Appendix: Comments on the Literature

Since the literature on necessary conditions for extrema, duality theory in convex programming, minimax problems and numerical techniques for solving various extremal problems includes thousands of references, it is impossible to undertake any form of comparative analysis here. Thus, we shall only refer to books and papers mentioned in the text which are directly related to the subject matter of this book.

As far as the theory of necessary conditions for extrema and that of duality in convex programming are concerned, we based our study in Section 1.2 on the books by A.D. Ioffe and V.M. Tikhomirov [7], B.N. Pshenichnyj [23,28], R.T. Rockafellar [33], and I. Ekeland and R. Temam [47]. A detailed theoretical analysis of penalty function techniques is given in the book by K. Grosman and A.A. Kaplan [2] which includes further references to the literature. However, we based our study of the conditions under which the minima of penalty functions coincide with the solutions of the original problem (Section 1.2.7) on the paper by F. Clarke [9].

Quadratic programming problems are fundamental to the use of the linearization method. The speed and economy (from the point of view of the computations and the computer storage involved) with which they are solved determines the effectiveness of the method, particularly as far as large problems are concerned. In this connection, Section 1.3 was based on a technique for solving quadratic programming problems which generalizes the simplex method of linear programming (see G.B. Dantzig [3]). Here, we considered the need for economic use of computer resources when working with sparse matrices, and used the approaches developed by B.A. Murtagh and M.A. Saunders [11] together with a multiplicative representation of inverse matrices. This representation is not the only one possible nor is it always optimal. More details of this may be found in R. Tewarson [36]. In addition to the methods for solving quadratic programming problems studied in Section 1.3, there are also other methods which are described in the paper by V.A. Daugavet [5], in the books by H.P. Künzi and W. Krelle [10] and B.N. Pshenichnyj and Yu.M. Danilin [28] and in the collected papers [46]. It should be said that most of these methods carry over to the problem of minimizing an arbitrary function subject to linear constraints, as is shown in the aforementioned articles [11,46] and in the paper by B.N. Pshenichnyj and I.F. Ganzhela [27]. A very detailed study of conjugate gradient methods in relation to the minimization of quadratic functions is given in [28] and in the book by E. Polak [19].

The linearization method in the form described in this book was first studied by B.N. Pshenichnyj in [24] and, in relation to equations and inequalities, in [25]. Modifications of it, with differences in the auxiliary problem, are considered by U.M. Garcia-Palomares and O.L. Mangasarian [1], S.M. Robinson [31, 32], and S.P. Han [39]. Quite rightly, these only consider local convergence. Later, in the papers [40–42], S.P. Han considered questions of global convergence. The papers by V.M. Panin [13–16] and M. Powell [17,18] are devoted to the same problem area. We note that in these articles, a superlinear rate of convergence was achieved as a result of the (explicit and implicit) use of the second derivatives of the Lagrange function in the auxiliary problem. However, in fact, this required the matrix of second derivatives of the Lagrange functions to be strictly positive definite. This shortcoming was overcome by B.N. Pshenichnyj and L.A. Sobolenko [30].

Resolution of minimax problems by the linearization method (other techniques for solving these problems are considered in the book by V.F. Dem'yanov and V.N. Malozemov [6]) was studied in the book by B.N. Pshenichnyj and Yu.M. Danilin [28] and in the paper by B.N. Pshenichnyj [26]. The fact that the linearization method guarantees quadratic convergence for Chebyshev points was proved by V.A. Daugavet and V.N. Malozemov [4]. Solutions of various generalized problems using minimax methods which are modifications of the linearization method were considered by K. Kiwiel [8], V.M. Panin [13,14], S.P. Han [42] and R. Fletcher [38].

All the calculations using the linearization method given in Section 3.3 were carried out by L.A. Sobolenko. In addition to problems from the literature and a number of real problems were also solved. It is impossible to describe all this material in the book and thus Section 3.3 only includes examples which are, in the author's opinion, in some sense characteristic. In the absence of a standard against which to assess the results of calculations given in the literature, it is difficult to make an accurate comparison; however, it is safe to say that the results of numerical computations using the linearization method or its accelerated modification are no worse than those of any other method.

References

1. Garcia-Palomares, U.M. and Mangasarian, O. (1976): Superlinearly convergent quasi-Newton methods for nonlinearly constrained optimization problems. Math. Program. **11** (1976) 1–13
2. Grosman, K. and Kaplan, A.A. (1981): Nonlinear Programming Based on Unconstrained Optimization. Nauka, Novosibirsk
3. Dantzig, G.B. (1962): Linear Programming and its Extensions. Princeton University Press, Princeton NJ
4. Daugavet, V.A. and Malozemov, V.N. (1981): Quadratic rate of convergence of a linearization method for solving discrete minimax problems. Zh. Vychisl. Mat. Mat. Fiz. **21** 4 (1981) 835–843
5. Daugavet, V.A. (1981): Modification of a method of Wulf. Zh. Vychisl. Mat. Mat. Fiz. **21** 2 (1981) 504–508
6. Dem'yanov, V.F. and Malozemov, V.N. (1972): Introduction to Minimax. Nauka, Moscow. English transl.: John Wiley and Sons (1974)
7. Ioffe, A.D. and Tikhomirov, V.M. (1974): Theory of Extremal Problems. Nauka, Moscow. English transl.: Academic Press (1979)
8. Kiwiel, K.C. (1981): A globally convergent quadratic approximation for inequality constrained minimax problems. IIASA Collab. Proc. Ser. **81-9** (1981) 1–24
9. Clarke, F. (1976): A new approach to Lagrange multipliers. Math. Oper. Res. **1** 2 (1976) 165–174
10. Künzi, H.P. and Krelle, W. (1962): Nichtlineare Programmierung. Springer-Verlag
11. Murtagh, B.A. and Saunders, M.A. (1978): Large-scale linearly constrained optimization. Math. Program. **14** (1978) 41–72
12. Ostrovskij, A.M. (1963): Solution of Equations and Systems of Equations. IL, Moscow. English transl.: Academic Press (1966)
13. Panin, V.M. (1980): The linearization method for the discrete minimax problem. Kibernetika **3** (1980) 86–90
14. Panin, V.M. (1981): The linearization method for the continuous minimax problem. Kibernetika **2** (1981) 75–78
15. Panin, V.M. (1981): Methods for solving convex programming problems. Zh. Vychisl. Mat. Mat. Fiz. **21** 2 (1981) 315–328
16. Panin, V.M. (1982): Global convergence of the damped Newtonian method in convex programming problems. Dokl. Akad. Nauk. SSSR **261** 4 (1982) 811–814
17. Powell, M. (1977): The convergence of variable matrix methods for nonlinearly constrained optimization calculations. Dept. Appl. Math. and Theoret. Phys.
18. Powell, M. (1978): Algorithms for nonlinear constraints that use Lagrangian functions. Math. Program. **14** (1978) 224–228

19. Polak, E. (1971): Computational Methods in Optimization. A Unified Approach. Academic Press, New York, London
20. Polyak, B.T. and Tret'yakov, N.V. (1972): An iterative linear programming method and its economic interpretation. Ehkon. Mat. Metody **VIII** 5 (1972) 740–751
21. Polyak, B.T. and Tret'yakov, N.V. (1973): A penalty function method for conditional extremum problems. Zh. Vychisl. Mat. Mat. Fiz. **13** 1 (1973) 34–46
22. Popkov, N.V., Tulupchuk, Yu.M. and Khilyuk, L.F. (1978). Three-criterion control of the water management system of the Dnieper basin. Avtomatika **2** (1978) 44–53
23. Pshenichnyj, B.N. (1969): Necessary Conditions for an Extremum. Nauka, Moscow. English transl.: Marcel Dekker Inc., New York (1971)
24. Pshenichnyj, B.N. (1970): Algorithm for a general mathematical programming problem. Kibernetika **5** (1970) 120–125
25. Pshenichnyj, B.N. (1970): Newton's method for solving systems of equations and inequalities. Mat. Zametki **8** 5 (1970) 635–640
26. Pshenichnyj, B.N. (1978). Nonsmooth optimization and nonlinear programming. In Lemarechal, V.C. and Mifflin, R. (eds.) (1978): Nonsmooth Optimization. Pergamon Press
27. Pshenichnyj, B.N. and Ganzhela, I.F. (1970): An algorithm for solving convex programming problems with linear constraints. Kibernetika **3** (1970) 81–85
28. Pshenichnyj, B.N. and Danilin, Yu.M. (1978): Numerical Methods in Extremal Problems. Mir, Moscow
29. Pshenichnyj, B.N. (1980): Convex Analysis and Extremal Problems. Nauka, Moscow
30. Pshenichnyj, B.N. and Sobolenko, L.A. (1980): Acceleration of the convergence of the linearization method for problems of conditional minimization. Zh. Vychisl. Mat. Mat. Fiz. **20** 3 (1980) 605–614
31. Robinson, S.M. (1972): A quadratically convergent algorithm for general nonlinear programming problems. Math. Program. **3** (1972) 145–156
32. Robinson, S.M. (1974): Perturbed Kuhn–Tucker points and rates of convergence for a class of nonlinear programming algorithm. Math. Program. **4** (1974) 1–16
33. Rockafellar, R.T. (1970): Convex Analysis. Princeton University Press, Princeton NJ
34. Rockafellar, R.T. (1976): Augmented Lagrangians and applications of the proximal point algorithm in convex programming. Math. Oper. Res. **1** 2 (1976) 97–115
35. Tret'yakov, N.V. (1973): Penalty function method for convex programming problems, Ehkon. Mat. Metody **IX** 3 (1973) 525–540
36. Tewarson, R. (1973): Sparse Matrices. Academic Press
37. Fedorenko, R.P. (1978): Approximate Solution of Optimal Control Problems. Nauka, Moscow
38. Fletcher, R. (1981): Second order corrections for nondifferentiable optimization. Numer. Anal. Rep. **50** (1981) 1–30
39. Han, S.P. (1976): Superlinearly convergent variable metric algorithms for general nonlinear programming problems. Math. Program. **11** (1976) 263–282

40. Han, S.P. (1977): A globally convergent method for nonlinear programming. J. Optimization Theory Appl. **22** (1977) 297–309.
41. Han, S.P. (1977): Dual variable metric method for constrained optimization, SIAM J. Control Optimization **15** (1977) 546–565
42. Han, S.P. (1981): Variable metric method for minimizing a class of nondifferentiable functions. Math. Program. **20** (1981) 1–13
43. Khenkin, Eh.I. and Volynskij, M.Z. (1976): Search algorithm for solving a general mathematical programming problem, Zh. Vychisl. Mat. Mat. Fiz. **10** 1 (1976) 61–71
44. Hestenes, M.R. (1969): Multiplier and gradient methods, J. Optimization Theory Appl. **4** (1969) 303–320
45. Himmelblau, D. (1972): Applied Nonlinear Programming. McGraw-Hill
46. Gill, P., Murray, W. and Wright, M. (1981): Practical Optimization. Academic Press, New York, London
47. Ekeland, I. and Temam, R. (1976): Convex Analysis and Variational Problems. North-Holland Elsevier, Amsterdam

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