A Fundamental Equations for Hypersurfaces

In this appendix we consider submanifolds with codimension 1 of a pseudo-Riemannian manifold and derive, among other things, the important equations of Gauss and Codazzi-Mainardi. These have many useful applications, in particular for a $3+1$ splitting of the Einstein equations (see Sect. 2.9).

For definiteness, we consider a spacelike hypersurface Σ of a Lorentz manifold (M, g). Other situations, e.g., timelike hypersurfaces can be treated in the same way, with sign changes here and there. The formulation can also be generalized to submanifolds of codimension larger than 1 of arbitrary pseudo-Riemannian manifolds (see, for instance, Vol.II of [43]), a situation that occurs in string theory.

Let $\iota : \Sigma \longrightarrow M$ denote the embedding and let $\bar{g} = \iota^* g$ be the **induced metric** on Σ. The pair $(\Sigma, \bar{g})$ is a three-dimensional Riemannian manifold. Other quantities which belong to Σ will often be indicated by a bar. The basic results of this appendix are most efficiently derived with the help of **adapted** moving frames and the structure equations of Cartan. So let $\{e_\mu\}$ be an 'orthonormal' frame on an open region of M, with the property that the e_i, for $i = 1, 2, 3$, in points of Σ are tangent to Σ. The dual basis of $\{e_\mu\}$ is denoted by $\{\theta^\mu\}$. On Σ the $\{e_i\}$ can be regarded as a triad of Σ that is dual to the restrictions $\bar{\theta}^i := \theta^j | T\Sigma$ (i.e., the restrictions to tangent vectors of Σ). Indices for objects on Σ always refer to this pair of dual basis.

Formulas of Gauss and Weingarten

Consider the restriction of the first structure equation

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \tag{A.1}$$

to $T\Sigma$. Since $\theta^0 | T\Sigma = 0$ we obtain

$$d\bar{\theta}^i + \omega^i{}_j \wedge \bar{\theta}^j = 0 \qquad \text{on } T\Sigma\,, \tag{A.2a}$$

$$\omega^0{}_k \wedge \bar{\theta}^k = 0 \qquad \text{on } T\Sigma\,. \tag{A.2b}$$

Because the connection forms $\bar{\omega}^i{}_j$ of $(\Sigma, \bar{g})$ also satisfy (A.2a), and have the same symmetry properties, we conclude (see Sect. 13.7) that

$$\bar{\omega}^i{}_j = \omega^i{}_j \qquad \text{on } T\Sigma\,. \tag{A.3}$$

This has a simple geometrical meaning. Let ∇ be the Levi-Cività connection on (M, g) and $\bar{\nabla}$ that of $(\Sigma, \bar{g})$, then we get from (A.3)

$$\langle \nabla_X e_j, e_i \rangle = \omega_{ij}(X) = \bar{\omega}_{ij}(X) = \langle \bar{\nabla}_X e_j, e_i \rangle \tag{A.4}$$

for $X \in T\Sigma$. This shows that for any $X \in T\Sigma$ and any vector field Y of M tangent to Σ, the tangential projection of $\nabla_X Y$ on $T\Sigma \subset TM$ is equal to $\bar{\nabla}_X Y$.

From (A.2b) we obtain information about the component of $\nabla_X Y$ normal to $T\Sigma$. This equation tells us that the $\omega^0_{\ i} = \omega^i_{\ 0}$ satisfy the hypothesis of the following

Lemma A.1 (Cartan). *If $\alpha^1, \dots, \alpha^n$ are linearly independent 1-forms on a manifold M of dimension $n' \geq n$, and $\beta_1, \dots, \beta_n$ are one forms on M satisfying*

$$\sum_{i=1}^{n} \alpha^i \wedge \beta_i = 0\,, \tag{A.5}$$

then there are smooth functions f_{ij} on M such that

$$\beta_i = \sum_{j=1}^{n} f_{ij}\, \alpha^j\,; \tag{A.6}$$

moreover

$$f_{ij} = f_{ji}\,. \tag{A.7}$$

Proof. In a neighborhood of any point we can choose 1-forms $\alpha^{n+1}, \dots, \alpha^{n'}$ so that $\alpha^1, \dots, \alpha^{n'}$ are everywhere linearly independent. Then there are smooth functions f_{ij} $(i \leq n, j \leq n')$ with

$$\beta_i = \sum_{j=1}^{n'} f_{ij}\, \alpha^j\,.$$

Now Equation (A.5) implies

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{n'} f_{ij}\, \alpha^i \wedge \alpha^j = \sum_{1 \leq i \leq j \leq n} (f_{ij} - f_{ji}) \alpha^i \wedge \alpha^j + \sum_{i=1}^{n} \sum_{j>n} f_{ij}\, \alpha^i \wedge \alpha^j\,.$$

Since the $\alpha^i \wedge \alpha^j$ for $i < j$ are linearly independent, we conclude $f_{ij} = f_{ji}$ for $i, j \leq n$ and $f_{ij} = 0$ for $j > n$. □

According to the lemma and (A.2b) there are functions K_{ij} on Σ such that

$$\omega^0_{\ i} = -K_{ij}\, \theta^j \qquad \text{on } T\Sigma\,, \tag{A.8a}$$

$$K_{ij} = K_{ji}\,. \tag{A.8b}$$

From this and $\nabla_{e_\mu} e_\nu = \omega^\lambda{}_\nu(e_\mu)e_\lambda$ we obtain

$$\langle \nabla_{e_i} e_j, e_0 \rangle = -\omega^0{}_j(e_i) = K_{ij} = K_{ji} = \langle \nabla_{e_j} e_i, e_0 \rangle \,. \tag{A.9}$$

So the bilinear form $K(X, Y)$ on $T\Sigma$ belonging to K_{ij} gives the normal component of $\nabla_X Y$:

$$\nabla_X Y = \bar{\nabla}_X Y - K(X, Y)N \tag{A.10}$$

for $X \in T\Sigma$, $Y \in \mathcal{X}(M)$ tangent to Σ, where $N = e_0$ (normalized normal vector on Σ, $\langle N, N \rangle = -1$).

Equation (A.10) is the **Gauss formula**. The symmetric bilinear form K, called the **second fundamental form** or **extrinsic curvature**, satisfies according to (A.9) the **Weingarten equations**

$$K(X, Y) = \langle N, \nabla_X Y \rangle = -\langle \nabla_X N, Y \rangle \,. \tag{A.11}$$

Remarks. The reader should be warned that some authors use as extrinsic curvature the negative of our K. The linear transformation of the tangent spaces of Σ corresponding to K is known as the **Weingarten map** of the second fundamental tensor. Its real eigenvalues are called **principle curvatures**.

Equations of Gauss and Codazzi-Mainardi

Next, we look for relations between the curvature forms $\Omega^\mu{}_\nu$ and $\bar{\Omega}^i{}_j$ of (M, g), respectively $(\Sigma, \bar{g})$. For this we restrict the second structure equation

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu \tag{A.12}$$

to Σ. Consider first the restriction of

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j + \omega^i{}_0 \wedge \omega^0{}_j$$

to $T\Sigma$. Using (A.3) and (A.8a) this gives

$$\Omega^i{}_j = \bar{\Omega}^i{}_j + K^i{}_k K_{jl}\, \bar{\theta}^k \wedge \bar{\theta}^l \qquad \text{on } T\Sigma \,. \tag{A.13}$$

The other components give on $T\Sigma$

$$\begin{aligned} \Omega^0{}_j &= d\omega^0{}_j + \omega^0{}_i \wedge \omega^i{}_j = -d(K_{ij}\bar{\theta}^j) - K_{ik}\, \bar{\theta}^k \wedge \bar{\omega}^i{}_j \\ &= -dK \wedge \bar{\theta}^j - K_{ij} d\bar{\theta}^i - K_{ik}\, \bar{\theta}^k \wedge \bar{\omega}^i{}_j \end{aligned}$$

or, using the first structure equation on $(\Sigma, \bar{g})$,

$$\Omega^0{}_j = -\bar{D}K_{ij} \wedge \bar{\theta}^i \qquad \text{on } T\Sigma \,. \tag{A.14}$$

Here $\bar{D}$ denotes the absolute exterior differential of the tensor field K.

The relations (A.13) and (A.14) are the famous **equations of Gauss** and **Codazzi-Mainardi** in terms of differential forms. We want to rewrite them in a more useful form. Consider for tangential X, Y

$$\begin{aligned}\langle R(X,Y)e_j, e_i\rangle &= \Omega_{ij}(X,Y) \\ &= \bar{\Omega}_{ij}(X,Y) + K_{ik}K_{jl}(\bar{\theta}^k \wedge \bar{\theta}^l)(X,Y) \\ &= \left\langle \bar{R}(X,Y)e_j, e_i\right\rangle + \\ &\quad K(e_i,X)K(e_j,Y) - K(e_i,Y)K(e_j,X)\,.\end{aligned}$$

This shows that for tangent vectors X, Y, Z and W of Σ we have

$$\begin{aligned}\langle R(X,Y)Z, W\rangle = \left\langle \bar{R}(X,Y)Z, W\right\rangle - \\ -K(X,Z)K(Y,W) + K(Y,Z)K(X,W)\,. \quad \text{(A.15)}\end{aligned}$$

One calls this relation often **Gauss' Theorema Egregium**. (Note that some of the signs are different from the ones for the Riemannian case, that is usually treated in mathematics books.)

Similarly, we obtain from (A.14)

$$\begin{aligned}\langle R(X,Y)e_j, N\rangle &= \Omega_{0j}(X,Y) = \left(\bar{D}K_{ij} \wedge \bar{\theta}^i\right)(X,Y) \\ &= \bar{\nabla}_k K_{ij}\left(\bar{\theta}^k \wedge \bar{\theta}^i\right)(X,Y) \\ &= X^k Y^i \bar{\nabla}_k K_{ij} - (X \longleftrightarrow Y) \\ &= \left(\bar{\nabla}_X K\right)(Y, e_j) - (X \longleftrightarrow Y)\end{aligned}$$

for $X, Y \in T\Sigma$. Hence, Eq. (A.14) is equivalent to the following standard form of the Codazzi-Mainardi equation:

$$\langle R(X,Y)Z, N\rangle = (\bar{\nabla}_X K)(Y,Z) - (\bar{\nabla}_Y K)(X,Z)\,, \qquad \text{(A.16)}$$

for $X, Y, Z \in T\Sigma$.

The basic equations of Gauss and Codazzi-Mainardi allow us to obtain interesting and useful expressions for the components R_{0i} and G_{00} of the Ricci and Einstein tensors on Σ. We first note that for any frame

$$i_{e_\alpha}\Omega^\alpha_{\ \beta} = \frac{1}{2} R^\alpha_{\ \beta\rho\sigma} i_{e_\alpha}(\theta^\rho \wedge \theta^\sigma) = R_{\beta\sigma}\,\theta^\sigma\,,$$

so that

$$R_{\beta\sigma} = \Omega^\alpha_{\ \beta}(e_\alpha, e_\sigma)\,. \qquad \text{(A.17)}$$

Especially, we obtain

$$R_{0i} = \Omega^j_{\ 0}(e_j, e_i)\,. \qquad \text{(A.18)}$$

Relative to our adapted basis we need $\Omega^j_{\ 0}$ on the right-hand side only on $T\Sigma$, and can thus use (A.14) to get

$$R_{0i} = \bar{\nabla}_i K^j_{\ j} - \bar{\nabla}_j K^j_{\ i}\,. \qquad \text{(A.19)}$$

Next, we consider

$$G_{00} = R_{00} + \frac{1}{2}\left(-R_{00} + R^i_{\ i}\right) = \frac{1}{2}\left(R_{00} + R^i_{\ i}\right).$$

From (A.17) we get

$$\begin{aligned} R_{00} &= \Omega^j_{\ 0}(e_j, e_0)\,, \\ R^i_{\ i} &= \Omega^{ji}(e_j, e_i) + \Omega^{0i}(e_0, e_i)\,, \end{aligned}$$

and thus

$$G_{00} = \frac{1}{2}\Omega^{ij}(e_i, e_j)\,. \tag{A.20}$$

Here we can use (A.13) to obtain

$$2G_{00} = \bar{R}^i_{\ i} + K^i_{\ i}K^j_{\ j} - K^j_{\ j}K^j_{\ i}\,,$$

or

$$G_{00} = \frac{1}{2}\bar{R} + \frac{1}{2}\left(K^i_{\ i}K^j_{\ j} - K^i_{\ j}K^j_{\ i}\right). \tag{A.21}$$

This form of Gauss' equation is particular useful in GR.

Consider, for illustration, a static metric as in Sect. 2.1. From (2.23) we see that the second fundamental form for any time slice vanishes, thus (A.21) gives

$$G_{00} = G_{\mu\nu}N^\mu N^\nu = \frac{1}{2}\bar{R}\,,$$

in agreement with (2.30). In the exercises we consider time-dependent metrics.

These formulas simplify if Σ is totally geodesic in the sense of

Definition A.1. *A hypersurface Σ is called* ***totally geodesic*** *if every geodesic $\gamma(s)$ of (M, g) with $\gamma(0) \in \Sigma$ and $\dot{\gamma}(0) \in T_{\gamma(0)}\Sigma$ remains in Σ for s in some open interval $(-\varepsilon, \varepsilon)$.*

This is equivalent to the property that $\nabla_X Y$ is tangent to Σ whenever X and Y are (see Exercise A.3). From (A.10) we conclude that $\Sigma \subset M$ is totally geodesic if and only if the second fundamental form K of Σ vanishes.

* * *

Null Hypersurfaces

Null hypersurfaces of Lorentz manifolds play an important role in GR.

Definition A.2. *A hypersurface Σ of a Lorentz manifold (M, g) is* ***null****, if the induced metric on Σ is degenerate.*

The reader may easily show that this is equivalent to the property that at any point $p \in \Sigma$ there exists a non-vanishing null vector in T_pM which is perpendicular to $T_p\Sigma$. As a consequence of property c) in Exercise 7.5, this null vector is unique up to a scale factor (up to a positive scale factor if the null vector is future directed). Hence, there exists locally a smooth non-vanishing null vector field l along Σ which is tangent to Σ and whose normal space at any point $p \in \Sigma$ coincides with T_pM. From the quoted exercise it also follows that tangent vectors of Σ not parallel to l are spacelike.

The null vector field l, which is unique up to rescaling, has the following remarkable property:

Proposition A.1. *The integral curves of l, when suitably parameterized, are null geodesics.*

Proof. We show below that $\nabla_l l$ is at each $p \in \Sigma$ perpendicular to T_pM. This implies that $\nabla_l l$ is proportional to l, which is equivalent to the proposition.

To establish the stated property, let $u \in T_pM$ and extend u locally to a vector field along Σ which is invariant under the flow belonging to l (show that this is possible). Then $0 = L_l u = [l, u] = \nabla_l u - \nabla_u l$. From $\langle l, u \rangle = 0$ we obtain

$$0 = l \langle l, u \rangle = \langle \nabla_l l, u \rangle + \langle l, \nabla_l u \rangle ,$$

thus

$$\langle \nabla_l l, u \rangle = -\langle l, \nabla_u l \rangle = -\frac{1}{2}\, u \langle l, l \rangle = 0 .$$

(These formulas should be interpreted in the sense of Sect. 14.3 for vector fields along the embedding map of Σ into M.) Since u is arbitrary in p, $\nabla_l l$ is indeed perpendicular to $T_p\Sigma$. □

The integral curves of l are called **null geodesic generators** of Σ.

* * *

Exercises.

Exercise A.1. Consider a metric of the form

$$g = -\varphi^2(t, x^k)\, dt^2 + g_{ij}(t, x^k)\, dx^i dx^j \tag{A.22}$$

and show that the second fundamental form of the slices $\{t = \text{const}\}$ is given by

$$K_{ij} = -\frac{1}{2\varphi} \partial_t g_{ij} . \tag{A.23}$$

Solution. The Ricci identity and $[\partial_t, \partial_i] = 0$ give

$$\begin{aligned} \partial_t g_{ij} &= \langle \nabla_{\partial_t} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_t} \partial_j \rangle \\ &= \langle \nabla_{\partial_i} \partial_t, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_j} \partial_t \rangle . \end{aligned}$$

Here we use $N = \frac{1}{\varphi}\partial_t$ and (A.11) to obtain

$$\begin{aligned}\partial_t g_{ij} &= \big\langle \nabla_{\partial_i}(\varphi N), \partial_j \big\rangle + \big\langle \partial_i, \nabla_{\partial_j}(\varphi N) \big\rangle \\ &= \varphi\big(\langle \nabla_{\partial_i} N, \partial_j\rangle + \langle \partial_i, \nabla_{\partial_j} N\rangle\big) \\ &= -\varphi(K_{ij} + K_{ji})\,.\end{aligned}$$

Exercise A.2. Use (A.23), the Gauss equation (A.21), and (2.78) to show that for the **Friedmann metric**

$$g = -dt^2 + a^2(t) h_{ij}\, dx^i \wedge dx^j \tag{A.24}$$

on $\mathbb{R} \times S^3$, where $h = h_{ij}\, dx^i \wedge dx^j$ is the standard metric on S^3, the time-time component of the Einstein tensor is given by

$$G_{00} = 3\left(\frac{\dot a^2}{a^2} + \frac{1}{a^2}\right)\,. \tag{A.25}$$

Hence, Einstein's field equation implies the following **Friedmann equation**

$$\left(\frac{\dot a}{a}\right)^2 + \frac{1}{a^2} = 8\pi G(\rho + \rho_\Lambda)\,, \tag{A.26}$$

where $\rho_\Lambda := \frac{\Lambda}{8\pi G}$ and ρ is the matter energy density.

Exercise A.3. Prove that a hypersurface Σ of (M, g) is totally geodesic if and only if $\nabla_X Y$ is tangent to Σ whenever X and Y are.

Solution. We already know that the second property holds if and only if the second fundamental form K of Σ vanishes. So let us show that $K = 0$ if and only if Σ is totally geodesic.

Let $\gamma : s \longmapsto \gamma(s)$ be a geodesic in (M, g), with $\gamma(0) \in \Sigma$, $\dot\gamma(0) \in T_{\gamma(0)}\Sigma$, and let $\bar\gamma$ be a geodesic in $(\Sigma, \bar g)$ with the same initial conditions. From (A.10) we get

$$\nabla_{\dot{\bar\gamma}}\dot{\bar\gamma} = \bar\nabla_{\dot{\bar\gamma}}\dot{\bar\gamma} - K(\dot{\bar\gamma}, \dot{\bar\gamma})N = -K(\dot{\bar\gamma}, \dot{\bar\gamma})N\,.$$

Hence, if K vanishes we see that $\bar\gamma$ is also a geodesic of (M, g) with the same initial conditions as γ, thus $\bar\gamma = \gamma$ on their common domain. This proves that a vanishing K implies that Σ is totally geodesic.

Conversely, let Σ be totally geodesic. Then γ stays in Σ for some open interval $(-\varepsilon, \varepsilon)$. From (A.10) we now conclude

$$0 = \nabla_{\dot\gamma}\dot\gamma = \bar\nabla_{\dot\gamma}\dot\gamma - K(\dot\gamma, \dot\gamma)N\,.$$

Therefore, $\bar\nabla_{\dot\gamma}\dot\gamma = 0$ and $K(\dot\gamma, \dot\gamma) = 0$. In particular, γ is also a geodesic of $(\Sigma, \bar g)$. Moreover, since $\gamma(0) \in \Sigma$ and $\dot\gamma(0) \in T_{\gamma(0)}\Sigma$ are arbitrary, we conclude that $K = 0$ (use the symmetry of K and polarization).

B Ricci Curvature of Warped Products

In GR one often encounters so-called warped products (see, e.g., Appendix 3.10). By definition, a pseudo-Riemannian manifold (M, g) is a **warped product** of two pseudo-Riemannian manifolds $(\tilde{M}, \tilde{g})$ and $(S, \hat{g})$ with an everywhere positive **warp function** $f : \tilde{M} \longrightarrow \mathbb{R}$, written as

$$M = \tilde{M} \times_f S\,, \tag{B.1}$$

if M is the product manifold $\tilde{M} \times S$ and the metric has the form

$$g = \pi^*(\tilde{g}) + (f \circ \pi)^2 \sigma^*(\hat{g})\,, \tag{B.2}$$

where π and σ are the projections from M onto $\tilde{M}$ and S, respectively. In the following we shall drop the pull-backs π^*, σ^*.

The calculation of the curvature quantities of (M, g) in terms of those of $(\tilde{M}, \tilde{g})$ and $(S, \hat{g})$ is best done within the framework of Cartan's calculus. We shall work with the orthonormal tetrad fields $\{\theta^\alpha\}$ of (M, g), which are adapted as follows: Let $\alpha = (a, A)$ and let

$$\theta^a = \tilde{\theta}^a\,, \qquad \theta^A = f\,\hat{\theta}^A\,, \tag{B.3}$$

where $\tilde{\theta}^a$, for $a = 1, \ldots, \dim(\tilde{M})$, and $\hat{\theta}^A$, for $A = \dim(\tilde{M}) + 1, \ldots, \dim(M)$, are orthonormal bases of $(\tilde{M}, \tilde{g})$ and $(S, \hat{g})$, respectively. The connection forms of the various spaces are denoted by $\omega^\alpha_{\ \beta}$, $\tilde{\omega}^a_{\ b}$ and $\hat{\omega}^A_{\ B}$, and for the corresponding curvature forms we use the symbols $\Omega^\alpha_{\ \beta}$, $\tilde{\Omega}^a_{\ b}$ and $\hat{\Omega}^A_{\ B}$.

To start, we determine the connection forms $\omega^\alpha_{\ \beta}$ from Cartan's first structure equation

$$d\theta^\alpha + \omega^\alpha_{\ \beta} \wedge \theta^\beta = 0\,. \tag{B.4}$$

We need

$$\begin{aligned} d\theta^A &= d(f\,\hat{\theta}^A) = f\,d\hat{\theta}^A - \hat{\theta}^A \wedge df \\ &= -f\,\hat{\omega}^A_{\ B} \wedge \hat{\theta}^B - f_b\,\hat{\theta}^A \wedge \tilde{\theta}^b \\ &= -\hat{\omega}^A_{\ B} \wedge \theta^B - \frac{f_b}{f}\,\theta^A \wedge \theta^b\,, \end{aligned}$$

where we have set $df = f_a\,\tilde{\theta}^a$. Comparing this result with (B.4) leads to the guess

$$\omega^A_{\ B} = \hat{\omega}^A_{\ B}\,, \qquad \omega^A_{\ b} = \frac{f_b}{f}\,\theta^A\,. \tag{B.5}$$

In addition, we also have to satisfy

$$\begin{aligned} d\theta^a &= -\omega^a_{\ B}\wedge\theta^B - \omega^a_{\ b}\wedge\theta^b \\ &= d\tilde{\theta}^a = -\tilde{\omega}^a_{\ b}\wedge\tilde{\theta}^b = -\tilde{\omega}^a_{\ b}\wedge\theta^b\,. \end{aligned}$$

Taking

$$\omega^a_{\ b} = \tilde{\omega}^a_{\ b}\,, \tag{B.6}$$

we see that the structure equation (B.4) is fulfilled for the connection forms (B.5) and (B.6). Therefore, these equations provide the correct (and unique) result.

Using the second structure equation,

$$\Omega^\alpha_{\ \beta} = d\omega^\alpha_{\ \beta} + \omega^\alpha_{\ \gamma}\wedge\omega^\gamma_{\ \beta}\,, \tag{B.7}$$

and the corresponding equations for $\tilde{\Omega}^a_{\ b}$ and $\hat{\Omega}^A_{\ B}$ as well, we find after a short routine calculation the curvature forms

$$\Omega^a_{\ b} = \tilde{\Omega}^a_{\ b}\,, \tag{B.8a}$$

$$\Omega^A_{\ B} = \hat{\Omega}^A_{\ B} - \frac{\langle df, df\rangle}{f^2}\,\theta^A\wedge\theta_B\,, \tag{B.8b}$$

$$\Omega^a_{\ B} = -\frac{1}{f}\tilde{\nabla}_b\tilde{\nabla}^a f\,\theta^b\wedge\theta_B\,, \tag{B.8c}$$

from which we can now extract the Ricci tensor. This is best done by taking advantage of the formula

$$R_{\beta\gamma} = \Omega^\alpha_{\ \beta}(e_\alpha, e_\gamma)\,, \tag{B.9}$$

which follows from $\Omega^\mu_{\ \nu} = R^\mu_{\ \nu\alpha\beta}\,\theta^\alpha\wedge\theta^\beta$ (e_α denotes the orthonormal frame dual to θ^α). One finally obtains

$$R_{ab} = \tilde{R}_{ab} - \frac{\dim(S)}{f}\,\tilde{\nabla}_b\tilde{\nabla}_a f\,, \tag{B.10a}$$

$$R_{aB} = 0\,, \tag{B.10b}$$

$$\begin{aligned} R_{AB} &= \hat{R}_{AB} \\ &\quad -\big(f\,\tilde{\Delta}f + \langle df, df\rangle(\dim(S)-1)\big)\hat{g}_{AB}\,. \end{aligned} \tag{B.10c}$$

This result obviously holds also for non-orthonormal adapted frames. (Note that all indices refer to the frame $\theta^\alpha = (\theta^a, \theta^A)$.)

Application: Friedmann Equations

Important examples of warped products are **Friedmann-Lemaître (-Robertson-Walker) spacetimes** in cosmology. For these $\tilde{M}$ is an open interval of $\mathbb{R}$ with the metric $-dt^2$, $(S, \hat{g})$ is a 3-dimensional space of constant curvature $k = \pm 1, 0$, and the warp function is the scale factor $a(t)$. The spacetime metric is thus

$$g = -dt^2 + a^2(t)\,\hat{g}\,. \tag{B.11}$$

We use the general results (B.10a)–(B.10c) to compute the Ricci tensor belonging to (B.11).

Since $(\tilde{M}, \tilde{g})$ is now a 1-dimensional Euclidean space, (B.10a) gives

$$R_{00} = -3\,\frac{\ddot{a}}{a}\,. \tag{B.12}$$

According to (B.10b) we have

$$R_{0i} = 0\,, \tag{B.13}$$

and (B.10c) implies

$$R_{ij} = \hat{R}_{ij} + \left[2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] a^2\,\hat{g}_{ij}\,.$$

According to (2.79), which holds for all Riemannian spaces of constant curvature, we have

$$\hat{R}_{ij} = 2\,k\,\hat{g}_{ij}\,, \tag{B.14}$$

hence

$$R_{ij} = \left[2\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{k}{a^2} + \frac{\ddot{a}}{a}\right] a^2\,\hat{g}_{ij}\,. \tag{B.15}$$

For an orthonormal tetrad $\{\theta^\alpha\}$ we have $a^2\,\tilde{g}_{ij} = \delta_{ij}$ (because of (B.3)). Then we find for the Einstein tensor

$$\begin{aligned}
G_{00} &= 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right], \\
G_{0i} &= 0\,, \\
G_{ij} &= -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\delta_{ij}\,.
\end{aligned} \tag{B.16}$$

In order to satisfy the field equations, the energy-momentum tensor must have the form

$$(T_{\mu\nu}) = \mathrm{diag}(\rho, p, p, p)\,. \tag{B.17}$$

From (B.16) and (B.17) we obtain the independent Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\,\rho + \frac{\Lambda}{3}\,, \tag{B.18a}$$

$$-\,2\,\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = 8\pi G\,p - \Lambda\,. \tag{B.18b}$$

Note also, that (B.12) gives the (dependent) equation

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}\,a\,. \tag{B.19}$$

This shows that $\ddot{a}$ depends on the combination $\rho + 3p$, and that a *positive* Λ leads to an **accelerated expansion**.

* * *

Exercise B.1. Use the connection forms (B.5) and (B.6) to show that $T^{\alpha\beta}{}_{;\beta} = 0$ implies for $\alpha = 0$:

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}(\rho + p) = 0\,. \tag{B.20}$$

Derive this equation also from (B.18a) and (B.18b).

C Frobenius Integrability Theorem

In this appendix we develop an important differential geometric tool of GR that is also one of the pillars of differential topology and the calculus on manifolds. The following problem is a natural generalization of finding integral curves for a vector field. Instead of a vector field we assign to each point p of a manifold M a subspace E_p of T_pM which depends on p in a smooth manner and ask: When is it possible to find for each point $p_0 \in M$ a submanifold N with $p_0 \in N$, which is tangent to E_p, i.e., such that $T_pN = E_p$ for each point $p \in N$? In general, such integral manifolds do not exist, even locally. For the local problem the so-called **Frobenius theorem** (due to A. Clebsch and F. Deahna) gives necessary and sufficient criteria for integral manifolds to exist.

We begin with some precisions.

Definition C.1. *A k-**dimensional distribution** (k-**plane**, k-**direction field**) on M is a map $p \longmapsto E_p$, where E_p is a k-dimensional subspace of T_pM. We call this distribution **smooth** if the following condition is satisfied: For each $p_0 \in M$ there is a neighborhood V and k differentiable vector fields $X_1, \ldots, X_k$ such that $X_1(p), \ldots, X_k(p)$ form a basis of E_p for all $p \in V$.*

Definition C.2. *A k-dimensional submanifold N of M is called an **integral manifold** of the distribution E if for every $p \in N$ we have*

$$T_p\,\iota(T_pN) = E_p\,,$$

*where $\iota : N \hookrightarrow M$ is the inclusion map. We say that the distribution E on M is **integrable** if, through each $p \in M$, there passes an integral manifold of E.*

We want to find out when E is integrable. For this we consider vector fields which belong to E: We say that the vector field X **belongs to** E if $X_p \in E_p$ for all p. Suppose now that N is an integrable manifold of E, and $\iota : N \hookrightarrow M$ is the inclusion map. If X, Y are two vector fields which belong to E, then for all $p \in N$ there are unique $\bar{X}_p$, $\bar{Y}_p \in T_pN$ such that

$$X_p = T_p\,\iota \cdot \bar{X}_p\,, \qquad Y_p = T_p\,\iota \cdot \bar{Y}_p\,.$$

It is not difficult to show[1] that the vector fields $\bar{X}$, $\bar{Y}$ are C^∞. In other words $\bar{X}$ and X, as well as $\bar{Y}$ and Y, are ι-related. Then we know (Theorem 11.6) that $[\bar{X}, \bar{X}]$ and $[X, Y]$ are ι-related. Thus

$$T_p\iota \cdot [\bar{X}, \bar{X}]_p = [X, Y]_p .$$

Since $[\bar{X}, \bar{X}]_p \in T_pN$ we conclude that $[X, Y]_p \in E_p$. We thus have shown: If E is integrable then $[X, Y]$ also belongs to E. This suggests to introduce the following concept:

Definition C.3. *We say that the k-distribution E on M is **involutive** if $[X, Y]$ belongs to E whenever the vector fields X and Y (defined on open sets of M) belong to E.*

Using this terminology, we have shown that if E is integrable then E is involutive. It turns out that the reverse is also true.

Theorem C.1 (Frobenius' Theorem (first version)). *Let E be a C^∞ k-dimensional distribution on M which is involutive. Then E is integrable. More precisely the following holds: For each point $p \in M$ there is a coordinate system $\{x^1, \ldots, x^n\}$ in a neighborhood U of p with*

$$x^i(p) = 0 , \quad x^i(U) \subset (-\varepsilon, \varepsilon) \qquad (\varepsilon > 0) ,$$

such that for each $(a^{k+1}, \ldots, a^n)$ with all $|a^i| < \varepsilon$ the set

$$\left\{q \in U \ : \ x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n\right\}$$

is an integral manifold of E. Any connected integral manifold of E restricted to U is contained in one of these sets.

We postpone the proof of this important theorem and first reformulate it in a useful manner in terms of differential forms. For this we need some concepts.

To each k-dimensional distribution E we associate the set $\mathcal{I}(E) \subset \bigwedge(M)$ of differential forms ω with the property that each homogeneous component $\omega^l \in \bigwedge^l(M)$ of ω ($\omega = \sum_l \omega^l$) annihilates the sets $(X_1, \ldots, X_l)$ of vector fields belonging to E:

$$\omega^l(X_1, \ldots, X_l) = 0 .$$

The set $\mathcal{I}(E)$ is called the **annihilator** of E. Clearly, $\mathcal{I}(E)$ is an ideal of the algebra $\bigwedge(M)$ of exterior forms: With ω_1, $\omega_2 \in \mathcal{I}(E)$ the sum $\omega_1 + \omega_2$ also belongs to E, and $\eta \wedge \omega \in \mathcal{I}(E)$ if $\omega \in \mathcal{I}(E)$ and $\eta \in \bigwedge(M)$. Locally, the ideal $\mathcal{I}(E)$ is generated by $n-k$ independent 1-forms $\omega^{k+1}, \ldots, \omega^n$. In fact, around each point $p \in M$ we can choose coordinates $\{x^i\}$ so that $\{\partial_1|_p, \ldots, \partial_k|_p\}$ span

[1] Since ι is an immersion one can use coordinates as in Definition 9.7. For details, see [43], Vol. I, p.258.

E_p. Then $dx^1(p) \wedge \ldots \wedge dx^k(p)$ is non-zero on E_p and by continuity also in a small neighborhood of p. This implies that $\{dx^1(q), \ldots, dx^k(q)\}$ are linearly independent in E_q. Therefore, if we restrict $dx^\alpha(q)$ for $\alpha = k+1, \ldots, n$ to E_q we have the expansion

$$dx^\alpha(q) = \sum_{\beta=1}^{k} f^\alpha_\beta(q)\, dx^\beta(q)\,.$$

Therefore

$$\omega^\alpha = dx^\alpha - \sum_{\beta=1}^{k} f^\alpha_\beta\, dx^\beta\,, \quad \alpha = k+1, \ldots, n$$

are linearly independent 1-forms which belong to $\mathcal{I}(E)$. These generate the ideal. To show this we complete $\omega^{k+1}, \ldots, \omega^n$ to a basis $\omega^1, \ldots, \omega^n$ of 1-forms and consider the dual basis $X_1, \ldots, X_n$ of vector fields. Then $X_1, \ldots, X_k$ span the k-dimensional distribution E. Suppose an l-form $\omega \in \mathcal{I}(E)$ contains in the expansion with respect to the basis $\omega^1, \ldots \omega^n$,

$$\omega = \sum_{i_1 < i_2 < \ldots < i_l} c_{i_1 \ldots i_l}\, \omega^{i_1} \wedge \ldots \wedge \omega^{i_l}\,,$$

a term $c_{j_1 \ldots j_l}\, \omega^{j_1} \wedge \ldots \wedge \omega^{j_l}$ which contains no factor from $\omega^{k+1}, \ldots, \omega^n$. Then we have $\omega(X_{j_1}, \ldots, X_{j_l}) \neq 0$, in contradiction to $\omega \in \mathcal{I}(E)$. This proves:

Proposition C.1. *Let $\mathcal{I}(E)$ be the ideal of $\bigwedge(M)$ belonging to the k-dimensional distribution E. Then $\mathcal{I}(E)$ is* ***locally generated*** *by $n-k$ independent 1-forms: For each point of M there is a neighborhood U and $n-k$ pointwise linearly independent 1-forms $\omega^{k+1}, \ldots, \omega^n \in \bigwedge^1(U)$ such that for each $\omega \in \mathcal{I}(E)$*

$$\omega|U = \sum_{i=k+1}^{n} \theta_i \wedge \omega^i$$

for some $\theta_i \in \bigwedge(U)$.

Now we can reformulate the condition in the Frobenius theorem.

Proposition C.2. *A distribution E on M is involutive if and only if $\mathcal{I}(E)$ is a* ***differential ideal****: $d\mathcal{I}(E) \subset \mathcal{I}(E)$.*

Proof. We use the same notation as above. It is easy to see that E is involutive if and only if there exist smooth functions $C^l{}_{ij}$ such that

$$[X_i, X_j] = \sum_{l=1}^{k} C^l{}_{ij}\, X_l$$

for $i, j = 1, \ldots k$. Now we have (see (12.14))

$$d\omega^\alpha(X_i, X_j) = X_i\big(\omega^\alpha(X_j)\big) - X_j\big(\omega^\alpha(X_i)\big) - \omega^\alpha\big([X_i, X_j]\big)\,.$$

For $1 \leq i, j \leq k$ and $\alpha > k$ the first two terms on the right vanish. So $d\omega^\alpha(X_i, X_j) = 0$ if and only if $\omega^\alpha\big([X_i, X_j]\big) = 0$. But the latter equation holds for all i, j if and only if each $[X_i, X_j]$ belongs to E (i.e., if E is involutive), while $d\omega^\alpha(X_i, X_j) = 0$ holds exactly when $d\omega^\alpha \in \mathcal{I}(E)$. □

Next we establish some equivalent conditions that a locally (finitely) generated ideal is a differential ideal.

Proposition C.3. *Let $\mathcal{I}$ be an ideal of $\bigwedge(M)$ locally generated by $n-k$ linearly independent forms $\omega^{k+1}, \ldots, \omega^n \in \bigwedge^1(U)$. Furthermore let $\omega \in \bigwedge^{n-k}(U)$ be the form $\omega = \omega^{k+1} \wedge \ldots \wedge \omega^n$. Then the following statements are equivalent:*

(i) $\mathcal{I}$ *is an differential ideal.*
(ii) $d\omega^\alpha = \sum_{\beta=k+1}^n \omega^\alpha_\beta \wedge \omega^\beta$ *for certain* $\omega^\alpha_\beta \in \bigwedge^1(U)$.
(iii) $\omega \wedge d\omega^\alpha = 0$ *for* $\alpha = k+1, \ldots, n$.
(iv) There exists $\theta \in \bigwedge^1(U)$ *such that* $d\omega = \theta \wedge \omega$.

Proof. The equivalence of (i) and (iii) follows immediately from the definitions. The same is true for the implication (i) $\Longrightarrow$ (iv). For the proof of (iv) $\Longrightarrow$ (iii) note that the condition (iv) means that

$$\sum_{\alpha=k+1}^{n} (-1)^\alpha \, d\omega^\alpha \wedge \omega^{k+1} \wedge \ldots \wedge \hat{\omega}^\alpha \wedge \ldots \wedge \omega^n = \theta \wedge \omega^{k+1} \wedge \ldots \wedge \omega^n\,.$$

Multiplying this equation with ω^α we get (iii). It remains to show that (iii) $\Longrightarrow$ (ii): Again, let $\omega^1, \ldots, \omega^n$ be a basis of $\bigwedge^1(U)$ such that $\omega^{k+1}, \ldots, \omega^n$ generate $\mathcal{I}$ over U. Then

$$d\omega^i = \sum_{j<l} f^i_{\ jl}\, \omega^j \wedge \omega^l\,, \tag{C.1}$$

where $f^i_{\ jl} \in \mathcal{F}(U)$. But

$$0 = d\omega^\alpha \wedge \omega = \sum_{1 \leq j < l \leq k} f^\alpha_{\ jl}\, \omega^j \wedge \omega^l \wedge \omega^{k+1} \wedge \ldots \wedge \omega^n\,.$$

Thus $f^\alpha_{\ jl} = 0$ for $\alpha = k+1, \ldots, n$ and $1 \leq j < l \leq k$. Hence the sum in (C.1) is of the form (ii). □

Now we assemble the preceding results in the following version of the Frobenius theorem.

Theorem C.2 (Frobenius' Theorem). *Let M be an n-dimensional manifold and E a k-dimensional distribution on M, and $\mathcal{I}(E)$ the associated ideal. The following statements are all equivalent:*

(i) *E is integrable.*
(ii) *E is involutive.*
(iii) *$\mathcal{I}(E)$ is a differential ideal locally generated by $n-k$ linearly independent 1-forms $\omega^{k+1}, \dots, \omega^n \in \bigwedge^1(U)$.*
(iv) *For every point of M there is a neighborhood U and $\omega^{k+1}, \dots, \omega^n \in \bigwedge^1(U)$ generating $\mathcal{I}(E)$ such that*

$$d\omega^\alpha = \sum_{\beta=k+1}^{n} \omega^\alpha_\beta \wedge \omega^\beta \tag{C.2}$$

for certain $\omega^\alpha_\beta \in \bigwedge^1(U)$.
(v)

$$d\omega^\alpha \wedge \omega^{k+1} \wedge \dots \wedge \omega^n = 0 \tag{C.3}$$

(in the notation of (iv)).
(vi) *There exists a $\theta \in \bigwedge^1(U)$ such that*

$$d\omega = \theta \wedge \omega\,, \qquad \omega := \omega^{k+1} \wedge \dots \wedge \omega^n\,. \tag{C.4}$$

Applications

Consider first a single timelike vector field X on a Lorentz manifold (M, g). The orthogonal complement of X_p in every $T_p(M)$ defines a 3-dimensional distribution E. Obviously, X is hypersurface orthogonal if and only if E is integrable. Since $\omega = X^\flat$ generates the ideal $\mathcal{I}(E)$ the Frobenius theorem tells us that E is integrable if and only if

$$\omega \wedge d\omega = 0\,. \tag{C.5}$$

(We have used the equivalence of (i) and (v).) This shows that (C.5) is necessary and sufficient for X being locally hypersurface orthogonal.

Our main application in GR will be the introduction of adapted coordinates if there are several Killing fields on (M, g), in particular, if there exist two commuting Killing fields. Let us, however, first introduce adapted coordinates for more general situations.

Let again, for a k-dimensional distribution E, $E|U = \text{span}\{X_1, \dots, X_k\}$ and $\omega^{k+1}, \dots, \omega^n$ the generating 1-forms of the ideal $\mathcal{I}(E)$. We use the following notation: Indices running from $1, 2, \dots, n$ are denoted by Greek letters; for the first k numbers we use small Latin letters, while for indices with values $k+1, \dots, n$ capital letters will be used.

Assume that E is involutive and use adapted coordinates $\{x^\mu\}$ as in the first version of the Frobenius theorem. The (local) integral manifolds are given by $\{x^A = \text{const}\}$. For the basis of vector fields X_a belonging to E we have

$$X_a(x^A) = 0 \qquad (X_a = X_a{}^b \frac{\partial}{\partial x^b}\,, \;\; X_a{}^A = 0). \tag{C.6}$$

Since $\langle \omega^A, X_a \rangle = 0$ we obtain, setting $\omega^A = \omega^A_{\ \mu}\, dx^\mu$,

$$0 = X_a^{\ b} \omega^A_{\ \mu} \langle dx^\mu, \partial_b \rangle = X_a^{\ b} \omega^A_{\ b} = 0$$

for all $a = 1, \ldots, k$. Hence $\omega^A_{\ b} = 0$ and thus

$$\omega^A = \omega^A_B \, dx^B \,. \tag{C.7}$$

Clearly, the matrices $(X_a^{\ b})$ and (ω^A_B) are non-singular.

Let us specialize this to commuting fields X_a. We show that one can choose adapted coordinates such that

$$X_a = \frac{\partial}{\partial x^a} \qquad (a = 1, \ldots, k) \,. \tag{C.8}$$

To prove this, consider the flows ϕ^a_t of X_a. We know that these commute (Theorem 11.10). Now choose the coordinates x^a for a point p of an integral manifold $\{x^A = \text{const}\}$ according to

$$p = \left(\phi^1_{x^1} \circ \ldots \circ \phi^k_{x^k} \right) (p_0) \,,$$

where p_0 is a fixed "origin". With this choice the representation (C.8) obviously holds.

Returning to the general case we assume now that M has a distinguished pseudo-Riemannian metric g. Keeping the previous notation we can then also introduce the 1-forms $\omega^a = X_a^\flat$. These generate an ideal, denoted by $\mathcal{I}(E^\perp)$, which is the annihilator of a distribution $E^\perp$. Let $E^\perp$ be spanned by the vector fields X_A, $E^\perp = \text{span}\{X_A\}$. Then we have

$$\langle \omega^a, X_A \rangle = 0 \Longleftrightarrow g(X_a, X_A) = 0 \, : \, E \perp E^\perp \,. \tag{C.9}$$

In what follows we assume that the restrictions $g|E$ and $g|E^\perp$ are non-singular. Then $E \cap E^\perp = \{0\}$, so

$$T_p M = E_p \oplus E_p^\perp \,. \tag{C.10}$$

Dually we have

$$T_p^* M = \text{span}\{\omega^a\} \oplus^\perp \text{span}\{\omega^A\} \,, \tag{C.11}$$

where $\omega^A = X_A^\flat$. We have thus constructed a basis of vector fields $\{X_\mu\}$ and the basis $\{\omega^\mu = X_\mu^\flat\}$ of 1-forms such that $\{\omega^A\}$ generates the ideal $\mathcal{I}(E)$, and $\{\omega^a\}$ the ideal $\mathcal{I}(E^\perp)$.

Let us now assume that E and $E^\perp$ are involutive. We use coordinates $\{u^\mu\}$ which are adapted to E and also coordinates $\{v^\mu\}$ that are adapted to $E^\perp$. Then we have according to (C.6) and (C.7)

$$X_a(u^A) = 0 \,, \quad \omega^A = \omega^A_B \, du^B \,, \tag{C.12a}$$

$$X_A(v^a) = 0 \,, \quad \omega^a = \omega^a_b \, dv^b \,. \tag{C.12b}$$

Now we define coordinates $\{x^\mu\}$ by

$$x^a = v^a\,, \qquad x^A = u^A\,. \tag{C.13}$$

The x^μ indeed form a coordinate system. For this we have to show that the dx^μ are linearly independent. Suppose there would be a linear dependence

$$f_a\, dx^a = g_A\, dx^A\,,$$

i.e., $f_a\, dv^a = g_A\, du^A$, then applying this to X_c leads to

$$f_a \langle dv^a, X_c\rangle = g_A \langle du^A, X_c\rangle = g_A\, X_c(u^A) = 0\,,$$

i.e., $\langle dv^a, f_a\, X_c\rangle = 0$ or, by (C.12b), $\langle \omega^a, f_a\, X_c\rangle = 0$. Thus $f_a\, X_c \in E \cap E^\perp$, whence $f_a\, X_c = 0$ thus $f_a = 0$, implying $g_A = 0$.

The coordinates $\{x^\mu\}$ have, as a result of (C.12a) and (C.12b), the following properties:

(i) $X_a(x^A) = 0$,
(ii) $g_{aA} = 0$.

The last property is obtained as follows: Let $\omega^a = \omega^a_{\ \mu}\, dx^\mu$ and use (C.12b) to conclude that $\omega^a_{\ A} = 0$. But by (i)

$$\omega^a_{\ A} = g_{A\mu} X^\mu_{\ a} = g_{Ab} X^b_{\ a}$$

which implies (ii).

Summarizing, we have the result:

Proposition C.4. *Let $E = span\{X_1, \dots, X_k\}$ be a k-dimensional distribution which is involutive. Assume in addition that the ideal generated by $\{\omega^a = X^\flat_a\}$ is differential. If this is the annihilator of $E^\perp$, then under the assumption that $E \cap E^\perp = \{0\}$ we can introduce coordinates $\{x^\mu\}$ with ($\mu = (a, A)$)*

$$g_{aA} = 0\,, \qquad X_a(x^A) = 0\,. \tag{C.14}$$

Assume next, that the generating vector fields $X_1, \dots, X_k$ in this Proposition are *Killing fields*, $L_{X_a} g = 0$. In adapted coordinates this becomes

$$X^b_{\ a}\, g_{\mu\nu,b} + g_{b\nu}\, X^b_{\ a,\mu} + g_{\mu b}\, X^b_{\ a,\nu} = 0\,. \tag{C.15}$$

If we apply this for $\mu = A$, $\nu = B$, and use (C.14), we obtain $g_{AB,b} = 0$. Thus g_{AB} *depends only on* x^C. Choosing, on the other hand, $\mu = A$, $\nu = c$ in (C.15) we obtain

$$X^b_{\ a}\, g_{Ac,b} + g_{bc}\, X^b_{\ a,A} + g_{Ab}\, X^b_{\ a,c} = 0\,. \tag{C.16}$$

Since $g_{aA} = 0$ the first and the last term vanish, and we conclude that $X^b_{\ a,A} = 0$. Thus $X^b_{\ a}$ *depends only on* x^c. Summarizing, we have the useful

Proposition C.5. *If in addition to the assumptions of the Proposition C.4 it is assumed that X_a are Killing fields of the pseudo-Riemannian manifold (M, g), then there are local coordinates $\{x^\mu\}$ such that (in the notation introduced above)*

$$g = g_{ab}(x^\mu)\, dx^a dx^b + g_{AB}(x^C)\, dx^A dx^B \,, \tag{C.17a}$$

$$X_a = X^b_{\ a}\, \partial_b \,, \tag{C.17b}$$

where $X^b_{\ a}$ are only functions of x^c.

Note, as a result of this, that the metric $g_{ab}\, dx^a dx^b$ on the integral manifolds $\{x^A = \text{const}\}$ has k Killing fields X_a, whence this submanifolds are homogeneous spaces.

Finally, we consider the important special case when the Killing fields X_a commute. We can then choose the adapted coordinates such that (C.8) holds. If we use this in (C.15) for $\mu = c$, $\nu = d$, we get

$$X^b_{\ a} g_{cd,b} + g_{bd} X^b_{\ a,c} + 0 = 0 \,.$$

Since the second term vanishes we see that $g_{cd,b} = 0$. Hence g_{cd} depends only on x^A. In this special case we can, therefore, find coordinates x^μ such that

$$g = g_{ab}(x^C)\, dx^a dx^b + g_{AB}(x^C)\, dx^A dx^B \,, \tag{C.18a}$$

$$X_a = \frac{\partial}{\partial x^a} \,. \tag{C.18b}$$

This is used as the starting point of the derivation of the Kerr solution in Chap. 7.

Proof of Frobenius' Theorem (in the first version)

Proof (of Theorem C.1). Since this theorem is a local statement, we can work in $\mathbb{R}^n$ and choose $p = \{0\}$. Moreover, we can assume that $E_0 \subset T_0\mathbb{R}^n$ is spanned by the basis vectors

$$\left.\frac{\partial}{\partial t^1}\right|_0, \ldots, \left.\frac{\partial}{\partial t^k}\right|_0, \tag{C.19}$$

where $(t^1, \ldots, t^n)$ are the standard coordinates of $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be the projection onto the first k factors. The corresponding tangential map $d\pi$ ($= T\pi$) maps E_0 isomorphically on $T_0\mathbb{R}^k$. By continuity, $d\pi$ is bijective on E_q for q near 0. So near 0, we can choose unique $X_1(q), \ldots, X_k(q) \in E_q$ so that

$$d\pi \cdot X_i(q) = \left.\frac{\partial}{\partial t^i}\right|_{\pi(q)} \qquad (i = 1, \ldots, k) \,. \tag{C.20}$$

Then (on a neighborhood U of $0 \in \mathbb{R}^n$) the vector fields X_i and $\frac{\partial}{\partial t^i}$ (on $\mathbb{R}^k$) are π-related. This then also holds for their Lie brackets:

$$d\pi \cdot [X_i, X_j]_q = \left[\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j}\right]_{\pi(q)} = 0\,. \tag{C.21}$$

Since E was assumed to be involutive, we have $[X_i, X_j]_q \in E_q$, thus with (C.21) $[X_i, X_j] = 0$ ($d\pi$ is on E_q injective). According to the proof of (C.8) we can find coordinates $\{x^i\}$ such that

$$X_a = \frac{\partial}{\partial x^a}\,, \quad a = 1, \dots, k\,. \tag{C.22}$$

The sets $\{q \in U \mid x^{k+1}(q) = a^{k+1}, \dots, x^n(q) = a^n\}$ are then obviously integral manifolds of E, since their tangent spaces are spanned by (C.22).

Conversely, if N is a connected integral manifold restricted to U, with the inclusion $\iota : N \hookrightarrow U$, then we have for $x^\alpha \circ \iota$ with $\alpha = k+1, \dots, n$ and $X_q \in T_qN$:

$$d(x^\alpha \circ \iota)(X_q) = X_q(x^\alpha \circ \iota) = d\iota \cdot X_q(x^\alpha) = 0\,,$$

since $d\iota \cdot X_q \in E_q$, which is spanned by the vectors (C.22) taken in q. Thus $d(x^\alpha \circ \iota) = 0$, implying that $x^\alpha \circ \iota$ is constant on the connected manifold N. □

* * *

Exercise C.1. Consider a 1-form ω on an open set $U \subset \mathbb{R}^n$. Use the Frobenius Theorem to show that the $(n-1)$-distribution $E = \{v \in TU \mid \omega(v) = 0\}$ is integrable if and only if ω is of the form

$$\omega = g\; df\,. \tag{C.23}$$

The smooth function g is called an **integrating factor**.

Since by the Frobenius Theorem integrability is equivalent to

$$\omega \wedge d\omega = 0\,, \tag{C.24}$$

we see that (C.24) is necessary and sufficient for the existence of an integrating factor. Note that (C.24) is always satisfied for $n = 2$, so that ω has always an integrating factor.

Solution. If (C.23) is satisfied, then $\{f = \text{const}\}$ are integral manifolds of E. Conversely, if E is integrable then according to the first version of the Frobenius theorem, the integral manifolds of E are given by $\{f = \text{const}\}$ for a smooth function, so that df annihilates E. Thus the ideal $\mathcal{I}(E)$ generated by ω is also generated by df, whence $\omega = g\; df$ for some smooth function g on U.

D Collection of Important Formulas

The following list summarizes some of the important formulas which have been obtained in the foregoing chapters and will constantly be used throughout this book.

Vector Fields, Lie Brackets

The set of smooth vector fields $\mathcal{X}(M)$ on a manifold M with the commutator product $[X, Y]$ form a $\mathbb{R}$-Lie algebra and a module over the associative algebra $\mathcal{F}(M)$ of C^∞-functions. In local coordinates we have

$$X = \xi^i \frac{\partial}{\partial x^i}\,, \qquad Y = \eta^i \frac{\partial}{\partial x^i}\,,$$
$$[X, Y] = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}\,.$$

Under a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ the Lie bracket transforms naturally:

$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y]\,.$$

Differential Forms

The differential forms $\bigwedge(M)$ on M form a real associative algebra with the exterior product $\wedge$ as multiplication; $\bigwedge^p(M)$ denotes the $\mathcal{F}(M)$-module of p-forms. The exterior product satisfies,

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha\,, \qquad \alpha \in \bigwedge^k(M)\,, \quad \beta \in \bigwedge^l(M)\,.$$

For a differentiable map φ,

$$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta\,.$$

Exterior Differential

The exterior differential d is an antiderivation of degree $+1$; in particular

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta\,, \qquad \text{for } \alpha \in \bigwedge^k(M)\,,\ \beta \in \bigwedge(M)\,.$$

Furthermore

$$d \circ d = 0\,.$$

If $\alpha \in \bigwedge^p(M)$ and $X_1, \ldots, X_{p+1} \in \mathcal{X}(M)$, then

$$d\alpha(X_1, \ldots, X_{p+1}) = \sum_{1 \le i \le p+1} (-1)^{i+1} X_i\, \alpha\big(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}\big) + \sum_{i<j} (-1)^{i+j}\, \alpha\big([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}\big)\,.$$

For a map φ,

$$\varphi^* d\alpha = d\varphi^* \alpha$$

Poincaré Lemma

If $d\alpha = 0$, then α is locally exact; that is, there is a neighborhood U of every $x \in M$ on which $\alpha = d\beta$.

Interior Product

The interior product i_X is an antiderivation of degree -1; in particular

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^p \alpha \wedge i_X\beta\,, \qquad \text{for } \alpha \in \bigwedge^p(M)\,,\ \beta \in \bigwedge(M)\,.$$

We obviously have

$$i_{fX}\alpha = f\, i_X\alpha = i_X f\, \alpha\,, \qquad \text{for } f \in \mathcal{F}(M)\,,$$

and

$$i_X \circ i_X = 0\,.$$

For a diffeomorphism φ (see Exercise D.1),

$$\varphi^* i_X \alpha = i_{\varphi^* X} \varphi^* \alpha\,.$$

Lie Derivative

The Lie derivative L_X with respect to the vector field X is a derivation of degree 0 of the tensor algebra $\mathcal{T}(M)$; it is $\mathbb{R}$-linear in X. For a diffeomorphism φ and $T \in \mathcal{T}(M)$

$$\varphi^* L_X T = L_{\phi^* X} \varphi^* T\,.$$

For a covariant tensor field $T \in \mathcal{T}^0_q(M)$ and $X_1, \ldots, X_q \in \mathcal{X}(M)$

$$(L_X\,T\,)(X_1,\dots,X_q) = X\big(T(X_1,\dots,X_q)\big) - \sum_{k=1}^{q} T\big(X_1,\dots,[X,X_k],\dots,X_q\big)\,.$$

In local coordinates the components of $L_X T$ for a tensor field $T \in \mathcal{T}^r_s(M)$ are given by

$$\begin{aligned}(L_X T)^{i_1\dots i_r}_{j_1\dots j_s} = X^i\, T^{i_1\dots i_r}_{j_1\dots j_s,i} & \\ -T^{k i_2\dots i_r}_{j_1\dots j_s}\cdot X^{i_1}_{,k} & - \text{all upper indices} \\ +T^{i_1\dots i_r}_{k j_2\dots j_s}\cdot X^{k}_{,j_1} & + \text{all lower indices}\,.\end{aligned}$$

Relations Between L_X, i_X and d

The following identities hold for differential forms

$$\begin{aligned}L_X\alpha &= d\, i_X\alpha + i_X d\alpha\,, \qquad (\textbf{Cartan's formula})\\ L_{fX}\alpha &= f\, L_X\alpha + df\wedge i_X\alpha\,,\\ L_{[X,Y]} &= [L_X, L_Y] = L_X\circ L_Y - L_Y\circ L_X\,,\\ i_{[X,Y]} &= [L_X, i_Y] = L_X\circ i_Y - i_Y\circ L_X\,,\\ L_X\circ d &= d\circ L_X\,,\\ L_X\circ i_X &= i_X\circ L_X\,.\end{aligned}$$

Volume Form

For an oriented n-dimensional pseudo-Riemannian manifold (M,g), with $\theta^1,\dots,\theta^n$ an oriented basis of one-forms, the canonical volume form can be represented as

$$\eta = \sqrt{|g|}\;\theta^1\wedge\dots\wedge\theta^n\,,\quad g = g_{ik}\,\theta^i\otimes\theta^k\,,\quad |g| = |\det(g_{ik})|\,.$$

Hodge-Star Operation

The Hodge-star operator for (M,g) is a linear isomorphism $* : \bigwedge^k(M) \longrightarrow \bigwedge^{n-k}(M)$ and satisfies for $\alpha,\beta\in\bigwedge^k(M)$:

$$\begin{aligned}\alpha\wedge *\beta = \beta\wedge *\alpha &= \langle\alpha,\beta\rangle\eta\,,\\ *(*\alpha) &= (-1)^{k(n-k)} sgn(g)\alpha\,,\\ \langle *\alpha, *\beta\rangle &= sgn(g)\langle\alpha,\beta\rangle\,.\end{aligned}$$

In local coordinates

$$*(\theta^{i_1}\wedge\dots\wedge\theta^{i_p}) = \frac{\sqrt{|g|}}{(n-p)!}\,\varepsilon_{j_1\dots j_n}\, g^{j_1 i_1}\dots g^{j_p i_p}\,\theta^{j_{p+1}}\wedge\dots\wedge\theta^{j_n}\,.$$

Codifferential

The codifferential for (M, g) is a linear isomorphism $\delta : \bigwedge^k(M) \longrightarrow \bigwedge^{k-1}(M)$ defined by

$$\delta := sgn(g)(-1)^{np+n} * d* \,,$$

and satisfies

$$\delta \circ \delta = 0\,.$$

In local coordinates, for $\alpha \in \bigwedge^k(M)$,

$$(\delta\alpha)^{i_1 \ldots i_{p-1}} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \alpha^{k i_1 \ldots i_{p-1}}\right)_{,k} \,.$$

Covariant Derivative

The covariant derivative ∇_X in the X-direction of an affine connection on M is a derivation of the tensor algebra $\mathcal{T}(M)$, which commutes with all contractions. For $S \in \mathcal{T}^p_q(M)$,

$$\begin{aligned}(\nabla_X S)(\,Y_1\,, \ldots, Y_q, \omega_1, \ldots, \omega_p)\\ &= X\big(S(Y_1, \ldots, Y_q, \omega_1, \ldots, \omega_p)\big) - S(\nabla_X Y_1, Y_2, \ldots, \omega_p) - \ldots - \\ &\quad - S(Y_1, \ldots, \nabla_X \omega_p)\,.\end{aligned}$$

In local coordinates, the components of $\nabla_X S$ are given by

$$\begin{aligned} S^{i_1 \ldots i_p}_{j_1 \ldots j_q ; k} = S^{i_1 \ldots i_p}_{j_1 \ldots j_q , k} &+ \Gamma^{i_1}{}_{kl}\, S^{l i_2 \ldots i_p}_{j_1 \ldots j_q} + \ldots \\ &- \Gamma^{l}{}_{k j_1}\, S^{i_1 \ldots i_p}_{l j_2 \ldots j_q} - \ldots \,. \end{aligned}$$

Connection Forms

Let $(e_1, \ldots, e_n)$ be a moving frame with the dual basis $(\theta^1, \ldots, \theta^n)$ of one-forms. The connection forms $\omega^i{}_j$ are given by

$$\nabla_X e_j = \omega^i{}_j(X) e_i \,.$$

or

$$\nabla_X \theta^i = -\omega^i{}_j(X)\theta^j \,.$$

Curvature Forms

The expansion coefficients of the curvature forms $\Omega^i{}_j$ are the components of the Riemann tensor relative to a moving frame:

$$\Omega^i{}_j = \frac{1}{2}\, R^i{}_{jkl}\, \theta^k \wedge \theta^l \,, \qquad R^i{}_{jkl} = -R^i{}_{jlk} \,.$$

Cartan's Structure Equations

The torsion forms Θ^i and curvature forms $\Omega^i{}_j$ satisfy Cartan's structure equations

$$\Theta^i = d\theta^i + \omega^i{}_j \wedge \theta^j \,,$$

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \,.$$

Riemannian Connection

For the Riemannian connection with metric $g = g_{ik}\, \theta^i \otimes \theta^k$

$$\omega_{ik} + \omega_{ki} = dg_{ik} \,, \qquad \omega_{ik} = g_{ij}\, \omega^j{}_k \,.$$

Coordinate Expressions

Riemann tensor

$$R^i{}_{jkl} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls} \,,$$

Ricci tensor

$$R_{jl} = R^i{}_{jil} = \Gamma^i{}_{lj,i} - \Gamma^i{}_{ij,l} + \Gamma^s{}_{lj}\Gamma^i{}_{is} - \Gamma^s{}_{ij}\Gamma^i{}_{ls} \,,$$

Scalar curvature

$$R = g^{ik} R_{ik} \,,$$

Einstein tensor

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik}\, R \,,$$

Christoffel symbols

$$\Gamma^l{}_{ij} = \frac{1}{2} g^{lk} \big(g_{ki,j} + g_{kj,i} - g_{ij,k} \big) \,.$$

Absolute Exterior Differential

The absolute exterior differential of a tensor valued p-form ϕ of type (r, s) has the components

$$\begin{aligned} (D\phi)^{i_1 \dots i_r}_{j_1 \dots j_s} = d\, \phi^{i_1 \dots i_r}_{j_1 \dots j_s} &+ \omega^{i_1}{}_l \wedge \phi^{l i_2 \dots i_r}_{j_1 \dots j_s} + \dots \text{for all upper indices} \\ &- \omega^l{}_{j_1} \wedge \phi^{i_1 \dots i_r}_{l j_2 \dots j_s} - \dots \text{for all lower indices} \,. \end{aligned}$$

The second derivative is given by

$$(D^2\phi)^{i_1 \dots i_r}_{j_1 \dots j_s} = \Omega^{i_1}{}_l \wedge \phi^{l i_2 \dots i_r}_{j_1 \dots j_s} + \dots - \Omega^l{}_{j_1} \wedge \phi^{i_1 \dots i_r}_{l j_2 \dots j_s} - \dots \,.$$

Bianchi Identities

In terms of the coordinate components of the Riemann tensor the Bianchi identities read for a symmetric affine connection:

$$\sum_{(jkl)} R^i{}_{jkl} = 0 \qquad (\textbf{1st Bianchi identity})\,,$$
$$\sum_{(klm)} R^i{}_{jkl;m} = 0 \qquad (\textbf{2nd Bianchi identity})\,.$$

In terms of the torsion and curvature forms they are

$$D\Theta^i = \Omega^i{}_j \wedge \theta^j$$
$$D\Omega^i{}_j = 0\,.$$

* * *

Exercise D.1. Let $\varphi : M \longrightarrow N$ be a differentiable mapping between the manifolds M and N, $\alpha \in \bigwedge(N)$, $X \in \mathcal{X}(N)$, $Y \in \mathcal{Y}(M)$ such that Y and X are φ-related. Show that

$$i_Y \varphi^* \alpha = \varphi^* i_X \alpha\,.$$

In particular, if φ is a diffeomorphism, then

$$i_{\varphi^* X}\, \varphi^* \alpha = \alpha^* i_X \alpha\,.$$

Remark. This naturalness property of the interior product follows directly from the definitions.

References

Textbooks on General Relativity

Classical Texts

1. W. Pauli, *Theory of Relativity*. Pergamon Press 1958; Relativitätstheorie, In: *Enzyklopädie der Mathematischen Wissenschaften*, Band 5, Teil 2, Teubner, Leipzig 1921; neu herausgegeben und kommentiert von D. Giulini, Springer-Verlag 2000
2. H. Weyl, *Space-Time-Matter*. Dover, New York 1952 (translated from the 3rd German edition by H.L. Brose); *Raum-Zeit-Materie*, Vorlesungen über allgemeine Relativitätstheorie, Springer, Berlin 1918; neu herausgegeben und ergänzt von J. Ehlers, Springer-Verlag 1993 (8th edition)
3. A.S. Eddington, *The Mathematical Theory of Relativity*. Chelsea Publishing Company 1975
4. V. Fock, *The Theory of Space Time and Gravitation*. Pergamon Press, Oxford 1959
5. J.L. Synge, *Relativity, The General Theory*. North-Holland 1971

Selection of (Graduate) Textbooks

6. C.M. Will, *Theory and Experiment in Gravitational Physics*. Cambridge University Press, Cambridge 1993 (2nd edition)
7. R.M. Wald, *General Relativity*. The University of Chicago Press, Chicago 1984
8. G. Ellis and S. Hawking, *The Large Scale Structure of Space-Time*. Cambridge Univeristy Press 1973
9. J.L. Anderson, *Principles of Relativity Physics*. Academic Press 1967
10. L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*. Addison Wesley 1987 (4th edition)
11. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*. McGraw-Hill 1975 (2nd edition)
12. B. Schutz, *A First Course in General Relativity*. Cambridge University Press 1985
13. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*. Freeman 1973
14. N. Straumann, *General Relativity and Relativistic Astrophysics*, Texts and Monographs in Physics, Springer-Verlag Berlin 1984
15. F. de Felice and C. Clarke, *Relativity on Curved Manifolds*. Cambridge University Press 1990

General Physics and Astrophysics Books

16. L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*. Pergamon Press 1987 (2nd edition)
17. G. Wentzel, *Quantum Theory of Fields*. Interscience Publishers 1949
18. J.D. Jackson, *Classical Electrodynamics*. John Wiley and Sons 1999 (3rd edition)
19. P. Schneider, J. Ehlers and E.E. Falco, *Gravitational Lenses*. Springer Heidelberg 1992
20. J. Binney and S. Tremaine, *Galactic Dynamics*. Princeton University Press, Princeton 1987
21. S.L. Shapiro and S.A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars*. John Wiley and Sons, Inc. 1983
22. N.K. Glendenning, *Compact Stars*. Springer 1997
23. F. Weber, *Pulsars as Astrophysical Laboratories for Nuclear and Particle Physics*, Institute of Physics Publishing. Bristol and Philadelphia 1999
24. S. Chandrasekhar, *Structure and Evolution of the Stars*. University of Chicago Press 1939
25. P.A. Charles and F.D. Seward, *Exploring the X-ray Universe*. Cambridge University Press 1995
26. P. Mészáros, *High-Energy Radiation from Magnetized Neutron Stars*. University of Chicago Press 1992
27. J. Frank, A. King and D. Raine, *Accretion Power in Astrophysics*. Cambridge University Press 2002 (3rd edition)
28. V.I. Arnold, *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics, Vol. 60, Springer-Verlag 1989
29. N. Straumann, *Relativistische Quantentheorie*. Springer-Verlag 2004

Mathematical Tools

Modern Treatments of Differential Geometry for Physicists

30. W. Thirring, *A Course in Mathematical Physics I and II: Classical Dynamical Systems and Classical Field Theory*. Springer-Verlag 1992 (2nd edition)
31. Y. Choquet-Bruhat, C. De Witt-Morette and M. Dillard-Bleick, *Analysis, Manifolds and Physics*. North-Holland 1982 (rev. edition)
32. G. von Westenholz, *Differential Forms in Mathematical Physics*. North-Holland 1978
33. R. Abraham and J.E. Marsden, *Foundations of Mechanics*. Benjamin 1978 (2nd edition)
34. T. Frankel, *The Geometry of Physics*. Cambridge University Press 1997

Selection of Mathematical Books

35. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I and II*. Interscience Publishers 1963/69
36. Y. Matsushima, *Differentiable Manifolds*. Marcel Dekker, New York 1972

37. R. Sulanke and P. Wintgen, *Differentialgeometrie und Faserbündel*. Birkhäuser 1972
38. R. L. Bishop and R. J. Goldberg, *Tensor Analysis on Manifolds*. McMillan, New York 1968
39. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, Inc. 1983
40. S. Lang, *Differential and Riemannian Manifolds*. Graduate Texts in Mathematics 160, Springer 1995
41. J. Jost, *Riemannian Geometry and Geometric Analysis*. Springer 2002 (3rd edition)
42. B. O'Neill, *Elementary Differential Geometry*. Academic Press 1997
43. M. Spivak, *A Comprehensive Introduction to Differential Geometry I-V*, Publish or Perish 1979
44. P.K. Raschewski, *Riemannsche Geometrie und Tensoranalysis*. Deutscher Verlag der Wissenschaften 1959
45. L.H. Loomis and S. Sternberg, *Advanced Calculus*. Addison-Wesley 1968
46. L.C. Evans, *Partial Differential Equations*. Graduate Studies in Mathematics, Vol. 19, American Mathematical Society 1998
47. F. John, *Partial Differential Equations*. Springer-Verlag 1982 (4th edition)
48. M.E. Taylor, *Partial Differential Equations*. Springer Verlag 1996

Historical Sources

49. G. Nordström, *Phys. Zeitschrift* **13**, 1126 (1912); *Ann. Phys. (Leipzig)* **40**, 856 (1913); *Ann. Phys. (Leipzig)* **42**, 533 (1913)
50. A. Einstein and A.D. Fokker, *Ann. Phys. (Leipzig)* **44**, 321 (1914)
51. L. O'Raifeartaigh and N. Straumann, *Rev. Mod. Phys.* **72**, 1 (2000)
52. E. Cartan, *Annales Scientifiques de l'École Normale Supérieure* **40**, 325 (1923), and **41**, 1 (1924)
53. K. Friedrichs, *Mathematische Annalen* **98**, 566 (1928)
54. H. Hertz, *Über die Constitution der Materie*. Springer Verlag 1999
55. A. Pais, *Subtle is the Lord, The Science and the Life of Albert Einstein*. Oxford University Press 1982
56. J. Stachel, *Einstein from "B" to "Z"*. Einstein Studies, Vol. 9, Birkhäuser 2002
57. A. Einstein, *Collected Papers*. Vol. 4, Doc. 10
58. A. Einstein, *Collected Papers*. Vol. 6, Doc. 25, Doc. 24, Doc. 32
59. A. Einstein, *Collected Papers*. Vol. 7, Doc. 9, Doc. 1
60. A. Einstein, *Collected Papers*. Vol. 8
61. L. Corry, J. Renn and J. Stachel, *Science* **278**, 1270-1273 (1997)
62. D. Hilbert, *Nach. Ges. Wiss. Göttingen*, 395 (1916)
63. A. Einstein, *Sitzungsber. Preuss. Akad. Wiss.* Phys.-math. Klasse VI, 142 (1917)
64. J. Norton, *General Covariance and the Foundations of General Relativity: Eight Decades of Dispute*. Reports on Progress in Physics **56**, 791-858 (1993)
65. M. Friedmann, *Foundations of Space-Time Theories, Relativistic Physics and Philosophy of Science*. Princeton University Press 1983
66. A. Einstein, *The Meaning of Relativity*. Princeton 1956 (5th edition)
67. W. Israel, Dark Stars: The Evolution of an Idea. In: S. Hawking and W. Israel (eds.), *300 Years of Gravitation*. Cambridge University Press 1987

Recent Books on Cosmology

Although cosmology is not treated in this book, we give references to some useful recent textbooks.

68. P.J.E. Peebles, *Principles of Physical Cosmology*. Princeton University Press 1993
69. J.A. Peacock, *Cosmological Physics*. Cambridge University Press 1999
70. A.R. Liddle and D.H. Lyth, *Cosmological Inflation and Large Scale Structure*. Cambridge University Press 2000
71. S. Dodelson, *Modern Cosmology*. Academic Press 2003
72. G. Börner, *The Early Universe*. Springer-Verlag 2003 (4th edition)

Research Articles, Reviews and Specialized Texts

Chapter 1

73. J. Ehlers and W. Rindler, *Gen. Rel. Grav.* **29**, 519 (1997)
74. C.M. Will, *Living Reviews in Relativity* **4** (2001); gr-qc/9811036
75. N. Straumann, *Fundamental Physics in Space & Related Topics*. ESA-CERN Workshop 5-7 April 2000, European Space Agency, SP-469, 55 (2001); astro-ph/0006423
76. W. Pauli and M. Fierz, *Helv. Phys. Acta* **12**, 287 (1939); *Proc. Roy. Soc. (London) Ser. A* **173**, 211 (1939)
77. S. Deser, *Gen. Rel. Grav.* **1**, 9 (1970)

Chapter 2

78. D. Lovelock, *J. Math. Phys.* **13**, 874 (1972)
79. N. Straumann, On the Cosmological Constant Problems and the Astronomical Evidence for a Homogenous Energy Density with Negative Pressure. In: B. Duplantier and V. Rivasseau (eds.), *Vacuum Energy-Renormalization*, Séminaire Poincaré (2002). Birkhäuser Verlag 2003; astro-ph/0203330; and The History of the Cosmological Constant. In: *On the Nature of Dark Energy*, Proceedings of the 18th IAP Astrophysics Colloquium, Frontier Group 2002; gr-qc/0208027
80. D.E. Rowe, *Phys. Perspect.* **3**, 379-424 (2001)
81. A.E. Fischer and J.E. Marsden, The Initial Value Problem and the Dynamical Formulation of General Relativity. In: S.W. Hawking and W. Israel (eds.), *General Relativity, an Einstein Centenary Survey*. Cambridge University Press 1979
82. Y. Choquet-Bruhat and J.W. York, The Cauchy Problem. In: A. Held (ed.), *General Relativity and Gravitation*, Vol. 1, Plenum Press 1980
83. S. Klainerman and F. Nicolò, *The Evolution Problem in General Relativity*. Birkhäuser 2002
84. R. Schoen and S.T. Yau, *Commun. Math. Phys.* **65**, 45 (1976); *Phys. Rev. Lett.* **43**, 1457 (1979); *Commun. Math. Phys.* **79**, 231 (1981); *Commun. Math. Phys.* **79**, 47 (1981)
85. L. Rosenfeld, *Mem. Roy. Acad. Belg. Cl. Sci.* **18**, 6 (1940)

86. D. Christodoulou and S. Klainermann, *The Global, Nonlinear Stability of the Minkowski Space*. Princeton University Press, Princeton 1993
87. M. Shibata and T. Nakamura, *Phys. Rev.* D **52**, 5428 (1995)
88. T.W. Baumgarte and S.L. Shapiro, *Phys. Rev.* D **59**, 024007 (1999)
89. M. Alcubierre et al., *Phys. Rev.* D **62**, 044034 (2000)
90. S. Frittelli and O. Reula, *Commun. Math. Phys.* **166**, 221 (1994)
91. C. Bona, J. Massó, E. Seidel and J. Stela, *Phys. Rev. Lett.* **75**, 600 (1995)
92. A. Abrahams, A. Anderson, Y. Choquet-Bruhat and J.W. York Jr., *Phys. Rev. Lett.* **75**, 3377 (1996)
93. M.H.P.M. van Putten and D.M. Eardley, *Phys. Rev.* D **53**, 3056 (1996)
94. H. Friedrich, *Class. Quantum Grav.* **13**, 1451 (1996)
95. A. Anderson, Y. Choquet-Bruhat, and J.W. York,Jr., *Top. Methods Nonlinear Anal.* **10**, 353 (1997)
96. http://cargese.univ-course.fr/Pages/Ecoles2002.html

Chapter 3

97. R.A. Hulse, *Rev. Mod. Phys.* **66**, 699 (1994)
98. J.H. Taylor, *Rev. Mod. Phys.* **66**, 711 (1994)
99. P. Coles, Einstein, Eddington and the 1919 Eclipse. In: V.J. Martinez, V. Trimble and M.J. Pons-Borderia (eds.), Proceedings on International School on *The Historical Development of Modern Cosmology*, Valencia 2000, ASP Conference Series (to appear); astro-ph/0102462
100. M. Froeschlé, F. Miguard and F. Arenon, in Proceedings of the Hipparcos Venice 1997 Symposium, to appear
101. I. Shapiro et al., *Phys. Rev. Lett.* **26**, 1137 (1971)
102. R.D. Reasonberg et al., *Astrophys. J.* **234**, L 219 (1979)
103. J.G. Williams et al., *Phys. Rev.* D **53**, 6730 (1996)
104. J.M. Weisberg and J.H. Taylor, *Astrophys. J.* **576**, 942 (2002)
105. S. Chandrasekhar, *The Mathematical Theory of Black Holes*. Clarendon Press, Oxford University Press 1983
106. J.B. Hartle, *Phys. Reports* **46**, 201 (1978)
107. C.V. Vishveshwara, *Phys. Rev.* D **1**, 2870 (1970)
108. F.J. Zerilli, *Phys. Rev. Lett.* **24**, 737 (1970)
109. R. Schödel et al., *Nature* **419**, 694 (2002); astro-ph/0210426
110. B. Bertotti, L. Iess and P. Tortora, *Nature* **425**, 374-376 (2003)

Chapter 4

111. S. Rowan and J. Hough, *Gravitational Wave Detection by Interferometry*; http://www.livingreviews.org/Articles/Volume3/2000-3hough/.
112. *System and Technology Study Report* ESA/SCI (2000) 11, by the LISA Science Team
113. B.F. Schutz, *Gravitational Wave Astronomy*, Class. Quant. Grav. **16**, A131-156 (1999).
114. C. Cutler and K.S. Thorne, gr-qc/0204090
115. N. Straumann, Lectures on Gravitational Lensing. In: R. Durrer and N. Straumann (eds.), *New Methods for the Determination of Cosmological Parameters*. Troisième Cycle de la Physique en Suisse Romande (1999)

116. M.J. Irvin et al., *Astrophys. J.* **98**, 1989 (1989)
117. A. Milsztajn, The Galactic Halo from Microlensing. In: *Matter in the Universe*, Space Science Series of ISSI, Vol. 14 (Reprinted in from *Space Science Reviews*, Vol. 100, Nos. 1-4, 2002)
118. N. Kaiser and G. Squires, *Astrophys. J.* **404**, 441 (1993)
119. M. Bartelmann and P. Schneider, *Weak Gravitational Lensing*, Physics Reports **340**, 291-472 (2001)
120. N. Kaiser, *Astrophys. J.* **439** (1995)
121. R.A. Isaacson, *Phys. Rev.* **166**, 1263 (1968); *Phys. Rev.* **166**, 1272 (1968)
122. S. Seitz and P. Schneider, *Astron. Astrophys.* **305**, 383 (1996)

Chapter 5

123. T. Damour, The Problem of Motion in Newtonian and Einsteinian Gravity. In: S.W. Hawking and W. Israel (eds.), *300 Years of Gravitation*. Cambridge University Press, London 1987
124. R. Penrose, The Light Cone at Infinity. In: L. Infeld (ed.), *Relativistic Theories of Gravitation*. Pergamon Press, Oxford 1964
125. H. Friedrich, Conformal Einstein Evolution. In: J. Frauendiener and H. Friedrich (eds.), *The Conformal Structure of Spacetime*. Springer-Verlag, Berlin Heidelberg 2002
126. J. Frauendiener, *Conformal Infinity*, Living Reviews in Relativity, **3**, 2000; http://www.livingreviews.org/Articles/Volume3/2000-4frauendiener/.
127. H. Friedrich , *Proc. Roy. Soc. London* A**378**, 401 (1981)
128. H. Bondi, M.G.J. Van der Burg and A.W.K. Metzner, *Proc. Roy. Soc. London* A**289**, 21 (1962)
129. J. Nester and W. Israel, *Phys. Lett.* **85A**, 259 (1981)
130. A. Ashtekar and A. Magnon-Ashtekar, *Phys. Rev. Lett.* **43**, 181 (1979)
131. L. Blanchet, T. Damour and G. Schäfer, *MNRAS* **242**, 289 (1990)
132. S. Chandrasekhar, *Astrophys. J.* **142**, 1488 (1965)
133. T. Damour, Gravitational Radiation and the Motion of Compact Bodies. In: N. Deruelle and T. Piran (eds.), *Gravitational Radiation*. North-Holland, Amsterdam 1983
134. L. Blanchet, *Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries*, Living Reviews in Relativity, **5**, 2002; http://www.livingreviews.org/Articles/Volume/2002-3Blanchet/.
135. M.E. Pati and C.M. Will, *Phys. Rev.* D **62**, 124015 (2000)
136. M.E. Pati and C.M. Will, *Phys. Rev.* D **65**, 104008 (2002); gr-qc/0201001
137. G. Schäfer, *Gen. Relativ. Gravit.* **18**, 255 (1986)
138. T. Damour, P. Jaranowski and G. Schäfer, *Pys. Rev.* D **62**, 021501 (2000)
139. T. Damour, Gravitational Radiation Reaction in the Binary Pulsar and the Quadrupole Formula Controversy. In: S. Benenti, M. Ferraris and M. Francavighia (eds.), *Proceedings of Journées Relativistes 1983*. Pitagora Editrice 1985; *Phys. Rev. Lett.* **51**, 1019 (1983)
140. R.A. Hulse and J.H. Taylor, *Astropys. J.* **195**, L 51 (1975)
141. J.H. Taylor, *Millisecond pulsars: Natur's mos stable clocks* Proc. IEEE **79**, 1054 (1991)
142. C.M. Will, *Ann. Phys.* **155**, 133 (1984)

143. R. Blandford and S.A. Teukolsky, *Astrophys. J.* **205**, 580 (1976)
144. T. Damour and N. Deruelle, *Ann. Inst. H. Poincaré (Physique Théorique)* **43**, 107 (1985)
145. T. Damour and N. Deruelle, *Ann. Inst. H. Poincaré (Physique Théorique)* **44**, 263 (1986)
146. T. Damour and J.H. Taylor, *Phys. Rev.* D **45**, 1840 (1992)
147. J.H. Taylor, *Phil. Trans. R. Soc.* **A 341**, 117 (1992)
148. J.M. Weisberg and J.H. Taylor, The Relativistic Binary Pulsar B 1913+16. In: D.J. Nice and S.E. Thorsett (eds.), Proceedings of *Binary Pulsars*, Chania, Crete 2002 ASP Conference Series; astro-ph/0211217
149. T. Damour and J.H. Taylor, *Astrophys. J.* **366**, 501 (1991)
150. L.L. Smarr and R. Blandford, *Astrophys. J.* **207** 574 (1976)
151. A. Wolszczan, *Nature* **350**, 688 (1991)
152. I.H. Stairs, S.E. Thorsett, J.H. Taylor and A. Wolszczan, *Astrophys. J.* **581**, 501 (2002); astro-ph/0208357
153. I.H. Stairs, E.M. Splaver, S.E. Thorsett, D.J. Nice and J.H. Taylor, *MNRAS* **314**, 459 (2000)
154. J.H. Taylor and J.M. Cordes, *Astrophys. J.* **411**, 674 (1993)
155. T. Damour and G. Esposito-Farèse, *Phys. Rev* D **54**, 1474 (1996); D **58**, 042001 (1998)
156. A.G. Lyne, et al., *Science* **303**, 1153 (2004)
157. M. Burgay, et al., *Nature* **426**, 531 (2003)

Chapter 6

158. J.L. Provencal, H.L. Shpiman, E. Hog and P. Thejll, *Astrophys.* **494**, 759 (1998)
159. L.D. Landau, *Phys. Z. Sowjetunion* **1**, 285 (1932)
160. N. Straumann, Physics of Type II Supernova Explosions. In: H. Mitter and F. Widder (eds.), *Particle Physics and Astrophysics, Current Viewpoints.* Springer-Verlag 1983
161. H.A. Bethe, *Rev. Mod. Phys.* **62**, 801 (1990)
162. J.R. Oppenheimer and G.M. Volkoff *Phys. Rev* **55**, 374 (1939)
163. H. Heiselberg and V. Pandharipande, *Ann. Rev. Nucl. and Part. Science* **50**, 481-524 (2000)
164. S. Balberg and S.L. Shapiro, *The Properties of Matter in White Dwarfs and Neutron Stars*; astro-ph/0004317
165. R.B. Wiringa, V.G.J. Stoks and R. Schiavilla, *Phys. Rev.* **C53**, R1483 (1996)
166. A. Akmal and V.R. Pandharipande, *Phys. Rev.* C **56**, 2261 (1997)
167. J.L. Forest, V.R. Pandharipande and J.L. Friar, *Phys. Rev.* C **52**, 568 (1995)
168. A. Akmal, V.R. Pandharipande and R. Ravenhall, *Phys. Rev.* C **58**, 1804 (1998)
169. H. Heiselberg and M. Hjorth-Jensen, *Phys. Rep.* **328**, 237 (2000)
170. F.M. Walter and L.D. Mathews, *Nature* **389**, 358 (1997)
171. D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact Solutions of Einstein's Field Equations.* Cambridge Monographs on Mathematical Physics 2003
172. M. Ansorg, A. Leinwächter and R. Meinel, *Astron. Astrophys.*, **381** L49 (2002)

173. D.G. Yakovlev, A.D. Kaminker, O.Y. Gnedin and P. Haensel, *Neutrino Emission from Neutron Stars*, Phys. Rept. **354**, 1 (2001); astro-ph/0012122
174. H.-Th. Janka, K. Kifonidis and M. Rampp, Supernova Explosions and Neutron Star Formation. In: (eds.) Proc. of International Workshop on *Physics of Neutron Star Interiors*, Trento 2000. Springer Lecture Notes in Physics, Vol. 578, Springer Berlin 2001
175. K.S. Thorne, The General Relativistic Theory of Stellar Structure and Dynamics. In: C. DeWitt, E. Schatzmann and P. Veron (eds.), *High Energy Astrophysics*. Gardon and Breach 1967
176. G. Baym et al., *Nucl. Phys.* **A 175**, 225 (1971); *Astrophys. J.* **170**, 99 (1972)
177. E. Flowers and N. Itoh, *Astrophys. J.* **206**, 218 (1976)
178. S. Ayasli and P.C. Joss, *Astrophys. J.* **256**, 637 (1982)
179. P. Giannone and A. Weigert, *Zs. f. Ap.* **67**, 41 (1967)
180. T.M. Tauris and E.P.J. van den Heuvel, Formation and Evolution of Compact Stellar X-ray Sources. In: W.H.G. Lewin and M. van der Klis (eds.), *Compact Stellar X-Ray Sources*. Cambridge University Press (in preparation); astro-ph/0303456
181. W.H.G. Lewin and M. van der Klis, *Compact Stellar X-Ray Sources*. Cambridge University Press (in preparation)
182. E.P.J. van den Heuvel, M. Livio and S.N. Shore, *Interacting Binaries*, Saas-Fee Advanced Course 22 (1994), Swiss Society for Astrophysics and Astronomy
183. S. Chandrasekhar, *Phil. Mag.* **11**, 592 (1931); *Astrophys. J.* **74**, 81 (1931)
184. S. Chandrasekhar, *Observatory* **57**, 373 (1934)
185. R.H. Fowler, *Mon. Not. R. Astron. Soc.*, **87**, 114 (1926)
186. B. Paczynski, *Ann. Rev. Astron. Astrophys.* **9**, 183 (1971)
187. R. Schoen and S.-T. Yau, *Comm. Math. Phys.* **90**, 575 (1983)

Chapter 7

188. W. Israel, *Phys. Rev.* **164**, 1776 (1967); *Commun. Math. Phys.* **8**, 245 (1968)
189. M. Heusler, *Black Hole Uniqueness Theorems*. Cambridge University Press 1996
190. M. Heusler, *Stationary Black Holes: Uniqueness and Beyond*, Living Reviews, **1**, 1998; http://www.livingrews.org
191. M.S. Volkov and D.V. Gal'tsov, *Phys. Rep.* **319**, 1-83 (1999)
192. R.P. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963)
193. E.T. Newman and A.I. Janis, *J. Math. Phys.* **6**, 915 (1965)
194. E.T. Newman et al., *J. Math. Phys.* **6**, 918 (1965)
195. G.C. Debney et al., *J. Math. Phys.* **10**, 1842 (1969)
196. R.D. Blandford and R.L. Znajek, *Mon. Not. R. Astron. Soc.* **179**, 433 (1977)
197. K.S. Thorne, R.H. Price and D.A. MacDonald, *Black Holes: The Membrane Paradigm*. Yale University Press, New Haven and London (1986)
198. N. Straumann, The Membrane Model of Black Holes and Applications. In: F.W. Hehl, C. Kiefer and R.J.K. Metzler (eds.), *Black Holes: Theory and Observation*. Springer-Verlag 1998
199. H.K. Lee, R.A. Wijers and G.E. Brown, *Phys. Rep.* **325**, 83-114 (2000)
200. B. Carter, *Commun. Math. Phys.* **10**, 280 (1968)
201. J.M. Bardeen, W.H. Press and S.A. Teucholsky, *Astrophys. J.* **178**, 347 (1972)

202. M. Heusler and N. Straumann, *Class. Quantum Grav.* **10**, 1299 (1993); *Phys. Lett. B* **315**, 55 (1993)
203. V. Iyer and R.M. Wald, *Phys. Rev. D* **50**, 846 (1994)
204. R.M. Wald, Black Holes and Thermodynamics. In: R.M. Wald (ed.), *Black Holes and Relativistic Stars*. University of Chicago Press 1998
205. B. Carter, *J. Math. Phys.* **10**, 70 (1969)
206. D. Giulini, *J. Math. Phys.* **39**, 6603 (1998)
207. D.R. Gies and C.T. Bolton *Astrophys. J.* **304**, 371 (1986)
208. A.P. Cowley et al. *Astrophys. J.* **272**, 118 (1983)
209. J.E. McClintock and R.A. Remillard *Astrophys. J.* **308**, 110 (1986)
210. J.A. Orosz, Inventory of Black Hole Binaries. In: K.A. van der Hucht, A. Herraro and C. Esteban (eds.), IAU Symposium No. 212, *A Massive Star Odyssey, from Main Sequence to Supernova*, Lanzarote 2002, ASP Conference Series (to appear); astro-ph/0209041
211. R.M. Wagner et al. *Astrophys. J* **556**, 42 (2001)
212. J.E. McClintock et at. *Astrophys. J.* **551**, L147 (2001)
213. M. Miyoshi, J. Moran, J. Herrnstein, L. Greenhill, N. Nakai, P. Diamonnd and M. Inone *Nature* **373**, 127 (1995)
214. T. Ott et al. *ESO Messenger* **111**, 1 (2003); astro-ph/0303408
215. M. Kozlowski, M. Jaroszynski and M.A. Abramowicz, *Astron. Astrophys.* **63**, 209 (1978); M. Sikore, M. Jaroszynski and M.A. Abramowicz, *Astron. Astrophys.* **63**, 221 (1978)
216. D. Giulini, *J. Math. Phys.* **39**, 6603 (1998)
217. P.T. Chrusciel, E. Delay, G. Galloway and R. Howard, *Annales Henri Poincaré* **2**, 109 (2001)
218. P.T. Chrusciel, *Helv. Phys. Acta* **69**, 529 (1996)
219. J.H. Krolik, *Active Galactic Nuclei*. Princeton University Press 1999
220. Ch.J. Willott, R.J. McLure and M. Jarvis, *Astrophys. J.* **587**, L15 (2003)
221. R. Penrose, Structure of Spacetime. In: C. DeWitt and J.A. Wheeler (eds.), *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*. W.A. Benjamin New York 1968
222. R. Genzel et al., *Nature* **425**, 934 (2003)

Chapter 8

223. E. Witten, *Commun. Math. Phys.* **80**, 381 (1981)
224. J. Stewart, *Advanced General Relativity*. Cambridge University Press, 2003
225. J. Nester, *Phys. Lett.* **83A**, 241 (1981)
226. R. Penrose and W. Rindler, *Spinors and Space-Time*. Cambridge University Press 1984
227. R.P. Geroch, *J. Math. Phys.* **9**, 1739 (1968)
228. O. Reula, *J. Math. Phys.* **23**, 810 (1982)
229. T. Parker and C.H. Taubes, *Comm. Math. Phys.* **84**, 223 (1982)
230. G.W. Gibbons, S.W. Hawking, G.T. Horowitz and M.J. Perry, *Comm. Math. Phys.* **88**, 295 (1983)
231. R. Penrose, *Ann. N.Y. Acad. Sci.* **224**, 125 (1973)
232. G. Huisken and T. Ilmanen, *J. Diff. Geom.* **59**, 353 (2001)

Index

Texts and Monographs in Physics

Series Editors: R. Balian W. Beiglböck H. Grosse E. H. Lieb
N. Reshetikhin H. Spohn W. Thirring

Essential Relativity Special, General, and Cosmological Revised 2nd edition
By W. Rindler

The Elements of Mechanics
By G. Gallavotti

Generalized Coherent States and Their Applications
By A. Perelomov

Quantum Mechanics II
By A. Galindo and P. Pascual

Geometry of the Standard Model of Elementary Particles
By. A. Derdzinski

From Electrostatics to Optics
A Concise Electrodynamics Course
By G. Scharf

Finite Quantum Electrodynamics
The Causal Approach 2nd edition
By G. Scharf

Path Integral Approach to Quantum Physics An Introduction
2nd printing By G. Roepstorff

Supersymmetric Methods in Quantum and Statistical Physics By G. Junker

Relativistic Quantum Mechanics and Introduction to Field Theory
By F. J. Ynduráin

Local Quantum Physics Fields, Particles, Algebras 2nd revised and enlarged edition
By R. Haag

The Mechanics and Thermodynamics of Continuous Media By M. Šilhavý

Quantum Relativity A Synthesis of the Ideas of Einstein and Heisenberg
By D. R. Finkelstein

Scattering Theory of Classical and Quantum *N*-Particle Systems
By. J. Derezinski and C. Gérard

Effective Lagrangians for the Standard Model By A. Dobado, A. Gómez-Nicola, A. L. Maroto and J. R. Peláez

Quantum The Quantum Theory of Particles, Fields, and Cosmology By E. Elbaz

Quantum Groups and Their Representations
By A. Klimyk and K. Schmüdgen

Multi-Hamiltonian Theory of Dynamical Systems By M. Błaszak

Renormalization An Introduction
By M. Salmhofer

Fields, Symmetries, and Quarks
2nd, revised and enlarged edition By U. Mosel

Statistical Mechanics of Lattice Systems
Volume 1: Closed-Form and Exact Solutions
2nd, revised and enlarged edition
By D. A. Lavis and G. M. Bell

Statistical Mechanics of Lattice Systems
Volume 2: Exact, Series and Renormalization Group Methods
By D. A. Lavis and G. M. Bell

Conformal Invariance and Critical Phenomena By M. Henkel

The Theory of Quark and Gluon Interactions
3rd revised and enlarged edition
By F. J. Ynduráin

Quantum Field Theory in Condensed Matter Physics By N. Nagaosa

Quantum Field Theory in Strongly Correlated Electronic Systems
By N. Nagaosa

Information Theory and Quantum Physics
Physical Foundations for Understanding the Conscious Process By H.S. Green

Magnetism and Superconductivity
By L.-P. Lévy

The Nuclear Many-Body Problem
By P. Ring and P. Schuck

Perturbative Quantum Electrodynamics and Axiomatic Field Theory By O. Steinmann

Quantum Non-linear Sigma Models
From Quantum Field Theory to Supersymmetry, Conformal Field Theory, Black Holes and Strings By S. V. Ketov

Texts and Monographs in Physics

The Statistical Mechanics of Financial Markets 2nd Edition By J. Voit

Statistical Mechanics A Short Treatise
By G. Gallavotti

Statistical Physics of Fluids
Basic Concepts and Applications
By V. I. Kalikmanov

Many-Body Problems and Quantum Field Theory An Introduction 2nd Edition
By Ph. A. Martin and F. Rothen

Foundations of Fluid Dynamics
By G. Gallavotti

High-Energy Particle Diffraction
By E. Barone and V. Predazzi

Physics of Neutrinos
and Applications to Astrophysics
By M. Fukugita and T. Yanagida

Relativistic Quantum Mechanics
By H. M. Pilkuhn

The Geometric Phase in Quantum Systems
Foundations, Mathematical Concepts, and Applications in Molecular and Condensed Matter Physics
By A. Bohm, A. Mostafazadeh, H. Koizumi, Q. Niu and J. Zwanziger

The Atomic Nucleus as a Relativistic System
By L.N. Savushkin and H. Toki

The Frenkel–Kontorova Model
Concepts, Methods, and Applications
By O. M. Braun and Y. S. Kivshar

Aspects of Ergodic, Qualitative and Statistical Theory of Motion
By G. Gallavotti, F. Bonetto and G. Gentile

Quantum Entropy and Its Use
By M. Ohya and D. Petz

General Relativity
With Applications to Astrophysics
By N. Straumann

Coherent Dynamics of Complex Quantum Systems By V. M. Akulin