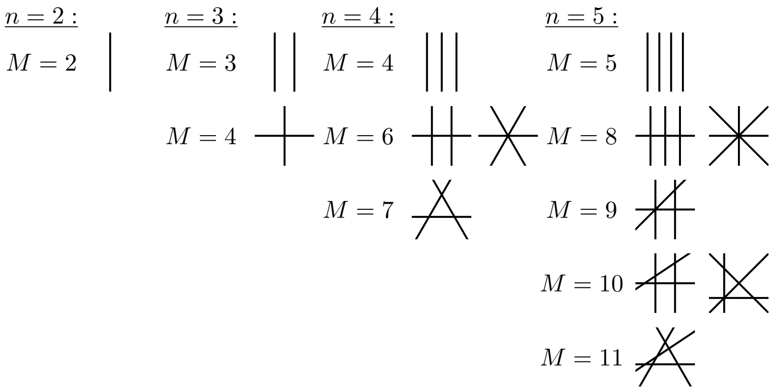
Appendix A*
Into How Many Parts Do *n* Lines Divide the Plane?

Consider n distinct lines in the real projective plane. They divide it into (convex) parts. The question is this: how many parts can be obtained (under all possible arrangements of the lines)?

For small n, the answer is clear; the possible number M of parts is given by the following table:

n	1	2	3	4	5
M	1	2	3	4	5
			4	6	8
				7	9
					10
					11

If we consider one of the lines to be the line at infinity, then we obtain $n-1$ lines in $\mathbb{R}^2$. The values of M in the table are provided by the following configurations of $n-1$ lines:



* This appendix is a translation of the author's article in *Mat. Prosveshchenie, Ser. 3* 12 (2008), 95–104. It was not part of the original Russian edition of this book. It has been appended here at the author's request.

V.I. Arnold, *Real Algebraic Geometry*, UNITEXT – La Matematica per il 3+2 66,
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The number of connected components of the complement of the collection of lines in the affine plane $\mathbb{R}^2$ is studied in similar fashion. This is the same problem, since one can declare one of the n lines to be the line at infinity and study the complement of the $n-1$ lines in the affine plane (which coincides with the complement of the n lines in the projective plane).

Looking at the previous examples, we observe that the smallest number of parts of the complement of n lines in $\mathbb{R}P^2$ is $M = n$, while the largest is

$$M = 1 + (1 + 2 + 3 + \cdots + (n-1)) = 1 + \frac{n(n-1)}{2}.$$

However, only some (and not all) of the intermediate numbers between these limits can be attained. Starting from some point, all sufficiently large values of M are attained, but the beginning of the list of attainable values contains gaps (or holes).

The aim of this paper is to describe these holes. Here is the first hole.

Theorem 1. *The value $M = 2(n-1)$ is attainable, but no other value of M in the interval*

$$n < M < 2(n-1)$$

is attainable.

For any choice of $n > 2$ lines in $\mathbb{R}P^2$, their complement cannot consist of such a number M of connected components.

Proof. Denote by k the greatest number of lines (among our n lines) passing through one point.

Lemma 1. *If $k = n$, then $M = n$.*

Proof. We assume that one of these n lines is the line at infinity. Then the remaining lines are parallel. They divide the affine plane $\mathbb{R}^2$ (complementary to the first line) into n parts, since there are $n-1$ such parallel lines.

Lemma 2. *If $k = n-1$, then $M = 2(n-1)$.*

Proof. We assume that the remaining nth line is the line at infinity. Its complement is $\mathbb{R}^2$. The family of $n-1$ lines passing through one point divides the plane $\mathbb{R}^2$ into $2(n-1)$ parts, as required.

Lemma 3. *If $k \leq n-1$, then $M \geq 2(n-1)$.*

Proof. Note that for $n > 1$ we have $k \geq 2$ (since any two lines in $\mathbb{R}P^2$ intersect). We choose one of the k lines passing through one point as the line at infinity. Its complement is the affine plane $\mathbb{R}^2$, which contains $k-1$ parallel lines of the chosen pencil together with a further $n-k \geq 1$ remaining lines.

These parallel lines divide the plane $\mathbb{R}P^2$ into k parts. By adding the remaining lines one by one, we shall gradually increase the number of parts. Here, if the sth line to be added is intersected by the lines already existing in

x_s points, then it is divided by them into x_s segments, each of which divides one of the parts that existed before the sth line was added in two. Therefore, when the sth line is added, the number of parts of the complement is increased by exactly x_s.

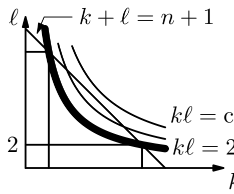
We now observe that $x_s \geq k$ (since the line being added intersects all k parallel lines of the original pencil in k distinct points). Therefore the total number of added parts is

$$x_1 + \cdots + x_{n-k} \geq k(n-k)\,.$$

By adding these parts to the k parts that we already had before dealing with the $n-k$ "additional" lines, we conclude that

$$M \geq k + k(n-k) = k(n-k+1)\,. \qquad (1)$$

The sum of both factors on the right-hand side is equal to $n+1$. When two positive numbers whose sum is equal to $n+1$ are multiplied together, the greater the smaller factor is, the greater their product:



If $2 \leq k \leq n-1$, then

$$\min_{k+l=n+1}(k,l) \geq 2,$$

so that

$$M \geq 2(n-2+1) = 2(n-1).$$

This completes the proof of Lemma 3

Theorem 1 now follows from Lemmas 1, 2, and 3 (since any number $k \leq n$ is either equal to n or equal to $n-1$ or less than $n-1$).

The first hole has now been described. Next we describe the second hole. Let $n \geq 3$.

Theorem 2. *The value $M = 3n-6$ is attainable, but no other value of M in the interval*

$$2(n-1) < M < 3(n-2)$$

is attainable.

For any choice of $n > 2$ lines in $\mathbb{R}P^2$, their complement cannot consist of such a number M of connected components.

Proof. As before, we denote by k the maximum number of lines (among the given n lines) passing through one point, and we call one of them the line at infinity. We regard the remaining $n-1$ lines in the affine plane $\mathbb{R}^2$, which is the complement of the chosen line, as the pencil of $k-1$ parallel lines supplemented by the $n-k$ remaining lines not parallel to these $k-1$ lines of the pencil.

If $k = n-1$, then $M = 2(n-1)$ by Lemma 2. If, on the other hand, $k \leq n-2$, then from formula (1) in the proof of Lemma 3 we obtain

$$M \geq k(n-k+1) \geq (n-2)((n+1)-(n-2)) = 3(n-2),$$

provided that $k \geq 3$ (for which $\min_{k+l=n+1}(k,l) \geq 3$).

Thus, the theorem is proved for all arrangements of n lines such that $k > 2$.

We now prove that $M \geq 3(n-2)$ in the remaining case, in which $k = 2$.

If $k = 2$, that is, no three lines are concurrent, then all our n lines divide the plane into the maximum possible number of parts for n lines (and attainable for n lines in general position), namely

$$M = 1 + \frac{n(n-1)}{2}.$$

Lemma. *We have the inequality*

$$\frac{n(n-1)}{2} + 1 \geq 3(n-2).$$

Proof. This inequality has the form

$$n^2 - n - 6(n-2) + 2 \geq 0,$$

that is,

$$n^2 - 7n + 14 \geq 0,$$

which is true because the discriminant of the quadratic trinomial on the left-hand side,

$$49 - 4 \cdot 14,$$

is negative.

Thus the lemma is proved, so that the inequality $M \geq 3(n-2)$ holds in the case $k = 2$ as well.

Having proved Theorem 2, we have described the second hole. It first appears when $n = 6$ (when in the interval defining the hole there are integer points: $3(n-2) - 2(n-1) > 1$ for $n > 5$).

The subsequent holes that we now study are under the assumption that the number n of lines is sufficiently large (in comparison with the number of the hole).

Theorem 3. *Suppose that the greatest number of lines (among our n lines) passing through one point is k. Then these n lines divide the plane* $\mathbb{R}P^2$ *into M parts, where M lies in the interval*

$$k(n+1-k) \leq M \leq k(n+1-k) + \frac{r(r-1)}{2}, \quad \text{where } r = n-k.$$

(Here all the numbers M in this interval are attainable for a suitable choice of n lines, provided that n is sufficiently large.)

Proof. Choose a pencil of k lines. The lines of this pencil divide the plane $\mathbb{R}P^2$ into k parts. The remaining $n-k=r$ lines add to k the following number of parts:

$$M' = x_1 + x_2 + \cdots + x_{n-k},$$

where x_s is the number of points of the sth added line intersected by the previous lines.

Among these previous lines there are k lines of the chosen pencil (and $s-1$ added lines). The points of intersection with the lines of the pencil are all distinct (since the only common point of two lines of the pencil is the point of intersection of the k lines of the pencil originally chosen, so that no other of our n lines can pass through this point).

Hence $x_s \geq k$, $M' \geq k(n-k)$, $M \geq k(n-k+1)$, which proves the first inequality in Theorem 3.

On the other hand, $x_s \leq k + (s-1)$. Consequently,

$$M' \leq k(n-k) + (0+1+\cdots+n-k-1) = k(n-k) + \frac{r(r-1)}{2},$$

$$M \leq k(n+1-k) + \frac{r(r-1)}{2}.$$

This proves the second inequality in Theorem 3.

All the values of M in the interval described by both inequalities in the above theorem are attained (for sufficiently large n) for the following reason.

The greatest number of parts is provided by choosing the r additional lines in general position. For them all the points of intersection (with the lines of the pencil and with each other) are distinct; as a result, this gives us $k(n+1-k) + r(r-1)/2$ parts.

If n is sufficiently large in comparison with r (for example,[27] if $n \geq r(r+1)/2$), then one can choose the additional lines so that any chosen points in the pairwise intersection of the lines lie on the lines of the pencil (so that $x_s = k$ for the corresponding s).

In fact, we can, for example, start with r lines in general position in the affine plane $\mathbb{R}^2$ and draw, through any collection of $r(r-1)/2 - S$ points of their pairwise intersection, lines that are parallel to each other but not parallel to these r lines. By including these parallel lines along with the line at infinity in the pencil of $k = n - r$ parallel lines, we obtain a collection of lines for which M', the sum of x_s over all $s = 1, \ldots, r$, exceeds kr precisely by S. In this case,[28] $M' = kr + S$, $M = k(n - k + 1) + S$, and Theorem 3 is proved.

Theorem 4. *Suppose that the greatest number of lines (among our n lines) passing through one point is $k > 2$. Then these n lines divide the plane $\mathbb{R}P^2$ into M parts, where*

$$M \geq \frac{n(n-1)}{2(k-1)}.$$

Here it is important that the numerator increases with the number n of lines at the same rate as n^2, while the denominator is independent of the number n. As a result, the right-hand side becomes larger than any linear function of n for sufficiently large n (when k is fixed).

To prove Theorem 4 we arrange the given n lines in some order. By an "event" we mean the intersection of some line with the lowest-numbered line. Thus, the number of events is $0 + 1 + 2 + \cdots + (n-1) = n(n-1)/2$ (independently of how many distinct points of intersection there are).

By a "partition" we mean a division into parts of some line (say the sth) by lines with lower numberings. We denote by x_s the number of points in the partition of the sth line. These x_s points divide that projective line into x_s parts.

We add the lines one by one, and at each stage, we increase the number of components of the complement of the lines by the number x_s of parts added by the sth line (which divides each of the already existing x_s components intersected by it into two parts).

Therefore the total number of components of the complement of the union of the n lines in the projective plane $\mathbb{R}P^2$ is given by

$$M = \sum_{s=1}^{n} x_s,$$

supposing formally that $x_1 = 1$: although the first line does not divide any "preceding" lines, one needs to take into account the (single) component of the complement of one line in the *projective* plane.

At each point of the partition, the maximum number of occurring events is $k - 1$ (the intersection of the sth line with the preceding ones), since there cannot be more than k lines of our collection passing through one point. Therefore, *the number of all events does not exceed* $M(k-1)$. And since the latter is equal to $n(n-1)/2$, we conclude that

$$\frac{n(n-1)}{2} \leq M(k-1),$$

that is,

$$M \geq \frac{n(n-1)}{2(k-1)},$$

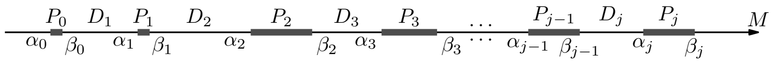
which completes the proof of Theorem 4.

For the investigation of "stable" holes (the jth stable hole D_j will be investigated under the assumption that the number n exceeds some constant depending on j), we introduce the following notation:

$$\alpha_j = (n-j)(j+1), \quad \beta_j = \frac{(n-j)(j+1)+j(j-1)}{2}.$$

For sufficiently large n, the first terms of these two sequences are arranged in the following order:

$$\alpha_0 = \beta_0 < \alpha_1 = \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \cdots < \alpha_{j-1} < \beta_{j-1} < \alpha_j.$$



We denote by $P_0, P_1, \ldots$ the closed intervals

$$P_0 = [\alpha_0 \le M \le \beta_0], \quad P_j = [\alpha_1 \le M \le \beta_1], \quad \ldots, \quad P_1 = [\alpha_j \le M \le \beta_j],$$

and by $D_1, D_2, \ldots, D_j$ the complementary open intervals

$$D_1 =]\beta_0 < M < \alpha_1[, \quad D_2 =]\beta_1 < M < \alpha_2[, \quad \ldots, \quad D_j =]\beta_{j-1} < M < \alpha_j[.$$

The stable hole D_j is described as follows.

Theorem 5. *If the number of lines n is sufficiently large, then the number M of components of their complement in the projective plane $\mathbb{R}P^2$ cannot take values in the intervals D_j. In other words, no value of M for which*

$$\beta_{j-1} = j(n+1-j) + \frac{(j-1)(j-2)}{2} < M < (j+1)(n-j) = \alpha_j$$

is possible.

Proof. Denote by k the greatest number of lines (among our n lines) passing through one point. We shall prove that M cannot lie in an interval D_j for any k, but this proof will be based on different considerations in the following three cases:

I. $k > n - j$;

II. II $j + 1 \le k \le n - j$;

III. $k \le j$.

Here we suppose that $n - j \ge j + 1$ (which holds for n sufficiently large).

Case I. Suppose that k takes one of the values $\{n, n-1, \ldots, n-j+1\}$, and put $r = n-k$.[29]

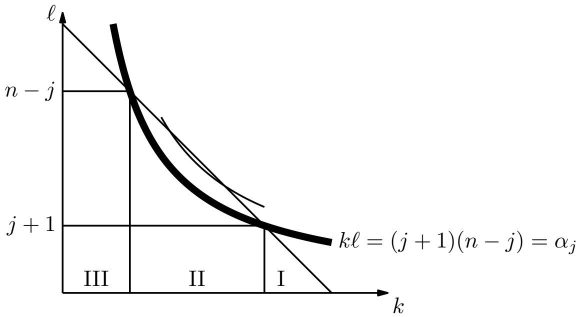
According to Theorem 3, the number M lies in the interval

$$\alpha_r = (n-r)(r+1) \leq M \leq (n-r)(r+1) + r(r-1)/2 = \beta_r\,,$$

that is, $M \in P_r$, $0 \leq r \leq j-1$.

None of these r segments intersects the interval D_j, so Theorem 5 is proved for case I.

Case II. Suppose that $j+1 \leq k \leq n-j$. Then $\ell = n+1-k$ also satisfies the inequalities $j+1 \leq \ell \leq n-j$. In this case, $\min_{k+\ell=n+1}(k,\ell) \geq j+1$:



Therefore, again by Theorem 3,

$$M \geq (n-j)(j+1) = \alpha_j\,.$$

However, $M < \alpha_j$ in the interval D_j. Thus Theorem 5 is proved for Case II as well.

Case III. Suppose that $2 < k \leq j$. According to Theorem 4,

$$M \geq \frac{n(n-1)}{2(k-1)} \geq \frac{n(n-1)}{2(j-1)}\,.$$

The right-hand side of this inequality is greater than α_j if n is sufficiently large. Indeed,

$$\frac{n(n-1)}{2(j-1)} \geq (n-j)(j+1)$$

for sufficiently large n, because then

$$\frac{n(n-1)}{n-j} > 2(j^2-1)\,.$$

For example,

$$\frac{n(n-1)}{n-j} \geq n\,,$$

so that the condition $n > 2(j^2 - 1)$ suffices for the inequailty $M > \alpha_j$, which does not allow M to lie in the interval D_j (between β_{j-1} and α_j).

In the only case not yet disposed of, $k = 2$, our n lines in general position divide the projective plane $\mathbb{R}P^2$ into $M = 1 + n(n-1)/2$ parts.

This number M is greater than the limit $\alpha_j = (n-j)(j+1)$, since for $j + 1 < n/2$ (which we have supposed), the inequalities

$$\alpha_j < \frac{n(n-1)}{2} < M$$

hold for $j > 0$.

Thus Theorem 5 is proved for $k = 2$ as well, and it is therefore proved for all k (the case $k = 1$ for $n > 1$ is not realized, since every pair of lines in the projective plane intersect).

Thus Theorem 5 is completely proved, so that (for a sufficiently large number of lines n) all the stable holes

$$D_1, D_2, \ldots, D_j, \ldots$$

exist in the sequence of numbers M of components into which n lines divide the real projective plane.

Remark 1. I do not know whether the unstable holes (for values of n smaller than those indicated above) are given by the same formulas ($\beta_{j-1} < M < \alpha_j$) as the stable holes. To begin with, instability does not manifest itself for small j.

The first unclear case is the third hole for $n = 9$. In this case, the formulas give $\alpha_3 = 24$, $\beta_2 = 22$.

Nine lines can divide the projective plane into 22 regions and into 24 regions. Whether they can divide it into 23 regions or, on the contrary, whether $M = 23$ for $n = 9$ is the third hole, is unknown. Such an arrangement of lines (if it is possible) would be possible only in the case that no four of these nine lines pass though a single point (case $k = 3$ in the proof of Theorem 5).

Remark 2. The motivation for this article was the publication in Berkeley of A.B. Givental's translation into English of A.P. Kiselev's book *Geometry*. When in April 2007 in California I looked at this translation, I was unable to solve one of the problems in this book (I had resolved them all in my youth).

This problem was as follows: how many lines divide the plane $\mathbb{R}^2$ into five convex parts?

I asked Givental how this question was formulated in Kiselev's original book; he confessed that this was by no means the case: the problem was added by the translator (who improved Kiselev in other places as well).

Every mathematical problem admits two versions: the Russian version, which nobody can simplify (without losing the essence of the problem), and the French version, which nobody can generalize any further (since it has already been formulated in such a general form that it contains all possible generalizations).

On arriving from Paris to Berkeley I decided to formulate the French version of Givental's problem. With this in mind, I changed five regions to any number M of regions. This is how the present article came about.

I have been unable to solve this general problem. One should be able to describe all holes for all values of n; but even the third hole was calculated by me explicitly only for $n \geq 14$ (when it becomes stable). By lecturing local school pupils in Berkeley, Stanford, San José, and Santa Clara (where heroic leaders of the Moscow-style mathematical circles have taught the pupils how to solve difficult problems better than I), I hoped that they would come up with a description of the unstable holes, which is still awaited.

It seems that the question whether 9 lines can divide the projective plane into 23 regions remains unsolved.[30]

Editors' Comments on Gudkov's Conjecture

It is difficult to overestimate Arnold's role in the revolutionary explosive developments in the study of topological properties of real algebraic varieties (and in particular, in solving Hilbert's sixteenth problem). A whole series of works was inspired by Arnold's foundational paper "On the Arrangements of Ovals of Real Plane Algebraic Curves, Involutions of Four-Dimensional Smooth Manifolds, and the Arithmetic of Integer Quadratic Forms" (*Funk. Anal. i Prilozh.* 5 (1971), no. 3, 1–9). The very title of this important paper indicated a new direction of investigation in this field. A number of new results in this paper were immediately named after Arnold, such as Arnold's congruence, Arnold's inequalities, and so on.

Arnold's paper was stimulated by (private and public) communications with D.A. Gudkov and V.A. Rokhlin. Gudkov (in his 1971 note in *Doklady AN SSSR* 200 (1971), no. 6, 1269–1272, and a 1970 talk in Moscow) had formulated as a conjecture the congruence modulo 8 for maximal curves* of even degree. Rokhlin (in his 1971 paper *Funk. Anal. i Prilozh.* 5 (1971), no. 1, 48–60) had obtained his genus bounds in 4-dimensional topology by means of the Atiyah–Singer–Hirzebruch formula for signatures of ramified coverings (he presented these results at a 1970 talk in Moscow; it may also be worth mentioning a slightly earlier talk given by Rokhlin in Moscow about his attempts to prove the Whitney conjecture using congruences modulo 16). Arnold came to his foundational discoveries through his detailed knowledge of Gudkov's habilitation thesis, in which Gudkov had completed the classification of real nonsingular plane projective curves of degree 6, and his awareness of both Gudkov's conjecture and Rokhlin's results. (Arnold himself, in the abstract of his talk given at the Moscow Mathematical Society meeting of April 6, 1971, mentioned both Gudkov's authorship in stating the conjecture and the role of Rokhlin's results on signatures of ramified coverings.)

* A *maximal curve* (also called an *M-curve*) is a smooth real algebraic curve whose number of real components is equal to $g + 1$, where g is the genus of the curve.

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As to the history of Gudkov's conjecture, Rokhlin told us, and Gudkov confirmed, the following story (letters corroborating this story were found by G. M. Polotovsky in Gudkov's personal archive). During the preparation of his habilitation thesis, Gudkov showed a preliminary version of part of it to V. V. Morozov, a professor at Kazan University. This version contained detailed proofs of a complete classification of smooth degree-6 curves, which, as was discovered later, contained a few errors at this stage. Morosov (who later was one of the official referees of Gudkov's habilitation) pointed out to Gudkov a strange—and doubtful in Morosov's opinion—asymmetry in a diagram expressing the classification obtained. It is in rectifying this asymmetry that Gudkov came to the correct classification and thence to his conjecture.

The first traces of the conjecture (in its complete form) are found in a letter of Gudkov to Morosov (winter 1969–1970, Gudkov's archive, communicated by Polotovsky), in the verbatim record of the habilitation defense (1970, Gudkov's archive, communicated by Polotovsky), and in his paper "Construction of New Series of *M*-Curves," *Doklady AN SSSR 200* (1971), no. 6, 1269–1272.

Additional documentary confirmation that Gudkov was the author of the conjecture for *M*-curves is a letter from Gudkov to Arnold dated October 15, 1972 (Gudkov's personal archive, communicated by Polotovsky), in which Gudkov proposes an additional conjecture and writes to Arnold that "*I already had this in mind when I formulated the conjecture for M-curves, but perfidiously concealed it from you and Rokhlin.*"

Let us conclude by observing that as Arnold wrote in "On the Arrangements of Ovals of Real Plane Algebraic Curves, Involutions of Four-Dimensional Smooth Manifolds, and the Arithmetic of Integer Quadratic Forms," that paper would not exist if Gudkov had not communicated his conjecture to Arnold.

Notes

1. Questions concerning the topology of real algebraic varieties form the first part of Hilbert's sixteenth problem. Below we present an English translation of the text of this part. (Hilbert's lecture was translated by Mary Winston Newson for the *Bulletin of the American Mathematical Society* 8 (1902), 437–479.) It is exactly this part, under the title "Problem der Topologie algebraischer Curven und Flächen," that was presented by Hilbert in 1900 as the sixteenth problem in his renowned lecture at the second International Congress of Mathematicians. For the lecture he selected 10 problems. The famous list of 23 problems appeared in its final form only later and was published in 1901 in *Archiv der Mathematik und Physik*:

> The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack (*Mathematische Annalen* 10). There arises the further question as to the relative position of the branches in the plane.
>
> As to curves of the 6th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely. A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4th order in three-dimensional space can really have. (cf. Rohn, "Flächen vierter Ordnung," *Preisschriften der Fürstlich Jablonowskischen Gesellschaft*, Leipzig 1886).

Hilbert's questions on curves of degree 6 in the real projective plane and surfaces of degree 4 in three-dimensional real projective space have found their complete answers through the work of D. A. Gudkov, V. I. Arnold, V. A. Rokhlin, and V. M. Kharlamov in the breakthrough of 1969–1976; see `http://www.pdmi.ras.ru/~olegviro/H16-e.pdf` for a detailed discussion.

2. In principle, one could add to this list some other real forms of curves described by a second-degree equation, namely pairs of intersecting complex conjugate imaginary lines (such as the pair given by the equation $x^2 + y^2 = 0$), pairs of parallel complex conjugate imaginary lines (such as the pair given by the equation $x^2 + 1 = 0$), and empty "ellipses" (such as the one given by the equation $x^2 + y^2 + 1 = 0$). A possible reason not to include them in the list is the fact that they do not represent a curve in the usual purely real sense, but either a point or the empty set.

3. Arnold is not very precise in this comment. Perhaps he is intentionally forcing his readers to think further and recall that every point that can be constructed with straightedge and compass can be also constructed with compass alone.

4. Presumably, Arnold has here in mind the following beautiful lines from Goethe's poem "Gott, Gemüt und Welt" (God, Soul, and World): "Willst du ins Unendliche schreiten; Geh nur im Endlichen nach allen Seiten." This can be translated literally as "If you would step into the infinite, you have only to walk in the finite in all directions." Or more poetically, "If to the Infinite you want to stride, Just walk in the Finite to every side."

5. A *semicubic cusp* (or an *ordinary cusp*) is an isolated singularity that in appropriate local coordinates is given by the equation $y^2 = x^3$.

6. Even if the precise meaning of such a philosophical statement is unclear, it may push us to think about a kind of asymptotic real algebraic geometry. Such a field does not exist yet, but see V. M. Kharlamov, S. Yu. Orevkov, "The Number of Trees Half of Whose Vertices Are Leaves and Asymptotic Enumeration of Plane Real Algebraic Curves," *J.*

Combin. Theory Ser. A 1051 (2004), no. 1, 127–142, where it is shown that the only known asymptotically significant results on the topology of real plane algebraic curves are some consequences of Bézout's theorem and certain Arnold inequalities (and not the Gudkov congruence, contrary to the impression given by Arnold's remarks on p. 45).

7. The discovery of the Möbius strip is usually cited as originating with a paper by Möbius on another subject: "On the Determination of the Volume of a Polyhedron."

Drawings representing such a shape date back to ancient times; it is for, example, evident in the famous Ouroboros drawing from the early alchemical text the *Chrysopoeia of Cleopatra* from the second century.

8. If by Möbius's assertion one understands the statement that for a smooth projective curve obtained from a projective line by a deformation in the class of embedded circles, the number of inflection points is odd and greater than or equal to 3, then such a statement is proved, for example, by S. Sasaki in *Tôhoku Math. J.* 2 (1957), no. 9, 113–117 (another proof and a bit more geometric information can be found in R. Pignoni, *Manuscripta Math* 72 (1991) 223–249). Indeed, even in a stronger form, such an inflection property holds for every embedded smooth curve that is not contractible to a point. In all these statements, it is more convenient to define an inflection point as a point where locally the curve goes from one side of the tangent line to the other.

On the other hand, there is another question, which was raised by Arnold himself and which is still open. Namely, Arnold conjectured a statement that reinforces Möbius's theorem. It is based on the notion of *dangerous self-tangencies* of an immersed circle. Following Arnold, a self-tangency is called dangerous if an orientation of the circle induces, by means of the tangent branches, the same direction on the tangent at the tangency point. The conjecture states that every projective curve obtained from a projective line by deformation in the class of immersed circles without dangerous self-tangencies has at least three inflection points. Some partial results in this direction can be found in D. Panov, *Funct. Analysis and Its Appl.* 32 (1998), no. 1, 23–31.

9. In fact, in the same volume of *Mathematische Annalen* that contains the paper by A. Harnack, there appeared a paper by F. Klein (at that time editor of *Mathematische Annalen*). Harnack's paper was submitted in January, and Klein's in April. Formally, Klein's paper concerns a related but different topic. Klein, recognizing that the topological technique from his paper allowed him to give another proof of Harnack's bound for the number of connected components of a real algebraic curve by $g + 1$ (this gives half of Harnack's theorem; the other half is the realizability of this bound by plane curves in any degree), inserted such a proof as an additional section.

Let us observe also that Harnack's proof of the inequality is different. He never explicitly manipulates the Riemann surface of the curve, using instead Bézout's theorem. As to the genus, it appears in Harnack's proof as the standard expression $(d-1)(d-2)/2$ diminished by the number of nodes and cusps (Harnack does not consider more complicated singularities).

10. This discussion is in fact postponed to Chapter 5.

11. Neither Newton nor Descartes carried out or even intended a "full" investigation of real plane curves of degree 4. Such an investigation was launched by A. Cayley and continued, in what concerns at least the nonsingular quartics, by F. Klein and H.-G. Zeuthen. Stratifications (of the same type as Newton's stratifications in the case of degree 3) of the space of real plane curves of degree 4 were produced in a series of papers by D. A. Gudkov and his collaborators; see references and further information in A. B. Korchagin and D. A. Weinberg, *Rocky Mountain J. Math.* 32 (2002), no. 1, 255–347.

12. This is probably a misprint, and Arnold meant the opposite statement. The topology of nonsingular real curves of degree 5 and 7 is somehow less complicated than that of curves of degree 6. It seems to be a general rule: the real curves of degree $2k+1$ are, in a sense, not more complicated than the real curves of degree $2k$.

13. This sentence is a bit confusing. The situation was in fact as follows. It is in his PhD thesis that Gudkov "proved" Hilbert's statement on the arrangements of 11 ovals of real

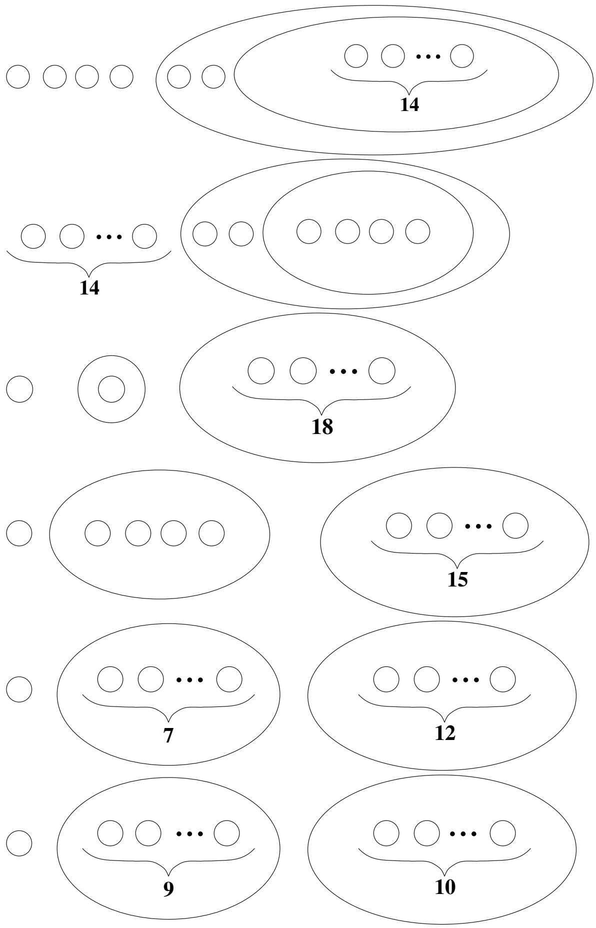
sextics. A little while later, he found a mistake in the proof, and after correcting the mistake, completed the classification of all possible arrangements of nonsingular sextics (even with any number of ovals). This correct classification (which refutes Hilbert's theorem) became the core of Gudkov's habilitation thesis.

14. Gudkov's result on M-sextics excludes only a small part of these billion possibilities. A majority is in fact excluded by Bézout's-theorem arguments. For curves of higher degrees as well, Bézout's-theorem arguments forbid many more oval arrangements than Gudkov's congruence (cf., endnote 6).

15. Arnold's discussion of Gudkov's conjecture may give the impression that it was Arnold and not Gudkov who conjectured the congruence. We refer the reader to the comments on Gudkov's conjecture at the end of this book, where we present a short review that refutes such an attribution of the conjecture.

16. Rokhlin's proof is based on different tools, but it uses the same 4-manifold as in Arnold's proof.

17. At present, 83 configurations have been realized, and 6 remain uncertain; see S. Yu. Orevkov, *Funct. Analysis and Appl.* 36 (2002), 247–249, and *Geometric and Functional Analysis* 12 (2002), no. 4, 723–755. The questionable arrangements are shown in the following figures:



18. The expression "the numbers of points of intersection with lines" means the restrictions that arise from intersecting the curves with lines (as explained a few pages above in the case of quartics), as well as with conics.
19. In what follows, this definition is understood in such a way that it is necessary to assume, in addition, that $g_0 = h_0 = \mathrm{id}$.
20. In an equivalent form, that is, regarding $K(n)$ as the number of up–down sequences, this theorem is an old result of D. André ("Développement de sec x et de tan x," *C. R. Acad. Sci. Paris* 88 (1879), 965–967).
21. Additional information and references can be found in J. Millar, N. J. A. Sloane, and N. E. Young, *J. Combinatorial Theory, Series A* 76 (1996), 44–54.
22. The lower bound given by Ortiz-Rodriguez was improved by 2 in E. Brugallé and B. Bertrand, *C. R. Math. Acad. Sci. Paris* 348 (2010), no. 5–6, 287–289; compare the next note.
23. This lower bound, similar to that on p. 52, was also improved by 2 in E. Brugallé and B. Bertrand, *C. R. Math. Acad. Sci. Paris* 348 (2010), no. 5–6, 287–289.
24. In fact, it has already been established above that "complex projective circles" are homeomorphic (and even diffeomorphic) to a 2-dimensional sphere, which gives the same result for complex projective lines, since they are diffeomorphic to complex circles. Arnold gives below an independent and more straightforward explanation.
25. This is an illustration of the "Arnold principle": *If a mathematical notion bears a personal name, then that name is not the name of the discoverer.*
26. More information on these problems, as well as principal proofs (based on the Picard–Lefschetz monodromy theory), can be found in V. A. Vassiliev, *Ramified Integrals, Singularities, and Lacunas*, Kluwer Academic Publishers, 1995.
27. In the original, Arnold requires the inequality $n \geq r(r-1)/2$, but in fact, he uses $n \geq r(r+1)/2$.
28. Here the proof is slightly corrected in order to cover all possible situations, while in the original, Arnold presents only one situation.
29. In the original, Arnold writes $k = n - r$, but we have written it as $r = n - k$ to stress the fact that this formula actually defines the parameter r.
30. R. Cordovil showed in "Sur l'evaluation $t(M;2,0)$ du polynôme de Tutte d'un matroïde et une conjecture de B. Grünbaum relative aux arrangements de droites du plan," *European J. Combin.* 1 (1980), 317–322, that 9 lines cannot divide the projective plane into 23 regions. A complete answer to the general problem was discovered by N. Martinov: "Classification of Arrangements by the Number of Their Cells," *Discrete Comput. Geom.* 9 (1993), 39–46. An induction step in his proof is apparently questionable. For a new proof, see I. Shnurnikov, "Into How Many Regions Do n Lines Divide the Plane If at Most k of Them Are Concurrent?" *Moscow Univ. Math. Bull.* 65 (2010), no. 5, 208–212.

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