

A Fundamental Equations for Hypersurfaces

In this appendix we consider submanifolds with codimension 1 of a pseudo-Riemannian manifold and derive, among other things, the important equations of Gauss and Codazzi-Mainardi. These have many useful applications, in particular for a 3 + 1 splitting of the Einstein equations (see Sect. 2.9).

For definiteness, we consider a spacelike hypersurface Σ of a Lorentz manifold (M, g) . Other situations, e.g., timelike hypersurfaces can be treated in the same way, with sign changes here and there. The formulation can also be generalized to submanifolds of codimension larger than 1 of arbitrary pseudo-Riemannian manifolds (see, for instance, Vol.II of [43]), a situation that occurs in string theory.

Let $\iota : \Sigma \rightarrow M$ denote the embedding and let $\bar{g} = \iota^*g$ be the **induced metric** on Σ . The pair (Σ, \bar{g}) is a three-dimensional Riemannian manifold. Other quantities which belong to Σ will often be indicated by a bar. The basic results of this appendix are most efficiently derived with the help of **adapted** moving frames and the structure equations of Cartan. So let $\{e_\mu\}$ be an ‘orthonormal’ frame on an open region of M , with the property that the e_i , for $i = 1, 2, 3$, in points of Σ are tangent to Σ . The dual basis of $\{e_\mu\}$ is denoted by $\{\theta^\mu\}$. On Σ the $\{e_i\}$ can be regarded as a triad of Σ that is dual to the restrictions $\bar{\theta}^i := \theta^j|_{T\Sigma}$ (i.e., the restrictions to tangent vectors of Σ). Indices for objects on Σ always refer to this pair of dual basis.

Formulas of Gauss and Weingarten

Consider the restriction of the first structure equation

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \tag{A.1}$$

to $T\Sigma$. Since $\theta^0|_{T\Sigma} = 0$ we obtain

$$d\bar{\theta}^i + \omega^i{}_j \wedge \bar{\theta}^j = 0 \quad \text{on } T\Sigma, \tag{A.2a}$$

$$\omega^0{}_k \wedge \bar{\theta}^k = 0 \quad \text{on } T\Sigma. \tag{A.2b}$$

Because the connection forms $\bar{\omega}^i{}_j$ of (Σ, \bar{g}) also satisfy (A.2a), and have the same symmetry properties, we conclude (see Sect. 13.7) that

$$\bar{\omega}^i{}_j = \omega^i{}_j \quad \text{on } T\Sigma. \tag{A.3}$$

This has a simple geometrical meaning. Let ∇ be the Levi-Civita connection on (M, g) and $\bar{\nabla}$ that of (Σ, \bar{g}) , then we get from (A.3)

$$\langle \nabla_X e_j, e_i \rangle = \omega_{ij}(X) = \bar{\omega}_{ij}(X) = \langle \bar{\nabla}_X e_j, e_i \rangle \tag{A.4}$$

for $X \in T\Sigma$. This shows that for any $X \in T\Sigma$ and any vector field Y of M tangent to Σ , the tangential projection of $\nabla_X Y$ on $T\Sigma \subset TM$ is equal to $\bar{\nabla}_X Y$.

From (A.2b) we obtain information about the component of $\nabla_X Y$ normal to $T\Sigma$. This equation tells us that the $\omega_i^0 = \omega_0^i$ satisfy the hypothesis of the following

Lemma A.1 (Cartan). *If $\alpha^1, \dots, \alpha^n$ are linearly independent 1-forms on a manifold M of dimension $n' \geq n$, and β_1, \dots, β_n are one forms on M satisfying*

$$\sum_{i=1}^n \alpha^i \wedge \beta_i = 0, \tag{A.5}$$

then there are smooth functions f_{ij} on M such that

$$\beta_i = \sum_{j=1}^n f_{ij} \alpha^j; \tag{A.6}$$

moreover

$$f_{ij} = f_{ji}. \tag{A.7}$$

Proof. In a neighborhood of any point we can choose 1-forms $\alpha^{n+1}, \dots, \alpha^{n'}$ so that $\alpha^1, \dots, \alpha^{n'}$ are everywhere linearly independent. Then there are smooth functions f_{ij} ($i \leq n, j \leq n'$) with

$$\beta_i = \sum_{j=1}^{n'} f_{ij} \alpha^j.$$

Now Equation (A.5) implies

$$0 = \sum_{i=1}^n \sum_{j=1}^{n'} f_{ij} \alpha^i \wedge \alpha^j = \sum_{1 \leq i < j \leq n} (f_{ij} - f_{ji}) \alpha^i \wedge \alpha^j + \sum_{i=1}^n \sum_{j > n} f_{ij} \alpha^i \wedge \alpha^j.$$

Since the $\alpha^i \wedge \alpha^j$ for $i < j$ are linearly independent, we conclude $f_{ij} = f_{ji}$ for $i, j \leq n$ and $f_{ij} = 0$ for $j > n$. □

According to the lemma and (A.2b) there are functions K_{ij} on Σ such that

$$\omega_i^0 = -K_{ij} \theta^j \quad \text{on } T\Sigma, \tag{A.8a}$$

$$K_{ij} = K_{ji}. \tag{A.8b}$$

From this and $\nabla_{e_\mu} e_\nu = \omega^\lambda_\nu(e_\mu)e_\lambda$ we obtain

$$\langle \nabla_{e_i} e_j, e_0 \rangle = -\omega^0_j(e_i) = K_{ij} = K_{ji} = \langle \nabla_{e_j} e_i, e_0 \rangle. \tag{A.9}$$

So the bilinear form $K(X, Y)$ on $T\Sigma$ belonging to K_{ij} gives the normal component of $\nabla_X Y$:

$$\nabla_X Y = \bar{\nabla}_X Y - K(X, Y)N \tag{A.10}$$

for $X \in T\Sigma, Y \in \mathcal{X}(M)$ tangent to Σ , where $N = e_0$ (normalized normal vector on $\Sigma, \langle N, N \rangle = -1$).

Equation (A.10) is the **Gauss formula**. The symmetric bilinear form K , called the **second fundamental form** or **extrinsic curvature**, satisfies according to (A.9) the **Weingarten equations**

$$K(X, Y) = \langle N, \nabla_X Y \rangle = -\langle \nabla_X N, Y \rangle. \tag{A.11}$$

Remarks. The reader should be warned that some authors use as extrinsic curvature the negative of our K . The linear transformation of the tangent spaces of Σ corresponding to K is known as the **Weingarten map** of the second fundamental tensor. Its real eigenvalues are called **principle curvatures**.

Equations of Gauss and Codazzi-Mainardi

Next, we look for relations between the curvature forms Ω^μ_ν and $\bar{\Omega}^i_j$ of (M, g) , respectively (Σ, \bar{g}) . For this we restrict the second structure equation

$$\Omega^\mu_\nu = d\omega^\mu_\nu + \omega^\mu_\lambda \wedge \omega^\lambda_\nu \tag{A.12}$$

to Σ . Consider first the restriction of

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i_0 \wedge \omega^0_j$$

to $T\Sigma$. Using (A.3) and (A.8a) this gives

$$\Omega^i_j = \bar{\Omega}^i_j + K^i_k K_{jl} \bar{\theta}^k \wedge \bar{\theta}^l \quad \text{on } T\Sigma. \tag{A.13}$$

The other components give on $T\Sigma$

$$\begin{aligned} \Omega^0_j &= d\omega^0_j + \omega^0_i \wedge \omega^i_j = -d(K_{ij}\bar{\theta}^j) - K_{ik} \bar{\theta}^k \wedge \bar{\omega}^i_j \\ &= -dK \wedge \bar{\theta}^j - K_{ij} d\bar{\theta}^i - K_{ik} \bar{\theta}^k \wedge \bar{\omega}^i_j \end{aligned}$$

or, using the first structure equation on (Σ, \bar{g}) ,

$$\Omega^0_j = -\bar{D}K_{ij} \wedge \bar{\theta}^i \quad \text{on } T\Sigma. \tag{A.14}$$

Here \bar{D} denotes the absolute exterior differential of the tensor field K .

The relations (A.13) and (A.14) are the famous **equations of Gauss** and **Codazzi-Mainardi** in terms of differential forms. We want to rewrite them in a more useful form. Consider for tangential X, Y

$$\begin{aligned}\langle R(X, Y)e_j, e_i \rangle &= \Omega_{ij}(X, Y) \\ &= \bar{\Omega}_{ij}(X, Y) + K_{ik}K_{jl}(\bar{\theta}^k \wedge \bar{\theta}^l)(X, Y) \\ &= \langle \bar{R}(X, Y)e_j, e_i \rangle + \\ &\quad K(e_i, X)K(e_j, Y) - K(e_i, Y)K(e_j, X).\end{aligned}$$

This shows that for tangent vectors X, Y, Z and W of Σ we have

$$\begin{aligned}\langle R(X, Y)Z, W \rangle &= \langle \bar{R}(X, Y)Z, W \rangle - \\ &\quad -K(X, Z)K(Y, W) + K(Y, Z)K(X, W).\end{aligned}\quad (\text{A.15})$$

One calls this relation often **Gauss' Theorema Egregium**. (Note that some of the signs are different from the ones for the Riemannian case, that is usually treated in mathematics books.)

Similarly, we obtain from (A.14)

$$\begin{aligned}\langle R(X, Y)e_j, N \rangle &= \Omega_{0j}(X, Y) = (\bar{D}K_{ij} \wedge \bar{\theta}^i)(X, Y) \\ &= \bar{\nabla}_k K_{ij}(\bar{\theta}^k \wedge \bar{\theta}^i)(X, Y) \\ &= X^k Y^i \bar{\nabla}_k K_{ij} - (X \longleftrightarrow Y) \\ &= (\bar{\nabla}_X K)(Y, e_j) - (X \longleftrightarrow Y)\end{aligned}$$

for $X, Y \in T\Sigma$. Hence, Eq. (A.14) is equivalent to the following standard form of the Codazzi-Mainardi equation:

$$\langle R(X, Y)Z, N \rangle = (\bar{\nabla}_X K)(Y, Z) - (\bar{\nabla}_Y K)(X, Z), \quad (\text{A.16})$$

for $X, Y, Z \in T\Sigma$.

The basic equations of Gauss and Codazzi-Mainardi allow us to obtain interesting and useful expressions for the components R_{0i} and G_{00} of the Ricci and Einstein tensors on Σ . We first note that for any frame

$$i_{e_\alpha} \Omega^\alpha_\beta = \frac{1}{2} R^\alpha_{\beta\rho\sigma} i_{e_\alpha} (\theta^\rho \wedge \theta^\sigma) = R_{\beta\sigma} \theta^\sigma,$$

so that

$$R_{\beta\sigma} = \Omega^\alpha_\beta(e_\alpha, e_\sigma). \quad (\text{A.17})$$

Especially, we obtain

$$R_{0i} = \Omega^j_0(e_j, e_i). \quad (\text{A.18})$$

Relative to our adapted basis we need Ω^j_0 on the right-hand side only on $T\Sigma$, and can thus use (A.14) to get

$$R_{0i} = \bar{\nabla}_i K^j_j - \bar{\nabla}_j K^j_i. \quad (\text{A.19})$$

Next, we consider

$$G_{00} = R_{00} + \frac{1}{2}(-R_{00} + R^i_i) = \frac{1}{2}(R_{00} + R^i_i).$$

From (A.17) we get

$$\begin{aligned} R_{00} &= \Omega^j_0(e_j, e_0), \\ R^i_i &= \Omega^{ji}(e_j, e_i) + \Omega^{0i}(e_0, e_i), \end{aligned}$$

and thus

$$G_{00} = \frac{1}{2}\Omega^{ij}(e_i, e_j). \tag{A.20}$$

Here we can use (A.13) to obtain

$$2G_{00} = \bar{R}^i_i + K^i_i K^j_j - K^j_j K^i_i,$$

or

$$G_{00} = \frac{1}{2}\bar{R} + \frac{1}{2}(K^i_i K^j_j - K^j_j K^i_i). \tag{A.21}$$

This form of Gauss' equation is particular useful in GR.

Consider, for illustration, a static metric as in Sect. 2.1. From (2.23) we see that the second fundamental form for any time slice vanishes, thus (A.21) gives

$$G_{00} = G_{\mu\nu}N^\mu N^\nu = \frac{1}{2}\bar{R},$$

in agreement with (2.30). In the exercises we consider time-dependent metrics.

These formulas simplify if Σ is totally geodesic in the sense of

Definition A.1. A hypersurface Σ is called **totally geodesic** if every geodesic $\gamma(s)$ of (M, g) with $\gamma(0) \in \Sigma$ and $\dot{\gamma}(0) \in T_{\gamma(0)}\Sigma$ remains in Σ for s in some open interval $(-\varepsilon, \varepsilon)$.

This is equivalent to the property that $\nabla_X Y$ is tangent to Σ whenever X and Y are (see Exercise A.3). From (A.10) we conclude that $\Sigma \subset M$ is totally geodesic if and only if the second fundamental form K of Σ vanishes.

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Null Hypersurfaces

Null hypersurfaces of Lorentz manifolds play an important role in GR.

Definition A.2. A hypersurface Σ of a Lorentz manifold (M, g) is **null**, if the induced metric on Σ is degenerate.

The reader may easily show that this is equivalent to the property that at any point $p \in \Sigma$ there exists a non-vanishing null vector in $T_p M$ which is perpendicular to $T_p \Sigma$. As a consequence of property c) in Exercise 7.5, this null vector is unique up to a scale factor (up to a positive scale factor if the null vector is future directed). Hence, there exists locally a smooth non-vanishing null vector field l along Σ which is tangent to Σ and whose normal space at any point $p \in \Sigma$ coincides with $T_p M$. From the quoted exercise it also follows that tangent vectors of Σ not parallel to l are spacelike.

The null vector field l , which is unique up to rescaling, has the following remarkable property:

Proposition A.1. *The integral curves of l , when suitably parameterized, are null geodesics.*

Proof. We show below that $\nabla_l l$ is at each $p \in \Sigma$ perpendicular to $T_p M$. This implies that $\nabla_l l$ is proportional to l , which is equivalent to the proposition.

To establish the stated property, let $u \in T_p M$ and extend u locally to a vector field along Σ which is invariant under the flow belonging to l (show that this is possible). Then $0 = L_l u = [l, u] = \nabla_l u - \nabla_u l$. From $\langle l, u \rangle = 0$ we obtain

$$0 = l \langle l, u \rangle = \langle \nabla_l l, u \rangle + \langle l, \nabla_l u \rangle,$$

thus

$$\langle \nabla_l l, u \rangle = -\langle l, \nabla_u l \rangle = -\frac{1}{2} u \langle l, l \rangle = 0.$$

(These formulas should be interpreted in the sense of Sect. 14.3 for vector fields along the embedding map of Σ into M .) Since u is arbitrary in p , $\nabla_l l$ is indeed perpendicular to $T_p \Sigma$. \square

The integral curves of l are called **null geodesic generators** of Σ .

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Exercises.

Exercise A.1. Consider a metric of the form

$$g = -\varphi^2(t, x^k) dt^2 + g_{ij}(t, x^k) dx^i dx^j \quad (\text{A.22})$$

and show that the second fundamental form of the slices $\{t = \text{const}\}$ is given by

$$K_{ij} = -\frac{1}{2\varphi} \partial_t g_{ij}. \quad (\text{A.23})$$

Solution. The Ricci identity and $[\partial_t, \partial_i] = 0$ give

$$\begin{aligned} \partial_t g_{ij} &= \langle \nabla_{\partial_t} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_t} \partial_j \rangle \\ &= \langle \nabla_{\partial_i} \partial_t, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_j} \partial_t \rangle. \end{aligned}$$

Here we use $N = \frac{1}{\varphi} \partial_t$ and (A.11) to obtain

$$\begin{aligned} \partial_t g_{ij} &= \langle \nabla_{\partial_i}(\varphi N), \partial_j \rangle + \langle \partial_i, \nabla_{\partial_j}(\varphi N) \rangle \\ &= \varphi (\langle \nabla_{\partial_i} N, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_j} N \rangle) \\ &= -\varphi (K_{ij} + K_{ji}). \end{aligned}$$

Exercise A.2. Use (A.23), the Gauss equation (A.21), and (2.78) to show that for the **Friedmann metric**

$$g = -dt^2 + a^2(t) h_{ij} dx^i \wedge dx^j \tag{A.24}$$

on $\mathbb{R} \times S^3$, where $h = h_{ij} dx^i \wedge dx^j$ is the standard metric on S^3 , the time-time component of the Einstein tensor is given by

$$G_{00} = 3 \left(\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} \right). \tag{A.25}$$

Hence, Einstein’s field equation implies the following **Friedmann equation**

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{1}{a^2} = 8\pi G(\rho + \rho_\Lambda), \tag{A.26}$$

where $\rho_\Lambda := \frac{\Lambda}{8\pi G}$ and ρ is the matter energy density.

Exercise A.3. Prove that a hypersurface Σ of (M, g) is totally geodesic if and only if $\nabla_X Y$ is tangent to Σ whenever X and Y are.

Solution. We already know that the second property holds if and only if the second fundamental form K of Σ vanishes. So let us show that $K = 0$ if and only if Σ is totally geodesic.

Let $\gamma : s \mapsto \gamma(s)$ be a geodesic in (M, g) , with $\gamma(0) \in \Sigma$, $\dot{\gamma}(0) \in T_{\gamma(0)}\Sigma$, and let $\bar{\gamma}$ be a geodesic in (Σ, \bar{g}) with the same initial conditions. From (A.10) we get

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \bar{\nabla}_{\dot{\gamma}} \dot{\gamma} - K(\dot{\gamma}, \dot{\gamma})N = -K(\dot{\gamma}, \dot{\gamma})N.$$

Hence, if K vanishes we see that $\bar{\gamma}$ is also a geodesic of (M, g) with the same initial conditions as γ , thus $\bar{\gamma} = \gamma$ on their common domain. This proves that a vanishing K implies that Σ is totally geodesic.

Conversely, let Σ be totally geodesic. Then γ stays in Σ for some open interval $(-\varepsilon, \varepsilon)$. From (A.10) we now conclude

$$0 = \nabla_{\dot{\gamma}} \dot{\gamma} = \bar{\nabla}_{\dot{\gamma}} \dot{\gamma} - K(\dot{\gamma}, \dot{\gamma})N.$$

Therefore, $\bar{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0$ and $K(\dot{\gamma}, \dot{\gamma}) = 0$. In particular, γ is also a geodesic of (Σ, \bar{g}) . Moreover, since $\gamma(0) \in \Sigma$ and $\dot{\gamma}(0) \in T_{\gamma(0)}\Sigma$ are arbitrary, we conclude that $K = 0$ (use the symmetry of K and polarization).

B Ricci Curvature of Warped Products

In GR one often encounters so-called warped products (see, e.g., Appendix 3.10). By definition, a pseudo-Riemannian manifold (M, g) is a **warped product** of two pseudo-Riemannian manifolds (\tilde{M}, \tilde{g}) and (S, \hat{g}) with an everywhere positive **warp function** $f : \tilde{M} \rightarrow \mathbb{R}$, written as

$$M = \tilde{M} \times_f S, \quad (\text{B.1})$$

if M is the product manifold $\tilde{M} \times S$ and the metric has the form

$$g = \pi^*(\tilde{g}) + (f \circ \pi)^2 \sigma^*(\hat{g}), \quad (\text{B.2})$$

where π and σ are the projections from M onto \tilde{M} and S , respectively. In the following we shall drop the pull-backs π^* , σ^* .

The calculation of the curvature quantities of (M, g) in terms of those of (\tilde{M}, \tilde{g}) and (S, \hat{g}) is best done within the framework of Cartan's calculus. We shall work with the orthonormal tetrad fields $\{\theta^\alpha\}$ of (M, g) , which are adapted as follows: Let $\alpha = (a, A)$ and let

$$\theta^a = \tilde{\theta}^a, \quad \theta^A = f \hat{\theta}^A, \quad (\text{B.3})$$

where $\tilde{\theta}^a$, for $a = 1, \dots, \dim(\tilde{M})$, and $\hat{\theta}^A$, for $A = \dim(\tilde{M}) + 1, \dots, \dim(M)$, are orthonormal bases of (\tilde{M}, \tilde{g}) and (S, \hat{g}) , respectively. The connection forms of the various spaces are denoted by ω_β^α , $\tilde{\omega}_b^a$ and $\hat{\omega}_B^A$, and for the corresponding curvature forms we use the symbols Ω_β^α , $\tilde{\Omega}_b^a$ and $\hat{\Omega}_B^A$.

To start, we determine the connection forms ω_β^α from Cartan's first structure equation

$$d\theta^\alpha + \omega_\beta^\alpha \wedge \theta^\beta = 0. \quad (\text{B.4})$$

We need

$$\begin{aligned} d\theta^A &= d(f \hat{\theta}^A) = f d\hat{\theta}^A - \hat{\theta}^A \wedge df \\ &= -f \hat{\omega}_B^A \wedge \hat{\theta}^B - f_b \hat{\theta}^A \wedge \tilde{\theta}^b \\ &= -\hat{\omega}_B^A \wedge \theta^B - \frac{f_b}{f} \theta^A \wedge \theta^b, \end{aligned}$$

where we have set $df = f_a \tilde{\theta}^a$. Comparing this result with (B.4) leads to the guess

$$\omega_B^A = \hat{\omega}_B^A, \quad \omega_b^A = \frac{f_b}{f} \theta^A. \quad (\text{B.5})$$

In addition, we also have to satisfy

$$\begin{aligned} d\theta^a &= -\omega_B^a \wedge \theta^B - \omega_b^a \wedge \theta^b \\ &= d\tilde{\theta}^a = -\tilde{\omega}_b^a \wedge \tilde{\theta}^b = -\tilde{\omega}_b^a \wedge \theta^b. \end{aligned}$$

Taking

$$\omega_b^a = \tilde{\omega}_b^a, \quad (\text{B.6})$$

we see that the structure equation (B.4) is fulfilled for the connection forms (B.5) and (B.6). Therefore, these equations provide the correct (and unique) result.

Using the second structure equation,

$$\Omega_\beta^\alpha = d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma, \quad (\text{B.7})$$

and the corresponding equations for $\tilde{\Omega}_b^a$ and $\hat{\Omega}_B^A$ as well, we find after a short routine calculation the curvature forms

$$\Omega_b^a = \tilde{\Omega}_b^a, \quad (\text{B.8a})$$

$$\Omega_B^A = \hat{\Omega}_B^A - \frac{\langle df, df \rangle}{f^2} \theta^A \wedge \theta_B, \quad (\text{B.8b})$$

$$\Omega_B^a = -\frac{1}{f} \tilde{\nabla}_b \tilde{\nabla}^a f \theta^b \wedge \theta_B, \quad (\text{B.8c})$$

from which we can now extract the Ricci tensor. This is best done by taking advantage of the formula

$$R_{\beta\gamma} = \Omega_\beta^\alpha(e_\alpha, e_\gamma), \quad (\text{B.9})$$

which follows from $\Omega_\nu^\mu = R_{\nu\alpha\beta}^\mu \theta^\alpha \wedge \theta^\beta$ (e_α denotes the orthonormal frame dual to θ^α). One finally obtains

$$R_{ab} = \tilde{R}_{ab} - \frac{\dim(S)}{f} \tilde{\nabla}_b \tilde{\nabla}_a f, \quad (\text{B.10a})$$

$$R_{aB} = 0, \quad (\text{B.10b})$$

$$\begin{aligned} R_{AB} &= \hat{R}_{AB} \\ &\quad - (f \tilde{\Delta} f + \langle df, df \rangle (\dim(S) - 1)) \hat{g}_{AB}. \end{aligned} \quad (\text{B.10c})$$

This result obviously holds also for non-orthonormal adapted frames. (Note that all indices refer to the frame $\theta^\alpha = (\theta^a, \theta^A)$.)

Application: Friedmann Equations

Important examples of warped products are **Friedmann-Lemaître (-Robertson-Walker) spacetimes** in cosmology. For these \tilde{M} is an open interval of \mathbb{R} with the metric $-dt^2$, (S, \hat{g}) is a 3-dimensional space of constant curvature $k = \pm 1, 0$, and the warp function is the scale factor $a(t)$. The spacetime metric is thus

$$g = -dt^2 + a^2(t) \hat{g}. \tag{B.11}$$

We use the general results (B.10a)–(B.10c) to compute the Ricci tensor belonging to (B.11).

Since (\tilde{M}, \tilde{g}) is now a 1-dimensional Euclidean space, (B.10a) gives

$$R_{00} = -3 \frac{\ddot{a}}{a}. \tag{B.12}$$

According to (B.10b) we have

$$R_{0i} = 0, \tag{B.13}$$

and (B.10c) implies

$$R_{ij} = \hat{R}_{ij} + \left[2 \left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] a^2 \hat{g}_{ij}.$$

According to (2.79), which holds for all Riemannian spaces of constant curvature, we have

$$\hat{R}_{ij} = 2k \hat{g}_{ij}, \tag{B.14}$$

hence

$$R_{ij} = \left[2 \left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{k}{a^2} + \frac{\ddot{a}}{a} \right] a^2 \hat{g}_{ij}. \tag{B.15}$$

For an orthonormal tetrad $\{\theta^\alpha\}$ we have $a^2 \tilde{g}_{ij} = \delta_{ij}$ (because of (B.3)). Then we find for the Einstein tensor

$$\begin{aligned} G_{00} &= 3 \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right], \\ G_{0i} &= 0, \\ G_{ij} &= - \left[2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \delta_{ij}. \end{aligned} \tag{B.16}$$

In order to satisfy the field equations, the energy-momentum tensor must have the form

$$(T_{\mu\nu}) = \text{diag}(\rho, p, p, p). \tag{B.17}$$

From (B.16) and (B.17) we obtain the independent Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}, \quad (\text{B.18a})$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = 8\pi G p - \Lambda. \quad (\text{B.18b})$$

Note also, that (B.12) gives the (dependent) equation

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}a. \quad (\text{B.19})$$

This shows that \ddot{a} depends on the combination $\rho + 3p$, and that a *positive* Λ leads to an **accelerated expansion**.

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Exercise B.1. Use the connection forms (B.5) and (B.6) to show that $T^{\alpha\beta}_{;\beta} = 0$ implies for $\alpha = 0$:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \quad (\text{B.20})$$

Derive this equation also from (B.18a) and (B.18b).

C Frobenius Integrability Theorem

In this appendix we develop an important differential geometric tool of GR that is also one of the pillars of differential topology and the calculus on manifolds. The following problem is a natural generalization of finding integral curves for a vector field. Instead of a vector field we assign to each point p of a manifold M a subspace E_p of T_pM which depends on p in a smooth manner and ask: When is it possible to find for each point $p_0 \in M$ a submanifold N with $p_0 \in N$, which is tangent to E_p , i.e., such that $T_pN = E_p$ for each point $p \in N$? In general, such integral manifolds do not exist, even locally. For the local problem the so-called **Frobenius theorem** (due to A. Clebsch and F. Deahna) gives necessary and sufficient criteria for integral manifolds to exist.

We begin with some precisions.

Definition C.1. A *k -dimensional distribution* (*k -plane, k -direction field*) on M is a map $p \mapsto E_p$, where E_p is a k -dimensional subspace of T_pM . We call this distribution **smooth** if the following condition is satisfied: For each $p_0 \in M$ there is a neighborhood V and k differentiable vector fields X_1, \dots, X_k such that $X_1(p), \dots, X_k(p)$ form a basis of E_p for all $p \in V$.

Definition C.2. A k -dimensional submanifold N of M is called an **integral manifold** of the distribution E if for every $p \in N$ we have

$$T_p \iota(T_pN) = E_p,$$

where $\iota : N \hookrightarrow M$ is the inclusion map. We say that the distribution E on M is **integrable** if, through each $p \in M$, there passes an integral manifold of E .

We want to find out when E is integrable. For this we consider vector fields which belong to E : We say that the vector field X **belongs to E** if $X_p \in E_p$ for all p . Suppose now that N is an integrable manifold of E , and $\iota : N \hookrightarrow M$ is the inclusion map. If X, Y are two vector fields which belong to E , then for all $p \in N$ there are unique $\bar{X}_p, \bar{Y}_p \in T_pN$ such that

$$X_p = T_p \iota \cdot \bar{X}_p, \quad Y_p = T_p \iota \cdot \bar{Y}_p.$$

It is not difficult to show¹ that the vector fields \bar{X}, \bar{Y} are C^∞ . In other words \bar{X} and X , as well as \bar{Y} and Y , are ι -related. Then we know (Theorem 11.6) that $[\bar{X}, \bar{X}]$ and $[X, Y]$ are ι -related. Thus

$$T_p\iota \cdot [\bar{X}, \bar{X}]_p = [X, Y]_p.$$

Since $[\bar{X}, \bar{X}]_p \in T_pN$ we conclude that $[X, Y]_p \in E_p$. We thus have shown: If E is integrable then $[X, Y]$ also belongs to E . This suggests to introduce the following concept:

Definition C.3. We say that the k -distribution E on M is *involutive* if $[X, Y]$ belongs to E whenever the vector fields X and Y (defined on open sets of M) belong to E .

Using this terminology, we have shown that if E is integrable then E is involutive. It turns out that the reverse is also true.

Theorem C.1 (Frobenius' Theorem (first version)). Let E be a C^∞ k -dimensional distribution on M which is involutive. Then E is integrable. More precisely the following holds: For each point $p \in M$ there is a coordinate system $\{x^1, \dots, x^n\}$ in a neighborhood U of p with

$$x^i(p) = 0, \quad x^i(U) \subset (-\varepsilon, \varepsilon) \quad (\varepsilon > 0),$$

such that for each (a^{k+1}, \dots, a^n) with all $|a^i| < \varepsilon$ the set

$$\{q \in U : x^{k+1}(q) = a^{k+1}, \dots, x^n(q) = a^n\}$$

is an integral manifold of E . Any connected integral manifold of E restricted to U is contained in one of these sets.

We postpone the proof of this important theorem and first reformulate it in a useful manner in terms of differential forms. For this we need some concepts.

To each k -dimensional distribution E we associate the set $\mathcal{I}(E) \subset \bigwedge(M)$ of differential forms ω with the property that each homogeneous component $\omega^l \in \bigwedge^l(M)$ of ω ($\omega = \sum_l \omega^l$) annihilates the sets (X_1, \dots, X_l) of vector fields belonging to E :

$$\omega^l(X_1, \dots, X_l) = 0.$$

The set $\mathcal{I}(E)$ is called the **annihilator** of E . Clearly, $\mathcal{I}(E)$ is an ideal of the algebra $\bigwedge(M)$ of exterior forms: With $\omega_1, \omega_2 \in \mathcal{I}(E)$ the sum $\omega_1 + \omega_2$ also belongs to $\mathcal{I}(E)$, and $\eta \wedge \omega \in \mathcal{I}(E)$ if $\omega \in \mathcal{I}(E)$ and $\eta \in \bigwedge(M)$. Locally, the ideal $\mathcal{I}(E)$ is generated by $n - k$ independent 1-forms $\omega^{k+1}, \dots, \omega^n$. In fact, around each point $p \in M$ we can choose coordinates $\{x^i\}$ so that $\{\partial_1|_p, \dots, \partial_k|_p\}$ span

¹ Since ι is an immersion one can use coordinates as in Definition 9.7. For details, see [43], Vol. I, p.258.

E_p . Then $dx^1(p) \wedge \dots \wedge dx^k(p)$ is non-zero on E_p and by continuity also in a small neighborhood of p . This implies that $\{dx^1(q), \dots, dx^k(q)\}$ are linearly independent in E_q . Therefore, if we restrict $dx^\alpha(q)$ for $\alpha = k + 1, \dots, n$ to E_q we have the expansion

$$dx^\alpha(q) = \sum_{\beta=1}^k f_\beta^\alpha(q) dx^\beta(q).$$

Therefore

$$\omega^\alpha = dx^\alpha - \sum_{\beta=1}^k f_\beta^\alpha dx^\beta, \quad \alpha = k + 1, \dots, n$$

are linearly independent 1-forms which belong to $\mathcal{I}(E)$. These generate the ideal. To show this we complete $\omega^{k+1}, \dots, \omega^n$ to a basis $\omega^1, \dots, \omega^n$ of 1-forms and consider the dual basis X_1, \dots, X_n of vector fields. Then X_1, \dots, X_k span the k -dimensional distribution E . Suppose an l -form $\omega \in \mathcal{I}(E)$ contains in the expansion with respect to the basis $\omega^1, \dots, \omega^n$,

$$\omega = \sum_{i_1 < i_2 < \dots < i_l} c_{i_1 \dots i_l} \omega^{i_1} \wedge \dots \wedge \omega^{i_l},$$

a term $c_{j_1 \dots j_l} \omega^{j_1} \wedge \dots \wedge \omega^{j_l}$ which contains no factor from $\omega^{k+1}, \dots, \omega^n$. Then we have $\omega(X_{j_1}, \dots, X_{j_l}) \neq 0$, in contradiction to $\omega \in \mathcal{I}(E)$. This proves:

Proposition C.1. *Let $\mathcal{I}(E)$ be the ideal of $\Lambda(M)$ belonging to the k -dimensional distribution E . Then $\mathcal{I}(E)$ is **locally generated** by $n - k$ independent 1-forms: For each point of M there is a neighborhood U and $n - k$ point-wise linearly independent 1-forms $\omega^{k+1}, \dots, \omega^n \in \Lambda^1(U)$ such that for each $\omega \in \mathcal{I}(E)$*

$$\omega|_U = \sum_{i=k+1}^n \theta_i \wedge \omega^i$$

for some $\theta_i \in \Lambda(U)$.

Now we can reformulate the condition in the Frobenius theorem.

Proposition C.2. *A distribution E on M is involutive if and only if $\mathcal{I}(E)$ is a **differential ideal**: $d\mathcal{I}(E) \subset \mathcal{I}(E)$.*

Proof. We use the same notation as above. It is easy to see that E is involutive if and only if there exist smooth functions C_{ij}^l such that

$$[X_i, X_j] = \sum_{l=1}^k C_{ij}^l X_l$$

for $i, j = 1, \dots, k$. Now we have (see (12.14))

$$d\omega^\alpha(X_i, X_j) = X_i(\omega^\alpha(X_j)) - X_j(\omega^\alpha(X_i)) - \omega^\alpha([X_i, X_j]).$$

For $1 \leq i, j \leq k$ and $\alpha > k$ the first two terms on the right vanish. So $d\omega^\alpha(X_i, X_j) = 0$ if and only if $\omega^\alpha([X_i, X_j]) = 0$. But the latter equation holds for all i, j if and only if each $[X_i, X_j]$ belongs to E (i.e., if E is involutive), while $d\omega^\alpha(X_i, X_j) = 0$ holds exactly when $d\omega^\alpha \in \mathcal{I}(E)$. \square

Next we establish some equivalent conditions that a locally (finitely) generated ideal is a differential ideal.

Proposition C.3. *Let \mathcal{I} be an ideal of $\Lambda(M)$ locally generated by $n - k$ linearly independent forms $\omega^{k+1}, \dots, \omega^n \in \Lambda^1(U)$. Furthermore let $\omega \in \Lambda^{n-k}(U)$ be the form $\omega = \omega^{k+1} \wedge \dots \wedge \omega^n$. Then the following statements are equivalent:*

- (i) \mathcal{I} is an differential ideal.
- (ii) $d\omega^\alpha = \sum_{\beta=k+1}^n \omega_\beta^\alpha \wedge \omega^\beta$ for certain $\omega_\beta^\alpha \in \Lambda^1(U)$.
- (iii) $\omega \wedge d\omega^\alpha = 0$ for $\alpha = k + 1, \dots, n$.
- (iv) There exists $\theta \in \Lambda^1(U)$ such that $d\omega = \theta \wedge \omega$.

Proof. The equivalence of (i) and (iii) follows immediately from the definitions. The same is true for the implication (i) \implies (iv). For the proof of (iv) \implies (iii) note that the condition (iv) means that

$$\sum_{\alpha=k+1}^n (-1)^\alpha d\omega^\alpha \wedge \omega^{k+1} \wedge \dots \wedge \hat{\omega}^\alpha \wedge \dots \wedge \omega^n = \theta \wedge \omega^{k+1} \wedge \dots \wedge \omega^n.$$

Multiplying this equation with ω^α we get (iii). It remains to show that (iii) \implies (ii): Again, let $\omega^1, \dots, \omega^n$ be a basis of $\Lambda^1(U)$ such that $\omega^{k+1}, \dots, \omega^n$ generate \mathcal{I} over U . Then

$$d\omega^i = \sum_{j<l} f_{jl}^i \omega^j \wedge \omega^l, \tag{C.1}$$

where $f_{jl}^i \in \mathcal{F}(U)$. But

$$0 = d\omega^\alpha \wedge \omega = \sum_{1 \leq j < l \leq k} f_{jl}^\alpha \omega^j \wedge \omega^l \wedge \omega^{k+1} \wedge \dots \wedge \omega^n.$$

Thus $f_{jl}^\alpha = 0$ for $\alpha = k + 1, \dots, n$ and $1 \leq j < l \leq k$. Hence the sum in (C.1) is of the form (ii). \square

Now we assemble the preceding results in the following version of the Frobenius theorem.

Theorem C.2 (Frobenius' Theorem). *Let M be an n -dimensional manifold and E a k -dimensional distribution on M , and $\mathcal{I}(E)$ the associated ideal. The following statements are all equivalent:*

- (i) E is integrable.
- (ii) E is involutive.
- (iii) $\mathcal{I}(E)$ is a differential ideal locally generated by $n - k$ linearly independent 1-forms $\omega^{k+1}, \dots, \omega^n \in \Lambda^1(U)$.
- (iv) For every point of M there is a neighborhood U and $\omega^{k+1}, \dots, \omega^n \in \Lambda^1(U)$ generating $\mathcal{I}(E)$ such that

$$d\omega^\alpha = \sum_{\beta=k+1}^n \omega_\beta^\alpha \wedge \omega^\beta \tag{C.2}$$

for certain $\omega_\beta^\alpha \in \Lambda^1(U)$.

- (v)
$$d\omega^\alpha \wedge \omega^{k+1} \wedge \dots \wedge \omega^n = 0 \tag{C.3}$$

(in the notation of (iv)).

- (vi) There exists a $\theta \in \Lambda^1(U)$ such that

$$d\omega = \theta \wedge \omega, \quad \omega := \omega^{k+1} \wedge \dots \wedge \omega^n. \tag{C.4}$$

Applications

Consider first a single timelike vector field X on a Lorentz manifold (M, g) . The orthogonal complement of X_p in every $T_p(M)$ defines a 3-dimensional distribution E . Obviously, X is hypersurface orthogonal if and only if E is integrable. Since $\omega = X^\flat$ generates the ideal $\mathcal{I}(E)$ the Frobenius theorem tells us that E is integrable if and only if

$$\omega \wedge d\omega = 0. \tag{C.5}$$

(We have used the equivalence of (i) and (v).) This shows that (C.5) is necessary and sufficient for X being locally hypersurface orthogonal.

Our main application in GR will be the introduction of adapted coordinates if there are several Killing fields on (M, g) , in particular, if there exist two commuting Killing fields. Let us, however, first introduce adapted coordinates for more general situations.

Let again, for a k -dimensional distribution E , $E|U = \text{span}\{X_1, \dots, X_k\}$ and $\omega^{k+1}, \dots, \omega^n$ the generating 1-forms of the ideal $\mathcal{I}(E)$. We use the following notation: Indices running from $1, 2, \dots, n$ are denoted by Greek letters; for the first k numbers we use small Latin letters, while for indices with values $k + 1, \dots, n$ capital letters will be used.

Assume that E is involutive and use adapted coordinates $\{x^\mu\}$ as in the first version of the Frobenius theorem. The (local) integral manifolds are given by $\{x^A = \text{const}\}$. For the basis of vector fields X_a belonging to E we have

$$X_a(x^A) = 0 \quad (X_a = X_a^b \frac{\partial}{\partial x^b}, \quad X_a^A = 0). \tag{C.6}$$

Since $\langle \omega^A, X_a \rangle = 0$ we obtain, setting $\omega^A = \omega^A_\mu dx^\mu$,

$$0 = X_a^b \omega^A_\mu \langle dx^\mu, \partial_b \rangle = X_a^b \omega^A_b = 0$$

for all $a = 1, \dots, k$. Hence $\omega^A_b = 0$ and thus

$$\omega^A = \omega^A_B dx^B. \quad (\text{C.7})$$

Clearly, the matrices (X_a^b) and (ω^A_B) are non-singular.

Let us specialize this to commuting fields X_a . We show that one can choose adapted coordinates such that

$$X_a = \frac{\partial}{\partial x^a} \quad (a = 1, \dots, k). \quad (\text{C.8})$$

To prove this, consider the flows ϕ_t^a of X_a . We know that these commute (Theorem 11.10). Now choose the coordinates x^a for a point p of an integral manifold $\{x^A = \text{const}\}$ according to

$$p = (\phi_{x^1}^1 \circ \dots \circ \phi_{x^k}^k)(p_0),$$

where p_0 is a fixed “origin”. With this choice the representation (C.8) obviously holds.

Returning to the general case we assume now that M has a distinguished pseudo-Riemannian metric g . Keeping the previous notation we can then also introduce the 1-forms $\omega^a = X_a^b$. These generate an ideal, denoted by $\mathcal{I}(E^\perp)$, which is the annihilator of a distribution E^\perp . Let E^\perp be spanned by the vector fields X_A , $E^\perp = \text{span}\{X_A\}$. Then we have

$$\langle \omega^a, X_A \rangle = 0 \iff g(X_a, X_A) = 0 : E \perp E^\perp. \quad (\text{C.9})$$

In what follows we assume that the restrictions $g|E$ and $g|E^\perp$ are non-singular. Then $E \cap E^\perp = \{0\}$, so

$$T_p M = E_p \oplus E_p^\perp. \quad (\text{C.10})$$

Dually we have

$$T_p^* M = \text{span}\{\omega^a\} \oplus^\perp \text{span}\{\omega^A\}, \quad (\text{C.11})$$

where $\omega^A = X_A^b$. We have thus constructed a basis of vector fields $\{X_\mu\}$ and the basis $\{\omega^\mu = X_\mu^b\}$ of 1-forms such that $\{\omega^A\}$ generates the ideal $\mathcal{I}(E)$, and $\{\omega^a\}$ the ideal $\mathcal{I}(E^\perp)$.

Let us now assume that E and E^\perp are involutive. We use coordinates $\{u^\mu\}$ which are adapted to E and also coordinates $\{v^\mu\}$ that are adapted to E^\perp . Then we have according to (C.6) and (C.7)

$$X_a(u^A) = 0, \quad \omega^A = \omega^A_B du^B, \quad (\text{C.12a})$$

$$X_A(v^a) = 0, \quad \omega^a = \omega^a_b dv^b. \quad (\text{C.12b})$$

Now we define coordinates $\{x^\mu\}$ by

$$x^a = v^a, \quad x^A = u^A. \tag{C.13}$$

The x^μ indeed form a coordinate system. For this we have to show that the dx^μ are linearly independent. Suppose there would be a linear dependence

$$f_a dx^a = g_A dx^A,$$

i.e., $f_a dv^a = g_A du^A$, then applying this to X_c leads to

$$f_a \langle dv^a, X_c \rangle = g_A \langle du^A, X_c \rangle = g_A X_c(u^A) = 0,$$

i.e., $\langle dv^a, f_a X_c \rangle = 0$ or, by (C.12b), $\langle \omega^a, f_a X_c \rangle = 0$. Thus $f_a X_c \in E \cap E^\perp$, whence $f_a X_c = 0$ thus $f_a = 0$, implying $g_A = 0$.

The coordinates $\{x^\mu\}$ have, as a result of (C.12a) and (C.12b), the following properties:

- (i) $X_a(x^A) = 0$,
- (ii) $g_{aA} = 0$.

The last property is obtained as follows: Let $\omega^a = \omega^a_\mu dx^\mu$ and use (C.12b) to conclude that $\omega^a_A = 0$. But by (i)

$$\omega^a_A = g_{A\mu} X^mu_a = g_{Ab} X^b_a$$

which implies (ii).

Summarizing, we have the result:

Proposition C.4. *Let $E = \text{span}\{X_1, \dots, X_k\}$ be a k -dimensional distribution which is involutive. Assume in addition that the ideal generated by $\{\omega^a = X^b_a\}$ is differential. If this is the annihilator of E^\perp , then under the assumption that $E \cap E^\perp = \{0\}$ we can introduce coordinates $\{x^\mu\}$ with ($\mu = (a, A)$)*

$$g_{aA} = 0, \quad X_a(x^A) = 0. \tag{C.14}$$

Assume next, that the generating vector fields X_1, \dots, X_k in this Proposition are *Killing fields*, $L_{X_a}g = 0$. In adapted coordinates this becomes

$$X^b_a g_{\mu\nu,b} + g_{b\nu} X^b_{a,\mu} + g_{\mu b} X^b_{a,\nu} = 0. \tag{C.15}$$

If we apply this for $\mu = A, \nu = B$, and use (C.14), we obtain $g_{AB,b} = 0$. Thus g_{AB} depends only on x^C . Choosing, on the other hand, $\mu = A, \nu = c$ in (C.15) we obtain

$$X^b_a g_{Ac,b} + g_{bc} X^b_{a,A} + g_{Ab} X^b_{a,c} = 0. \tag{C.16}$$

Since $g_{aA} = 0$ the first and the last term vanish, and we conclude that $X^b_{a,A} = 0$. Thus X^b_a depends only on x^c . Summarizing, we have the useful

Proposition C.5. *If in addition to the assumptions of the Proposition C.4 it is assumed that X_a are Killing fields of the pseudo-Riemannian manifold (M, g) , then there are local coordinates $\{x^\mu\}$ such that (in the notation introduced above)*

$$g = g_{ab}(x^\mu) dx^a dx^b + g_{AB}(x^C) dx^A dx^B, \quad (\text{C.17a})$$

$$X_a = X_a^b \partial_b, \quad (\text{C.17b})$$

where X_a^b are only functions of x^c .

Note, as a result of this, that the metric $g_{ab} dx^a dx^b$ on the integral manifolds $\{x^A = \text{const}\}$ has k Killing fields X_a , whence this submanifolds are homogeneous spaces.

Finally, we consider the important special case when the Killing fields X_a commute. We can then choose the adapted coordinates such that (C.8) holds. If we use this in (C.15) for $\mu = c$, $\nu = d$, we get

$$X_a^b g_{cd,b} + g_{bd} X_{a,c}^b + 0 = 0.$$

Since the second term vanishes we see that $g_{cd,b} = 0$. Hence g_{cd} depends only on x^A . In this special case we can, therefore, find coordinates x^μ such that

$$g = g_{ab}(x^C) dx^a dx^b + g_{AB}(x^C) dx^A dx^B, \quad (\text{C.18a})$$

$$X_a = \frac{\partial}{\partial x^a}. \quad (\text{C.18b})$$

This is used as the starting point of the derivation of the Kerr solution in Chap. 7.

Proof of Frobenius' Theorem (in the first version)

Proof (of Theorem C.1). Since this theorem is a local statement, we can work in \mathbb{R}^n and choose $p = \{0\}$. Moreover, we can assume that $E_0 \subset T_0\mathbb{R}^n$ is spanned by the basis vectors

$$\left. \frac{\partial}{\partial t^1} \right|_0, \dots, \left. \frac{\partial}{\partial t^k} \right|_0, \quad (\text{C.19})$$

where (t^1, \dots, t^n) are the standard coordinates of \mathbb{R}^n . Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be the projection onto the first k factors. The corresponding tangential map $d\pi$ ($= T\pi$) maps E_0 isomorphically on $T_0\mathbb{R}^k$. By continuity, $d\pi$ is bijective on E_q for q near 0. So near 0, we can choose unique $X_1(q), \dots, X_k(q) \in E_q$ so that

$$d\pi \cdot X_i(q) = \left. \frac{\partial}{\partial t^i} \right|_{\pi(q)} \quad (i = 1, \dots, k). \quad (\text{C.20})$$

Then (on a neighborhood U of $0 \in \mathbb{R}^n$) the vector fields X_i and $\frac{\partial}{\partial t^i}$ (on \mathbb{R}^k) are π -related. This then also holds for their Lie brackets:

$$d\pi \cdot [X_i, X_j]_q = \left[\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \right]_{\pi(q)} = 0. \tag{C.21}$$

Since E was assumed to be involutive, we have $[X_i, X_j]_q \in E_q$, thus with (C.21) $[X_i, X_j] = 0$ ($d\pi$ is on E_q injective). According to the proof of (C.8) we can find coordinates $\{x^i\}$ such that

$$X_a = \frac{\partial}{\partial x^a}, \quad a = 1, \dots, k. \tag{C.22}$$

The sets $\{q \in U \mid x^{k+1}(q) = a^{k+1}, \dots, x^n(q) = a^n\}$ are then obviously integral manifolds of E , since their tangent spaces are spanned by (C.22).

Conversely, if N is a connected integral manifold restricted to U , with the inclusion $\iota : N \hookrightarrow U$, then we have for $x^\alpha \circ \iota$ with $\alpha = k + 1, \dots, n$ and $X_q \in T_q N$:

$$d(x^\alpha \circ \iota)(X_q) = X_q(x^\alpha \circ \iota) = d\iota \cdot X_q(x^\alpha) = 0,$$

since $d\iota \cdot X_q \in E_q$, which is spanned by the vectors (C.22) taken in q . Thus $d(x^\alpha \circ \iota) = 0$, implying that $x^\alpha \circ \iota$ is constant on the connected manifold N . □

* * *

Exercise C.1. Consider a 1-form ω on an open set $U \subset \mathbb{R}^n$. Use the Frobenius Theorem to show that the $(n - 1)$ -distribution $E = \{v \in TU \mid \omega(v) = 0\}$ is integrable if and only if ω is of the form

$$\omega = g \, df. \tag{C.23}$$

The smooth function g is called an **integrating factor**.

Since by the Frobenius Theorem integrability is equivalent to

$$\omega \wedge d\omega = 0, \tag{C.24}$$

we see that (C.24) is necessary and sufficient for the existence of an integrating factor. Note that (C.24) is always satisfied for $n = 2$, so that ω has always an integrating factor.

Solution. If (C.23) is satisfied, then $\{f = \text{const}\}$ are integral manifolds of E . Conversely, if E is integrable then according to the first version of the Frobenius theorem, the integral manifolds of E are given by $\{f = \text{const}\}$ for a smooth function, so that df annihilates E . Thus the ideal $\mathcal{I}(E)$ generated by ω is also generated by df , whence $\omega = g \, df$ for some smooth function g on U .

D Collection of Important Formulas

The following list summarizes some of the important formulas which have been obtained in the foregoing chapters and will constantly be used throughout this book.

Vector Fields, Lie Brackets

The set of smooth vector fields $\mathcal{X}(M)$ on a manifold M with the commutator product $[X, Y]$ form a \mathbb{R} -Lie algebra and a module over the associative algebra $\mathcal{F}(M)$ of C^∞ -functions. In local coordinates we have

$$X = \xi^i \frac{\partial}{\partial x^i}, \quad Y = \eta^i \frac{\partial}{\partial x^i},$$
$$[X, Y] = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Under a diffeomorphism $\varphi \in \text{Diff}(M)$ the Lie bracket transforms naturally:

$$\varphi_*[X, Y] = [\varphi_*X, \varphi_*Y].$$

Differential Forms

The differential forms $\bigwedge(M)$ on M form a real associative algebra with the exterior product \wedge as multiplication; $\bigwedge^p(M)$ denotes the $\mathcal{F}(M)$ -module of p -forms. The exterior product satisfies,

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha, \quad \alpha \in \bigwedge^k(M), \quad \beta \in \bigwedge^l(M).$$

For a differentiable map φ ,

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta.$$

Exterior Differential

The exterior differential d is an antiderivation of degree +1; in particular

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta, \quad \text{for } \alpha \in \bigwedge^k(M), \beta \in \bigwedge(M).$$

Furthermore

$$d \circ d = 0.$$

If $\alpha \in \bigwedge^p(M)$ and $X_1, \dots, X_{p+1} \in \mathcal{X}(M)$, then

$$\begin{aligned} d\alpha(X_1, \dots, X_{p+1}) &= \sum_{1 \leq i \leq p+1} (-1)^{i+1} X_i \alpha(X_1, \dots, \hat{X}_i, \dots, X_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}). \end{aligned}$$

For a map φ ,

$$\varphi^* d\alpha = d\varphi^* \alpha$$

Poincaré Lemma

If $d\alpha = 0$, then α is locally exact; that is, there is a neighborhood U of every $x \in M$ on which $\alpha = d\beta$.

Interior Product

The interior product i_X is an antiderivation of degree -1 ; in particular

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge i_X \beta, \quad \text{for } \alpha \in \bigwedge^p(M), \beta \in \bigwedge(M).$$

We obviously have

$$i_{fX} \alpha = f i_X \alpha = i_X f \alpha, \quad \text{for } f \in \mathcal{F}(M),$$

and

$$i_X \circ i_X = 0.$$

For a diffeomorphism φ (see Exercise D.1),

$$\varphi^* i_X \alpha = i_{\varphi^* X} \varphi^* \alpha.$$

Lie Derivative

The Lie derivative L_X with respect to the vector field X is a derivation of degree 0 of the tensor algebra $\mathcal{T}(M)$; it is \mathbb{R} -linear in X . For a diffeomorphism φ and $T \in \mathcal{T}(M)$

$$\varphi^* L_X T = L_{\varphi^* X} \varphi^* T.$$

For a covariant tensor field $T \in \mathcal{T}_q^0(M)$ and $X_1, \dots, X_q \in \mathcal{X}(M)$

$$\begin{aligned} (L_X T)(X_1, \dots, X_q) &= X(T(X_1, \dots, X_q)) - \sum_{k=1}^q T(X_1, \dots, [X, X_k], \dots, X_q). \end{aligned}$$

In local coordinates the components of $L_X T$ for a tensor field $T \in \mathcal{T}_s^r(M)$ are given by

$$\begin{aligned} (L_X T)_{j_1 \dots j_s}^{i_1 \dots i_r} &= X^i T_{j_1 \dots j_s, i}^{i_1 \dots i_r} \\ &\quad - T_{j_1 \dots j_s}^{k i_2 \dots i_r} \cdot X_{,k}^{i_1} - \text{all upper indices} \\ &\quad + T_{k j_2 \dots j_s}^{i_1 \dots i_r} \cdot X_{,j_1}^k + \text{all lower indices}. \end{aligned}$$

Relations Between L_X , i_X and d

The following identities hold for differential forms

$$\begin{aligned} L_X \alpha &= d i_X \alpha + i_X d \alpha, & \text{(Cartan's formula)} \\ L_f \alpha &= f L_X \alpha + df \wedge i_X \alpha, \\ L_{[X, Y]} &= [L_X, L_Y] = L_X \circ L_Y - L_Y \circ L_X, \\ i_{[X, Y]} &= [L_X, i_Y] = L_X \circ i_Y - i_Y \circ L_X, \\ L_X \circ d &= d \circ L_X, \\ L_X \circ i_X &= i_X \circ L_X. \end{aligned}$$

Volume Form

For an oriented n -dimensional pseudo-Riemannian manifold (M, g) , with $\theta^1, \dots, \theta^n$ an oriented basis of one-forms, the canonical volume form can be represented as

$$\eta = \sqrt{|g|} \theta^1 \wedge \dots \wedge \theta^n, \quad g = g_{ik} \theta^i \otimes \theta^k, \quad |g| = |\det(g_{ik})|.$$

Hodge-Star Operation

The Hodge-star operator for (M, g) is a linear isomorphism $*$: $\bigwedge^k(M) \rightarrow \bigwedge^{n-k}(M)$ and satisfies for $\alpha, \beta \in \bigwedge^k(M)$:

$$\begin{aligned} \alpha \wedge * \beta &= \beta \wedge * \alpha = \langle \alpha, \beta \rangle \eta, \\ *(* \alpha) &= (-1)^{k(n-k)} \text{sgn}(g) \alpha, \\ \langle * \alpha, * \beta \rangle &= \text{sgn}(g) \langle \alpha, \beta \rangle. \end{aligned}$$

In local coordinates

$$*(\theta^{i_1} \wedge \dots \wedge \theta^{i_p}) = \frac{\sqrt{|g|}}{(n-p)!} \varepsilon_{j_1 \dots j_n} g^{j_1 i_1} \dots g^{j_p i_p} \theta^{j_{p+1}} \wedge \dots \wedge \theta^{j_n}.$$

Codifferential

The codifferential for (M, g) is a linear isomorphism $\delta : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)$ defined by

$$\delta := \text{sgn}(g)(-1)^{np+n} * d*,$$

and satisfies

$$\delta \circ \delta = 0.$$

In local coordinates, for $\alpha \in \Lambda^k(M)$,

$$(\delta\alpha)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} \alpha^{k i_1 \dots i_{p-1}} \right)_{,k}.$$

Covariant Derivative

The covariant derivative ∇_X in the X -direction of an affine connection on M is a derivation of the tensor algebra $\mathcal{T}(M)$, which commutes with all contractions. For $S \in \mathcal{T}_q^p(M)$,

$$\begin{aligned} (\nabla_X S)(Y_1, \dots, Y_q, \omega_1, \dots, \omega_p) \\ = X(S(Y_1, \dots, Y_q, \omega_1, \dots, \omega_p)) - S(\nabla_X Y_1, Y_2, \dots, Y_q) - \dots - \\ - S(Y_1, \dots, \nabla_X \omega_p). \end{aligned}$$

In local coordinates, the components of $\nabla_X S$ are given by

$$\begin{aligned} S_{j_1 \dots j_q; k}^{i_1 \dots i_p} = S_{j_1 \dots j_q, k}^{i_1 \dots i_p} + \Gamma_{kl}^{i_1} S_{j_1 \dots j_q}^{l i_2 \dots i_p} + \dots \\ - \Gamma_{kj_1}^l S_{l j_2 \dots j_q}^{i_1 \dots i_p} - \dots \end{aligned}$$

Connection Forms

Let (e_1, \dots, e_n) be a moving frame with the dual basis $(\theta^1, \dots, \theta^n)$ of one-forms. The connection forms ω_j^i are given by

$$\nabla_X e_j = \omega_j^i(X) e_i.$$

or

$$\nabla_X \theta^i = -\omega_j^i(X) \theta^j.$$

Curvature Forms

The expansion coefficients of the curvature forms Ω_j^i are the components of the Riemann tensor relative to a moving frame:

$$\Omega_j^i = \frac{1}{2} R_{jkl}^i \theta^k \wedge \theta^l, \quad R_{jkl}^i = -R_{jlk}^i.$$

Cartan's Structure Equations

The torsion forms Θ^i and curvature forms Ω^i_j satisfy Cartan's structure equations

$$\begin{aligned}\Theta^i &= d\theta^i + \omega^i_j \wedge \theta^j, \\ \Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j.\end{aligned}$$

Riemannian Connection

For the Riemannian connection with metric $g = g_{ik} \theta^i \otimes \theta^k$

$$\omega_{ik} + \omega_{ki} = dg_{ik}, \quad \omega_{ik} = g_{ij} \omega^j_k.$$

Coordinate Expressions

Riemann tensor

$$R^i_{jkl} = \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \Gamma^s_{lj} \Gamma^i_{ks} - \Gamma^s_{kj} \Gamma^i_{ls},$$

Ricci tensor

$$R_{jl} = R^i_{jil} = \Gamma^i_{lj,i} - \Gamma^i_{ij,l} + \Gamma^s_{lj} \Gamma^i_{is} - \Gamma^s_{ij} \Gamma^i_{ls},$$

Scalar curvature

$$R = g^{ik} R_{ik},$$

Einstein tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R,$$

Christoffel symbols

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} (g_{ki,j} + g_{kj,i} - g_{ij,k}).$$

Absolute Exterior Differential

The absolute exterior differential of a tensor valued p -form ϕ of type (r, s) has the components

$$\begin{aligned}(D\phi)_{j_1 \dots j_s}^{i_1 \dots i_r} &= d\phi_{j_1 \dots j_s}^{i_1 \dots i_r} + \omega^i_l \wedge \phi_{j_1 \dots j_s}^{li_2 \dots i_r} + \dots \text{ for all upper indices} \\ &\quad - \omega^l_{j_1} \wedge \phi_{lj_2 \dots j_s}^{i_1 \dots i_r} - \dots \text{ for all lower indices.}\end{aligned}$$

The second derivative is given by

$$(D^2\phi)_{j_1 \dots j_s}^{i_1 \dots i_r} = \Omega^i_l \wedge \phi_{j_1 \dots j_s}^{li_2 \dots i_r} + \dots - \Omega^l_{j_1} \wedge \phi_{lj_2 \dots j_s}^{i_1 \dots i_r} - \dots$$

Bianchi Identities

In terms of the coordinate components of the Riemann tensor the Bianchi identities read for a symmetric affine connection:

$$\sum_{(jkl)} R^i{}_{jkl} = 0 \quad (\text{1st Bianchi identity}),$$

$$\sum_{(klm)} R^i{}_{jkl;m} = 0 \quad (\text{2nd Bianchi identity}).$$

In terms of the torsion and curvature forms they are

$$D\theta^i = \Omega^i{}_j \wedge \theta^j$$

$$D\Omega^i{}_j = 0.$$

* * *

Exercise D.1. Let $\varphi : M \rightarrow N$ be a differentiable mapping between the manifolds M and N , $\alpha \in \wedge(N)$, $X \in \mathcal{X}(N)$, $Y \in \mathcal{Y}(M)$ such that Y and X are φ -related. Show that

$$i_Y \varphi^* \alpha = \varphi^* i_X \alpha.$$

In particular, if φ is a diffeomorphism, then

$$i_{\varphi^* X} \varphi^* \alpha = \alpha^* i_X \alpha.$$

Remark. This naturalness property of the interior product follows directly from the definitions.

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