Appendix : Spence Functions

In this appendix we follow essentially the paper by *K.Mitchell, Phil.Mag. 40, 351 (1949)*. For complex $z \notin [1, +\infty)$, one defines the Spence function by

$$L(z) = \int_0^z \frac{\log(1-t)}{t} dt, \tag{A1}$$

where

$$\log z = \log |z| + i \arg z \quad , \quad |\arg z| < \pi. \tag{A2}$$

It is directly related to the Euler dilogarithm (see *L.Lewin, Dilogarithms and associated functions, Mc Donald, London 1958*). The path of integration in (A1) is the straight line from 0 to z or some other contour avoiding $(1, +\infty)$. The integral appearing in the applications corresponds to real $z = x$

$$\int_0^x \frac{\log|1-t|}{t} dt = \frac{1}{2}\big(L(x+i0) + L(x-i0)\big)$$

$$\stackrel{\text{def}}{=} L(x). \tag{A3}$$

This definition is useful because by taking the arithmetic mean in (A3), the arguments of the logarithms cancel out.

Inserting the logarithmic series

$$\log(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \quad , \quad |t| < 1,$$

into (A1), we obtain the power series expansion

$$L(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \tag{A4}$$

which is valid for $|z| \leq 1$. It yields the special values

$$L(1) = -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{6} \tag{A5}$$

$$L(-1) = \sum_{n=1}^{\infty} (-)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}. \tag{A6}$$

We now turn to functional equations satisfied by $L(z)$. For z not real, we write ($A1$) in the form

$$L(z) = \left(\int_0^1 + \int_1^z\right) \frac{\log(1-t)}{t} dt \tag{$A7$}$$

and integrate the second integral by parts

$$= -\frac{\pi^2}{6} + \log z \log(1-z) + \int_1^z \frac{\log t}{1-t} dt, \tag{$A8$}$$

where ($A5$) has been used. Introducing the new integration variable $s = 1 - t$ in the remaining integral, we find

$$L(z) = -L(1-z) + \log z \log(1-z) - \frac{\pi^2}{6}. \tag{$A9$}$$

For real $z = x$, it follows from ($A3$) that

$$L(1-x) = -L(x) + \log|x| \log|1-x| - \frac{\pi^2}{6}. \tag{$A10$}$$

In order to calculate $L(1/z)$, we proceed for z not real as follows:

$$L\left(\frac{1}{z}\right) = L(1) + \int_1^{1/z} \frac{\log(1-t)}{t} dt = L(1) - \int_1^z \frac{\log(1-\frac{1}{s})}{s} ds$$

$$= L(1) - \int_1^z \frac{ds}{s}\big(\log(1-s) - \log(-s)\big) = 2L(1) - L(z) + \int_1^z \frac{\log(-s)}{s} ds.$$

The last integral can be easily carried out

$$L\left(\frac{1}{z}\right) = -L(z) - \frac{\pi^2}{3} + \frac{1}{2}\log^2(-z) - \frac{1}{2}\log^2(-1)$$

$$= -L(z) + \frac{1}{2}\log^2(-z) + \frac{\pi^2}{6}. \tag{$A11$}$$

Here, $\log^2 z \overset{\text{def}}{=} (\log z)^2$. The relation ($A11$) holds also for real $x < 0$. For positive x we must use ($A3$). Then the term $\log^2(-z)$ gives a contribution $-\pi^2$:

$$L\left(\frac{1}{x}\right) = -L(x) + \frac{1}{2}\log^2|x| + \begin{cases} \frac{\pi^2}{6}, & x < 0 \\ -\frac{\pi^2}{3}, & x > 0. \end{cases} \tag{$A12$}$$

From ($A10$) and ($A12$) one obtains

$$L\left(\frac{x-1}{x}\right) = L\left(1 - \frac{1}{x}\right) = -L\left(\frac{1}{x}\right) - \log|x| \log\left|1 - \frac{1}{x}\right| - \frac{\pi^2}{6} =$$

$$= L(x) + \frac{1}{2}\log^2|x| - \log|x|\log|x-1| + \begin{cases} -\frac{\pi^2}{3}, & x < 0 \\ \frac{\pi^2}{6}, & x > 0. \end{cases} \tag{A13}$$

Again from (A12) we get

$$L\left(\frac{x}{x-1}\right) = -L\left(\frac{x-1}{x}\right) + \frac{1}{2}\log^2\left|\frac{x-1}{x}\right| + \begin{cases} \frac{\pi^2}{6} & \text{(a)} \\ -\frac{\pi^2}{3} & \text{(b)}. \end{cases}$$

The case (a) corresponds to $0 < x < 1$, whereas (b) corresponds to the values $x < 0$ or $x > 1$. We find with the aid of (A13)

$$L\left(\frac{x}{x-1}\right) = -L(x) + \frac{1}{2}\log^2|x-1| + \begin{cases} 0, & x < 1 \\ -\frac{\pi^2}{2}, & x > 1. \end{cases} \tag{A14}$$

Likewise we shall obtain from (A12) and (A10)

$$L\left(\frac{1}{1-x}\right) = L(x) - \frac{1}{2}\log|1-x|\log\frac{x^2}{|1-x|} + \begin{cases} \frac{\pi^2}{3}, & x > 1 \\ -\frac{\pi^2}{6}, & x < 1. \end{cases} \tag{A15}$$

The formulas used in the main text are special cases of (A10) and (A13–15), using the variable $\xi = -x > 0$:

$$L\left(\frac{\xi}{1+\xi}\right) = -L(-\xi) + \frac{1}{2}\log^2(1+\xi) \tag{A16}$$

$$L\left(\frac{1}{1+\xi}\right) = L(-\xi) - \frac{1}{2}\log(1+\xi)\log\frac{\xi^2}{1+\xi} - \frac{\pi^2}{6} \tag{A17}$$

$$L(1+\xi) = -L(-\xi) + \log\xi\log(1+\xi) - \frac{\pi^2}{6} \tag{A18}$$

$$L\left(\frac{1+\xi}{\xi}\right) = L(-\xi) - \frac{1}{2}\log\xi\log\frac{(1+\xi)^2}{\xi} - \frac{\pi^2}{3}. \tag{A19}$$

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