a decimal expansion $a_0.a_1a_2a_3\ldots$: this is the sum $a_0 10^0 + a_1 10^{-1} + a_2 10^{-2} + a_3 10^{-3} + \cdots$.

We know that a finite expansion corresponds to a rational number, and the representation of an irrational number requires an infinite number of decimal digits, and therefore requires the study of an infinite number of terms and infinite sums, i.e., series.

If we truncate a series at some point, we get a rational number. We usually work such numbers. What happened here? We have simplified the object, allowing some error. This means that we are approximating a complex object (an irrational number in this case) through some other objects (the rational numbers here), while allowing some error, which we call the degree of precision of the approximation. An improvement in the precision leads to the complication of the object that we use as an approximation. A compromise has to be found depending on the concrete circumstances.

A.5.2 Expansion of a Vector in a Basis and Some Analogies with Series

In linear algebra and in geometry, we decompose vectors in terms of a basis. For mathematical analysis, the traditional representation

$$f(x) = f(0) + \frac{1}{1!} f'(0)x + \frac{1}{2!} f''(0)x^2 + \cdots$$

actually means the same thing if we consider that the basis is the set of functions $\mathrm{e}_n = x^n$. This is the Taylor series of the function f at the point $x_0 = 0$.

Analogously, if some periodic signal or process $f(t)$ is subjected to spectral analysis, then one is interested in its decomposition $f(t) = \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt$ into the simplest harmonic oscillations. Such series are called classical (or trigonometric) Fourier series.

What is new in this situation, in comparison with that in linear algebra, is that we consider here an infinite sum, which is understood as the limit of finite sums.

Thus in the space of our objects one must define the concept of proximity between the objects, in addition to the structure of a linear space, allowing one to be able to consider the limit of the sequence of the objects themselves or their sum.

A.5.3 Distance

The proximity between objects is determined by the presence of a particular concept, the concept of neighborhood of an object (neighborhood of a point in the space). This is the same as specifying a topology in the space. In topological spaces it is possible to speak about limits and continuity.

If in a space, a distance between objects, i.e., the points of the space, is somehow introduced, then the neighborhoods of a point are automatically defined, and even more specifically, the δ-neighborhoods of a point.

The distance between points of the same space can be measured in different ways. For example, the distance between two continuous functions over an interval can be measured by the maximum of the absolute value of the difference between the values of the functions on this interval (uniform metric), and it is also possible to measure it by the integral of the absolute value of the difference of the functions over this interval (integral metric). The choice of the metric is dictated by the problem under consideration.

Appendix B
Change of Variables in Multiple Integrals (Deduction and First Discussion of the Change of Variables Formula)[1]

B.1 Formulation of the Problem and a Heuristic Derivation of the Change of Variables Formula

By studying the integral in the one-dimensional case, at some moment we obtained an important change of variables formula for such an integral. Our task now is to find a change of variables formula in the general case. We formulate the problem more precisely.

Let D_x be a set in $\mathbb{R}^n$, f an integrable function on D_x, and $\varphi : D_t \to D_x$ a mapping $t \mapsto \varphi(t)$ from the set $D_t \subset \mathbb{R}^n$ to D_x. The question is, what is the law, assuming that we know f and φ, that allows us to find a function ψ on D_t such that we have the equality

$$\int_{D_x} f(x)\,\mathrm{d}x = \int_{D_t} \psi(t)\,\mathrm{d}t,$$

which reduces the computation of an integral over D_x to an integral over D_t?

We suppose first that D_t is an n-dimensional interval $I \subset \mathbb{R}^n$ and $\varphi : I \to D_x$ is a diffeomorphic mapping from I onto D_x. To every partition of the interval I into subintervals $I_1, I_2, \dots, I_k$ corresponds a partition of D_x into subsets $\varphi(I_i)$, $i = 1, \dots, k$. If all these sets are measurable and intersect pairwise only on sets of measure zero, then by the additivity of the integral,

$$\int_{D_x} f(x)\,\mathrm{d}x = \sum_{i=1}^{k} \int_{\varphi(I_i)} f(x)\,\mathrm{d}x. \tag{B.1}$$

If the function f is continuous on D_x, then the mean value theorem implies

$$\int_{\varphi(I_i)} f(x)\,\mathrm{d}x = f(\xi_i)\mu\bigl(\varphi(I_i)\bigr),$$

[1]Fragment of a lecture with an alternative and independent proof of the change of variables formula.

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where $\xi_i \in \varphi(I_i)$. Since $f(\xi_i) = f(\varphi(\tau_i))$, with $\tau_i = \varphi^{-1}(\xi_i)$, then it remains for us to link $\mu(\varphi(I_i))$ with $\mu(I_i) = |I_i|$.

If φ is a linear transform, then $\varphi(I_i)$ is a parallelepiped, whose volume we know from analytical geometry and algebra and is equal to $|\det\varphi'|\mu(I_i)$. But a diffeomorphism is locally almost a linear map. Therefore, if the size of the intervals I_i is sufficiently small, then it can be assumed, with a small relative error, that $\mu(\varphi(I_i)) \approx |\det\varphi'(\tau_i)|\mu(I_i)$ (it is possible to prove that with a proper choice of the point $\tau_i \in I_i$, one has the exact equality). In this way,

$$\sum_{i=1}^{k} \int_{\varphi(I_i)} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{k} f\big(\varphi(\tau_i)\big)\big|\det\varphi'(\tau_i)\big| \cdot |I_i|. \tag{B.2}$$

However, on the right-hand side of this approximate equality there is the integral sum of the function $f(\varphi(t))|\det\varphi'(t)|$ over the interval I, corresponding to the partition P of this interval with marked points τ. In the limit $\lambda(P) \to 0$, from equations (B.1) and (B.2) we get

$$\int_{D_x} f(x)\,\mathrm{d}x = \int_{D_t} f\big(\varphi(t)\big)\big|\det\varphi'(t)\big|\,\mathrm{d}t. \tag{B.3}$$

This is the required formula together with its explanation. Note that it is possible to justify rigorously each step of this deduction, which led us to the formula. Strictly speaking, we need to prove only the validity of the last passage to the limit, that the integral on the right-hand side of (B.3) exists, and also to explain the approximation $\mu(\varphi(I_i)) \approx |\det\varphi'(\tau_i)| \cdot |I_i|$.

Let us do it.

B.2 Some Properties of Smooth Mappings and Diffeomorphisms

a) Recall that a smooth mapping φ from a closed and bounded interval $I \subset \mathbb{R}^n$ (or from any other convex compact subset) is a Lipschitz function. This follows from the mean value theorem and the boundedness of φ' (because of the continuity) over a compact set

$$\big|\varphi(t_2) - \varphi(t_1)\big| \le \sum_{\tau\in[t_1,t_2]} \big\|\varphi'(\tau)\big\| \cdot |t_2 - t_1| \le L|t_2 - t_1|. \tag{B.4}$$

b) Thus, the distance between the images of the points under the mapping φ cannot exceed L times the distance between the points.

For instance, if some subset $E \subset I$ has diameter d, then the diameter of its image $\varphi(E)$ is not more than Ld, and the set $\varphi(E)$ can be covered with (n-dimensional) cubes with edges of size Ld and volume $(Ld)^n$.

Thus if E is a cube with edges of size δ and volume δ^n, then its image is covered by a standard coordinate cube of volume $(L\sqrt{n}\delta)^n$.

c) It follows from this that the image under smooth mappings of 0-measure sets have also measure 0 (in the sense of n-dimensional objects). [After all, in the definition of a set of measure zero, it is possible to consider coverings by cubes, instead of a covering with general n-dimensional intervals, i.e., “rectangular parallelepipeds”, as we can easily see.]

If a smooth mapping $\varphi : D_t \to D_x$ has also an inverse smooth mapping $\varphi^{-1} : D_x \to D_t$, i.e., if φ is a diffeomorphism, then it is clear that the pre-image of a set with measure zero also has measure zero.

d) Since under a diffeomorphism, the Jacobian of the mapping $\det \varphi'$ is everywhere different from zero, and the mapping itself is bijective, then (due to the inverse function theorem) the interior points of any set under such a mapping are transformed into the interior points of the image of this set, and the boundary points are transformed into the boundary points of the image.

Recall the definition of an admissible (Jordan-measurable) set, as a bounded set whose boundary set has measure zero; thus we can conclude that under diffeomorphisms, the image of a measurable set is again a measurable set.

(This is also true for any smooth mapping. However, for diffeomorphisms it is even true that the pre-image of a measurable set is also a measurable set.)

e) This latter in particular means that if $\varphi : D_t \to D_x$ is a diffeomorphism, then from the existence of the integral on the left-hand side of formula (B.3) there follows (based on Lebesgue’s criterion) the existence of the integral on the right-hand side.

B.3 Relation Between the Measures of the Image and the Pre-image Under Diffeomorphisms

We shall show that if $\varphi : I \to \varphi(I)$ is a diffeomorphism, then

$$\mu\big(\varphi(I)\big) = \int_I \det \varphi'(t)\, \mathrm{d}t, \tag{B.5}$$

under the assumption that the integrand $\det \varphi'$ is positive.

Hence, by the mean value theorem, in particular, we find that there is a point $\tau \in I$ such that

$$\mu\big(\varphi(I)\big) = \det \varphi'(\tau)|I|. \tag{B.6}$$

Formula (B.5) is actually a particular case of (B.3), when $f \equiv 1$.

For linear mappings, this formula is already known, although perhaps without discussing those details related to the fact that it is valid (for linear maps) not only for simple parallelepipeds but for all measurable sets. Let us clarify this. We know that a linear map is the composite of elementary linear mappings, which, up to a possible permutation of a pair of coordinates, are reduced to a change in only one of these coordinates: multiplying or adding a number of any one of the coordinates to another one. Fubini’s theorem allows us to determine that in the first case, the volume of any measurable set is multiplied by the same factor that multiplies the coordinate (more precisely, its absolute value if we consider nonoriented volume). In the second case, although the face changes, its volume remains the same, since the corresponding one-dimensional section only moves, keeping its linear measure. Finally, a permutation of a pair of coordinates changes the orientation of the spatial frame (the determinant of such a linear transformation is -1), but it does not change the nonoriented volume of the face. (In the language of Fubini's theorem, this is just a change in the order of two integrations.)

It now remains to recall that the determinant of the composition of linear mappings is the product of the determinants of the factors.

Thus, considering that for linear and affine mappings the formula (B.5) is already established, we prove it for an arbitrary diffeomorphism with positive Jacobian.

a) We use again the finite-increment theorem, but now to estimate the possible deviation of the mapping $\varphi : I \to \varphi(I)$ from the affine mapping $t \mapsto A(t) = \varphi(a) + \varphi'(a)(t-a)$, where t is a variable, and a is a fixed point in the interval I. The mapping $A : I \to A(I)$ is simply the linear part of the Taylor expansion of the function φ at the point $a \in I$.

If we apply the finite-increment (mean value) theorem to the function $t \to \varphi(t) - \varphi'(a)(t-a)$, we obtain

$$\left|\varphi(t) - \varphi(a) - \varphi'(a)(t-a)\right| \le \sup_{\tau \in [a,t]} \left\|\varphi'(\tau) - \varphi'(a)\right\| \cdot |t-a|. \tag{B.7}$$

Given the uniform continuity of the continuous function φ' on the compact set I, from equation (B.7) we conclude that there is a nonnegative function $\delta \to \varepsilon(\delta)$, tending to zero as $\delta \to +0$, such that for any two points $t, a \in I \subset \mathbb{R}^n$,

$$|t-a| \le \sqrt{n}\delta \Longrightarrow \left|\varphi(t) - A(t)\right| = \left|\varphi(t) - \varphi(a) - \varphi'(a)(t-a)\right| \le \varepsilon(\delta)\delta. \tag{B.8}$$

b) Now we go back to the proof of formula (B.5). First we shall carry out a small technical simplification: we shall assume that the lengths of the edges of the parallelepiped I are commensurable and that therefore, they can be divided into equal cubes $\{I\}$ with arbitrarily small (as necessary) edges $\delta_i = \delta$ and volume $\delta_i^n = \delta^n$, i.e., $I = \bigcup_i I_i$ and $|I| = \sum_i |I_i| = \sum_i \delta_i^n$.

In every cube I_i, we fix a point a_i, we build the corresponding affine mapping $A_i(t) = \varphi(a_i) - \varphi'(a_i)(t-a_i)$, we consider the image $A_i(\partial I_i)$ of the cube's I_i boundary ∂I_i under the mapping A_i, and we consider the $\varepsilon(\delta)\delta$-neighborhood of this image, which we denote by Δ_i. By (B.8), the image $\varphi(\partial I_i)$ of the boundary ∂I_i of the cube I_i lies in Δ_i under the diffeomorphism φ. Thus, one has the following inclusions and inequalities:

$$A_i(I_i) \setminus \Delta_i \subset \varphi(I_i) \subset A_i(I_i) \cup \Delta_i,$$
$$\left|A_i(I_i)\right| - |\Delta_i| \le \left|\varphi(I_i)\right| \le \left|A_i(I_i)\right| + |\Delta_i|.$$

When we take the sum over all indices, we have

$$\sum_i \left|A_i(I_i)\right| - \sum_i |\Delta_i| \le \left|\varphi(I)\right| = \sum_i \left|\varphi(I_i)\right| \le \sum_i \left|A_i(I_i)\right| + \sum_i |\Delta_i|. \tag{B.9}$$

As $\delta \to +0$,

$$\sum_i \big|A_i(I_i)\big| = \sum_i \det \varphi'(a_i)|I_i| \to \int_I \det \varphi'(t)\,\mathrm{d}t.$$

Therefore, to prove formula (B.5) in our case, it remains to verify that $\sum_i |\Delta_i| \to 0$ if $\delta \to +0$.

c) We estimate from above the volume $|\Delta_i|$, based on the estimates (B.4) and (B.8). According to (B.4), the edges of the parallelepiped $A_i(I_i)$ have length not greater than $L\delta$, where $\delta = \delta_i$ is the length of the edge of a cube I_i. Thus the $(n-1)$-dimensional "area" of any of the $2n$ faces of the parallelepiped $A_i(I_i)$ is not greater than $(L\delta)^{n-1}$. We take an $\varepsilon(\delta)\delta$-neighborhood of such a face. Its volume is estimated with the value $(2+2)\varepsilon(\delta)\delta(L\delta)^{n-1}$, where the second 2 appearing in the formula is the absorption contribution of the rounded parts of this neighborhood, occurring near the boundary of the face. In this way, $|\Delta_i| < 2n \cdot 4L^{n-1}\varepsilon(\delta)\delta^n$; therefore,

$$\sum_i |\Delta_i| < 8nL^{n-1} \sum_i \varepsilon(\delta)\delta_i^n = 8nL^{n-1}\varepsilon(\delta)|I|,$$

and we see that $\sum_i |\Delta_i| \to 0$ for $\delta \to +0$.

d) The estimated values for $|\Delta_i|$ show at the same time that no matter how arbitrarily small the reduction of the edges of the original interval I becomes, which one might need in order to obtain their commensurability, in the limit this does not affect the result.

B.4 Some Examples, Remarks, and Generalizations

Thus formula (B.3) for the case $D_t = I$ and a continuous function f is already proved. We shall consider and discuss some examples. These will show at the same time that in fact, we have already proved formula (B.3) not only for the case $D_t = I$ and not only for a continuous function f.

a) *Negligible sets*. As it is used in practice, replacing variables or the use of a coordinate transformation formula sometimes has several special features (for example, somewhere there might be a violation of mutual uniqueness, vanishing of the Jacobian, or lack of differentiability). Typically, these special features occur on sets of measure zero, and are therefore relatively easy to overcome.

For example, if you need to go from an integral over a circle to an integral over a rectangle, we often make the change of variables

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{B.10}$$

These are the well-known formulas for the transition from polar coordinates to Cartesian coordinates in the plane. The rectangle $I = \{(r,\varphi) \in \mathbb{R}^2 \mid 0 \le r \le R, 0 \le \varphi \le 2\pi\}$ under this mapping is transformed into the circle $K = \{(x,y) \in \mathbb{R}^2 \mid$

$x^2 + y^2 \le R^2\}$. This mapping is smooth, but it is not a diffeomorphism: the whole side of the rectangle I on which $r = 0$ is transformed under this mapping into the point $(0, 0)$; the images of the points $(r, 0)$ and $(r, 2\pi)$ coincide. However, if we consider, for example, the sets $I \setminus \partial I$ and $K \setminus E$, where E is the union of the boundary ∂K of the circle K and the radius going to the point $(0, R)$, then the restriction of the mapping (B.10) to the domain $I \setminus \partial I$ is a diffeomorphism with the set $K \setminus E$. Therefore, if instead of the rectangle I, we take a slightly smaller rectangle I_δ lying strictly in the interior of I, then we can apply formula (B.10) to this rectangle I_δ and its image K_δ. And then, exhausting the rectangle I with such rectangles I_δ and noticing that their images exhaust the circle K, that $|I_\delta| \to |I|$ and $|K_\delta| \to |K|$, in the limit we obtain formula (B.3) applied to the original pair K, I.

This applies, of course, to the general polar (spherical) coordinates system in $\mathbb{R}^n$.

We shall now develop these observations.

b) *Exhaustions and limit transitions.* We define an *exhaustion* of a set $E \subset \mathbb{R}^n$ to be a sequence of measurable sets $\{E_n\}$ such that $E_n \subset E_{n+1} \subset E$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} E_n = E$.

Lemma 1 *If $\{E_n\}$ is an exhaustion of a measurable set E, then*

a) $\lim_{n\to\infty} \mu(E_n) = \mu(E)$;
b) *for every function $f \in \mathcal{R}(E)$, one has $f|_{E_n} \in \mathcal{R}(E_n)$ and*

$$\lim_{n\to\infty} \int_{E_n} f(x)\,\mathrm{d}x = \int_E f(x)\,\mathrm{d}x.$$

Proof a) Since $E_n \subset E_{n+1} \subset E$, then $\mu(E_n) \le \mu(E_{n+1}) \le \mu(E)$ and $\lim_{n\to\infty} \mu(E_n) \le \mu(E)$. For proving the equality in a), we shall show that the inequality $\lim_{n\to\infty} \mu(E_n) \ge \mu(E)$ also holds.

The boundary ∂E of the set E is compact and has measure zero. Therefore, it can be covered with a finite number of open intervals such that the sum of their volumes is less than ε for a given $\varepsilon > 0$. Let Δ be the union of these open intervals. Then the set $O = E \cup \Delta$ is open in $\mathbb{R}^m$; by construction, O contains the closure $\overline{E}$ of the set E; and $\mu(O) \le \mu(E) + \mu(\Delta) < \mu(E) + \varepsilon$.

For every set E_n of the exhaustion $\{E_n\}$ we repeat the construction above with $\varepsilon_n = \varepsilon/2^n$. We obtain then a sequence of open sets $O_n = E_n \cup \Delta_n$ such that $E_n \subset O_n$, $\mu(O_n) \le (E_n) + \mu(\Delta_n) < \mu(E_n) + \varepsilon_n$ and $\bigcup_{n=1}^{\infty} O_n \supset \bigcup_{n=1}^{\infty} E_n \supset E$.

The system of open sets $\Delta, O_1, O_2, \ldots$ is an open cover of the compact set $\overline{E}$.

Let $\Delta, O_1, O_2, \ldots, O_k$ be a finite open subcover of the compact set $\overline{E}$. Since $E_1 \subset E_2 \subset \cdots \subset E_k$, the sets $\Delta, \Delta_1, \ldots, \Delta_k, E_k$ are also a cover of $\overline{E}$, and then

$$\mu(E) \le \mu(\overline{E}) + \mu(\Delta) + \mu(\Delta_1) + \cdots + \mu(\Delta_k) < \mu(E_k) + 2\varepsilon.$$

It follows from this that $\mu(E) \le \lim_{n\to\infty} \mu(E_n)$.

b) The fact that $f|_{E_n} \in \mathcal{R}(E_n)$ is known to us, and it follows from Lebesgue's criterion for the existence of the integral over a measurable set. By the hypothesis $f \in \mathcal{R}(E)$, there exists a constant M such that $|f(x)| \le M$ over E. From the additivity of the integral and the general estimates for the integral, we get

$$\left|\int_E f(x)\,\mathrm{d}x - \int_{E_n} f(x)\,\mathrm{d}x\right| = \left|\int_{E\setminus E_n} f(x)\,\mathrm{d}x\right| \le M\mu(E \setminus E_n).$$

Hence, taking into account what we proved in a), we conclude that assertion b) holds. □

The additivity of the integral and the possibility of exhausting the domain of integration with the domains where the change of variables formula works (i.e., it is directly applicable) allow us to apply the formula to the original domain. In general, the idea of exhaustion lies at the heart of many constructions in analysis. In particular, it is fundamental in the definition of improper integrals.

Appendix C
Multidimensional Geometry and Functions of a Very Large Number of Variables (Concentration of Measures and Laws of Large Numbers)

C.1 An Observation

Almost the entire volume of a multidimensional body is concentrated in a small neighborhood of the boundary of the body.

Problem 1 a) Check this in the examples of the cube and the ball. Show that if we remove the shell with thickness 1 cm from a 1000-dimensional watermelon with 1 meter radius, then there remains less than a thousandth of the original watermelon.

b) If we project the sphere $S^{n-1}(r) \subset \mathbb{R}^n$ orthogonally onto a hyperplane passing through the center of the sphere, then we obtain a ball (double covered) with the same dimension $n-1$ and the same radius r. Considering what we obtain above, notice (still on a qualitative level), that almost all the area of the sphere $S^{n-1}(r)$ for $n \gg 1$ is concentrated in a small neighborhood of the equator, the intersection of the sphere with the former hyperplane.

C.2 Sphere and Random Vectors

Problem 2 a) The sphere $S^{n-1}(r)$ with radius r and center at the origin of the n-dimensional Euclidean space $\mathbb{R}^n$ is projected orthogonally onto a coordinate axis. We get the interval $[-r, r]$. We fix another interval $[a, b] \subset [-r, r]$. Let $S[a, b]$ be the area of the part $S^{n-1}_{[a,b]}(r)$ of the sphere $S^{n-1}(r)$ that is projected onto the interval $[a, b]$. Find the quotient $\frac{S[a,b]}{S[-r,r]}$, i.e., the probability $\Pr_n[a, b]$ that a randomly chosen point on the sphere will be on the layer $S^{n-1}_{[a,b]}(r)$ over the interval $[a, b]$, considering that the points are uniformly distributed over the sphere.

Answer:

$$\Pr_n[a, b] = \frac{\int_a^b (1 - (x/r)^2)^{\frac{n-3}{2}}\,\mathrm{d}x}{\int_{-r}^r (1 - (x/r)^2)^{\frac{n-3}{2}}\,\mathrm{d}x}.$$

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b) Let $\delta \in (0, 1)$ and $[a, b] = [\delta r, r]$. Show that as $n \to \infty$,

$$\Pr_n[\delta r, r] \sim \frac{1}{\delta\sqrt{2\pi n}} e^{-\frac{1}{2}\delta^2 n}.$$

Hint: You can use Laplace's method for obtaining asymptotics of the integral over a large parameter.

c) The result obtained in b) implies that the vast majority of the area of a multidimensional sphere is concentrated in a small neighborhood of the equatorial plane, in the layer $S^{n-1}_{[-\delta r, \delta r]}(r)$ over the interval $[-\delta r, \delta r]$.

Deduce from this that if we take independently and randomly a pair of vectors in $\mathbb{R}^n$, then for $n \gg 1$, it is very likely that they will be almost orthogonal, i.e., their scalar product will be close to zero. Estimate the probability that the scalar product is greater than $\varepsilon > 0$ and calculate its variance for $n \gg 1$.

d) Prove, based on the result proved in a), that for $r = \sigma\sqrt{n}$ and $n \to \infty$, one has

$$\Pr_n[a, b] \to \frac{1}{\sqrt{2\pi}\sigma} \int_a^b e^{-\frac{x^2}{2\sigma^2}}\, \mathrm{d}x.$$

e) Considering the result obtained in b), prove now Gauss's law on the distribution of measurement errors and Maxwell's laws on the distribution of gas molecules according to speed and energy (considering in the first case that the observations are independent and their mean square stabilizes as the number of observations increases, and in the second case considering that the gas is homogeneous and that the total energy of the molecules in a portion of the gas is proportional to the number of molecules in this portion).

C.3 Multidimensional Sphere, Law of Large Numbers, and Central Limit Theorem

By solving this problem, you will discover the following fact, important in many aspects and manifested in many areas (for example, in statistical physics).

Let S^m be the unit sphere in the Euclidean space $\mathbb{R}^{m+1}$ with a very large dimension $m + 1$. Suppose also that we are given a sufficiently regular real-valued function on the sphere (for example, from a fixed Lipschitz class). We take randomly and independently two points and calculate the value of the function at these points. With a high probability, the values will almost coincide and they will be close to a certain number M_f.

(This, still hypothetical, number M_f is called the *median value of the function* or *function median*. It is also called the *average value of the function in the sense of Lévy*.[1] The motivation for these terms will soon be clear, together with a precise definition of the number M_f.)

[1] P. Lévy (1886–1971) – famous French mathematician, student of J. Hadamard.

We introduce some notation and conventions. We define the distance between two points on the sphere $S^m \subset \mathbb{R}^{m+1}$, understood in terms of its geodesic metric ρ. We denote by A_δ a δ-neighborhood in S^m of the set $A \subset S^m$. We replace the standard mass of the sphere with a uniformly distributed probability measure μ, i.e., $\mu(S^m) = 1$.

We have the following assertion proved by Paul Lévy, commonly called *Lévy's isoperimetric inequality*.

For every $0 < a < 1$ *and* $\delta > 0$, *there exists* $\min\{\mu(A_\delta) \mid A \subset S^m, \mu(A) = a\}$, *and it is attained on the spherical cap* A^0 *with measure* a.

Here $A^0 = B(r)$, where $B(r) = B(x_0, r) = \{x \in S^m \mid \rho(x_0, x) < r\}$ and $\mu(B(r)) = a$.

Problem 3 a) For $a = 1/2$, i.e., when A^0 is a hemisphere, obtain the following result:

If the subset $A \subset S^{n+1}$ *is such that* $\mu(A) \geq 1/2$, *then* $\mu(A_\delta) \geq 1 - \sqrt{\pi/8}e^{-\delta^2 n/2}$. (If $n \to \infty$, we can change here $\sqrt{\pi/8}$ for $1/2$.)

b) We denote by M_f the number such that

$$\mu\{x \in S^n \mid f(x) \leq M_f\} \geq 1/2 \quad \text{and} \quad \mu\{x \in S^n \mid f(x) \geq M_f\} \geq 1/2.$$

It is called the *median* or *average value in the sense of Lévy of the function* $f : S^n \to \mathbb{R}$. (If the M_f-level of the function f on the sphere has measure zero, then the measure of each of these two sets mentioned above will be equal to exactly half of the μ-area of the sphere S^m.)

Obtain the following lemma due to Lévy:

If $f \in C(S^{n+1})$ *and* $A = \{x \in S^{n+1} \mid f(x) = M_f\}$, *then* $\mu(A_\delta) \geq 1 - \sqrt{\pi/2} \times e^{-\delta^2 n/2}$.

c) Let $\omega_f(\delta) = \sup\{|f(x) - f(y)| \mid \rho(x, y) \leq \delta\}$ be the *modulus of continuity of the function* f.

The values of the function f on the set A_δ are close to M_f. More precisely, if $\omega_f(\delta) \leq \varepsilon$, then $|f(x) - M_f| \leq \varepsilon$ on A_δ. Thus Lévy's lemma shows that "good" functions are actually almost constant in almost their entire domain of definition S^m when the dimension m is very large.

Considering that $f \in \mathrm{Lip}(S^{n-1}, \mathbb{R})$ and L is the Lipschitz constant of the function f, estimate the probability $\Pr\{|f(x) - M_f| > \varepsilon\}$ and the dispersion value $|f(x) - M_f|$ for $n \gg 1$.

d) Obtain, as above, estimates in the case that the function f is not defined on the unit sphere but in the sphere $S^{n-1}(r)$ with radius r.

e) If f is a smooth function, then we can clearly take the maximum modulus of its gradient as the Lipschitz constant L. For example, the linear function $S_n = \frac{1}{n}(x_1 + \cdots + x_n)$ has $L = L_n = \frac{1}{\sqrt{n}}$. Suppose that we have a sequence of Lipschitz functions $f_n \in \mathrm{Lip}(S^{n-1}(r_n), \mathbb{R})$, for which $L_n = O(\frac{1}{\sqrt{n}})$ and $r_n = \sqrt{n}$.

Estimate $\Pr\{|f_n(x) - M_{f_n}| > \varepsilon\}$ and the dispersion value $|f_n(x) - M_{f_n}|$ for $n \gg 1$.

In particular, for $f_n = S_n$ deduce the standard law of large numbers.

f) Let $f_n = x_1 + \cdots + x_n$. The levels of this function are hyperplanes in $\mathbb{R}^n$ orthogonal to the vector $(1, \dots, 1)$. The same can be said about the linear function $\Sigma_n = \frac{1}{\sqrt{n}}(x_1 + \cdots + x_n)$, with the only difference that under the movement from the origin in the direction of $(1, \dots, 1)$, its values coincide with the distances to the origin. For this reason, its values are distributed on the sphere $S^{n-1}(r_n)$ exactly as they are on each of the coordinates.

Using this discussion and the result of Problem 2.d), setting $r_n = \sigma\sqrt{n}$, obtain your own version of the central limit theorem.

C.4 Multidimensional Intervals (Multidimensional Cubes)

Problem 4 a) Let I be the standard unit interval [0, 1] of the real line $\mathbb{R}$, and I^n the standard n-dimensional interval in $\mathbb{R}^n$, usually called the n-dimensional unit cube. This is a unit of volume in $\mathbb{R}^n$, but its diameter $\sqrt{n}$ for $n \gg 1$ is extremely huge. Thus, even Lipschitz functions on I^n with Lipschitz constant L can have values spread within $L\sqrt{n}$.

Yet here, as in the above case of a sphere, there is a phenomenon of asymptotic stabilization (concentration) of values of such functions in the limit $n \to \infty$.

Now, try to find the proper formulations of the problem and study the phenomenon, up to the level of your ability (then check Sect. C.5 of this appendix).

b) Suppose we have n independent random variables x_i, taking values in the unit interval [0, 1] and having distribution probabilities $p_i(x)$, which are uniformly separated from zero (in particular, all $p_i(x)$ may coincide). Then as n grows, the large majority of the random points $(x_1, \dots, x_n) \in I^n$ will lie in close proximity to the border of the cube.

Explain this, and considering the result in a), obtain your own general law of large numbers.

c) Show with an example that if the probability density of the random variables in b) is concentrated in the vertices of the cube as point masses, then the asymptotic stabilization of values for Lipschitz functions in the limit $n \to \infty$ may not occur.

d) We noted above that although the volume of the cube I^n in $\mathbb{R}^n$ is equal to 1, its diameter $\sqrt{n}$ increases for $n \gg 1$, which creates difficulties. However, we have the following useful compensating observation: if each of two subsets A and B of the cube I^n has measure greater than an arbitrarily small fixed positive number ε, then the distance between A and B is bounded from above by a constant depending only on ε (and not depending on n). Prove this, and use this result if you need it.

e) Calculate the volume of the unit ball in $\mathbb{R}^n$ and show that the radius of the ball with volume one increases as $\sqrt{n/(2\pi e)}$ as $n \to \infty$. Go back to Sects. C.1 and C.2 and convince yourself again that the normal distribution and the laws related to it are closely linked in the geometric aspect with a simple multidimensional object, namely with the ball of unit volume.

C.5 Gaussian Measures and Their Concentration

Problem 5 a) We mentioned in Sect. C.2 of this appendix the isoperimetric inequality on the sphere, in connection with the discussion of the observed stabilization of values (constancy) of regular functions on the multidimensional sphere. The same problem about minimizing the measure of a δ-blowup of a set is important, and for the same reason it is also interesting in relation to other spaces that serve as natural domains for the relevant functions.

For example, in the case of the Gaussian probability measures defined by the normal probability distribution in the standard Euclidean space $\mathbb{R}^n$, the answer is also known (obtained by Borel). In this case, the extreme domain (with the fixed initial value of the Gaussian measure and a δ-blowup, understood in the sense of the Euclidean metric) turns out to be a half-space.

In particular, if we take the half-space with Gaussian measure $\frac{1}{2}$ and we directly calculate the value of the Gaussian measure of the complement in its Euclidean δ-blowup, then considering Borel's isoperimetric inequality, we can deduce that for any set A having a Gaussian measure $\frac{1}{2}$ in the space $\mathbb{R}^n$, the measure of its δ-blowup can be estimated from below with $\mu(A_\delta) \geq 1 - I_\delta$, where I_δ is the integral of the density $(2\pi)^{\frac{n}{2}} \exp(-\frac{|x|^2}{2})$ of the Gaussian measure of the half-space, given with Euclidean distance δ from the origin.

An estimate from above of the integral I_δ, for example, allows us to claim that $\mu(A_\delta) \geq 1 - 2\exp(-\frac{\delta^2}{2})$. Prove this.

b) This is a rough estimate, but it shows the rapid growth of $\mu(A_\delta)$, with an increase of δ, whatever the initial set A of measure $\frac{1}{2}$ is.

It is very interesting to notice (and considering the possible transition to infinite-dimensional spaces, even quite useful) that the last estimate does not depend on the dimension of the space. It may seem that this absence of the dimension is a great loss and weakness in the estimates within the context of concentration measures discussed and in the stabilization of values of functions of several variables. In fact, this estimate even contains the principle of the concentration of a measure on the unit sphere of large dimension, discussed above.

It is enough to prove (prove it) that the main part of the Gaussian probability measure of the Euclidean space $\mathbb{R}^n$ for $n \gg 1$ is concentrated in the vicinity of the unit Euclidean sphere of radius $\sqrt{n}$. This means that at the intersection of this neighborhood with the half-space, which is distant from the origin, the proportion of this measure is exponentially small. Therefore, the main part of the measure is in this neighborhood of the sphere of radius $\sqrt{n}$, which falls in the layer between two close parallel hyperplanes, symmetric with respect to the origin. If now we move through a homothety from the sphere of radius $\sqrt{n}$ to the unit sphere, then we obtain the principle of concentration of measure on the unit sphere, which we have already discussed (do the necessary calculations). In the statement of this principle, the dimension of the space occurs explicitly. This dimension was also present in the Gaussian case, but it was hidden in the size $\sqrt{n}$ of the sphere, and the main part of the measure of the whole space is concentrated in a neighborhood of this sphere.

C.6 A Little Bit More About the Multidimensional Cube

In the Euclidean space $\mathbb{R}^n$ we consider the n-dimensional unit interval ("cube")

$$I^n := \left\{x = \left(x^1, \dots, x^n\right) \in \mathbb{R}^n \mid \left|x^i\right| \le \frac{1}{2}, i = 1, 2, \dots, n\right\}.$$

Its volume is equal to one, although the diameter is $\sqrt{n}$. (Recall that the Euclidean ball of volume one in $\mathbb{R}^n$ has radius of order $\sqrt{n}$, as mentioned above.) We shall consider the standard probability measure uniformly distributed on the cube I^n.

Let $a = (a^1, \dots, a^n)$ be a unit vector, and $x = (x^1, \dots, x^n)$ an arbitrary point in the cube I^n.

The following inequality holds (probability estimate of Bernstein type):

$$\Pr_n\left\{\left|\sum_{i=1}^n a^i x^i\right| \ge t\right\} \le 2\exp\left(-6t^2\right).$$

If we interpret the sum $\sum_{i=1}^n a^i x^i$ as a scalar product $\langle a, x\rangle$, we notice that this can be large (of order $\sqrt{n}$) if the vector a is not directed along any edge of the cube, but along the main diagonal, mixing all coordinate directions equally. If we take $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ in the previous estimate, we deduce that the volume of the n-dimensional cube I^n concentrates, as n increases, in a small neighborhood of the hyperplane passing through the origin and orthogonal to the vector $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$.

In particular, if we consider a billiard in such a cube as a dynamical system (gas) composed with noninteracting particles, then for $n \gg 1$, the large majority of particle trajectories will go in a direction nearly perpendicular to the fixed vector $(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$, and they are a large part of the time in a neighborhood of the above hyperplane.

C.7 The Coding of a Signal in a Channel with Noise

We point out in conclusion another area where the functions with a very large number of variables also appear naturally and where the principle of concentration of a measure is shown and also used substantially.

We are already used to the digital (discrete) coding and transmission of signals (music, images, messages, information) on a communication channel. In this form, a message can be thought of as a vector $x = (x^1, \dots, x^n)$ in the space $\mathbb{R}^n$ with a very large dimension. The transmission of such messages requires an energy E, which is proportional to $\|x\|^2 = |x^1|^2 + \dots + |x^n|^2$ (like the total kinetic energy of the gas molecules, discussed above). If T is the duration of the transmitted message x, then $P = E/T$ is the average power required to transfer one character (a coordinate of the vector x). If Δ is the average time required to transfer a single coordinate of the vector x, then $T = n\Delta$ and $E = nP\Delta$.

The transmitting and receiving devices are aligned in a such a way that the transmitter transforms (encodes) the original message to be transmitted in the form of the vector x. It sends it over the communication channel, and the receiver, knowing the code, decrypts x, transforming it into the form of the original message.

If we need to transmit M messages $A_1, \ldots, A_M$ of length n, then it is enough to fix n points in the ball of radius $\sqrt{E}$, agreeing on this selection with the receiving end of the communication channel. If in the communication channel there is no interference, then having received the vector from the agreed set, the receiver decodes it correctly into the corresponding message A.

If in the channel we do have interference (which is often the case), then because of the interference, a random vector $\xi = (\xi^1, \ldots, \xi^n)$ shifts the transmitted vector a, and the vector $a + \xi$ arrives at the receiver, and this vector must be properly decoded.

If the points $a_1, \ldots, a_n$ were chosen in such a way that the balls of radius $\|\xi\|$ with these points as center do not intersect, then an unambiguous deciphering is still possible. But if we want to meet this requirement, then we cannot take just any points $a_1, \ldots, a_M$, and there is a problem of dense packing of spheres. This is a difficult problem, whose solution in the present situation can be avoided, as was shown by Shannon, given that here the dimension n of the space $\mathbb{R}^n$ is huge.

We shall allow ourselves sometimes to make mistakes while interpreting the received message. However, we require the probability of error to be arbitrarily small (less than any fixed positive number).

Shannon showed that even in the presence of random noise (white noise) in the communication channel with limited capacities, by choosing a long enough code (i.e., for a large value of n), it is possible to achieve velocities of transmission close to the velocities of transmission of information in channels without noise, with an arbitrarily small probability of error.

The geometric idea of Shannon's theorem is directly related to the characteristics discussed above of the distribution measures (volumes) of domains in a space with large dimension. Let us explain this.

Suppose that two identical balls in the space $\mathbb{R}^n$ intersect. If the received signal lies in this intersection, then it is possible to have errors in the interpretation of the message sent by the source. But if the probability of falling into such an area is considered proportional to the relative volume of the region, then it is natural to compare the volume of the intersection of the balls with the volume of a ball. We carry out the proper estimations. If the centers of two balls of radius 1 are separated by the distance ε $(0 < \varepsilon < 2)$, then the intersection of these balls is contained in a ball of radius $\sqrt{1 - (\varepsilon/2)^2}$ with center in the middle of the segment connecting the centers of the original balls. Hence, the ratio between the volume of the intersection of the two balls and their own original volume does not exceed $(1 - (\varepsilon/2)^2)^{n/2}$. It is clear now that for every fixed ε, this value can be made arbitrarily small by choosing a sufficiently large value of n.

Appendix D
Operators of Field Theory in Curvilinear Coordinates

Introduction

Almost any book with mathematical problems and even any textbook of mathematical analysis states something like the following. "Children, remember":

We call the *gradient* of a function $U(u, x, z)$ the vector

$$\operatorname{grad} U := \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right).$$

The *curl* of a vector field $A = (P, Q, R)(x, y, z)$ is the vector

$$\operatorname{curl} A := \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

The *divergence* of a vector field $B = (P, Q, R)(x, y, z)$ is the function

$$\operatorname{div} B := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The fact that this is true only in Cartesian coordinates is not usually discussed, as well as what should be done if the coordinate system is different. This is understandable, since the very formulation of this problem already requires some suitable definition of these objects.

D.1 Reminders of Algebra and Geometry

D.1.1 Bilinear Forms and Their Coordinate Representation

a. Scalar Product and General Linear Forms

We shall consider a vector space with a scalar product $\langle , \rangle$. We can still consider that $\langle , \rangle$ denotes an arbitrary bilinear form on an n-dimensional vector space X. If we

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choose a basis of the space $\xi_1, \dots, \xi_n$, then the objects of the space (in particular, vectors and forms) will have a coordinate representation. We recall the coordinate representation of the bilinear form $\langle , \rangle$.

If we take two vectors $x = x^i \xi_i$, $y = y^j \xi_j$ and their decomposition in terms of the basis, then we have $\langle x, y\rangle = \langle x^i \xi_i, y^j \xi_j\rangle = \langle \xi_i, \xi_j\rangle x^i y^j = g_{ij} x^i y^j$. As usual, summation over repeated indices is understood. Thus if a basis of the space is given, the choice of values $\langle \xi_i, \xi_j\rangle = g_{ij}$ completely defines the bilinear form.

If the form is a scalar product, then a basis is orthogonal if $g_{ij} = 0$ for $i \neq j$. It is assumed here that the form is nondegenerate, of course.

b. Nondegeneracy of Bilinear Forms

A bilinear form is called nondegenerate if once we fix a value in one of its arguments, then the bilinear form is identically zero with respect to the other argument if and only if the fixed value is zero (the zero vector).

The nondegeneracy of the form is equivalent to the fact that the determinant of the matrix (g_{ij}) is different from zero. Indeed, if the fixed vector $x = x^i \xi_i$ is such that $\langle x, y\rangle \equiv 0$ with respect to y, then $\langle \xi_i, \xi_j\rangle x^i = 0$ and $g_{ij} x^i = 0$ for every value $j \in \{i, \dots, n\}$. This homogeneous system of equations has a unique solution (zero) if and only if the determinant of the matrix (g_{ij}) of the system is nonzero.

D.1.2 Correspondence Between Forms and Vectors

a. 1-Forms in the Presence of 2-Forms and Their Correspondence with Vectors

If one has a 2-form $\langle , \rangle$, then each vector A can be associated with a 1-form, namely the linear form $\langle A, x\rangle$. If the 2-form is nondegenerate, then the correspondence is one-to-one. Indeed, if we are given such a linear function $a(x) = a_j x^j$ (where $a_j = a(\xi_j)$) and we want to represent it in the form $\langle A, x\rangle$, where $A = \xi_i A^i$, then in the coordinates of the vector A we have the system of equations $a(\xi) = \langle \xi_i, \xi_j\rangle A^i$, $j = 1, \dots, n$, which is uniquely solvable if the determinant of the matrix (g_{ij}) is different from zero.

Thus, the coordinates of the vector $A = A^i \xi_i$ and the coefficients of the 1-form a in the same basis $\{\xi_i\}$ are linked by the mutually inverse relations

$$a_j = g_{ij} A^i, \qquad A^i = g^{ij} a_j.$$

b. Correspondence Between a Vector and an $(n-1)$-Form

Similarly, if one has a nondegenerate n-form Ω, each vector B can be associated with an $(n-1)$-form, namely the form $\Omega(B, \dots)$.

We shall deal below with vector fields A, B and carry out this described method on the tangent space, for example in relation to the form of work $\omega_A^1 = \langle A, \cdot\rangle$ and the form flux $\omega_B^{n-1} = \Omega^n(B, \dots)$, in the presence of the inner product $\langle , \rangle$ and the volume form Ω^n, respectively.

D.1.3 Curvilinear Coordinates and Metric

a. Curvilinear Coordinates, Metric, and Volume Form

Suppose that in an n-dimensional surface (manifold) we have a metric, which in local coordinates $(t^1, \dots, t^n)$ (in the local charts) is given by the form $g_{ij}(t)\,\mathrm{d}t^i\,\mathrm{d}t^j$, determined by the scalar product $\langle , \rangle(t)$, with the corresponding parameter t of the tangent plane (tangent space) to the surface.

For example, if the surface (or curve) is given in a parametric form, it is embedded into the Euclidean space, and then the scalar product in the tangent planes (spaces) to the surface is naturally induced from that in the ambient space.

We even know how to find the area of such a surface (n-measure), i.e., it is necessary to integrate the volume form

$$\Omega = \sqrt{\det g_{ij}(t)}\,\mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^n.$$

b. Orthogonal Systems of Curvilinear Coordinates and Unit Vectors

Recall that a system of curvilinear coordinates $(t^1, \dots, t^n)$ is called orthogonal if $g_{ij} \equiv 0$ for $i \neq j$.

The length element in an orthogonal system of curvilinear coordinates is written in a particularly simple form:

$$\mathrm{d}s^2 = g_{11}(t)\big(\mathrm{d}t^1\big)^2 + \cdots + g_{nn}(t)\big(\mathrm{d}t^n\big)^2.$$

It is often rewritten in the more compact notation

$$\mathrm{d}s^2 = E_1(t)\big(\mathrm{d}t^1\big)^2 + \cdots + E_n(t)\big(\mathrm{d}t^n\big)^2.$$

The vectors $\xi_1 = (1, 0, \dots, 0), \dots, \xi_n = (0, \dots, 0, 1)$ of the coordinate directions form a basis of the tangent space, corresponding to the value of the parameter t. But the norm (length) of these vectors is, in general, not equal to one. We have always, independent of whether the system of coordinates is orthogonal, $\langle \xi_i, \xi_i\rangle(t) = g_{ii}(t)$, i.e., $\|\xi_i\| = \sqrt{g_{ii}(t)}$, $i \in \{1, \dots, n\}$.

Thus, the unit vectors $(e_1, \dots, e_n)$ (vectors of length one) of the coordinate directions have the following coordinate representation:

$$e_1 = \left(\frac{1}{\sqrt{g_{11}}}, 0, \dots, 0\right), \quad \dots, \quad e_n = \left(0, \dots, 0, \frac{1}{\sqrt{g_{nn}}}\right).$$

In particular, if the system of curvilinear coordinates is orthogonal, then the following system of vectors of coordinate directions will be an orthonormal basis in the corresponding tangent space:

$$e_1 = \left(\frac{1}{\sqrt{E_1}}, 0, \dots, 0\right), \quad \dots, \quad e_n = \left(0, \dots, 0, \frac{1}{\sqrt{E_n}}\right).$$

c. Cartesian, Cylindrical, and Spherical Coordinates

As examples of orthogonal coordinate systems we have the standard Cartesian, cylindrical, and spherical coordinates in $\mathbb{R}^3$.

Problem 1 Write down the metric $g_{ij}(t)\,\mathrm{d}t^i\,\mathrm{d}t^j$ in each of these coordinate systems and find an orthonormal basis (e_1, e_2, e_3).

Answer In Cartesian coordinates (x, y, z), cylindrical coordinates (r, φ, z), and spherical coordinates (R, φ, θ) of the Euclidean space $\mathbb{R}^3$, the quadratic form $g_{ij}(t)\,\mathrm{d}t^i\,\mathrm{d}t^j$ has the following form:

$$\begin{aligned}\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \\ &= \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2 + \mathrm{d}z^2 = \\ &= \mathrm{d}R^2 + R^2\cos^2\theta\,\mathrm{d}\varphi^2 + R^2\,\mathrm{d}\theta^2.\end{aligned}$$

In Cartesian, cylindrical, and spherical coordinates, the triples of unit vectors of coordinate directions are the following, respectively:

$$\begin{aligned}&e_x = (1, 0, 0), \qquad e_y = (0, 1, 0), \qquad e_z = (0, 0, 1);\\ &e_r = (1, 0, 0), \qquad e_\varphi = \left(0, \frac{1}{r}, 0\right), \qquad e_z = (0, 0, 1);\\ &e_R = (1, 0, 0), \qquad e_\varphi = \left(0, \frac{1}{R\cos\theta}, 0\right), \qquad e_\theta = \left(0, 0, \frac{1}{R}\right).\end{aligned}$$

D.2 Operators grad, curl, div in Curvilinear Coordinates

D.2.1 Differential Forms and Operators grad, curl, div

The differential $\mathrm{d}U$ of a function U is a 1-form. When one has a scalar product $\langle , \rangle$, as we know, to the 1-form $\mathrm{d}U$ corresponds a vector A such that $\mathrm{d}U = \langle A, \cdot\rangle$. This vector is called the *gradient of the function* U and is denoted by $\operatorname{grad} U$.

Thus, $\mathrm{d}U = \langle \operatorname{grad} U, \cdot\rangle$.

Suppose that in the Euclidean space $\mathbb{R}^3$ (or in any three-dimensional Riemannian manifold) we have the 1-form $\omega_A^1 = \langle A, \cdot \rangle$ corresponding to the field A. The differential $\mathrm{d}\omega_A^1$ of this form is a 2-form ω_B^2, corresponding, in the presence of a volume form Ω^3, to some vector field B (i.e., $\omega_B^2 = \Omega^3(B, \cdot, \cdot)$). Then the field B is called the *curl of the vector field* A, and is denoted by curl A.

Thus, $\mathrm{d}\omega_A^1 = \omega_{\mathrm{curl}\, A}^2$.

If one has a volume form Ω^n on an n-dimensional surface (for example on $\mathbb{R}^n$), then there is defined an $(n-1)$-form for the flux of a vector field B, namely the form $\omega_B^{n-1} = \Omega^n(B, \cdot, \cdot)$. The differential $\mathrm{d}\omega_B^{n-1}$ of this $(n-1)$-form is an n-form, which therefore has the type $\rho\Omega^n$. The proportionality factor, the function ρ, is called the *divergence of the vector field* B and is denoted by div B.

Thus, $\mathrm{d}\omega_B^{n-1} = (\operatorname{div} B)\Omega^n$.

D.2.2 Gradient of a Function and Its Coordinate Representation

a. Coordinate Representation for the Correspondence Between a Vector and a 1-Form

In Sect. D.1.2, we derived a relation between the coefficients of a 1-form $\omega_A^1 = \langle A, \cdot \rangle$ and the coordinates of the vector $A = A^i \xi_i$. If we take the unit vectors e_i instead of the vectors ξ_i, and since $\xi_i = \sqrt{g_{ii}} e_i$, then the coordinates of the vector $A = A_e^i e_i$ in the basis $\{e_i\}$ and its former coordinates are related through the equation $A_e^i = A^i \sqrt{g_{ii}}$ for $i \in \{1, \ldots, n\}$.

Hence all new related formulas have the form

$$a_j = g_{ij} \frac{A_e^i}{\sqrt{g_{ii}}}, \qquad \frac{A_e^i}{\sqrt{g_{ii}}} = g^{ij} a_j.$$

These formulas allow us to write, in terms of the vector $A = A_e^i e_i$, the corresponding form $\omega_A^1 = \langle A, \cdot \rangle = a_j \, \mathrm{d}t^j$ and conversely, to write the vector $A = A_e^i e_i$ in terms of the 1-form $\omega^1 = a_j \, \mathrm{d}t^j$.

Problem 2 Write down in Cartesian, cylindrical, and spherical coordinates of the Euclidean space $\mathbb{R}^3$ the explicit form of the 1-form $\omega_A^1 = \langle A, \cdot \rangle$, corresponding to the vector $A = A^i e_i$.

Answer The 1-form ω_A^1 has the following form, in Cartesian coordinates (x, y, z), cylindrical coordinates (r, φ, z), and spherical coordinates (R, φ, θ) of the Euclidean space $\mathbb{R}^3$, respectively:

$$\begin{aligned}
\omega_A^1 &= A_x \, \mathrm{d}x + A_y \, \mathrm{d}y + A_z \, \mathrm{d}z = \\
&= A_r \, \mathrm{d}r + A_\varphi r \, \mathrm{d}\varphi + A_z \, \mathrm{d}z = \\
&= A_R \, \mathrm{d}R + A_\varphi R \cos\varphi \, \mathrm{d}\varphi + A_\theta R \, \mathrm{d}\theta.
\end{aligned}$$

b. Differential of a Function and the Gradient

We shall apply the general formula relating the vector A and the form ω_A^1 in the case of the form $\mathrm{d}U = \langle \operatorname{grad} U, \cdot \rangle$, in order to find the decomposition $\operatorname{grad} U = A_e^i e_i$. Since $\mathrm{d}U = \frac{\partial U}{\partial t^j}\,\mathrm{d}t^j$, i.e., $a_j = \frac{\partial U}{\partial t^j}$, then we have $A_e^i = g^{ij}\sqrt{g_{ii}}\frac{\partial U}{\partial t^j}$.

In the case of an orthogonal system of curvilinear coordinates, the matrix (g_{ij}) is diagonal, as well as its inverse matrix (g^{ij}). Moreover, $g^{ii} = 1/g_{ii}$. Hence in this case,

$$\operatorname{grad} U = \frac{1}{\sqrt{g_{11}}}\frac{\partial U}{\partial t^1}e_1 + \cdots + \frac{1}{\sqrt{g_{nn}}}\frac{\partial U}{\partial t^n}e_n.$$

c. Gradient in Cartesian, Cylindrical, and Spherical Coordinates

Problem 3 Write down the vector $\operatorname{grad} U = A_e^i e_i$ in Cartesian, cylindrical, and spherical coordinates of the Euclidean space $\mathbb{R}^3$.

Answer The vector $\operatorname{grad} U$ has the following form in Cartesian (x, y, z), cylindrical (r, θ, z), and spherical (R, φ, θ) coordinates of the Euclidean space $\mathbb{R}^3$, respectively:

$$\begin{aligned}\operatorname{grad} U &= \frac{\partial U}{\partial x}e_x + \frac{\partial U}{\partial y}e_y + \frac{\partial U}{\partial z}e_z = \\ &= \frac{\partial U}{\partial r}e_r + \frac{1}{r}\frac{\partial U}{\partial \varphi}e_\varphi + \frac{\partial U}{\partial z}e_z = \\ &= \frac{\partial U}{\partial R}e_R + \frac{1}{R\cos\theta}\frac{\partial U}{\partial \varphi}e_\varphi + \frac{1}{R^2}\frac{\partial U}{\partial \theta}e_\theta.\end{aligned}$$

D.2.3 Divergence and Its Coordinate Representation

a. Coordinate Representation for the Correspondence Between a Vector and an $(n-1)$-Form

We know that if there exists a nondegenerate n-form Ω^n in an n-dimensional vector space, then one can establish a one-to-one correspondence between a vector B and the $(n-1)$-form $\omega_B^{n-1} = \Omega^n(B, \ldots)$. We wish to write down an explicit formula relating the coordinates of the vector $B = B^i\xi_i$ and the coefficients of the form $\omega_B^{n-1} = b_i x^1 \wedge \cdots \overset{\frown}{x^i} \wedge \cdots \wedge x^n$, considering that both objects are expressed in terms of the one basis $\{\xi_i\}$ of the space. Here, x^i is a linear function as usual, whose action is given by assigning the i-coordinate of a vector, i.e., $x^i(v) := v^i$; the symbol $\overset{\frown}{x^i}$ means that the corresponding factor is omitted. The n-form Ω^n in the n-dimensional vector space is $x^1 \wedge \cdots \wedge x^n$ or proportional to this standard volume form, equal to one on the set of the basis vectors $(\xi_1, \ldots, \xi_n)$.

In general, the value of the form $\Omega^1 = x^1 \wedge \cdots \wedge x^n$ on any vector set $(v_1, \ldots, v_n)$ is equal to the determinant of the matrix (v_i^j) consisting of the coordinates of these vectors. Hence if we consider the rule for the expansion of the determinant on a row, we can write

$$\Omega^n(B, \ldots) = \sum_{i=1}^{n} (-1)^{i-1} B^i x^1 \wedge \cdots \wedge \widehat{x^i} \wedge \cdots \wedge x^n.$$

However, $\omega_B^{n-1} = \Omega^n(B, \ldots)$; thus

$$\sum_{i=1}^{n} b_i x^1 \wedge \cdots \wedge \widehat{x^i} \wedge \cdots \wedge x^n = \sum_{i=1}^{n} (-1)^{i-1} B^i x^1 \wedge \cdots \wedge \widehat{x^i} \wedge \cdots \wedge x^n.$$

Therefore, $b_i = (-1)^{i-1} B^i$ for every $i \in \{1, \ldots, n\}$. If instead we had the form $c\omega^n = cx^1 \wedge \cdots \wedge x^n$, then we would have the equation $b_i = (-1)^{i-1} cB^i$ for every $i \in \{1, \ldots, n\}$.

Recall also that if there is an inner product $\langle , \rangle$ and a fixed basis $\{\xi_i\}$ in a vector space, then there is also a natural volume form $\sqrt{\det g_{ij}} x^1 \wedge \cdots \wedge x^n$ defined, as well as the scalar product itself, in terms of the values $g_{ij} = \langle \xi_i, \xi_j \rangle$.

Finally, recall that in this case, the unit vectors (with respect to the norm) are not in general the vectors $\{\xi_i\}$, but the vectors $e_i = \xi_i / \sqrt{g_{ii}}$. Since $\xi_i = \sqrt{g_{ii}} e_i$, the original decomposition of the vector $B = B^i \xi_i$ in the basis $\{e_i\}$ becomes $B = B_e^i e_i$, where $B_e^i = \sqrt{g_{ii}} B^i$.

Therefore, if one has a scalar product on the space, then there is a natural volume form $\Omega_g^n = \sqrt{\det g_{ij}} x^1 \wedge \cdots \wedge x^n$, and if $\omega_B^{n-1} = \Omega_g^n(B, \ldots)$, then the coefficients of the form $\omega_B^{n-1} = b_i x^1 \wedge \cdots \wedge \widehat{x^i} \wedge \cdots \wedge x^n$ and the coordinates of the vector B in the decomposition $B = B_e^i e_i$ in terms of the basis of unit vectors $e_i = \xi_i / \sqrt{g_{ii}}$ are related by the equations

$$b_i = (-1)^{i-1} \sqrt{\det g_{ij}} \frac{B_e^i}{\sqrt{g_{ii}}}.$$

In an orthogonal basis, $\det g_{ij} = g_{11} \cdots g_{nn}$. In this case,

$$b_i = (-1)^{i-1} \sqrt{g_{11} \cdots \widehat{g_{ii}} \cdots g_{nn}} B_e^i.$$

All of the above remains valid when it is applied to the case of the vector field $B(t)$ and the differential form $\omega_B^{n-1} = \Omega_g^n(B, \ldots)$ of the field generated by the volume form.

Thus if $\Omega_g^n = \sqrt{\det g_{ij}}(t)\, \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^n$,

$$\omega_B^{n-1} = b_i(t)\, \mathrm{d}t^1 \wedge \cdots \wedge \widehat{\mathrm{d}t^i} \wedge \cdots \mathrm{d}t^n,$$

and $B(t) = B_e^i(t)e_i(t)$ is the decomposition in terms of the unit vectors of the curvilinear coordinates $(t^1, \dots, t^n)$, then

$$b_i = (-1)^{i-1}\frac{\sqrt{\det g_{ij}}}{\sqrt{g_{ii}}}B_e^i, \qquad B_e^i = (-1)^{i-1}\frac{\sqrt{g_{ii}}}{\sqrt{\det g_{ij}}}b_i.$$

If the system of curvilinear coordinates is orthogonal, we come back to the relation $b_i = (-1)^{i-1}\sqrt{g_{11}\cdots\widehat{g_{ii}}\cdots g_{nn}}\,B_e^i$.

In particular, for a 3-dimensional orthogonal system of curvilinear coordinates (t^1, t^2, t^3), using the same notation $E_i = g_{ii}$ mentioned at the beginning, it is possible to write the following coordinate representation of the form ω_B^2 corresponding to the vector $B = B_e^1 e_1 + B_e^2 e_2 + B_e^3 e_3$:

$$\omega_B^2 = B_e^1\sqrt{E_2E_3}\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + B_e^2\sqrt{E_3E_1}\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + B_e^3\sqrt{E_1E_2}\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 =$$
$$= \sqrt{E_1E_2E_3}\left(\frac{B_e^1}{\sqrt{E_1}}\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + \frac{B_e^2}{\sqrt{E_2}}\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + \frac{B_e^3}{\sqrt{E_3}}\,\mathrm{d}t^1 \wedge \mathrm{d}t^2\right).$$

(Bear in mind that in the 3-dimensional case, the 2-form ω^2 is not usually written as $b_1\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + a_2\,\mathrm{d}t^1 \wedge \mathrm{d}t^3 + b_3\,\mathrm{d}t^1 \wedge \mathrm{d}t^2$, but as $a_1\,\mathrm{d}t^2 \wedge \mathrm{d}t^3 + a_2\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + a_3\,\mathrm{d}t^1 \wedge \mathrm{d}t^2$; for example, $P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y$.)

Problem 4 Specify the explicit form of the 2-form $\omega_B^2 = \Omega_g^3(B, \dots)$ corresponding to the vector field $B = B_e^i e_i$ in Cartesian, cylindrical, and spherical coordinates of the Euclidean space $\mathbb{R}^3$.

Answer The form ω_B^2 has the following form in Cartesian (x, y, z), cylindrical (r, θ, z), and spherical (R, φ, θ) coordinates of the Euclidean space $\mathbb{R}^3$:

$$\omega_B^2 = B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y =$$
$$= B_r r\,\mathrm{d}\varphi \wedge \mathrm{d}z + B_\varphi\,\mathrm{d}z \wedge \mathrm{d}r + B_z r\,\mathrm{d}r \wedge \mathrm{d}\varphi =$$
$$= B_R R^2\cos\theta\,\mathrm{d}\varphi \wedge \mathrm{d}\theta + B_\varphi R\,\mathrm{d}\theta \wedge \mathrm{d}R + B_\theta R\cos\theta\,\mathrm{d}R \wedge \mathrm{d}\varphi.$$

b. The Differential Form of a Flux and the Divergence of the Velocity Field

The form $\omega_B^{n-1} = \Omega_g^n(B, \dots)$ is often called a form of a flux, since when B is the flux velocity field (at least for $n = 3$), one has to integrate exactly this form to find the outflow (flux) through a surface.

The differential of the form of a flux ω_B^{n-1} is an n-form, proportional to the volume form. The coefficients of proportionality are called the divergence field B, as we know. Thus $\mathrm{d}\omega_B^{n-1} = \operatorname{div} B \cdot \Omega_g^n$.

We want to study the field $B = B_e^i e_i$ itself and find its divergence $\operatorname{div} B$. We already know how to find the form of a flux ω_B^{n-1} from the field $B = B_e^i e_i$. We shall find it, compute its differential, and obtain an n-form, proportional to the volume form, whose coefficients of proportionality are the divergence of the field B.

Let us show this. We write the $(n-1)$-form ω_B^{n-1} in the following form:

$$\omega_B^{n-1} = b_1(t)\, \mathrm{d}t^1 \wedge \cdots \wedge \widehat{\mathrm{d}t^i} \wedge \cdots \wedge \mathrm{d}t^n.$$

We compute its differential

$$\mathrm{d}\omega_B^{n-1} = \left(\sum_{n=1}^{n} \frac{\partial b_i}{\partial t^i} (-1)^{i-1} \right) \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^n.$$

We express the coefficients b_i of the form ω_B^{n-1} through the coordinates B_e^i of the vector $B = B_e^i e_i$:

$$\mathrm{d}\omega_B^{n-1} = \left(\sum_{n=1}^{n} \frac{\partial}{\partial t^i} \left(\frac{\sqrt{\det g_{ij}}}{\sqrt{g_{ii}}} B_e^i \right) \right) \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^n.$$

We compare this form with the volume form

$$\Omega_g^n = \sqrt{\det g_{ij}}(t)\, \mathrm{d}t^1 \wedge \cdots \wedge \mathrm{d}t^n,$$

and we obtain

$$\operatorname{div} B = \frac{1}{\sqrt{\det g_{ij}}} \left(\sum_{n=1}^{n} \frac{\partial}{\partial t^i} \left(\frac{\sqrt{\det g_{ij}}}{\sqrt{g_{ii}}} B_e^i \right) \right).$$

In an orthogonal system of curvilinear coordinates, this formula takes the form

$$\operatorname{div} B = \frac{1}{\sqrt{g_{11} \cdots g_{nn}}} \left(\sum_{n=1}^{n} \frac{\partial}{\partial t^i} \left(\frac{\sqrt{g_{11} \cdots g_{nn}}}{\sqrt{g_{ii}}} B_e^i \right) \right).$$

c. Divergence in Cartesian, Cylindrical, and Spherical Coordinates

Problem 5 Write down formulas to calculate the divergence of a vector field $B = B_e^i e_i$ in Cartesian, cylindrical, and spherical coordinates of the Euclidean space $\mathbb{R}^3$.

Answer In Cartesian coordinates (x, y, z), cylindrical coordinates (r, φ, z), and spherical coordinates (R, φ, θ) of the Euclidean space $\mathbb{R}^3$, the divergence $\operatorname{div} B$ of the vector field $B = B_e^i e_i$ can be calculated according to the formula

$$\begin{aligned}\operatorname{div} B &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \\ &= \frac{1}{r}\left(\frac{\partial r B_r}{\partial r} + \frac{\partial B_\varphi}{\partial \varphi}\right) + \frac{\partial B_z}{\partial z} = \\ &= \frac{1}{R^2 \cos\varphi}\left(\frac{\partial R^2 \cos\theta B_R}{\partial R} + \frac{\partial R B_\varphi}{\partial \varphi} + \frac{\partial R\cos\theta B_\theta}{\partial \theta}\right).\end{aligned}$$

D.2.4 Curl of a Vector Field and Its Coordinate Representation

a. Correspondence Between a Vector Field A and the Vector Field $B = \operatorname{curl} A$

We shall now consider the special 3-dimensional case. We shall assume, as before, that we are given a metric $g_{ij}(t)\,\mathrm{d}t^i\,\mathrm{d}t^j$ in the curvilinear coordinates (t^1, t^2, t^3), generating at the same time the volume form $\Omega_g^3 = \sqrt{\det g_{ij}}(t)\,\mathrm{d}t^1 \wedge \mathrm{d}t^2 \wedge \mathrm{d}t^3$.

In this case the vector field $A = A_e^i e_i$ corresponds to the 1-form ω_A^1, and the differential $\mathrm{d}\omega_A^1$ of this form, as a 2-form ($(n-1)$-form), corresponds to a vector field $B = B_e^i e_i$ such that $\mathrm{d}\omega_A^1 = \omega_B^2$. This vector field B is called, as we know, the curl of the original field A and is denoted by $\operatorname{curl} A$.

b. The Coordinate Representation of the Correspondence Between Vector Fields A and $B = \operatorname{curl} A$

We wish to learn how to calculate the coordinates of the field $B = \operatorname{curl} A$ in terms of the coordinates of the vector field A. According to the procedure described above, from the vector field $A = A_e^i e_i$ we build its corresponding 1-form $\omega_A^1 = \langle A, \cdot\rangle$:

$$\omega_A^1 = a_i\,\mathrm{d}t^i = \frac{g_{ij}}{\sqrt{g_{jj}}} A_e^j\,\mathrm{d}t^i.$$

We take its differential

$$\begin{aligned}\mathrm{d}\omega_A^1 &= \frac{\partial}{\partial t^k}\left(\frac{g_{ij}}{\sqrt{g_{jj}}} A_e^j\right)\mathrm{d}t^k \wedge \mathrm{d}t^i = \\ &= \left(\frac{\partial}{\partial t^2}\left(\frac{g_{3j}}{\sqrt{g_{jj}}} A_e^j\right) - \frac{\partial}{\partial t^3}\left(\frac{g_{2j}}{\sqrt{g_{jj}}} A_e^j\right)\right)\mathrm{d}t^2 \wedge \mathrm{d}t^3 + \\ &\quad + \left(\frac{\partial}{\partial t^3}\left(\frac{g_{1j}}{\sqrt{g_{jj}}} A_e^j\right) - \frac{\partial}{\partial t^1}\left(\frac{g_{3j}}{\sqrt{g_{jj}}} A_e^j\right)\right)\mathrm{d}t^3 \wedge \mathrm{d}t^1 + \\ &\quad + \left(\frac{\partial}{\partial t^1}\left(\frac{g_{2j}}{\sqrt{g_{jj}}} A_e^j\right) - \frac{\partial}{\partial t^2}\left(\frac{g_{1j}}{\sqrt{g_{jj}}} A_e^j\right)\right)\mathrm{d}t^1 \wedge \mathrm{d}t^2,\end{aligned}$$

considering this form a form of type ω_B^2. By comparing the coefficients, we have $\omega_B^2 = \mathrm{d}\omega_A^1 = b_1\,\mathrm{d}t^2 \wedge b_2\,\mathrm{d}t^3 \wedge \mathrm{d}t^1 + b_3\,\mathrm{d}t^1 \wedge \mathrm{d}t^2$. We obtain the coordinates $B_e^i = \frac{\sqrt{g_{ii}}}{\sqrt{\det(g_{ij})}} b_i$ of the vector $B = \operatorname{curl} A$.

In the case of a 3-dimensional orthogonal system of curvilinear coordinates (t^1, t^2, t^3), the formula simplifies. In this case,

$$\begin{aligned}
\mathrm{d}\omega_A^1 &= \frac{\partial}{\partial t^k}\big(\sqrt{g_{ii}}A_e^i\big)\,\mathrm{d}t^k \wedge \mathrm{d}t^i = \\
&= \left(\frac{\partial}{\partial t^2}\big(\sqrt{g_{33}}A_e^3\big) - \frac{\partial}{\partial t^3}\big(\sqrt{g_{22}}A_e^2\big)\right)\mathrm{d}t^2 \wedge \mathrm{d}t^3 + \\
&\quad + \left(\frac{\partial}{\partial t^3}\big(\sqrt{g_{11}}A_e^1\big) - \frac{\partial}{\partial t^1}\big(\sqrt{g_{33}}A_e^3\big)\right)\mathrm{d}t^3 \wedge \mathrm{d}t^1 + \\
&\quad + \left(\frac{\partial}{\partial t^1}\big(\sqrt{g_{22}}A_e^2\big) - \frac{\partial}{\partial t^2}\big(\sqrt{g_{11}}A_e^1\big)\right)\mathrm{d}t^1 \wedge \mathrm{d}t^2,
\end{aligned}$$

and using the notation $E_i = g_{ii}$, it is possible to write the coordinates of the vector $\operatorname{curl} A = B = B_e^1 e_1 + B_e^2 e_2 + B_3^3 e_3$:

$$\begin{aligned}
B_e^1 &= \frac{1}{\sqrt{E_2E_3}}\left(\frac{\partial A_e^3\sqrt{E_3}}{\partial t^2} - \frac{\partial A_e^2\sqrt{E_2}}{\partial t^3}\right), \\
B_e^2 &= \frac{1}{\sqrt{E_3E_1}}\left(\frac{\partial A_e^1\sqrt{E_1}}{\partial t^3} - \frac{\partial A_e^3\sqrt{E_3}}{\partial t^1}\right), \\
B_e^3 &= \frac{1}{\sqrt{E_1E_2}}\left(\frac{\partial A_e^2\sqrt{E_2}}{\partial t^1} - \frac{\partial A_e^1\sqrt{E_1}}{\partial t^2}\right),
\end{aligned}$$

which means that

$$\operatorname{curl} A = \frac{1}{\sqrt{E_1E_2E_3}}\begin{vmatrix} \sqrt{E_1}e_1 & \sqrt{E_2}e_2 & \sqrt{E_3}e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ \sqrt{E_1}A_e^1 & \sqrt{E_2}A_e^2 & \sqrt{E_3}A_e^3 \end{vmatrix}.$$

c. Curl in Cartesian, Cylindrical, and Spherical Coordinates

Problem 6 Write down the formula to calculate the curl of a vector field $A = A_e^1 e_1 + A_e^2 e_2 + A_3^3 e_3$ in Cartesian, cylindrical, and spherical coordinates of the Euclidean space $\mathbb{R}^3$.

Answer In Cartesian (x, y, z), cylindrical (r, φ, z), and spherical (R, φ, θ) coordinates of the Euclidean space, the curl $(\operatorname{curl} A)$ of the vector field $A = A_e^1 e_1 + A_e^2 e_2 +$

$A_3^3 e_3$ is calculated according to the formula

$$
\begin{aligned}
\operatorname{curl} A &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)e_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)e_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)e_z = \\
&= \frac{1}{r}\left(\frac{\partial A_z}{\partial \varphi} - \frac{\partial r A_\varphi}{\partial z}\right)e_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)e_\varphi + \frac{1}{r}\left(\frac{\partial r A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)e_z = \\
&= \frac{1}{R\cos\theta}\left(\frac{\partial A_\theta}{\partial \varphi} - \frac{\partial A_\varphi \cos\theta}{\partial \theta}\right)e_R + \frac{1}{R}\left(\frac{\partial A_R}{\partial \theta} - \frac{\partial R A_\theta}{\partial R}\right)e_\varphi + \\
&\quad + \frac{1}{R}\left(\frac{\partial R A_\varphi}{\partial R} - \frac{1}{\cos\theta}\frac{\partial A_R}{\partial \varphi}\right)e_\theta .
\end{aligned}
$$

Appendix E
Modern Formula of Newton–Leibniz and the Unity of Mathematics (Final Survey)

E.1 Reminders

E.1.1 Differential, Differential Form, and the General Stokes's Formula

a. What Happened and Was the Reason That Brought Us to This Kind of Life

We already began the ascent to the modern Newton–Leibniz formula at the very beginning of this course of mathematical analysis, when we defined the differential $\mathrm{d}f(x)$ of a function $f : X \to Y$ at the point x. By analyzing this concept gradually in detail, we found that it is a linear function operating on a linear vector space $T_x X$ of displacements from the point under consideration with values in the space $T_y Y$ of displacements from the point $y = f(x)$. The spaces $T_x X$ and $T_y Y$ are called *tangent spaces* to X and Y at the corresponding points. The differential itself is also called the *tangent mapping* or *total derivative* with respect to the original mapping (function) $f : X \to Y$ at the point x.

Once one has become acquainted with the concept of tangent line or tangent plane to a surface, one understands the origin and the geometric meaning of this terminology.

Passing to functions of several variables and mappings of multidimensional objects, we left the definition of the differential unchanged, but every time, we explicitly deciphered the coordinate representation of the differential. In this way, the notion of the Jacobian matrix of a mapping appeared.

We know that the differential of a function $f : \mathbb{R}^n \to \mathbb{R}$ has the form

$$\mathrm{d}f(x) = \frac{\partial f}{\partial x^1}\,\mathrm{d}x^1 + \cdots + \frac{\partial f}{\partial x^n}\,\mathrm{d}x^n,$$

i.e., it is a linear combination of differentials of simple functions, the coordinate functions, and the value of the differential $\mathrm{d}f(x)(\xi)$ at the vector $\xi \in T_x\mathbb{R}^n$ coincides with the value of the derivative $D_\xi f(x)$ of the function on this vector, and

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since $\mathrm{d}x^i(\xi) = \xi^i$, one has

$$\mathrm{d}f(x)(\xi) = \frac{\partial f}{\partial x^1}\xi^1 + \cdots + \frac{\partial f}{\partial x^n}\xi^n.$$

If you are acquainted with the linear algebra of linear, multilinear, and skew-symmetric forms and the operation of their external product, you could, by applying this to differentials, write a differential form of the type

$$\omega^k(x) = a_{i_1\dots i_k}(x)\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

realizing that this is a skew-symmetric k-form on the tangent space whose value on the set of vectors $(\xi_1, \dots, \xi_k)$ can be calculated if the value of $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}(\xi_1, \dots, \xi_k)$ is known. Lastly, this is equal to the determinant of the matrix

$$\begin{pmatrix} \xi_1^{i_1} & \cdots & \xi_1^{i_k} \\ \vdots & \ddots & \vdots \\ \xi_k^{i_1} & \cdots & \xi_k^{i_k} \end{pmatrix},$$

as we know from algebra (given that $\mathrm{d}x^i(\xi) = \xi^i$).

Recall that we were led to differential forms by the change of variables formula for a multiple integral. For a one-dimensional integral, the form $f(x)\,\mathrm{d}x$, standing under the integral sign, dictated the correct change of variable formula $f(\varphi(t))\,\mathrm{d}\varphi(t)$. We were concerned, as Euler was, about the fact that this was not the case for higher-dimensional integrals. We wanted to correct this deficiency and at the same time understand what we are actually integrating, since the result should not depend on the choice of the system of coordinates.

Analyzing this problem, we also had to figure out a number of concepts, not only in algebra but also in geometry. We understood what a k-dimensional surface is, curvilinear coordinates, local charts, local maps and atlas, what the orientation of a surface is, and how it is specified, what the border of a surface and the induced orientation on the border are, and finally what all of this looks like in the general case of manifolds of dimension k.

We had to analyze what occurs with our objects and operations under a change of coordinate system. We also had to figure out the direction in which points, vectors, and functions on those objects are transferred, in particular forms under smooth mappings, and how exactly to implement the corresponding transfer in the coordinates. At the same time, we convinced ourselves that the operation of differentiation on forms is indeed invariant with respect to the choice of coordinate system. The differentiation of forms, in the coordinate representation, is realized in the most simple and natural way,

$$\mathrm{d}\omega^k(x) = \mathrm{d}a_{i_1\dots i_k}(x)\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

which it is often taken, for this reason, as the original definition of this operation.

Appealing to some suggestions from physics (computation of work, flux), we realized that we integrate differential forms not only because they solve the original problem about the change of variables formula in multiple integrals, but also they lead to the following far-reaching generalization of the classical Newton–Leibniz formula:

$$\int_{M_+^k} \mathrm{d}\omega^{k-1} = \int_{\partial M_+^k} \omega^{k-1}.$$

This formula, frequently called the *general Stokes's formula*, rightfully should be called the Newton–Leibniz–Gauss–Ostrogradskii–Green–Maxwell–Cartan–Poincaré formula.

b. The Problem of Primitives Yesterday and Today

One of the very first questions in classical mathematical analysis is the question about the inversion of the operation of differentiation, more precisely, the question of whether every function f (for example, continuous) is the derivative of some other function, and if so, how to find the antiderivative or primitive F of the given function. In the language of forms, this question is whether a 1-form $f(x)\,\mathrm{d}x$ is the differential $\mathrm{d}F$ of some 0-form, i.e., a function F.

We gave a positive answer to this question, considering everything over a numerical interval. We did not even consider any other situation. If you ask yourself the same question, for example, for a function identically equal to one on the circle or for an appropriate form $\mathrm{d}\varphi$, you will immediately realize that the answer is negative. There is no differentiable function on the circle whose derivative everywhere is equal to one.

This is one of the manifestations of a relation between a question of global analysis and the topology of the domain, where the question is posed and solved.

A significant part of the following text is devoted to a deeper, although not complete, discussion of this relation.

Generalizing the classical situation, we shall ask the following question: *Given a differential k-form ω^k, we look for a $(k-1)$-form ω^{k-1} such that $\omega^k = \mathrm{d}\omega^{k-1}$.*

c. Closed and Exact Differential Forms

Differential forms ω^k having a primitive (i.e., being the differential of some form ω^{k-1}: $\omega^k = \mathrm{d}\omega^{k-1}$) are called *exact forms*.

We shall easily prove that an obvious necessary differential condition for the exactness of a form ω^k is the equality $\mathrm{d}\omega^k = 0$, due to the fact that the external redifferentiation of any differential form is identically zero.

If the differential of a form is equal to zero, the form is called *closed*.

Thus, closedness is a necessary condition for the exactness of a form.

Previously, we considered in all details and interpretations the case of 1-forms. We also convinced ourselves that although closedness is a necessary condition for exactness, this condition is not sufficient, and it is significantly associated with the topology of the domain in which the problem is posed.

In physics, potential vector fields play an important role. If we have a scalar product $\langle , \rangle$ (or a nondegenerate bilinear form) in some space, then there arises a correspondence between linear functions (forms) and vector fields, defined by the equality $\omega_A^1(x)(\xi) = \langle A(x), \xi \rangle$. Incidentally, when we want to calculate the work that should be done by a vector field along a path γ, then we just integrate the form ω_A^1, called a *work form*. The remarkable characteristic of potential vector fields is that the work on those fields depends only on the beginning and the end of the path of transition and is equal to the difference between the values of the potential generating this field. In particular, the work on a closed contour (a cycle) with such a vector field is zero.

In the language of vector fields, the differential characteristic of a potential vector field is, as we know, that they have no rotation (their curl vanishes). We also know that irrotational vector fields are not always potential vector fields, and it depends on the topology of the domain on which they act. In a simply connected domain, this necessary characteristic is also sufficient. For example, in a three-dimensional ball or a ball with deleted center, or in a cut-out ball, every irrotational field is a potential field; in the two-dimensional disk this is also the case, but in the disk with the center deleted, it is no longer the case. (Recall the typical example: in writing the form $\mathrm{d}\varphi$ in Cartesian coordinates (x, y), we considered the vector field $(-y, x)/(x^2 + y^2)$ corresponding to it.)

Along with the necessary differential condition of exactness of a form, which “feels” the form locally, we had an integral criterion for exactness of 1-forms, consisting in the fact that the integral of a form over any cycle (closed path) lying in the considered domain is always equal to zero.

This *integral criterion for the exactness of forms* remains true with respect to forms of any degree, with the proper understanding of what the cycle of the corresponding dimension should be.

This is one of de Rham’s theorems, which has as a consequence a much older theorem, also called *Poincaré’s lemma*, asserting that in the space $\mathbb{R}^n$, in a ball, or on any other domain homeomorphic to it, every closed form is exact.

E.1.2 Manifolds, Chains, and the Boundary Operator

a. Cycles and Boundaries

In the previous Stokes’s formula we have geometric objects (curves, surfaces, manifolds, and their boundary, i.e., the border), on which we integrate the corresponding differential forms.

Similar to the operator d of differentiation, we have the operator ∂, which maps surfaces to their boundary. The boundary ∂M^k of a manifold M^k is also a manifold, but with one dimension fewer. Moreover, the variety ∂M^k no longer has a boundary, i.e., the reapplication of the operator ∂ always gives the empty set. In this sense, the operators d and ∂ are similar. But if the operator d increases the dimension of the object by 1, the operator ∂ reduces the dimension by 1.

The concepts of closedness and exactness in forms correspond here to the concepts of cycles and boundaries.

A compact surface, a manifold M^k (later we shall say also chain) of dimension k, is called *a cycle of dimension k* if $\partial M = \emptyset$, i.e., M does not have any boundary points.

Thus, the sphere of dimension k is a cycle of dimension k.

A surface, manifold M^k (a chain), is called a *boundary* if it has a "primitive" in the sense that there is a surface or manifold M^{k+1} (chain) such that $\partial M^{k+1} = M^k$.

It is clear that if the surface or manifold is the boundary of some other compact manifold, then it must be a cycle. However, the situation here is similar to that of forms, where the conditions are necessary but in general not sufficient to ensure that in the domain where this cycle lies, there is also a manifold such that the cycle is the boundary of that manifold.

Take, for example, a circular ring, or annulus, in the plane. Then every circle containing the hole is a cycle, but it is not the boundary of a manifold lying on the annulus. But if instead of an annulus we consider a disk, then the situation is radically different.

Let us consider the boundary of the annulus, and we shall recall the following fact. The operator ∂ acting on boundaries is not a simple set-theoretic transformation. On an atlas of the surface or manifold, this operator gives an atlas of the boundary, which is called the *induced atlas of the boundary*. If the original atlas consists of compatible charts, then under this operator, the induced atlas will also have this property. Thus if the manifold is orientable, then its boundary possesses an orientation, which is called the *induced orientation* or *agreed* or *compatible orientation of the boundary.*

If the annulus G that we just discussed is oriented with the standard left frame of the Cartesian coordinates in the plane, then its boundary, consisting of two circles γ_1, γ_2, will be oriented such that the outer circle γ_2 goes in the positive direction (counterclockwise) and the inner circle is negatively oriented (clockwise). The integral in such a boundary is reduced to the difference between the integrals over γ_1 and γ_2. It is useful to write that as $\partial G = \gamma_{2+} - \gamma_{1+}$.

For example, if you need to calculate the work that is accomplished by five turns along the path γ_{2+}, then three along the path γ_{1+}, and finally two along γ_{2-}, then you have to integrate over the chain $5\gamma_{2+} + 3\gamma_{1+} + 2\gamma_{2-} = 5\gamma_{2+} + 3\gamma_{1+} - 2\gamma_{2+} = 3\gamma_{2+} + 3\gamma_{1+}$. The integration over such chain corresponds, of course, to a linear combination of the integrals over γ_{1+} and γ_{2+}.

This discussion illustrates why it is useful to consider linear combinations of geometric objects. These are called *chains*. We have explained here only where the concept of chains comes from, what are they in general, and where and why they are useful. We are not going into general and formal definitions, since we do not need them here in the more general form, and they can be found in the book. Analogously, just as in analysis, when we are forced to go from the usual ordinary functions to generalized functions, in geometry one goes from the simplest objects like cubes and chains of cubes to their generalizations like singular cubes and chains of singular cubes. Moreover, we then do the next extension and invent the concept of flux, which combines differential forms, generalized functions, and manifolds.

b. Homological Cycles

We shall see below that it is sometimes possible to calculate the integral of a form over a cycle by going to some other cycle, sometimes significantly simpler, which is in some way associated with the original cycle. This is a remarkable, important, and useful fact, which is used in different areas of mathematics and its applications.

In order to understand the relation between cycles, we have to consider the following fact: their difference must be the boundary of an object lying on the domain we are considering. We say that such cycles are *homologous* in this domain.

For example, two closed oriented paths γ_{1+}, γ_{2+} on a domain D or on a manifold M are homologous if we can find an orientable surface $S^2_+ \subset D$ ($S^2_+ \subset M$) such that $\partial S^2_+ = \gamma_{2+} - \gamma_{1+}$.

Thus, the circles γ_{1+}, γ_{2+} considered above are homologous in the annulus G_+.

Since the operator ∂ acts on boundaries and is extended by linearity over chains, it is possible to determine the homology of chains.

For instance, the chains γ_{1+} and $2\gamma_{2+}$ are not homologous on the annulus G_+.

We shall discuss the role and applications of the concept of homology of cycles in the context of the integration of differential forms.

E.2 Pairing

E.2.1 The Integral as a Bilinear Function and General Stokes's Formula

a. The Integral of an Exact Form over a Cycle and of a Closed Form over a Boundary

We introduce first some useful notation.

Let $\Omega(M)$ denote the whole set of differential forms on a manifold (or surface) M, and let $\Omega^k(M)$ denote the subset of forms of order k (i.e., k-forms), $Z^k(M)$ its subset of closed k-forms, and $B^k(M)$ its subset of exact k-forms.

Analogously, let $C(M)$ be the set of chains on a manifold (or surface) M, and let $C_k(M)$ be the subset of chains of dimension k (k-chains), $Z_k(M)$ the subset of cycles (k-cycles), and $B_k(M)$ its subset of boundary cycles (k-boundaries).

Thus, $\Omega(M) \supset \Omega^k(M) \supset Z^k(M) \supset B^k(M)$ and $C(M) \supset C_k(M) \supset Z_k(M) \supset B_k(M)$.

As long as we do not change the manifold M on which we wish to calculate something, in order to simplify the notation we shall remove the symbol M whenever it does not lead to confusion, that is present in the just-discussed notation.

Now we shall make a concluding remark.

Consider the integral of an exact form $b^k \in B^k$ over the cycle $z_k \in Z_k$ and of a closed form $z^k \in Z^k$ over a boundary $b_k \in B_k$. Employing Stokes's formula, we find that

$$\int_{z_k} b^k = \int_{z_k} \mathrm{d}\omega^{k-1} = \int_{\partial z_k} \omega^{k-1} = \int_{\emptyset} \omega^{k-1} = 0$$

and

$$\int_{b_k} z^k = \int_{\partial c_{k+1}} z^k = \int_{c_{k+1}} \mathrm{d}z^k = \int_{c_{k+1}} 0 = 0.$$

b. Integral of a Closed Form over a Cycle and Its Invariance Under Certain Changes of the Form and the Cycle

The remark that we just made leads to the following important and very useful conclusion.

We shall consider now the integral of a closed form z^k over a cycle z_k. Given that the addition of an exact form b^k to a closed form z^k gives again a closed form (since $\mathrm{d}(z^k + b^k) = \mathrm{d}z^k + \mathrm{d}b^k = 0$), and the addition of a boundary cycle b_k to a cycle z_k gives again a cycle (since $\partial(b_k + z_k) = \partial b_k + \partial z_k = 0$), recalling the remark we just made, we can now write the following chain of equalities:

$$\int_{z_k} z^k = \int_{z_k} (z^k + b^k) = \int_{z_k + b_k} (z^k + b^k) = \int_{[z_k]} [z^k].$$

Here $[z^k]$ means the class of forms that differ from the original form z^k modulo an exact form, and $[z_k]$ is the class of cycles differing from the original one up to a boundary cycle.

Thus by calculating the integral of a closed form z^k over a cycle z_k, we can afford to choose, without changing the value of the integral, any cycle from the class $[z_k]$ and any form from the class $[z^k]$.

E.2.2 Equivalence Relations (Homology and Cohomology)

a. Toward Uniformity in Terminology: Cycles and Cocycles, Boundaries and Coboundaries

Along with the unification of notation, it is convenient to agree on the following standardization of terminology. Since the elements of the sets Z_k and B_k are called *cycles* and *boundaries*, respectively, we shall call the elements of Z^k and B^k *cocycles* and *coboundaries*, respectively.

Thus a cocycle is a closed differential form, and a coboundary is an exact differential form.

b. Homology and Cohomology

A class $[z_k]$, or more precisely a class $[z_k](M)$, is called a *homology class* of the cycle z_k on the manifold (or surface) M.

A class $[z^k]$, or more precisely a class $[z^k](M)$, is called a *cohomology class* of the cocycle z^k on the manifold (or surface) M.

The operator ∂ taking boundary chains is called a *boundary operator*, and the operator d acting on differential forms is called a *coboundary operator*.

Two cycles are *homologous on the manifold* (or surface) M if their difference is the boundary of a chain lying on M.

Two cocycles are *cohomologous on the manifold* (or surface) M if their difference is a coboundary on M (i.e., two closed forms are cohomologous on the manifold if their difference is an exact form on the manifold).

E.2.3 Pairing of Homology and Cohomology Classes

a. The Integral as a Bilinear Function

The integral $\int_{c_k} \omega^k$ of a k-form over a chain on some manifold M can be considered a pairing $\langle \omega^k, c_k \rangle$ of objects from two vector spaces, namely the linear space of k-forms Ω^k and the linear space of k-chains C_k.

We can conclude, knowing the properties of the integral, that the operation $\langle \omega^k, c_k \rangle$ is bilinear.

b. Nondegeneracy of the Bilinear Form of Pairing (de Rham Theorem)

When we considered the above pairing between cycles and cocycles, we obtained an important result, which can be stated now in the following form:

$$\left\langle z^k, z_k \right\rangle = \left\langle \left[z^k\right], [z_k] \right\rangle.$$

Recalling the definition of the cohomology and homology classes $[z^k]$, $[z_k]$, we can say that they are elements of the quotient space $H^k := Z^k/B^k$ and $H_k := Z_k/B_k$, respectively.

The vector spaces H^k and H_k, whose complete notation is $H^k(M)$ and $H_k(M)$, are called the *space of k-dimensional cohomology of the manifold M* and the *space of k-dimensional homology of the manifold M*, respectively.

Thus, the integral actually also pairs cohomology and homology classes. The pairing $\langle [z^k], [z_k] \rangle$ is clearly linear and is *nondegenerate*, as was shown by de Rham.

(Recall that a bilinear form $\langle , \rangle$ is called nondegenerate if once we fix one of the arguments with a nonzero value, the form is not identically zero with respect to the other argument.)

c. Integral Criterion for the Exactness of a Closed Form

De Rham's theorem that we just mentioned implies the following criterion of exactness of a closed form: *A closed form $z^k = \omega^k$ on a manifold (surface, domain) M is exact on M if and only if the integral of this form over every k-dimensional cycle lying on M is equal to zero.*

Indeed, if $\langle z^k, z_k \rangle = 0$ for every cycle z_k lying on M, then according to de Rham's theorem, $[z^k] = 0$ in $H^k = Z^k/B^k$. This means that $z^k \in B^k$.

We have examined in detail all aspects for the case of 1-forms, and we also proved this criterion in this case. We have now established this criterion in general.

In particular, you can now say by looking at a manifold or domain where there is an irrotational vector field or a divergence-free vector field whether the vector field is a potential, or it has a vector potential (i.e., it is the curl of some vector field), respectively.

We can also use de Rham's theorem on the second argument, of course. For example, if we know that on some manifold all the closed k-forms are exact, we can say that on this manifold every k-cycle is a boundary cycle (homologous to zero). Thus, we have a conclusion about the topology of the manifold.

E.2.4 Another Interpretation of Homology and Cohomology

a. Duality of Operators d and ∂

In the notation of the pairing $\langle \omega^k, c_k \rangle$, Stokes's formula has the form

$$\left\langle d\omega^{k-1}, c_k \right\rangle = \left\langle \omega^{k-1}, \partial c_k \right\rangle,$$

showing the duality between the operators d and ∂.

b. The Operators d and ∂ as Mappings

In some cases, it is useful to write the full notation of the operators d and ∂, for example, in the notation of the following sequences of linear mappings:

$$\cdots \xrightarrow{d_{k-2}} \Omega^{k-1} \xrightarrow{d_{k-1}} \Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \cdots,$$

$$\cdots \xleftarrow{\partial_{k-1}} C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}} \cdots.$$

Using the standard notations Ker and Im for the kernel and the image of a linear mapping, we can write, for example, that

$$Z^k = \operatorname{Ker} d_k, \qquad Z_k = \operatorname{Ker} \partial_k, \qquad B^k = \operatorname{Im} d_{k-1}, \qquad B_k = \operatorname{Im} \partial_{k+1},$$

and thus

$$H^k = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1} \quad \text{and} \quad H_k = \operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k+1}.$$

E.2.5 Remarks

A few words as a conclusion. I repeat that this is just an overview, an overview of the principles that does not go into details. The details are covered in the textbook, and numerous developments are given in the specialized literature, which is easier to read with an initial idea of the subject, of course.

In physics and mechanics, we often speak in the language of vector fields. However, you now know how to translate problems in the language of vector fields into the language of differential forms, and conversely you know how to relate standard operators like grad, curl, div with the operator d of the exterior differentiation of forms.

In continuum mechanics, the Hamiltonian operator ∇ is used. Some techniques that are used with it are presented in the text. There you will also find the answer to the question of how to represent and calculate the operators grad, curl, div in curvilinear coordinates.

All of this, including Stokes's formula, has numerous applications. For example, look at the deduction of Euler's equation in continuum mechanics, or write down Maxwell's equations for an electromagnetic field. I shall not mention the internal mathematical applications in analysis, especially complex analysis, geometry, algebraic topology...

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Index of Basic Notation

Logical symbols

$\Longrightarrow$	logical consequence (implication)
$\Longleftrightarrow$	logical equivalence
$:=$	equality by definition; colon
$=:$	on the side of the object defined

Sets

$\overline{E}$	closure of the set E
∂E	boundary of the set E
$\mathring{E} := E \setminus \partial E$	interior of the set E
$B(x, r)$	ball of radius r with center at x
$S(x, r)$	sphere of radius r with center at x

Spaces

(X, d)	metric space with metric d
(X, τ)	topological space with system τ of open sets
$\mathbb{R}^n (\mathbb{C}^n)$	n-dimensional real (complex) space
$\mathbb{R}^1 = \mathbb{R}$ $(\mathbb{C}^1 = \mathbb{C})$	set of real (complex) numbers
$x = (x^1, \dots, x^n)$	coordinate expression of a point of n-dimensional space
$C(X, Y)$	set (space) of continuous functions on X with values in Y
$C[a, b]$	abbreviation for $C([a, b], \mathbb{R})$ or $C([a, b], \mathbb{C})$
$C^{(k)}(X, Y)$	set of mappings from X into Y that are k times continuously differentiable
$C^{(k)}[a, b]$	abbreviation for $C^{(k)}([a, b], \mathbb{R})$ or $C^{(k)}([a, b], \mathbb{C})$
$C_p[a, b]$	space $C[a, b]$ endowed with norm $\|f\|_p$
$C_2[a, b]$	space $C[a, b]$ with Hermitian inner product $\langle f, g\rangle$ of functions or mean-square deviation norm
$\mathcal{R}(E)$	set (space) of functions that are Riemann integrable over the set E
$\mathcal{R}[a, b]$	space $\mathcal{R}(E)$ when $E = [a, b]$

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$\widetilde{\mathcal{R}}(E)$ space of classes of Riemann integrable functions on E that are equal almost everywhere on E

$\widetilde{\mathcal{R}}_p(E)(\mathcal{R}_p(E))$ space $\widetilde{\mathcal{R}}(E)$ endowed with norm $\|f\|_p$

$\widetilde{\mathcal{R}}_2(E)(\mathcal{R}_2(E))$ space $\widetilde{\mathcal{R}}(E)$ endowed with Hermitian inner product $\langle f, g\rangle$ or mean-square deviation norm

$\mathcal{R}_p[a,b], \mathcal{R}_2[a,b]$ spaces $\mathcal{R}_p(E)$ and $\mathcal{R}_2(E)$ when $E=[a,b]$

$\mathcal{L}(X;Y), (\mathcal{L}(X_1,\dots,X_n;Y))$ space of linear (n-linear) mappings from X (from $(X_1\times\cdots\times X_n)$) into Y

TM_p or $TM(p), T_pM, T_p(M)$ tangent space to the surface (manifold) M at the point $p\in M$

S Schwartz space of rapidly decreasing functions

$\mathcal{D}(G)$ space of fundamental functions of compact support in the domain G

$\mathcal{D}'(G)$ space of generalized functions on the domain G

$\mathcal{D}$ an abbreviation for $\mathcal{D}(G)$ when $G=\mathbb{R}^n$

$\mathcal{D}'$ an abbreviation for $\mathcal{D}'(G)$ when $G=\mathbb{R}^n$

Metrics, norms, inner products

$d(x_1,x_2)$ distance between points x_1 and x_2 in the metric space (X,d)

$|x|, \|x\|$ absolute value (norm) of a vector $x\in X$ in a normed vector space

$\|A\|$ norm of the linear (multilinear) operator A

$\|f\|_p := (\int_E |f|^p(x)\,\mathrm{d}x)^{1/p}$, $p\ge 1$ integral norm of the function f

$\|f\|_2$ mean-square deviation norm ($\|f\|_p$ when $p=2$)

$\langle \mathbf{a},\mathbf{b}\rangle$ Hermitian inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$

$\langle f,g\rangle := \int_E (f\cdot\overline{g})(x)\,\mathrm{d}x$ Hermitian inner product of the functions f and g

$\mathbf{a}\cdot\mathbf{b}$ inner product of $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$

$\mathbf{a}\times\mathbf{b}$ vector (cross) product of vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$

$(\mathbf{a},\mathbf{b},\mathbf{c})$ scalar triple product of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^3$

Functions

$g\circ f$ composition of functions f and g

f^{-1} inverse of the function f

$f(x)$ value of the function f at the points x; a function of x

$f(x^1,\dots,x^n)$ value of the function f at the point $x=(x^1,\dots,x^n)\in X$ in the n-dimensional space X; a function depending on n variables $x^1,\dots,x^n$

supp f support of the function f

$\lceil f(x)$ jump of the function f at the point x

$\{f_t\colon t\in T\}$ a family of functions depending on the parameter $t\in T$

$\{f_n;n\in\mathbb{N}\}$ or $\{f_n\}$ a sequence of functions

$f_t \underset{\mathcal{B}}{\longrightarrow} f$ on E convergence of the family of functions $\{f_t;t\in T\}$ to the function f on the set E over the base $\mathcal{B}$ in T

$f_t \underset{\mathcal{B}}{\rightrightarrows} f$ on E uniform convergence of the family of functions $\{f_t;t\in T\}$ to the function f on the set E over the base B in T

$f=o(g)$ over $\mathcal{B}$
$f=O(g)$ over $\mathcal{B}$
$f\sim g$ or $f\simeq g$ over $\mathcal{B}$
} asymptotic formulas (the symbols of comparative asymptotic behavior of the functions f and g over the base $\mathcal{B}$)

$f(x) \simeq \sum_{n=1}^{\infty} \varphi_n(x)$ over $\mathcal{B}$ expansion in an asymptotic series
$\mathcal{D}(x)$ Dirichlet function
$\exp(A)$ exponential of a linear operator A
$B(\alpha, \beta)$ Euler beta function
$\Gamma(\alpha)$ Euler gamma function
χ_E characteristic function of the set E

Differential calculus

$f'(x),\ f_x(x),\ \mathrm{d}f(x),\ Df(x)$ tangent mapping to f (differential of f) at the point x
$\frac{\partial f}{\partial x^i},\ \partial_i f(x),\ D_i f(x)$ partial derivative (partial differential) of a function f depending on variables $x^1, \dots, x^n$ at the point $x = (x^1, \dots, x^n)$ with respect to the variable x^i
$D_{\mathbf{v}} f(x)$ derivative of the function f with respect to the vector $\mathbf{v}$ at the point x
∇ Hamilton's nabla operator
$\operatorname{grad} f$ gradient of the function f
$\operatorname{div} \mathbf{A}$ divergence of the vector field $\mathbf{A}$
$\operatorname{curl} \mathbf{B}$ curl of the vector field $\mathbf{B}$

Integral calculus

$\mu(E)$ measure of the set E
$\int_E f(x)\,\mathrm{d}x$, $\int_E f(x^1, \dots, x^n)\,\mathrm{d}x^1 \cdots \mathrm{d}x^n$, $\underset{E}{\int \cdots \int} f(x^1, \dots, x^n)\,\mathrm{d}x^1 \cdots \mathrm{d}x^n$ integral of the function f over the set $E \subset \mathbb{R}^n$
$\int_Y \mathrm{d}y \int_X f(x, y)\,\mathrm{d}x$ iterated integral
$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$, $\int_\gamma \mathbf{F} \cdot \mathrm{d}s,\ \int_\gamma \langle \mathbf{F}, \mathrm{d}s \rangle$ curvilinear integral (of second kind) or the work of the field $\mathbf{F} = (P, Q, R)$ along the pathγ
$\int_\gamma f\,\mathrm{d}s$ curvilinear integral (of first kind) of the function f along the curve γ
$\iint_S P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y$, $\iint_S \mathbf{F} \cdot \mathrm{d}\boldsymbol{\sigma}$, $\iint_X \langle \mathbf{F}, \mathrm{d}\boldsymbol{\sigma} \rangle$ integral (of second kind) over the surface S in $\mathbb{R}^3$; flux of the field $\mathbf{F} = (P, Q, R)$ across the surface S
$\iiint_S f\,\mathrm{d}\sigma$ surface integral (of first kind) of f over the surface S

Differential forms

$\omega\ (\omega^p)$ a differential form (of degree p)
$\omega^p \wedge \omega^q$ exterior product of forms ω^p and ω^q
$\mathrm{d}\omega$ (exterior) derivative of the form ω
$\int_M \omega$ integral of the form ω over the surface (manifold) M
$\omega^1_{\mathbf{F}} := \langle \mathbf{F}, \cdot \rangle$ work form
$\omega^2_{\mathbf{V}} := (\mathbf{V}, \cdot, \cdot)$ flux form

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