Intermezzo I: Condition of Structured Data

The themes of Chaps. 3 and 5 introduced, *sotto voce*, the issue of structured data. In both cases we had a general set of data, the space $\mathbb{R}^{n\times n}$ of $n \times n$ real matrices, and a subset $\mathcal{S}$ whose elements are the valid inputs of a given algorithm: triangular matrices for FS and symmetric positive definite matrices for CGA.

It is apparent that the analysis pattern we have developed till now—an analysis of the relevant measure of performance for the considered algorithm (loss of precision or running time) in terms of a condition number, followed by a probabilistic analysis of the latter—needs to be adjusted. For the probabilistic analysis, the underlying measure will have to be chosen with support in $\mathcal{S}$. We have already done so in Chap. 3, by drawing from $N(0,1)$ only the matrix entries that are not fixed to be zero, as well as in Chap. 5, where the more elaborated family of Wishart distributions was imposed on the set of symmetric positive definite matrices.

As for the object of analysis itself, the condition number, its actual shape will have to depend on the situation at hand. Yet, even though there is no standard way to "structure" a condition number, a couple of ways occur frequently enough to be described in detail.

(a) **Structured perturbations**. When the analysis is based on data perturbations (e.g., in accuracy analyses), it is often the case that the only admissible perturbations are those respecting the structure of the data a, that is, those for which $\tilde{a} \in \mathcal{S}$ as well. This naturally leads to the following "structuring" of (O.1):

$$\mathsf{cond}^{\varphi}_{\mathcal{S}}(a) := \lim_{\delta\to 0} \sup_{\substack{\mathsf{RelError}(a)\le\delta \\ \tilde{a}\in\mathcal{S}}} \frac{\mathsf{RelError}(\varphi(a))}{\mathsf{RelError}(a)}. \tag{I.1}$$

In the case of triangular linear systems, the backward analysis of algorithm FS in Sect. 3.2 produced componentwise perturbation bounds that automatically force the perturbed matrix $\tilde{L}$ to be lower triangular as well. But this need not be the case.

(b) **Distance to structured ill-posedness**. We will soon see (in Chap. 6, after this intermezzo) that for a large class of problems (those having a discrete set of

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values, notably the decisional problems), the notion of condition given by (O.1) is inadequate and that a common, appropriate replacement is given by taking

$$\mathcal{Q}(a) := \frac{\|a\|}{d(a, \Sigma)}$$

for the condition number of a. Here Σ is a natural set of ill-posed data. It is therefore not surprising that in many of the situations in which such a condition number is considered and data are restricted to some subset $\mathcal{S}$, the useful way to structure $\mathcal{Q}(a)$ is by taking

$$\mathcal{Q}_{\mathcal{S}}(a) := \frac{\|a\|}{d(a, \Sigma \cap \mathcal{S})}. \tag{I.2}$$

The difference between $\mathcal{Q}$ and $\mathcal{Q}_{\mathcal{S}}$ can be large. A case at hand is that of triangular matrices. For any such matrix L, the condition number theorem (Theorem 1.7) shows that $d(L, \Sigma) = \|L^{-1}\|^{-1}$ and therefore $\mathcal{Q}(L) = \kappa(L)$. Theorem 3.1 then shows that $\mathbb{E} \log \mathcal{Q}(L) = \Omega(n)$. In contrast, we will see in Sect. 21.7 that $\mathbb{E} \log \mathcal{Q}_{\mathsf{Triang}}(L) = \mathcal{O}(\log n)$.

Occasionally, there is no need for a structuring of the condition number. This was the case, for instance, in the complexity analysis of the conjugate gradient method in Chap. 5. This analysis revealed a dependence of the number of iterations of Conj_Grad on the standard condition number $\kappa(A)$ of the input matrix A; the only influence of this matrix being symmetric positive definite was on the underlying distribution in the probabilistic analysis.